

# Tomography-free quantum diagnostics of dissipative quantum chaos

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## Research Article

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**Posted Date:** October 28th, 2026

**DOI:** <https://doi.org/10.21203/rs.3.rs-11442426/v1>

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**Additional Declarations:** The authors declare no competing interests.

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# Tomography-free quantum diagnostics of dissipative quantum chaos

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**Abstract.** Non-Hermitian generators encode both complex spectra and non-orthogonal eigenvectors, but strong non-normality alone does not distinguish chaotic from integrable dynamics. We study the propagator-norm form factor  $\Phi(t; \theta) = D^{-1} \|e^{te^{i\theta}L}\|_F^2$  and its excess over a baseline determined only by eigenvalues. A trace expansion relates the short-time excess to Henrici’s departure from normality. In the large-dimension limit, we derive the Ginibre curve  $X/\Phi_0 = tI_0(2t)/I_1(2t) - 1$  and an elliptic-family extension with asymptotically linear growth. A planar-order approximation connects the latter to finite-size non-Hermitian Sachdev–Ye–Kitaev models. Classical numerical benchmarks show that profile shape complements excess magnitude in distinguishing selected chaotic and integrable ensembles. A separately calibrated zero-angle rule gives 38 correct assignments and seven abstentions on 45 fresh instances, with no wrong-class assignments within the tested scope. Structured locally dissipative chains expose a limitation: spectral-spacing signatures can separate while the profile slopes do not. We relate  $\Phi$  to the normalized acceptance probability of a block-encoded propagator and, for physical channels at  $\theta = 0$ , to Choi-state purity. These identities provide tomography-free measurement routes for  $\Phi$ ; classification additionally requires an accurate spectral baseline. We distinguish the analytic results and classical sampling tests from the unimplemented gate-level synthesis and unresolved large-system baseline reconstruction. The resulting framework connects non-normal dynamics, random-matrix theory and quantum measurement without identifying non-normality with chaos.

*Keywords:* dissipative quantum chaos, non-Hermitian random matrices, block encoding, quantum eigenvalue transformation, Chalker–Mehlig overlaps, Choi purity, DQC1

Submitted to: *Physica Scripta*

## 1. Introduction

Whether a many-body quantum system is chaotic or integrable determines its ability to thermalize, scramble quantum information, and resist control. For isolated systems this dichotomy is diagnosed by now-canonical spectral signatures—level repulsion, random-matrix spacing statistics, and the dip-ramp-plateau of the spectral form factor (SFF). The past few years have extended this program to open quantum systems, whose dynamics are generated by non-Hermitian operators: effective Hamiltonians of monitored or lossy systems, and Liouvillians of Markovian master equations [1, 2, 3]. In this dissipative arena, chaos manifests in the statistics of *complex* eigenvalues—complex spacing ratios (CSR) with their characteristic “donut” distribution [4], the quadratic ramp of the dissipative spectral form factor (DSFF) [5], both established across random Lindbladians [6, 7] and non-Hermitian many-body models [8]—and, distinctively, in the *non-orthogonality of eigenvectors*, quantified by Chalker–Mehlig (CM) overlaps [9, 10, 11, 12], which has itself become an established signature of dissipative chaos with direct consequences for relaxation and entropy production [1, 13].

Experimentally, these signatures were detected for the first time only recently: complex spacing ratios of engineered dissipative circuits were measured on a superconducting processor, certifying both dissipative chaos and integrability, and revealing that intrinsic hardware noise is itself a dissipatively chaotic process [14]. Yet that landmark experiment obtained the spectrum by gradient-based *process tomography* of a four-qubit map followed by classical diagonalization. Tomography of an  $n$ -qubit channel requires  $O(16^n)$  parameters [15], and diagonalizing the reconstructed  $4^n \times 4^n$  map costs  $O(64^n)$  classical operations; neither step scales. On the algorithmic side, the quantum singular value transformation (QSVT) [16, 17] unified polynomial processing of *singular values*, but singular values of a non-normal generator have been shown to be uninformative about its chaos: singular-value statistics of non-Hermitian Sachdev–Ye–Kitaev (SYK) models fail to distinguish chaotic from integrable dynamics, in direct conflict with CSR analysis [18, 19]. The recently developed quantum eigenvalue transformation (QEVT) processes complex *eigenvalues* of non-normal matrices through Faber-polynomial expansions [20, 21, 22], but has not been connected to chaos diagnostics. However, the core obstruction is more basic: fine-grained spectral statistics such as CSR live at the mean level spacing, a scale that shrinks as  $D^{-1/2}$  with the Liouville dimension  $D$ , which motivates examining what can be inferred from inverse-polynomial-precision trace observables. The barrier discussed below is restricted to its stated access and spectral assumptions, rather than all structured-system protocols [23, 24].

In this study, we turn the eigenvector signature of dissipative chaos into an exactly solvable and directly measurable quantity. Here we introduce the propagator-norm form factor

$$\Phi(t; \theta) = \frac{1}{D} \|e^{te^{i\theta}L}\|_F^2, \quad (1)$$

where  $L$  is the generator and  $\theta$  rotates the complex plane [Fig. 1(a)], and its *overlap*

excess  $X(t; \theta) = \Phi - \Phi_0$  over the normal spectral baseline  $\Phi_0 = D^{-1} \sum_k e^{2t \operatorname{Re}(e^{i\theta} \lambda_k)}$ . Three analytic results organize the discussion: the short-time Henrici law (Lemma 1), the Ginibre planar-limit curve and its asymptotically linear ramp (Theorem 2), and the elliptic-family solution (Theorem 3). A planar-order approximation for finite-size non-Hermitian SYK reproduces the tested profiles to 3–5%; this comparison is not a controlled large- $N$  universality proof. We then ask whether profile shape, together with magnitude and angular dependence, distinguishes benchmark ensembles labelled by complex spacing ratios. A separately frozen  $\theta = 0$  rule gives 38 correct assignments, seven abstentions and no wrong-class assignments on 45 fresh instances, including three families absent from calibration. This finite test does not establish a universal classifier. In particular, locally dissipative chains exhibit a separation in spectral-spacing statistics without separation of the profile slopes, defining an important limit of the method.

The measurement interpretation has three complementary forms. (i) If a unitary block-encodes  $M/\beta$ , where  $M = e^{te^{i\theta}L}$  and  $\beta \geq \|M\|$ , its flag accepts on the maximally mixed input with probability  $p_{\text{acc}} = \Phi/\beta^2$ . The measured probability is therefore proportional to, rather than identical to,  $\Phi$ . For  $\theta = 0$  and a physical Lindbladian,  $\Phi = \operatorname{Tr}[J(\Lambda_t)^2]$  is the normalized Choi-state purity, accessible in principle through a two-copy SWAP test. Rotated generators are generally not physical channels; coherent synthesis provides one route to their propagators. (ii) The biorthogonal decomposition weights the evolution factors by CM overlaps. Its vector-norm counterpart already appears in Chalker–Mehlig’s treatment of transient evolution [9]; the present contribution combines the normalized trace, baseline subtraction, random-matrix formulas and diagnostic tests. (iii) The tested standardized profiles have order-unity excess ratios. Their statistical accessibility nevertheless depends on the acceptance probability, the synthesis cost and the accuracy of  $\Phi_0$ . Our numerical measurements are classical probability-level sampling tests, not a gate-level implementation or hardware experiment.

## 2. The estimable class and the propagator-norm form factor

*Setup.*—Let  $L$  act on a  $D$ -dimensional space ( $D = 2^n$  for effective Hamiltonians,  $4^n$  for vectorized Liouvillians); for the biorthogonal decomposition below we take  $L$  diagonalizable with eigenvalues  $\lambda_k$  and biorthogonal eigenvectors  $L|R_k\rangle = \lambda_k|R_k\rangle$ ,  $L^\dagger|L_k\rangle = \bar{\lambda}_k|L_k\rangle$ ,  $\langle L_j|R_k\rangle = \delta_{jk}$ . We assume block-encoded access to  $L$  with normalization  $\alpha = O(\text{poly}(n))\|L\|$  (Pauli-LCU [25, 26] for local  $H_{\text{eff}}$ ; LCU assembly from jump operators for Liouvillians; Appendix A). Throughout,  $t$  is measured in units of the inverse spectral radius.

**Theorem 1** (estimable class; informal). *Suppose normalized block encodings of  $f(L)/\beta_f$  and  $g(L^\dagger)/\beta_g$  are available to the required operator accuracy, with implementation costs  $Q_f$  and  $Q_g$  queries to the input oracle. Then  $T = D^{-1}\operatorname{Tr}[f(L)g(L^\dagger)]$  can be estimated to additive error  $\varepsilon\beta_f\beta_g$  with bounded failure probability using  $O((Q_f + Q_g)/\varepsilon)$  coherent input-oracle queries, up to confidence amplification. Efficiency requires polynomial implementation costs, normalizations and inverse target precision; analyticity alone does*

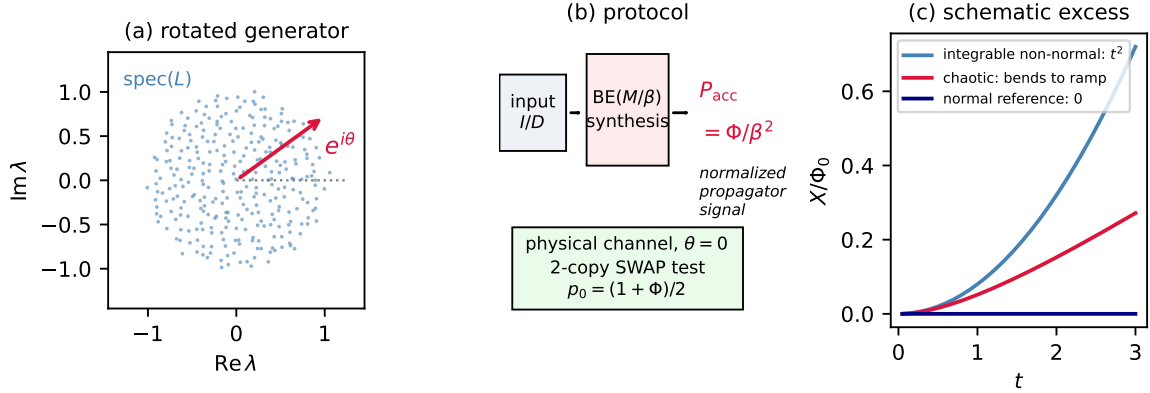


**Figure 1. A normalized acceptance probability provides access to the propagator norm.** (a) Complex spectrum and rotation angle  $\theta$ . (b) A block encoding of  $M/\beta$ , with  $M = e^{te^{i\theta}L}$ , accepts on  $I/D$  with probability  $p_{\text{acc}} = \Phi/\beta^2$ . For a physical channel at  $\theta = 0$ ,  $\Phi$  is instead its normalized Choi-state purity; the SWAP-test outcome obeys  $p_{\text{swap}}(0) = (1+\Phi)/2$ . These are measurement identities, not an implemented gate sequence. (c) Schematic excess profiles illustrate the short-time  $t^2$  law and subsequent shape differences in the tested families; the curves are not a universal classification boundary.

not guarantee these conditions.

The construction composes the two block encodings and uses normalized-trace estimation [27, 28, 16]. Analytic matrix functions can be supplied through Faber-QEVT under its approximation and access assumptions [20]; LCHS and related constructions provide alternatives for suitable propagators [29, 30, 31, 32]. The synthesis degree, coefficient preparation, subnormalization and approximation error remain part of the cost. For exponentials the logarithmic-norm bound  $\|e^{te^{i\theta}L}\| \leq e^{t\omega_\theta}$  controls operator growth [33]; it does not alone bound every overhead of a particular synthesis. Appendix A separates the general trace-estimation identity from these implementation assumptions.

Setting  $f(x) = e^{te^{i\theta}x}$  and  $g = \bar{f}$  yields  $\Phi(t; \theta)$ . To make the normalization explicit, let  $M = e^{te^{i\theta}L}$  and let  $\beta$  be the actual subnormalization supplied by its block encoding. Then

$$p_{\text{acc}} = \frac{\|M\|_F^2}{D\beta^2} = \frac{\Phi(t; \theta)}{\beta^2}, \quad \beta \geq \|M\|. \quad (2)$$

The logarithmic-norm bound  $\|M\| \leq e^{t\omega_\theta}$  supplies an operator-norm upper bound, but an implemented synthesis can have additional normalization overhead. The bare  $\Phi$  need not lie in  $[0, 1]$ . For  $\theta = 0$  and a completely positive trace-preserving channel, it equals  $\text{Tr}[J^2] = \|\Lambda_t\|_F^2/d^2$ , where  $D = d^2$ , and is bounded by one. A common scalar shift rescales  $\Phi$  and  $\Phi_0$  equally and leaves  $X/\Phi_0$  invariant. Thus the exact spectral-abscissa shift used for classical plotting is not necessary operationally: a certified norm bound can be used, at a possibly larger sampling cost. A variational estimate of the numerical abscissa alone is not a certified upper bound. The CM-weighted spectral sum

follows from the biorthogonal decomposition, with  $O_{kk} \geq 1$  and equality for normal generators [9, 12].

**Lemma 1** (short-time law). *For any square matrix  $L$  and any  $\theta$ ,*

$$X(t; \theta) = \frac{t^2}{D} \left( \|L\|_F^2 - \sum_k |\lambda_k|^2 \right) + O(t^3), \quad (3)$$

*i.e., the excess launches quadratically with a  $\theta$ -independent coefficient equal to Henrici's departure from normality [34] (squared, Frobenius); anisotropy enters only at higher order—at  $O(t^3)$  for a fixed matrix, and at  $O(t^4)$  after averaging in centrally symmetric ensembles (Appendix C.1). Diagonalizability is not required: the proof is a trace expansion, and the identity holds verbatim for defective  $L$  with eigenvalues counted by algebraic multiplicity (verified numerically on Jordan-block matrices to  $10^{-3}$ -order  $t$  corrections).*

(Proof: Appendix C.1; verified numerically to 4 digits across all five model families and two angles.) This anchors the short-time end of the profile analytically. For the Ginibre class we can go further and derive the entire curve:

**Theorem 2** (Ginibre universal curve). *For the complex Ginibre ensemble normalized to unit spectral radius, in the large- $D$  limit and for every  $\theta$ ,*

$$\frac{X(t; \theta)}{\Phi_0(t; \theta)} = \frac{t I_0(2t)}{I_1(2t)} - 1 \xrightarrow{t \gg 1} t - \frac{3}{4} + O(1/t), \quad (4)$$

*where  $I_\nu$  are modified Bessel functions: the late-time growth is asymptotically linear after taking the large- $D$  limit.*

*Proof sketch.*—Phase invariance  $G \rightarrow e^{i\varphi} G$  kills all mixed moments  $\text{Tr}[G^m (G^\dagger)^n]$  with  $m \neq n$  (and makes the curve  $\theta$ -independent); for  $m = n$ , Wick contractions of the word  $G^m (G^\dagger)^m$  admit exactly *one* non-crossing pairing (the fully nested one), so  $\lim_D \frac{1}{D} \mathbb{E} \text{Tr}[G^m (G^\dagger)^m] = 1$  and  $\Phi(t) = \sum_m t^{2m} / (m!)^2 = I_0(2t)$ ; the circular law gives  $\Phi_0(t) = \frac{1}{\pi} \int_{|z| \leq 1} e^{2t \text{Re } z} d^2 z = I_1(2t)/t$  (full stepwise proof with sources: Appendix C.2). The measured GinUE profiles agree with Theorem 2 to 2.9–3.9% at  $D = 2048$ , with deviations decreasing in  $D$  [Fig. 2(a)], and the theorem explains the observed fit-window slope: the log-log slope of  $t I_0(2t)/I_1(2t) - 1$  on  $t \in [2.0, 3]$  is 1.364, matching the measured  $s^* = 1.33$ –1.38 en route to the asymptotic ramp value 1. For random Liouvillians the bending toward the same ramp remains a numerically established conjecture; for the non-Hermitian SYK model it is suggested by the finite-size planar approximation following Theorem 3, without a uniform bound on the omitted contractions—and Theorem 2 fixes the universality-class anchor.

**Theorem 3** (elliptic interpolation). *For the elliptic Ginibre ensemble with Hermiticity parameter  $\tau \in [0, 1]$  (GinUE at  $\tau = 0$ , Hermitian at  $\tau = 1$ ), the planar-limit mixed moments are weighted non-crossing pairing sums,  $M_{mn}(\tau) = \sum_{\pi \in NC_2} \tau^{\#\{\text{same-type pairs}\}}$ ,*

with closed-form ordinary generating function

$$\sum_{m,n} M_{mn} x^m y^n = \frac{A(y)}{1 - \tau x^2 A(x) - xy A(y)},$$

$$A(v) = \frac{1 - \sqrt{1 - 4\tau v^2}}{2\tau v^2}, \quad (5)$$

whose double-Borel (Schwinger–Dyson) resummation is exact and closed:

$$\Phi_\tau(t; \theta) = \frac{1}{t^2} \sum_{k=1}^{\infty} \frac{k^2}{\tau^k} |I_k(2\sqrt{\tau} t e^{i\theta})|^2, \quad (6)$$

$$\Phi_0(t; \theta) = \frac{I_1(2b)}{b}, \quad b = t\sqrt{1 + 2\tau \cos 2\theta + \tau^2},$$

reducing to Theorem 2 at  $\tau = 0$  and collapsing to  $X \equiv 0$  at  $\tau = 1$ ; the late-time excess is an asymptotically linear ramp for each fixed  $0 \leq \tau < 1$ ,

$$\left. \frac{X}{\Phi_0} \right|_{\theta=0} \xrightarrow{t \rightarrow \infty} (1 - \tau)^2 (1 + \tau) t + O(1), \quad 0 \leq \tau < 1, \quad (7)$$

and the short-time anisotropy is

$$X(t; \theta) - X(t; \frac{\pi}{2} + \theta) \Big|_{\text{aniso}} = \frac{2}{3} \tau (1 - \tau^2) \cos(2\theta) t^4 + O(t^6). \quad (8)$$

(Derivation and validation in [Appendix C](#): the resummation (6) follows from a free-probability Schwinger–Dyson equation whose double Laplace transform closes as  $\hat{G} = \hat{\varphi}(p)\hat{\varphi}(q)/[1 - \hat{\varphi}(p)\hat{\varphi}(q)]$ ; it reproduces the mixed moments coefficient-by-coefficient ( $m, n \leq 8$ , exact), the direct series to  $10^{-5}$ , the Hermitian limit  $I_1(4t)/2t$  to  $4 \times 10^{-4}$ , and elliptic Monte Carlo at  $\tau = 0.5$  to 0.2–0.7%; the anisotropy formula holds to 0.1% at  $t = 0.05$  and the family-ramp ratio to within 2.4% at  $t = 150$  across the tested values of  $\tau$ .) Equation (7) extends the theory anchor of the decision rule from a single point to the whole family:  $s^* \rightarrow 1$  with amplitude  $(1 - \tau)^2(1 + \tau)$  vanishing at the normal limit, so the ramp is an analytic statement for the elliptic ensemble. Its application to non-Hermitian SYK at  $q = 4$  remains the finite-size planar-order approximation described below. Theorem 3 likewise turns the anisotropy feature  $A$  from a qualitative flag into a quantitative random-matrix prediction within the elliptic family [Fig. 4(b)]. The leading-order curve is moreover  $\beta$ -universal: the real Ginibre (GinOE) mixed moments converge to the same values ( $\frac{1}{D} \text{Tr}[G^2(G^T)^2] = 0.9999$  at  $D = 1024$ ), transpose corrections entering at  $O(1/D)$ .

*SYK planar reduction.*—Theorem 3 is not merely an interpolation device: it also supplies a planar-order approximation to the tested non-Hermitian SYK models. The complex couplings  $J = \sigma(g_1 + i\kappa g_2)$  have covariance ratio  $\mathbb{E}[JJ]/\mathbb{E}[J\bar{J}] = (1 - \kappa^2)/(1 + \kappa^2) \equiv \tau_{\text{eff}}$ —precisely the elliptic Hermiticity parameter ( $\tau_{\text{eff}} = 0.724$  at our  $\kappa = 0.4$ ; the measured moment ratio  $\text{Tr}[H^2]/\text{Tr}[HH^\dagger] = 0.710\text{--}0.718$  at  $N = 16\text{--}18$ ). The deviation from free (planar) combinatorics is controlled by the Q-Hermite crossing

weight  $\eta(N, q) = \binom{N}{q}^{-1} \sum_k (-1)^k \binom{q}{k} \binom{N-q}{q-k}$  [35, 36]: for  $q = 4$ ,  $\eta = -0.04$  to  $+0.05$  over  $N = 14$ – $20$  (exact combinatorics), motivating a planar approximation to the  $q=4$  mixed moments by the elliptic family in this finite-size range. The resulting *parameter-free* prediction—the Theorem 3 curve at  $\tau_{\text{eff}}$ , with time rescaled by the semi-major axis  $1 + \tau_{\text{eff}}$ —matches the measured  $q=4$  profile to 3–5% across the full window (Appendix C). Conversely  $\eta(N, 2) = 0.47$ – $0.62$ : the  $q=2$  model is strongly non-planar, explaining why its excess stays on the  $t^2$  law rather than joining the ramp family. GinUE, the elliptic ensembles, and SYK  $q=4$  thus form a single theoretical family at planar order. We state the epistemic status precisely: this is a leading-order reduction validated parameter-free at the 3–5% level, not a moment-by-moment large- $N$  proof—the crossing weight and finite-size numerical deviations have been evaluated, but they do not constitute a uniform remainder bound for the full propagator series. At fixed  $q$ ,  $\eta(N, q)$  tends to one as  $N \rightarrow \infty$ , so its smallness over the tested sizes does not establish a planar large- $N$  limit. A complete mixed-moment derivation with error bounds remains open.

*Baseline subtraction.*—The baseline  $\Phi_0$  depends on both  $\lambda$  and  $\bar{\lambda}$  and is not an analytic trace functional of  $L$ . At fixed standardized times it is a smooth-kernel integral against the spectral density. If a density approximation has a certified Wasserstein error, Appendix B bounds the resulting baseline error. Obtaining that density with sufficient accuracy is a separate requirement, not a consequence of smoothness alone. Hermitization and kernel-polynomial methods [37] offer a classical route with work depending on  $D$ , hence generally exponential in the number of qubits. Quantum singular-value estimation on the Hermitized matrix is a possible alternative, but its complete cost must include small-singular-value regularization, density reconstruction, integration and precision requirements. We do not establish an unconditional polynomial-in- $n$  algorithm for this baseline. The present classification benchmarks use classical spectral information; our large-system implementation did not attain the required late-window accuracy. Thus tomography-free access to  $\Phi$  is distinct from an efficient end-to-end classifier.

**Observation 1** (resolution barrier). *Individual-eigenvalue statistics (CSR) and the DSFF at Heisenberg times require resolving spectral scales  $\sim D^{-1/2}$ ; no protocol restricted to polynomially many additive-precision-1/poly estimates of the class of Theorem 1 accesses them for generic spectra (non-degenerate bulk with mean spacing  $\Theta(D^{-1/2})$  and no spectral gap structure to exploit; the restricted oracle result below has additional assumptions). Within the estimable class the barrier is a theorem: for normal generators, every  $T$  of Theorem 1 is a Lipschitz functional of the spectral measure, so spectra differing only at CSR scales—Wasserstein distance  $O(D^{-1/2} \text{polylog } D)$ —change  $T$  by the same amount, and resolving them requires  $\Omega(D)$  ordinary samples (proof: Appendix D; coherent amplitude-estimation access improves precision quadratically at best, leaving  $\Omega(\sqrt{D})$  coherent queries under standard estimation lower bounds—the two access models are distinct, and both remain exponential in  $n$ ). Beyond the class, the barrier is now an oracle-model theorem: for the canonical diagonal block encoding there exist pairs of normal generators—one with GinUE complex-spacing statistics,*

one with Poisson statistics—such that any protocol making  $T$  queries (adaptive, coherent, controlled queries and inverses included) distinguishes them with advantage at most  $2c_L T \delta / \alpha$ , where  $\delta$  is the minimax matching distance of the two spectra [ $\delta = O(D^{-1/2}(\log D)^{3/4})$  for such pairs] and  $c_L \leq 1 + 1/\sqrt{3}$ ; constant advantage therefore costs  $T = \Omega(\sqrt{D}/(\log D)^{3/4}) = \Omega(2^{n/2}/\text{poly } n)$  queries (hybrid argument [38]; proof and the matching construction: [Appendix D](#)). Two scope remarks keep this honest: the statement is worst-case over pairs—no universal “all  $W_1$ -close pairs” version can hold, since a single eigenvalue displaced by  $O(1)$  moves  $W_1$  by only  $O(1/D)$  yet is detectable by leading-eigenvalue estimation—and it is tied to the stated oracle; protocols exploiting additional structure of  $L$  beyond block-encoding access are not covered.

The stated oracle-model barrier is not a claim about every structured many-body problem or every SFF measurement scheme. Survival-probability filters [23] trade it for system-size restrictions. Our diagnostic therefore does not *reconstruct* CSR; it accesses the complementary, eigenvector-borne signature of dissipative chaos at  $O(1)$  observable scales.

*Relation to prior probes.*—Singular-value statistics of  $L$  do not distinguish the non-Hermitian models studied in Ref. [18];  $\Phi$  instead sums the squared singular values of the propagator  $e^{tL}$ , an object that entangles spectrum and eigenvectors, and the baseline subtraction isolates the latter. Unitarity randomized benchmarking estimates a quantity affinely related to the Choi purity of a *fixed physical channel* [39, 40] (the two coincide up to nonunital/trace-decreasing corrections)—our  $\theta = 0$  cross-section; the  $\theta$ -resolved family corresponds to non-CPTP generators and is reachable only through coherent synthesis. In the present classification the  $\theta$  grid supplies the isotropy feature  $A$  and the calibrated magnitude  $\chi$  (defined as a maximum over the grid), while the primary shape feature  $s^*$  lives on the near-term  $\theta = 0$  cross-section. A  $\theta = 0$ -only deployment is therefore *not* the calibrated three-feature rule, and we evaluated it as its own pre-registered protocol: freezing a  $\theta = 0$ -only rule from the existing aggregates *before* any evaluation run ( $\chi_{\theta=0} < 10^{-10}$  normal-integrable;  $\chi_{\theta=0} \geq 2$  chaotic; otherwise  $s^*$  with the standard cuts and abstention band) and applying it to 45 fresh-seed instances across nine families—three of them never used in any calibration (GinOE, elliptic  $\tau = 0.15$ , Gaussian-spectrum normal matrices)—yields *zero wrong-class assignments* (38 correct, 7 abstentions; per-instance table and script in the code package). The coverage cost lands exactly where the frozen aggregates predicted: random Liouvillians, whose magnitude rescue is  $\theta$ -grid-borne ( $\chi = 3.8 \pm 0.9$  falls to  $\chi_{\theta=0} = 0.79 \pm 0.10$ ), abstain 5/5, and  $q=2$  SYK coverage drops to 3/5, while a separate five-instance quasi-1D Hatano–Nelson stress test is excluded from that denominator: four instances abstain and one receives a binary call despite being outside the intended scope. Scope detection is therefore not guaranteed. The fault-tolerant  $\theta$ -synthesis is thus what buys *coverage* on anisotropic ensembles, not correctness, and remains a principled extension rather than a prerequisite for the shape diagnosis itself. Bi-Lanczos Krylov-complexity probes [41] access related eigenvector-sensitive information through classical recursion on moments; our observable can be viewed as their measurement-native counterpart, replacing ill-

conditioned classical three-term recurrences by direct propagator synthesis.

### 3. Chaos–integrability classification

*Decision rule.*—From the estimated profile  $X/\Phi_0$  on a  $(t, \theta)$  grid with  $t \in [0.25, 3]$ , we extract three features: the magnitude  $\chi = \max_{t, \theta} X/\Phi_0$ ; the late-time log-log slope  $s^*$  fitted on  $t \in [2.0, 3]$  at  $\theta = 0$ ; and the angular anisotropy  $A = \max_{\theta} X_{\text{late}} / \min_{\theta} X_{\text{late}}$ . The rule is: (1)  $\chi \approx 0$  (below estimation precision)  $\rightarrow$  integrable with normal generator; (2) otherwise classify in the  $(\chi, s^*)$  plane: chaotic dynamics populate either small late-time power ( $s^* \lesssim 1.3$ ; the excess bends from the universal short-time Henrici growth  $X \propto t^2$  toward a linear ramp) or large magnitude ( $\chi \gtrsim 2$ , RMT-class), while integrable non-normal dynamics cluster at  $s^* \gtrsim 1.6$  and  $\chi \lesssim 1$ —neither feature alone suffices (random-Liouvillian seeds reach  $s^* = 1.44$ , overlapping the single-threshold boundary, but carry  $\chi \geq 2.9$ ); (3)  $A \approx 1$  flags Ginibre-class isotropy and robustly overrides marginal calls. Of these thresholds only the chaotic ramp target  $s^* \rightarrow 1$  is theory-anchored (Theorems 2–3); the working values  $s_c$ , the abstention gap, and the  $\chi$  cut are empirically calibrated heuristics, and we present them as such. Ensembles with genuinely two-dimensional bulk spectra are the intended scope; quasi-1D spectra (e.g., disordered Hatano–Nelson chains [42]) are flagged by feature (3) and the  $D$ -trend of  $\chi$  (below). On the Liouvillian side our positive evidence is random-matrix-structured (random versus dephasing Lindbladians), and we characterized *structurally* why common physical families evade both CSR and our diagnostic at accessible sizes: (i) diagonal-dephasing chains (including the Bethe-solvable XX-dephasing model [43] and its chaotic XXZ partner) organize their spectra into discrete decay bands and keep the excess on the coherent  $t^2$  law for *both* members ( $s^* \approx 1.9$ – $2.0$ ); (ii) number-conserving chains with graded loss are block-triangular in magnetization, so the Liouvillian spectrum degenerates to unions of vertically shifted Hermitian-difference stripes; (iii) boundary-driven chains do not reach the dissipative-chaos regime even at  $n = 8$  ( $D = 12,870$ ;  $\langle r \rangle \leq 0.68$ , in the tested parameter sets). The fixed number of boundary jumps, compared with the growing number of bulk terms, suggests a boundary-to-bulk dilution mechanism; term counting alone does not prove a statement about spectral statistics. Dense, non-graded random Lindbladians provide the positive examples tested here; density of the jumps is not asserted to be necessary for a two-dimensional spectrum. Guided by this, replacing graded loss by mixed-charge *bulk* jumps  $K_i = \sqrt{\gamma}(\sigma_i^- + c\sigma_i^+)$  (which destroy the magnetization grading) restores genuine spectral chaos in a structured chain: the chaotic member separates cleanly from the comparison member in CSR ( $\langle r \rangle = 0.68$  versus  $0.52$  at  $n = 7$ ,  $D = 16,384$ ). Yet both members retain  $s^* \approx 2.0$  at these sizes: the spectral-spacing indicator and the profile-shape indicator disagree at the accessible sizes. This is a limitation of transferring the calibrated classifier to local dissipation, not proof of a delayed onset or of eventual convergence of the two indicators. Whether larger structured systems exhibit both signatures remains open (Sec. 5).

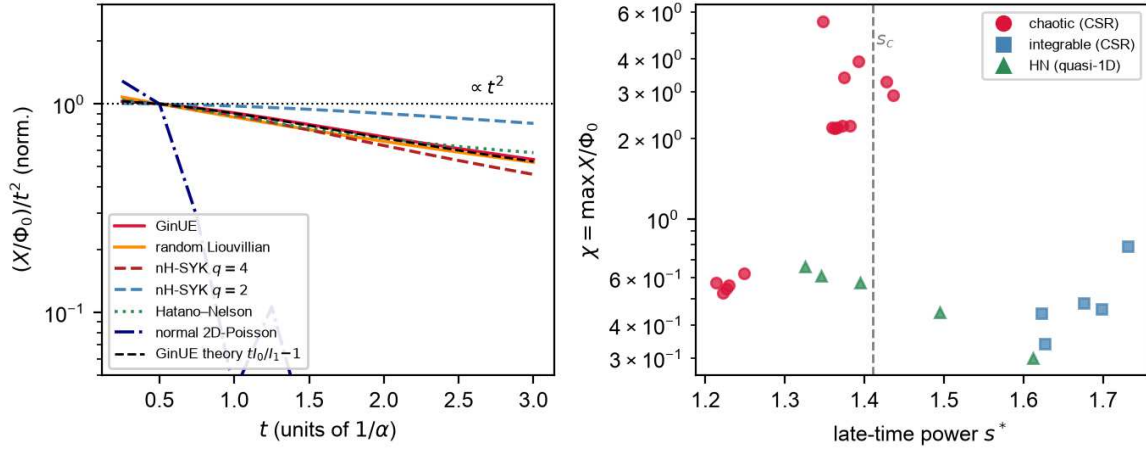


**Figure 2. The  $(\chi, s^*)$  plane separates chaotic from integrable non-normal dynamics in agreement with CSR ground truth.** (a) Compensated excess  $(X/\Phi_0)/t^2$  at  $\theta = 0$  (normalized at the second grid point): the integrable non-normal nH-SYK  $q=2$  stays on the Henrici  $t^2$  law, chaotic families (GinUE, random Liouvillian, nH-SYK  $q=4$ ) bend downward toward a linear ramp, and the normal 2D-Poisson reference drops to numerical noise. (b) All seeds with  $\chi > 10^{-10}$  in the  $(s^*, \chi)$  plane; circles: CSR-chaotic, squares: CSR-integrable, triangles: Hatano-Nelson stress test (quasi-1D spectrum, outside the CSR dichotomy). The dashed line is the SYK-family threshold  $s_c = 1.41$ .

*Numerical evidence (Fig. 2).*—Across five seeds each: Ginibre (GinUE,  $D = 1024$ ) versus normal matrices with 2D-Poisson spectra separate by *fourteen orders of magnitude* in  $\chi$  ( $2.22 \pm 0.02$  vs  $\lesssim 10^{-14}$ ); random Lindbladians versus dephasing Lindbladians ( $D = 1024$ ) reproduce this split ( $3.8 \pm 0.9$  vs  $10^{-15}$ ). For non-Hermitian SYK with  $N = 18$  Majoranas (even-parity sector,  $D = 256$ ), the chaotic  $q = 4$  model shows  $\langle r \rangle = 0.72 \pm 0.01$ ,  $-\langle \cos \theta \rangle = 0.20 \pm 0.06$  over five seeds while the  $q = 2$  CSR fluctuates around Poisson values (small-sector caveat:  $\sim 230$  bulk eigenvalues per instance); the two models are inseparable by  $\chi$  alone ( $0.57 \pm 0.03$  vs  $0.50 \pm 0.15$ )—but separate with *zero seed overlap* in  $s^*$ :  $1.23 \pm 0.01$  versus  $1.67 \pm 0.04$ , stable across  $N = 14$ – $18$  (max  $q4$  slope  $1.249 < \min q2$  slope  $1.445$  over all sizes and seeds; within-family threshold  $s_c = 1.41$ ). Ground truths are fixed by exact-diagonalization CSR with symmetry sectors resolved (fermion parity for SYK; the weak-U(1) sector  $\delta m = 0$  for Liouvillians), consistent with the benchmark values  $\langle r \rangle \approx 0.738$ ,  $-\langle \cos \theta \rangle \approx 0.241$  (GinUE) and  $2/3, 0$  (Poisson) [4] (measured  $-\langle \cos \theta \rangle = 0.21 \pm 0.02$ ,  $1.8\sigma$  below the asymptotic value at these sizes).

*Non-normality is not chaos.*—The failure of  $\chi$  alone is itself a physics result: integrable-but-non-normal systems generate order-unity overlap excess—in free-fermion ( $q=2$  SYK) systems because many-body eigenvector overlaps multiply across single-particle modes, and in skin-effect chains [42, 44, 45] because of coherent boundary amplification. The shape of the excess, not its size, carries the chaos information: chaotic classes bend to a linear ramp; coherent non-normality stays near the  $t^2$  Henrici

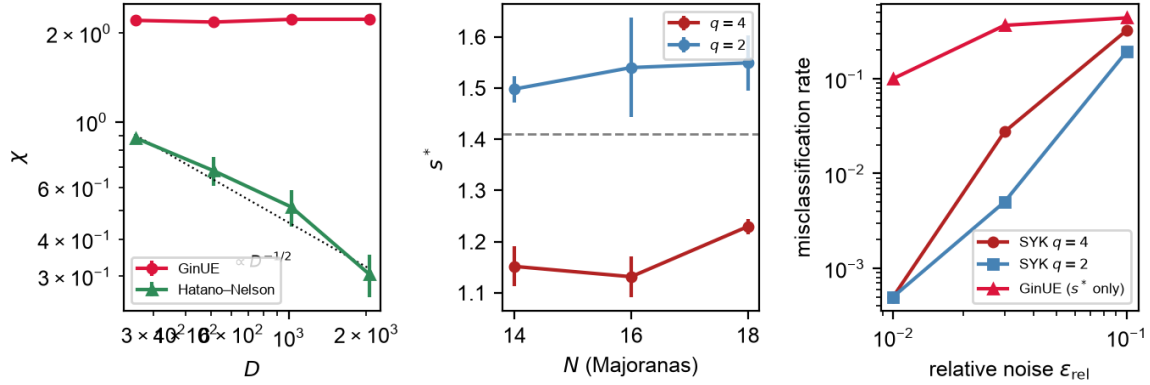


**Figure 3. Finite-size separation and sensitivity to imposed estimation noise.**

(a)  $\chi$  versus dimension: flat for GinUE, decaying  $\sim D^{-1/2}$  for the disordered Hatano–Nelson chain—the auxiliary feature that excludes quasi-1D coherent non-normality. (b)  $s^*$  versus SYK size  $N$ : the  $q = 4/q = 2$  gap persists with zero seed overlap. (c) Monte Carlo misclassification rate versus relative estimation noise: the SYK pair is error-free at  $\epsilon_{\text{rel}} = 1\%$ ; GinUE by  $s^*$  alone sits near the within-family boundary and is instead pinned by its isotropy  $A$ .

law. Disordered Hatano–Nelson chains, whose quasi-1D spectra render CSR itself ill-defined, overlap with the chaotic band in  $s^*$  but are excluded by two auxiliary features: their  $\chi$  *decays* with dimension (from 0.92 at  $D = 256$  to 0.29 at  $D = 2048$ , roughly  $D^{-1/2}$ ) while Ginibre’s stays flat at 2.2; and their anisotropy  $A = 1.1$ – $2.2$  contrasts with Ginibre’s isotropy  $A = 1.00$ – $1.04$ . We emphasize that  $A$  probes spectral shape rather than chaos per se—chaotic non-RMT models ( $q=4$  SYK, elliptic spectrum) also show  $A \approx 1.9$ , consistent with the elliptic anisotropy law of Theorem 3, which quantifies exactly this shape dependence.

*Robustness to estimation noise* (Fig. 3).—Injecting relative Gaussian error  $\epsilon_{\text{rel}}$  on every profile point (400 Monte Carlo realizations  $\times$  3 instances): the SYK pair classifies with error rates 2.8%/0.5% at  $\epsilon_{\text{rel}} = 3\%$  and 0%/0% at 1%. GinUE sits closer to the  $s^*$  boundary (margin 0.04) but is pinned by its isotropy, which as a ratio of simultaneously estimated quantities tolerates percent-level noise trivially. Widening the fit window degrades the classifier (finite-size saturation contaminates the integrable slope), so the window standardization above is part of the protocol. The rule includes a declared abstention gap  $1.30 < s^* < 1.60$  (at  $\chi < 1$ ): the chaos–integrability transition is a crossover, not a sharp boundary, and  $s^*$  drifts continuously through the gap as integrability is broken—the binary rule certifies endpoints, while the drift itself tracks the crossover. We verified this directly on the SYK interpolation  $L(\lambda) = (1 - \lambda)L_{q=2} + \lambda L_{q=4}$  [Fig. 4(a)]: the CSR ratio  $\langle r \rangle$  saturates at its Ginibre value already by  $\lambda \approx 0.1$ – $0.2$ —spectral statistics are hypersensitive to integrability breaking—whereas  $s^*$  drifts monotonically from 1.64 to 1.11 across the entire range, passing through the abstention gap precisely for  $\lambda \in [0.2, 0.5]$ :  $s^*$  meters chaos *strength* beyond the spectral onset. The anisotropy feature is grid-robust ( $A$  changes by  $< 0.1\%$

between 5-, 7-, and 9-point  $\theta$  grids for GinUE at  $D = 512$ ). To rule out circular calibration, we froze all thresholds and re-ran the full pipeline on 30 held-out instances with fresh random seeds: 29/30 were classified correctly and the single remaining  $q=2$  instance fell into the declared abstention gap—no instance was assigned to the wrong class. Extending the SYK scan to  $N = 20$  ( $D = 512$ ) preserves zero overlap with a widened gap ( $s^* = 1.30$  versus 1.77), though  $s^*$  itself drifts with  $N$ : the *relative* separation is robust while absolute thresholds are size-calibrated. An out-of-family test (frozen rule, ensembles never used in calibration) yields no wrong-class assignment either: Gaussian-density normal matrices classify correctly (3/3), while elliptic Ginibre at  $\tau = 0.3$  abstains (3/3;  $\chi \approx 1.2$ ,  $s^* \approx 1.33$  fall between the calibrated cuts)—the safe failure mode by design, and a concrete motivation for upgrading the threshold rule to direct fits of the Theorem 3 family curves. Beyond flat relative noise, we tested a classical error-budget model combining binomial sampling with imposed truncation, baseline-resolution and stochastic-trace errors. This is not an end-to-end QEVT or kernel-polynomial implementation. The dominant systematic is  $\eta$ : over-smoothed baselines ( $\eta = 0.1$ ) inflate  $\Phi_0$  at late times and masquerade as chaos (77% misclassification of the integrable member), while  $\eta \leq 0.02$ , truncation  $\leq 3 \times 10^{-3}$ , and  $10^6$  shots per grid point yield a  $\leq 0.5\%$  wrong-class rate with abstention rates of 13.5% ( $q=4$ ) and 26.5% ( $q=2$ )—we report correctness and coverage separately throughout, since a low wrong-class rate with high abstention certifies less than the error rate alone suggests—concrete resource targets for the protocol. Finally, we sampled ideal measurement probabilities at small scale ( $D = 64$ ) using classical dense matrices. The script forms a Taylor-truncated propagator  $M_K$ , computes its spectral-norm normalization  $\beta = \|M_K\|_2(1 + 10^{-7})$ , and draws binomial samples with probability  $\|M_K/\beta\|_F^2/D$ . A separate density-matrix calculation supplies the ideal SWAP-test probability for two Choi-state copies. No dilation unitary is explicitly assembled, no gate sequence is synthesized, and no hardware noise is simulated. The classifier uses an exact-eigenvalue baseline in this test. At  $t = 3$ , the tested acceptances  $p \approx 0.04$ – $0.07$  require about  $1.5$ – $2.5 \times 10^4$  independent samples for 3% relative standard error in  $p$  alone; this does not bound the total cost of the excess ratio or classifier. Over 100 Monte Carlo trials, the  $q = 4$  instance has no wrong-class assignments or abstentions at  $10^6$  samples per point. The  $q = 2$  instance has 19% wrong-class assignments at  $10^5$  samples, 2% wrong and 98% abstention at  $10^6$ , and 100% abstention at  $10^7$ . GinUE abstains in 92% of trials at  $10^5$  samples and all trials at the two larger budgets. These outcomes test ideal sampling statistics and the limits of size-calibrated thresholds, not a scalable circuit realization.

#### 4. Resource analysis and protocols

Two measurement routes follow from the identities above. In the synthesis route, the relevant quantities are the cost of implementing a block encoding, its actual

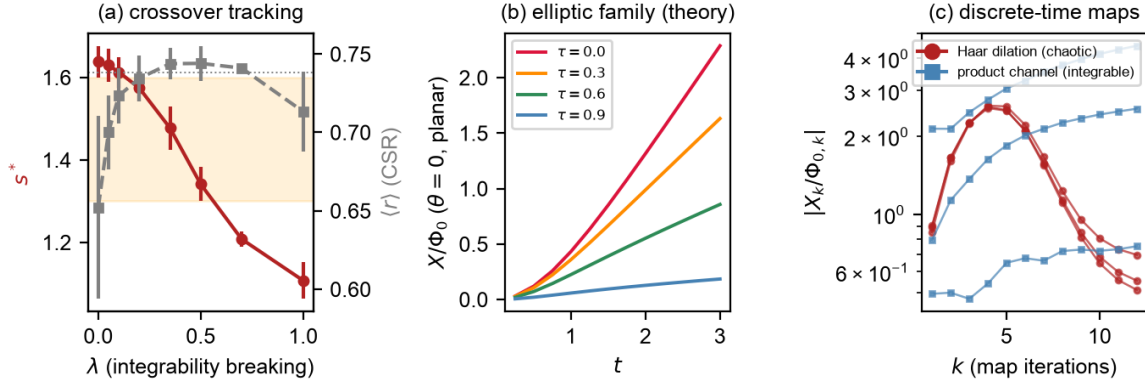


**Figure 4. The diagnostic tracks the chaos crossover, admits an exact elliptic interpolation, and extends to discrete-time maps.** (a) SYK interpolation  $L(\lambda)$ : CSR  $\langle r \rangle$  (gray, right axis) saturates by  $\lambda \approx 0.15$  while  $s^*$  (red) drifts monotonically through the abstention band (shaded), metering chaos strength beyond the spectral onset. (b) Planar-limit elliptic-family curves (Theorem 3), interpolating the Ginibre ramp ( $\tau = 0$ , Theorem 2) to the vanishing excess of the normal limit. (c) Discrete-map excess  $|X_k/\Phi_{0,k}|$  for Haar-dilation (red) versus product (blue) four-qubit channels: slopes separate with zero seed overlap.

subnormalization  $\beta$ , and the probability  $p = \Phi/\beta^2$ . Independent flag samples obey

$$\frac{\text{sd}(\hat{\Phi})}{\Phi} = \sqrt{\frac{1-p}{Np}}, \quad N \geq \frac{1-p}{p\varepsilon_{\text{rel}}^2} \quad (9)$$

for relative standard error  $\varepsilon_{\text{rel}}$ , assuming exact normalization and no synthesis bias. Coherent amplitude estimation has a different precision dependence and additionally requires coherent state preparation and inverse operations [28]; it must not be counted as independent shots. The total resource cost also includes the baseline. The sampled numerical abscissae are size-stable in the tested families (GinUE 1.36–1.38, Hatano–Nelson 1.07–1.09, random Liouvillians 1.18–1.22, and SYK 1.06–1.07 in the reported scans), and the  $N = 18$  SYK Pauli expansion has 3060 terms and an LCU normalization estimated as 55–80  $\|L\|$ . These finite-size checks do not constitute a compiled gate count or a general efficiency proof. In particular, an operator-norm bound alone does not determine the implemented synthesis normalization or the relative-precision cost.

For a physical channel at  $\theta = 0$ , a two-copy Choi-state SWAP test avoids propagator synthesis:  $p_{\text{swap}}(0) = (1 + \Phi)/2$ . The classical probability-level test gives a purity-estimator standard deviation of approximately  $8\text{--}9 \times 10^{-4}$  at  $10^6$  samples. An experimental realization still requires two channel copies, the reference systems and a suitable purity-measurement circuit; it is not demonstrated here. This route estimates  $\Phi$ , not automatically the baseline-subtracted classifier. Full process tomography requires exponentially many channel parameters [15], but avoiding tomography is not in itself proof of a quantum speedup over all classical methods. Although eigenvector-based evaluation becomes unstable for strongly non-normal matrices [33], direct exponential-

action and stochastic-trace methods provide classical alternatives and are used in our own implementation.

A separate worst-case complexity argument uses the fact that normalized-trace estimation is the defining DQC1-complete task of the one-clean-qubit model [27, 46]: for two-unitary mixed channels  $\Lambda = \frac{1}{2}(U \cdot U^\dagger + V \cdot V^\dagger)$  one has  $\text{Tr}[J(\Lambda)^2] = \frac{1}{2} + \frac{1}{2}|\text{Tr}(U^\dagger V)/d|^2$ , so additive-1/poly estimation of the Choi purity is DQC1-hard (a doubling/polarization circuit recovers  $\text{Re Tr} W/d$  from the squared modulus), while Theorem 1 places it in BQP; a polynomial-time classical estimator would thus imply  $\text{DQC1} \subseteq \text{BPP}$  (full algebra and the polarization circuit: Appendix D). This is a worst-case statement about the general estimation problem; the specific instances studied numerically here are, at these sizes, classically diagonalizable by construction—the near-term motivation is to measure a selected scalar without reconstructing the channel, not to claim hardness of our test set. The hardness transfers to generator access: the mixed-unitary channel defines the genuine Lindbladian  $L = \sum_i K_i \cdot K_i^\dagger - \text{id}$  with jumps  $K_1 = U/\sqrt{2}$ ,  $K_2 = V/\sqrt{2}$ , and the  $t^2$  Taylor coefficient of  $\Phi(t; \theta)$  at  $\theta = 0$  equals  $\frac{1}{D}(\text{Tr}[LL^\dagger] + \text{ReTr} L^2)$ —contaminated by the  $L^2$  term—but averaging the coefficients extracted at  $\theta = 0$  and  $\theta = \pi/2$  cancels  $\text{ReTr} L^2$  exactly and isolates  $\frac{1}{D}\text{Tr}[LL^\dagger] = \frac{3}{2} + \frac{1}{2}|\text{Tr}(U^\dagger V)/d|^2 - \frac{2}{D}\text{ReTr} \Lambda$  (both steps verified to  $10^{-6}$ ); finite differencing over polynomially many  $\Phi$  estimates at the two angles, with the auxiliary terms recovered from single-unitary instances, completes the reduction (which thus uses the synthesis-tier  $\theta$  access, not the  $\theta = 0$  tier alone). The diagnostic also extends directly to discrete-time quantum maps—the setting of circuit experiments—via  $\Phi_k = \frac{1}{D}\|\Lambda^k\|_F^2$ : for four-qubit channels, Haar-dilation (chaotic) and product-channel (integrable) maps separate with zero seed overlap in the  $\log k$  slope of the excess, with an inverted signature (decay, slope  $-1.9$  to  $-1.4$ , versus flat growth,  $+0.09$  to  $+0.43$ ) [Fig. 4(c)]; magnitudes again overlap, reconfirming that shape, not size, carries the chaos information.

## 5. Conclusion

The propagator-norm form factor links transient non-normal dynamics to CM overlaps and quantum measurement identities. The Henrici short-time law and the Ginibre and elliptic planar-limit formulas supply analytic reference curves. Finite-size SYK data are consistent with a planar-order approximation, while the empirical classifier distinguishes selected benchmark families with explicitly reported abstentions. These results support a limited but useful conclusion: excess magnitude alone is insufficient, and profile shape supplies complementary information within the tested domain.

The limitations are equally important. Normality does not by itself imply integrability, and a shape-based rule calibrated on random-matrix and SYK ensembles is not a universal chaos certificate. The structured mixed-charge chains already show spectral-spacing separation without corresponding slope separation. Our measurement tests sample ideal probabilities computed classically; gate-level synthesis and hardware validation remain future work. Moreover, accurate subtraction of  $\Phi_0$  is a separate

spectral problem, and the sparse implementation has not delivered the required late-window accuracy for large structured systems. We therefore distinguish tomography-free access to  $\Phi$  from an end-to-end efficient chaos classifier. Priorities are a controlled error analysis of the SYK approximation, baseline reconstruction with certified accuracy, tests of whether the two indicators converge or remain distinct in larger local systems, and an experimental Choi-purity measurement. The oracle-model lower bound in Appendix D.3 concerns its specified access model and does not exclude methods exploiting additional physical structure.

## Data availability statement

The numerical data and reproduction code are available at <https://github.com/deeptell-inc/simchaos> (release v1.0.0, MIT licence). The repository contains model generators, diagnostics, decision rules, driver scripts with fixed seeds and frozen result files (`refdata/`). The revised conceptual Fig. 1 contains no new physical data; its source, `make_fig1_physscr.py`, accompanies this revision separately from the public v1.0.0 release.

## Acknowledgments

(TODO: funding and acknowledgments)

## Appendix A. Proof of Theorem 1

*Statement and assumptions.*—Assume controlled block encodings  $U_f$  and  $U_g$  of  $A = f(L)/\beta_f$  and  $B = g(L^\dagger)/\beta_g$ , with  $\|A\|, \|B\| \leq 1$ . Each encoded block has operator-norm error at most  $\varepsilon/8$ , and the implementations and their inverses use  $Q_f$  and  $Q_g$  input-oracle queries. The normalized trace  $T = D^{-1}\text{Tr}[f(L)g(L^\dagger)]$  can then be estimated to additive error  $\varepsilon\beta_f\beta_g$  with failure probability at most  $p_{\text{fail}}$  using

$$O((Q_f + Q_g)\varepsilon^{-1} \log(1/p_{\text{fail}})) \quad (\text{A.1})$$

coherent queries. The costs  $Q_f, Q_g$  already include achieving the stated encoding accuracy.

*Step 1 (supply the function encodings).*—The required objects are the normalized function encodings, not merely an encoding of  $L/\alpha$ . Faber-QEVT [20] is one construction when its region, coefficient-access and approximation assumptions hold; for suitable exponentials LCHS is another [29]. These constructions determine  $Q_f, Q_g, \beta_f, \beta_g$ . For the adjoint side,  $g(L^\dagger) = (\bar{g}(L))^\dagger$  with  $\bar{g}(z) = \overline{g(\bar{z})}$ , so the adjoint of an encoding of  $\bar{g}(L)$  supplies the required block. No universal polynomial bound on these synthesis parameters is assumed here.

*Step 2 (product encoding).*—The block-encoding product construction [Ref. [16], Lemma 53] encodes  $AB$  using separate ancilla registers. If the encoded blocks are  $\tilde{A}$

and  $\tilde{B}$ , their being subblocks of unitaries implies norms at most one, and

$$\|\tilde{A}\tilde{B} - AB\| \leq \|\tilde{A} - A\| \|\tilde{B}\| + \|A\| \|\tilde{B} - B\| \leq \varepsilon/4. \quad (\text{A.2})$$

Normalized trace bias is no greater than this operator-norm error.

*Step 3 (trace measurement).*—Apply the controlled product unitary with its block ancillas initialized to zero and the system maximally mixed. The Hadamard-test mean is the real part of  $D^{-1}\text{Tr}[\tilde{A}\tilde{B}]$ ; a phase-shifted test yields the imaginary part [27]. A purification of the mixed input uses Bell pairs, allowing coherent preparation and its inverse. Ordinary independent tests cost  $O(\varepsilon^{-2} \log(1/p_{\text{fail}}))$  repetitions, while amplitude estimation [28] costs  $O(\varepsilon^{-1} \log(1/p_{\text{fail}}))$  coherent uses. Choose componentwise precision so that their combined statistical error is at most  $\varepsilon/2$ .

*Step 4 (rescale and count).*—Multiply the estimated normalized trace by  $\beta_f \beta_g$ . Combining product bias and statistical error gives the stated additive guarantee. Each use costs  $O(Q_f + Q_g)$  input-oracle queries, which yields the result. For an absolute target error  $\delta$ , substitute  $\varepsilon = \delta/(\beta_f \beta_g)$ . This makes the normalization dependence explicit. ■

*Specialization to  $\Phi$ .*—Take  $M = e^{te^{i\theta}L}$  and an encoding of  $M/\beta$ . Direct flag sampling on  $I/D$  gives Eq. (2), so  $\Phi = \beta^2 p_{\text{acc}}$ . The classical logarithmic-norm inequality bounds  $\|M\|$  by  $e^{t\omega_\theta}$  [33]; the Crouzeix–Palencia theorem separately bounds analytic functional calculus on the numerical range [47]. Neither replaces a resource analysis of the chosen encoding. Relative-error sampling depends on  $p_{\text{acc}}$  through Eq. (9). For a physical channel at  $\theta = 0$ , the Choi-purity identity supplies a synthesis-free alternative for  $\Phi$ . In both routes, obtaining  $\Phi_0$  remains an additional task.

## Appendix B. Baseline reconstruction and error propagation

The classifier consumes  $R = X/\Phi_0 = \Phi/\Phi_0 - 1$ . Suppose  $|\hat{\Phi} - \Phi| \leq \delta$  and  $|\hat{\Phi}_0 - \Phi_0| \leq \delta_0 < \Phi_0$ . Direct subtraction of the two ratios gives the finite-error bound

$$|\hat{R} - R| \leq \frac{\delta + (1 + R)\delta_0}{\Phi_0 - \delta_0}. \quad (\text{B.1})$$

Thus relative control of both inputs is required. Absolute precision independent of the baseline is insufficient when  $\Phi_0$  becomes small; near  $R = 0$ , relative precision in the excess itself requires still tighter control.

For  $\Phi_0 = \int \rho(z) K_{t,\theta}(z) d^2z$  with  $K_{t,\theta}(z) = e^{2t\text{Re}(e^{i\theta}z)}$  (shift-normalized so  $K \leq 1$  on the spectral disk), a density estimate  $\rho_\eta$  accurate in Wasserstein-1 distance  $W_1(\rho_\eta, \rho) \leq \eta$  yields

$$|\hat{\Phi}_0 - \Phi_0| \leq \eta \max_{|z| \leq \|L\|} |\nabla K_{t,\theta}(z)| = 2t\eta, \quad (\text{B.2})$$

so an error target requires  $\eta$  to scale with  $\Phi_0/t$ , not merely with system size. This is a conditional error-propagation statement: it assumes the density estimate has the stated Wasserstein accuracy. Hermitization and the kernel polynomial method [37] are candidate numerical tools, but a Chebyshev degree alone does not certify that

density accuracy. Small singular values of  $L - z$ , regularization of the log-potential, stochastic errors and numerical differentiation must all be controlled. Each sparse matrix–vector moment costs  $O(D \text{poly}(n))$  work; the number of moments, probes and grid points is additional. QSVT can process singular values of the Hermitized matrix, but counting them at a chosen threshold is not by itself a proof that a two-dimensional eigenvalue density or  $\Phi_0$  is recovered to the required accuracy. We leave an unconditional polynomial-in- $n$  baseline algorithm open.

*Implementation note (at scale).*—We implemented this baseline route matrix-free and validated it at  $D = 4096$  against exact diagonalization. Two practical obstructions dominate: (i) for gapped local Lindbladians the late-window baseline is dominated by *isolated* right-edge modes—the stationary state alone carries weight  $1/D$ —which can be missed by the tested coarse-density approximation; deflation of rightmost eigenvalues was therefore investigated. (ii) Plain Arnoldi fails at precisely this step on non-normal generators: at subspace dimension 120 its Ritz values land deep in the pseudospectrum, returning spectral-abscissa estimates of  $+19$  against a true value of 0, whereas shift-invert Arnoldi recovers the dense truth to four digits—the Krylov-side counterpart of the eigendecomposition instability above. The smooth remainder is best integrated against the kernel by parts (never numerically differentiating the stochastic log-potential, whose Laplacian amplifies trace noise by  $h^{-2}$ ). Even so, percent-level late-window accuracy of the full ratio at  $D \gtrsim 10^5$  remains open as a numerical-analysis problem; the scripts and the quantitative failure catalog accompany the code package. These failures concern the baseline route rather than the identity for measuring  $\Phi$ ; they nevertheless prevent an end-to-end large-system classification claim.

## Appendix C. Proofs of Lemma 1 and Theorems 2–3

### Appendix C.1. Proof of Lemma 1 (short-time law)

Start from the definitions in the main text,  $\Phi(t; \theta) = \frac{1}{D} \text{Tr}[e^{aL} e^{\bar{a}L^\dagger}]$  with  $a = te^{i\theta}$ , and  $\Phi_0(t; \theta) = \frac{1}{D} \sum_k e^{a\lambda_k + \bar{a}\bar{\lambda}_k}$ , where  $\lambda_k$  are the eigenvalues of  $L$  counted with algebraic multiplicity. Expanding both exponentials to second order in  $t$ ,

$$\begin{aligned} D\Phi &= \text{Tr}\left[\left(\mathbb{K} + aL + \frac{a^2}{2}L^2\right)\left(\mathbb{K} + \bar{a}L^\dagger + \frac{\bar{a}^2}{2}L^{\dagger 2}\right)\right] + O(t^3) \\ &= D + a\text{Tr}L + \bar{a}\text{Tr}L^\dagger + \frac{a^2}{2}\text{Tr}L^2 + \frac{\bar{a}^2}{2}\text{Tr}L^{\dagger 2} \\ &\quad + |a|^2 \text{Tr}[LL^\dagger] + O(t^3). \end{aligned} \tag{C.1}$$

Because  $\text{Tr}L^p = \sum_k \lambda_k^p$  for *any* square matrix (Schur triangularization; see, e.g., Ref. [33]), the identical expansion of  $\Phi_0$  reads

$$\begin{aligned} D\Phi_0 &= D + a\sum_k \lambda_k + \bar{a}\sum_k \bar{\lambda}_k + \frac{a^2}{2}\sum_k \lambda_k^2 \\ &\quad + \frac{\bar{a}^2}{2}\sum_k \bar{\lambda}_k^2 + |a|^2 \sum_k |\lambda_k|^2 + O(t^3). \end{aligned} \tag{C.2}$$

Every term of Eq. (C.1) except the  $|a|^2$  term cancels against Eq. (C.2) exactly; the difference is

$$X = \Phi - \Phi_0 = \frac{|a|^2}{D} \left( \text{Tr}[LL^\dagger] - \sum_k |\lambda_k|^2 \right) + O(t^3), \quad (\text{C.3})$$

and  $|a|^2 = t^2$  carries no  $\theta$ . For a *fixed* matrix,  $\theta$ -dependence first enters at  $O(t^3)$  through the  $a^2 \bar{a}$  terms, with coefficient  $\propto \text{Tr}[L^2 L^\dagger] - \sum_k \lambda_k^2 \bar{\lambda}_k$ —nonzero already for  $2 \times 2$  non-normal examples (e.g.,  $L = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$  gives  $X = \frac{1}{2}t^2 + \frac{1}{2}\cos\theta t^3 + O(t^4)$ , verified numerically). For ensembles whose matrix distribution is invariant under  $L \rightarrow -L$  combined with complex conjugation of couplings (Ginibre, elliptic), all odd-order terms vanish on average, and the leading *ensemble* anisotropy is the  $O(t^4)$  law of Theorem 3. The bracket is Henrici's departure from normality (squared Frobenius) [34]. No diagonalizability was used.  $\blacksquare$

### Appendix C.2. Proof of Theorem 2 (Ginibre universal curve)

*Step 1 (setup).* The complex Ginibre ensemble [48] has i.i.d. entries with

$$\mathbb{E}[G_{ij}] = 0, \quad \mathbb{E}[G_{ij}G_{kl}] = 0, \quad \mathbb{E}[G_{ij}\bar{G}_{kl}] = \frac{1}{D} \delta_{ik}\delta_{jl}. \quad (\text{C.4})$$

Expanding both exponentials in  $\Phi$ ,

$$\mathbb{E}\Phi(t; \theta) = \sum_{m,n \geq 0} \frac{a^m \bar{a}^n}{m! n!} M_{mn}, \quad M_{mn} := \frac{1}{D} \mathbb{E} \text{Tr}[G^m (G^\dagger)^n]. \quad (\text{C.5})$$

*Step 2 (phase invariance).* The measure defined by Eq. (C.4) is invariant under  $G \rightarrow e^{i\varphi} G$ , while  $\text{Tr}[G^m (G^\dagger)^n] \rightarrow e^{i(m-n)\varphi} \text{Tr}[G^m (G^\dagger)^n]$ . Averaging over  $\varphi$  gives  $M_{mn} = 0$  unless  $m = n$ ; then  $a^m \bar{a}^m = t^{2m}$ , so the curve is  $\theta$ -independent.

*Step 3 (Wick expansion and planarity).* By the Isserlis–Wick theorem for Gaussian variables [49, 50],  $M_{mm}$  is a sum over pairings of the  $m$  letters  $G$  with the  $m$  letters  $G^\dagger$  (same-type pairings vanish by Eq. (C.4)). In the genus expansion, a pairing of topological genus  $g$  contributes at order  $D^{-2g}$  (genus is not the number of chord crossings) [50]; only non-crossing pairings survive as  $D \rightarrow \infty$ . The case  $m = 2$  makes the mechanism explicit:

$$\begin{aligned} \mathbb{E} \text{Tr}[GGG^\dagger G^\dagger] &= \underbrace{\sum_{abcd} \frac{\delta_{bd}}{D} \frac{\delta_{bd}}{D}}_{\text{nested } (1 \leftrightarrow 4)(2 \leftrightarrow 3)} + \underbrace{\sum_{abcd} \frac{\delta_{ad}\delta_{bc}}{D} \frac{\delta_{ba}\delta_{cd}}{D}}_{\text{crossing } (1 \leftrightarrow 3)(2 \leftrightarrow 4)} \\ &= D + \frac{1}{D}, \end{aligned} \quad (\text{C.6})$$

so  $M_{22} = 1 + D^{-2} \rightarrow 1$ , *not* the Catalan number  $C_2 = 2$  obtained for the alternating word  $\text{Tr}[(GG^\dagger)^2]$  [50, 51].

*Step 4 (nested pairing is unique).* In the word  $G_1 \cdots G_m G_1^\dagger \cdots G_m^\dagger$ , letter  $G_1$  must pair with some  $G_j^\dagger$  (same-type covariances vanish). Non-crossing then forces the letters outside the pair,  $G_{j+1}^\dagger \cdots G_m^\dagger$ , to pair among themselves—impossible unless  $j = m$ . Hence  $G_1 \leftrightarrow G_m^\dagger$ , and induction on the enclosed word  $G_2 \cdots G_m G_1^\dagger \cdots G_{m-1}^\dagger$  yields

exactly one non-crossing pairing (the fully nested one), each pair contributing a free index sum:  $M_{mm} = 1 + O(D^{-2})$ .

*Step 5 (Bessel resummation).* Therefore

$$\mathbb{E} \Phi(t) \xrightarrow{D \rightarrow \infty} \sum_{m \geq 0} \frac{t^{2m}}{(m!)^2} = I_0(2t), \quad (\text{C.7})$$

by the defining series of the modified Bessel function [DLMF 10.25.2] [52].

*Step 6 (baseline from the circular law).* The eigenvalue density of  $G$  converges to the circular law  $\rho(z) = \pi^{-1} \mathbb{1}_{|z| \leq 1}$  [48, 53]. In polar coordinates  $z = re^{i\phi}$ , using  $\int_0^{2\pi} e^{x \cos \phi} d\phi = 2\pi I_0(x)$  [DLMF 10.32.1] [52] and  $\frac{d}{dx}[xI_1(x)] = xI_0(x)$  [DLMF 10.29.4] [52],

$$\Phi_0(t) = \frac{1}{\pi} \int_{|z| \leq 1} e^{2t \operatorname{Re} z} d^2 z = 2 \int_0^1 r I_0(2tr) dr = \frac{I_1(2t)}{t}. \quad (\text{C.8})$$

*Step 7 (ratio and asymptotics).* Combining Steps 5–6,

$$\frac{X}{\Phi_0} = \frac{t I_0(2t)}{I_1(2t)} - 1 \xrightarrow{t \gg 1} t - \frac{3}{4} + O(1/t), \quad (\text{C.9})$$

using  $I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} (1 - \frac{4\nu^2-1}{8x} + \dots)$  [DLMF 10.40.1] [52]. Both averages are annealed; self-averaging at large  $D$  (seed-to-seed spread  $< 1\%$ ) justifies the quenched comparison, and  $X/\Phi_0$  is invariant under the shift normalization used in the numerics. Finite- $D$  deviations (2.9% at  $D = 2048$ ) are dominated not by genus corrections but by the spectral-radius normalization: rescaling by  $\max_k |\lambda_k|$  overshoots the bulk edge by  $O(\sqrt{\log D/D})$ ; normalizing by the bulk radius  $\sqrt{2\langle |\lambda|^2 \rangle}$  collapses the deviation to 0.48%. ■

### Appendix C.3. Proof of Theorem 3 (elliptic interpolation)

*Step 1 (ensemble and weighted Wick).* The elliptic ensemble [54] deforms Eq. (C.4) by one additional covariance,

$$\mathbb{E}[G_{ij}G_{kl}] = \frac{\tau}{D} \delta_{il}\delta_{jk}, \quad \mathbb{E}[G_{ij}\bar{G}_{kl}] = \frac{1}{D} \delta_{ik}\delta_{jl}, \quad (\text{C.10})$$

interpolating GinUE ( $\tau = 0$ ) to GUE ( $\tau = 1$ ). Wick pairings of  $G^m(G^\dagger)^n$  now include same-type pairs, each contributing a factor  $\tau$ ; planarity again restricts to non-crossing pairings, so

$$M_{mn}(\tau) = \sum_{\pi \in NC_2(m,n)} \tau^{\#\{\text{same-type pairs in } \pi\}}. \quad (\text{C.11})$$

*Step 2 (recursion).* Decompose by the partner of the first letter. If  $G_1$  pairs with  $G_k$  ( $k \leq m$ , weight  $\tau$ ), non-crossing splits the word into the enclosed pure- $G$  block  $G_2 \cdots G_{k-1}$  and the remainder  $G_{k+1} \cdots G_m(G^\dagger)^n$ ; if  $G_1$  pairs with  $G_j^\dagger$  (weight 1), it

encloses  $G_2 \cdots G_m G_1^\dagger \cdots G_{j-1}^\dagger$  and leaves the pure- $G^\dagger$  tail. With the pure-word boundary values  $M_{m,0} = \tau^{m/2} C_{m/2}$  ( $C_k$  the Catalan numbers counting non-crossing pairings [51]),

$$M_{m,n} = \tau \sum_{\substack{k=2 \\ k \text{ even}}}^m M_{k-2,0} M_{m-k,n} + \sum_{j=1}^n M_{m-1,j-1} M_{0,n-j}. \quad (\text{C.12})$$

*Step 3 (generating function).* Multiplying Eq. (C.12) by  $x^m y^n$  and summing, with  $A(v) := \sum_k C_k (\tau v^2)^k = \frac{1-\sqrt{1-4\tau v^2}}{2\tau v^2}$  the Catalan generating function [51], the two convolution sums factor into  $\tau x^2 A(x) F(x, y)$  and  $xy A(y) F(x, y)$ , while the  $m = 0$  row supplies  $A(y)$ :

$$F(x, y) = A(y) + [\tau x^2 A(x) + xy A(y)] F(x, y), \quad (\text{C.13})$$

which is the closed form quoted in Theorem 3. At  $\tau = 0$ ,  $A \equiv 1$  and  $F = (1 - xy)^{-1}$ , whose Borel transform  $\sum_m t^{2m}/(m!)^2 = I_0(2t)$  recovers Theorem 2. (Verified against direct enumeration of Eq. (C.11) for all  $m, n < 8$  at three values of  $\tau$ ; symmetric in  $(x, y)$ .)

*Step 4 (baseline from the elliptic law).* The eigenvalue density is uniform on the ellipse with semi-axes  $1 \pm \tau$ , realized as the image of the uniform unit disk under  $z = u + \tau \bar{u}$  [54, 53]. Disk moments follow from one polar integral,  $\langle u^k \bar{u}^l \rangle = \delta_{kl} \frac{1}{\pi} \int_0^1 r^{2k} 2\pi r dr = \frac{\delta_{kl}}{k+1}$ , so  $\langle \lambda^m \bar{\lambda}^n \rangle$  are the binomial sums quoted in the main text, and  $\Phi_0$  follows term by term.

*Step 5 (anisotropy at  $O(t^4)$ ).* The  $a^3 \bar{a}$  coefficient of  $X$  is  $(M_{31} - \langle \lambda^3 \bar{\lambda} \rangle)/3!$ . From Eq. (C.11),  $M_{31} = 2\tau$  [the two non-crossing pairings  $(1 \leftrightarrow 4)(2 \leftrightarrow 3)$  and  $(1 \leftrightarrow 2)(3 \leftrightarrow 4)$ , each with one same-type pair]; from Step 4,  $\langle \lambda^3 \bar{\lambda} \rangle = \tau + \tau^3$ . With  $a^3 \bar{a} + a \bar{a}^3 = 2t^4 \cos 2\theta$ ,

$$X|_{\text{aniso}} = \frac{2\tau - \tau - \tau^3}{6} \cdot 2t^4 \cos 2\theta = \frac{\tau(1 - \tau^2)}{3} t^4 \cos 2\theta, \quad (\text{C.14})$$

verified numerically to 2.6% at  $t = 0.3$  (residual  $O(t^6)$ ). The full series matches elliptic Monte Carlo ( $D = 1024$ ,  $\tau = 0.5$ ) to 0.2–0.7% over  $t \leq 2$  at both tested angles, and the Henrici coefficient is  $(1 - \tau^2)/2$ , consistent with Lemma 1. ■

*Step 6 (finite-size SYK approximation checks).* The planar reduction of the main text was checked in three independent ways: (i) the crossing weight  $\eta(N, q)$  [Eq. (14) of Ref. [36], whose Q-Hermite approximation assigns a diagram with  $\alpha_p$  crossings the weight  $\eta^{\alpha_p}$ ; the  $q = 4$  case appears in Appendix B, Eq. (B5), of Ref. [35]] was evaluated by exact combinatorics, giving  $\eta(N, 4) \in [-0.039, +0.049]$  and  $\eta(N, 2) \in [0.47, 0.62]$  for  $N = 14$ –20; (ii) the measured moment ratio  $\text{Tr}[H^2]/\text{Tr}[HH^\dagger] = 0.710$ –0.718 (with imaginary part  $\approx 0.01$ ) agrees with the covariance prediction  $\tau_{\text{eff}} = 0.724$  to 1–2%; (iii) the Theorem 3 series at  $\tau = \tau_{\text{eff}}$ , with the single kinematic rescaling  $t \rightarrow t/(1 + \tau_{\text{eff}})$  and no fitted parameters, reproduces the seed-averaged  $q=4$  excess profile ( $N = 18$ , three seeds) with pointwise ratio 1.03–1.05 over the full window  $t \in [0.25, 3]$ ; the measured residual is not a certified bound on the neglected diagrams (reproduction script in the code package).

*Step 7 (Schwinger–Dyson resummation: proof of Eqs. (6)–(7)).* Realize the planar limit by a free elliptic element  $x$  with covariances  $\tau(xx) = \tau$ ,  $\tau(xx^*) = 1$  [51, 50], and set  $G(u, v) = \tau[e^{ux}e^{vx^*}]$ , so that  $\Phi_\tau(t; \theta) = G(a, \bar{a})$  with  $a = te^{i\theta}$ . The planar contraction rule  $\tau(xw_1 \cdots w_k) = \sum_i c(x, w_i) \tau(w_1 \cdots w_{i-1}) \tau(w_{i+1} \cdots w_k)$  (free Wick with non-crossing splits [51]), organized by exponential generating functions, yields the Volterra Schwinger–Dyson equation

$$\begin{aligned} \partial_u G(u, v) &= \tau \int_0^u \varphi(s) G(u-s, v) ds + \int_0^v G(u, s) \varphi(v-s) ds, \\ \varphi(s) &= \frac{I_1(2\sqrt{\tau} s)}{\sqrt{\tau} s}, \end{aligned} \quad (\text{C.15})$$

where  $\varphi(s) = \tau[e^{sx}] = \sum_k C_k \tau^k s^{2k}/(2k)!$  is the pure trace ( $C_k$  Catalan). Double Laplace transform ( $u \rightarrow p$ ,  $v \rightarrow q$ ) turns both convolutions into products; with  $\hat{\varphi}(p) = (p - \sqrt{p^2 - 4\tau})/(2\tau)$ , the Catalan equation  $p\hat{\varphi} = 1 + \tau\hat{\varphi}^2$  gives the key identity  $p - \tau\hat{\varphi}(p) = 1/\hat{\varphi}(p)$ , and the solution closes in manifestly  $(p, q)$ -symmetric form:

$$\hat{G}(p, q) = \frac{\hat{\varphi}(p) \hat{\varphi}(q)}{1 - \hat{\varphi}(p) \hat{\varphi}(q)} = \sum_{k \geq 1} [\hat{\varphi}(p) \hat{\varphi}(q)]^k. \quad (\text{C.16})$$

[Equivalently,  $F(x, y) = A(x)A(y)/(1 - xyA(x)A(y))$ , which the Catalan equation identifies with the generating function of Theorem 3—an independent second derivation of it.] Term-by-term inversion via the classical pair  $\mathcal{L}[(k/t) I_k(at)] = a^{-k} (p - \sqrt{p^2 - a^2})^k$  [52] with  $a = 2\sqrt{\tau}$  gives  $\mathcal{L}^{-1}[\hat{\varphi}^k](t) = k I_k(2\sqrt{\tau} t)/(\tau^{k/2} t)$ , hence  $G(u, v) = \sum_k f_k(u) f_k(v)$  with real-coefficient  $f_k$ , and Eq. (6) follows as  $\sum_k |f_k(a)|^2$ . The baseline closes by the same substitution  $\lambda = u + \tau \bar{u}$  as in Step 4:  $e^{a\lambda + \bar{a}\bar{\lambda}} = e^{2\text{Re}[(\bar{a} + \tau a)u]}$  reduces  $\Phi_0$  to the circular-law integral of Theorem 2 at effective argument  $b = |a + \tau \bar{a}| = t\sqrt{1 + 2\tau \cos 2\theta + \tau^2}$ . For Eq. (7), apply Debye (uniform large-order) asymptotics to  $I_k(2\sqrt{\tau} t)$  and evaluate the  $k$ -sum by saddle point: the saddle sits at  $k_* = (1 - \tau)t$ , where the exponent equals  $2t(1 + \tau)$ —matching the baseline’s exponent exactly, so the ratio grows only algebraically—and the prefactor ratio gives  $(1 - \tau)^2(1 + \tau)t$ . Validation (script `run_sd_closed_form.py`): mixed moments  $m, n \leq 8$  exact; direct series to  $10^{-5}$ ; Hermitian limit  $\Phi_{\tau \rightarrow 1} \rightarrow I_1(4t)/2t$  to  $4 \times 10^{-4}$ ; anisotropy to 0.1%; ramp ratio  $\rightarrow 1$  as 0.995–1.024 at  $t = 150$  across  $\tau \in \{0, 0.2, 0.5, 0.724\}$ . ■

## Appendix D. Hardness and barrier proofs

### Appendix D.1. Choi purity, DQC1-hardness, and generator transfer

*Choi purity identity.* For a channel  $\Lambda$  on  $d$  dimensions with superoperator matrix  $\Lambda_{(ij),(kl)}$ , the normalized Choi state [55] is  $J = \frac{1}{d} \sum_{ij} \Lambda(|i\rangle\langle j|) \otimes |i\rangle\langle j|$ ; its matrix elements are a reshuffling of  $\Lambda$ ’s [56], which preserves the sum of squared moduli:

$$\text{Tr}[J^2] = \frac{1}{d^2} \sum_{ijkl} |\Lambda_{(ij),(kl)}|^2 = \|\Lambda\|_F^2/d^2. \quad (\text{D.1})$$

*Mixture encodes a DQC1-complete quantity.* For  $\Lambda = \frac{1}{2}(U \cdot U^\dagger + V \cdot V^\dagger)$ , i.e.,  $\Lambda = \frac{1}{2}(U \otimes \bar{U} + V \otimes \bar{V})$  as a matrix,

$$\begin{aligned} \|\Lambda\|_F^2 &= \frac{1}{4}(\|U \otimes \bar{U}\|_F^2 + \|V \otimes \bar{V}\|_F^2 + 2\operatorname{Re}\langle U \otimes \bar{U}, V \otimes \bar{V} \rangle) \\ &= \frac{1}{4}(d^2 + d^2 + 2|\operatorname{Tr}(U^\dagger V)|^2), \end{aligned} \quad (\text{D.2})$$

using  $\langle U \otimes \bar{U}, V \otimes \bar{V} \rangle = \operatorname{Tr}(U^\dagger V) \overline{\operatorname{Tr}(U^\dagger V)}$ . Hence  $\operatorname{Tr}[J^2] = \frac{1}{2} + \frac{1}{2}|\operatorname{Tr}(U^\dagger V)/d|^2$  (verified to  $10^{-16}$ ). Estimating  $\operatorname{Re} \operatorname{Tr} W/d$  for a poly-size unitary circuit  $W$  is the defining complete problem of the one-clean-qubit class DQC1 [27, 46]; to recover it from the squared modulus, run the same estimate on the doubled unitary  $W' = W \oplus e^{i\varphi} \mathbb{K}$  (one ancilla control), for which

$$|\operatorname{Tr} W'/2d|^2 = \frac{1}{4d^2}(|\operatorname{Tr} W|^2 + d^2 + 2d \operatorname{Re} e^{-i\varphi} \operatorname{Tr} W), \quad (\text{D.3})$$

so two phase settings  $\varphi \in \{0, \pi/2\}$  isolate  $\operatorname{Re} \operatorname{Tr} W$  and  $\operatorname{Im} \operatorname{Tr} W$  by subtraction. Thus additive-1/poly estimation of the Choi purity is DQC1-hard, while Theorem 1 places it in BQP.

*Generator access.* The same channel arises from the Lindblad generator [57, 58]  $L = \sum_i K_i \cdot K_i^\dagger - \operatorname{id}$  with jumps  $K_1 = U/\sqrt{2}$ ,  $K_2 = V/\sqrt{2}$  (trace-preserving since  $\sum_i K_i^\dagger K_i = \mathbb{K}$ ). Expanding  $\Phi(t; \theta) = \frac{1}{D} \operatorname{Tr}[e^{aL} e^{\bar{a}L^\dagger}]$  ( $a = te^{i\theta}$ ) as in Lemma 1, the  $t^2$  coefficient at fixed  $\theta$  is  $\frac{1}{D}(\operatorname{Tr}[LL^\dagger] + \operatorname{Re}[e^{2i\theta} \operatorname{Tr} L^2])$ : at  $\theta = 0$  alone it is contaminated by  $\operatorname{Re} \operatorname{Tr} L^2$ , but the average of the coefficients extracted at  $\theta = 0$  and  $\theta = \pi/2$  cancels the  $L^2$  term exactly (the phase  $e^{2i\theta}$  flips sign) and isolates

$$\frac{1}{D} \operatorname{Tr}[LL^\dagger] = \frac{3}{2} + \frac{1}{2}|\operatorname{Tr}(U^\dagger V)/d|^2 - \frac{2}{D} \operatorname{Re} \operatorname{Tr} \Lambda \quad (\text{D.4})$$

(identity and  $\theta$ -average extraction both verified numerically to  $10^{-6}$ ). Finite differencing  $\Phi$  at polynomially many  $t$  values at the two angles extracts the averaged coefficient with polynomial error amplification, and the auxiliary terms  $\operatorname{Re} \operatorname{Tr} \Lambda/D \propto |\operatorname{Tr} U/d|^2 + |\operatorname{Tr} V/d|^2$  are recovered from single-unitary instances of the same protocol; the polarization step above completes the reduction. The reduction therefore uses the synthesis-tier  $\theta$  access ( $\theta = \pi/2$  evolution is not a physical channel), not the  $\theta = 0$  tier alone.

#### Appendix D.2. Wasserstein–Lipschitz resolution barrier

For normal  $L_i$  with spectral measures  $\mu_i = \frac{1}{D} \sum_k \delta_{\lambda_k^{(i)}}$ , every statistic of Theorem 1 is  $T_i = \int h d\mu_i$  with  $h(\lambda) = f(\lambda)g(\bar{\lambda})$ . By Kantorovich–Rubinstein duality [59],

$$|T_1 - T_2| \leq \operatorname{Lip}(h) W_1(\mu_1, \mu_2), \quad (\text{D.5})$$

and  $\operatorname{Lip}(h) = O(te^{2t\|L\|}) = O(1)$  in the standardized window. Point configurations drawn from the same limiting density with bulk spacing  $\Theta(D^{-1/2})$ —e.g., a Ginibre configuration versus a Poisson configuration of the same intensity—admit matchings that transport each point by  $O(D^{-1/2} \operatorname{polylog} D)$  [60], so  $W_1 = O(D^{-1/2} \operatorname{polylog} D)$  while their CSR statistics differ at  $O(1)$ . Resolving them through class- $T$  statistics therefore demands additive precision  $O(D^{-1/2} \operatorname{polylog})$ , i.e.,  $\Omega(D/\operatorname{polylog})$  samples. ■

## Appendix D.3. All-protocol query barrier

*Setup.* Let  $\Lambda = \{\lambda_k\}$  and  $\Lambda' = \{\lambda'_k\}$  be two spectra contained in the disk of radius  $\alpha/2$  and matched pointwise with  $\max_k |\lambda_k - \lambda'_k| \leq \delta$ . Fix any eigenbasis and encode diagonally: the canonical block encoding is  $V = \bigoplus_k V_1(\lambda_k/\alpha)$  with the one-ancilla dilation

$$V_1(z) = \begin{pmatrix} z & -\sqrt{1-|z|^2} \\ \sqrt{1-|z|^2} & \bar{z} \end{pmatrix}, \quad (\text{D.6})$$

unitary for  $|z| \leq 1$  with top-left block  $z$ ;  $V'$  is built identically from  $\Lambda'$ .

*Lemma D.1 (encoding continuity).*  $\|V - V'\| = \max_k \|V_1(\lambda_k/\alpha) - V_1(\lambda'_k/\alpha)\| \leq c_L \delta/\alpha$  with  $c_L \leq 1 + 1/\sqrt{3}$ . *Proof:* the diagonal difference is bounded by  $|z - z'|$  and the off-diagonal by  $|\sqrt{1-|z|^2} - \sqrt{1-|z'|^2}| \leq \sup_{r \leq 1/2} \frac{r}{\sqrt{1-r^2}} |z - z'| = |z - z'|/\sqrt{3}$  on the half-disk. (Measured worst case over  $2 \times 10^4$  random pairs: 1.15.)  $\square$

*Lemma D.2 (hybrid argument [38]).* Any protocol interleaving  $T$  queries to  $V$  (or controlled- $V$ , or  $V^\dagger$ —all with the same norm difference) with arbitrary  $V$ -independent unitaries, followed by any measurement, obeys  $|p_V - p_{V'}| \leq 2T\|V - V'\|$ . *Proof:* replacing queries one at a time, the final states differ by at most  $\sum_{i=1}^T \|(V - V')|\psi_i\rangle\| \leq T\|V - V'\|$ , and acceptance probabilities of states differ by at most twice their norm distance.  $\square$

*Proposition D.3 (matched CSR-distinct spectra).* A GinUE spectrum and an independent uniform (Poisson) configuration of equal intensity on the same disk admit a bottleneck matching with  $\delta = O(D^{-1/2}(\log D)^{3/4})$ , while their complex-spacing statistics differ at  $O(1)$ . For the intensity-matched i.i.d.-to-grid problem this is the Leighton–Shor minimax matching bound [61]; routing both configurations through a common grid bounds the pair distance by the sum, and the determinantal rigidity of GinUE only helps its side. We verified the combined scaling directly by exact bottleneck matchings (bisection over the threshold with Hopcroft–Karp feasibility):  $\delta\sqrt{D}/(\log D)^{3/4} = 0.77\text{--}0.97$  across  $D = 256\text{--}2048$ , with  $\langle r \rangle = 0.73$  versus  $0.62\text{--}0.70$  on the same configurations (script `run_barrier_matching.py`).  $\square$

*Theorem D.4 (all-protocol barrier).* For the pairs of Proposition D.3, any protocol distinguishing the two generators with constant advantage through the canonical oracle requires

$$T \geq \frac{\text{advantage} \cdot \alpha}{2c_L \delta} = \Omega\left(\frac{\sqrt{D}}{(\log D)^{3/4}}\right) = \Omega\left(\frac{2^{n/2}}{\text{poly } n}\right) \quad (\text{D.7})$$

queries. *Proof:* chain Lemmas D.1–D.2 with Proposition D.3.  $\blacksquare$

The bound is quadratically weaker than the  $\Omega(D)$  sampling bound of the preceding subsection—the intrinsic amplitude-estimation gap—and both are exponential in  $n$ . It is worst-case over pairs and tied to block-encoding access, as discussed in the main text.

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