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When entanglement implies two-way EPR steering: a sharp rank theorem

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For general mixed states, entanglement does not guarantee steerability. But this separation disappears when the density matrix has sufficiently low rank. We establish a unifying theorem: every bipartite entangled state whose rank does not exceed the larger local dimension is steerable in both directions. At every higher rank, there exist entangled states that are unsteerable in both directions under arbitrary measurements. This bound covers all rank-2 entangled states. To prove this, we construct a finite steering inequality based on the relation between coherence and measurement probabilities, and show that every entangled state satisfying the rank bound can violate it in both directions. Testing this inequality requires only two fixed measurement bases on the trusted side, while the other party can use either projective measurements or a fixed interferometer with controlled attenuation.

Introduction.— Since Einstein, Podolsky, and Rosen (EPR) questioned the completeness of quantum mechanics [1] and Schrödinger introduced the term "steering" in response [2], the distinction between quantum theory and classical local descriptions has attracted sustained attention. Bell first showed that quantum correlations can violate constraints that every local hidden-variable (LHV) model must satisfy, establishing an experimentally testable, fundamental distinction between quantum theory and classical local descriptions [3]. Wiseman et al. later gave steering a rigorous operational definition and established its position between entanglement and Bell nonlocality [4]. Quantum correlations thus exhibit three levels of departure from a local description: entanglement, steering, and Bell nonlocality. Bell nonlocality implies steering, and steering implies entanglement, whereas the converse implications generally fail [4–6]. As quantum information theory developed, these different forms of correlation also came to be recognized as quantum resources [6]. For example, steering enables quantum key distribution in which only one party's devices are trusted [7] and provides an advantage in quantum process discrimination [8].

These levels do not always remain distinct. Gisin's theorem shows that pure-state entanglement necessarily implies Bell nonlocality, so all three levels coincide in this case. The development from Gisin's original two-qubit proof [9], through its extension to higher-dimensional bipartite systems [10], to a complete proof in the general multipartite setting [13] spanned more than two decades. A pure state contains only one pure-state component. Mixing several pure states with classical probabilities, however, can preserve entanglement while removing steering or Bell nonlocality [5, 11], and can even separate the two steering directions [12]. This raises a natural question: how many pure-state components can the mixing involve while entanglement and steering still coincide? The rank r of a density matrix is the minimum number of pure states needed to represent the state as a mixture. Let m, n denote the effective local dimensions. For given m, n, we ask how large a rank range can guarantee that every entangled state is steerable in both directions.

Several pioneering studies have addressed this question [14–16]. For entangled two-qubit states, the all-versus-nothing (AVN) proof of Chen et al. established that a single projective measurement preparing two distinct pure conditional states is sufficient to demonstrate steering [14]. Related work also extended the equivalence between entanglement and Bell nonlocality to a class of rank-two mixed states [17]. Song et al. studied conditions for two-setting steering when only one pure conditional state is available [18], and Zhang et al. proved that all rank-two entangled two-qubit states are steerable [19]. Subsequently, in contemporaneous studies, Song et al. [20] and Zhang et al. [21] showed that one pure conditional state suffices to guarantee two-way steering of an entangled two-qubit state. The latter work also used the different scaling orders of coherence and measurement probabilities near the boundary to establish a steering criterion valid in arbitrary dimensions. This gives the form of the present criterion for a continuous family of measurement settings. Zhang et al. further extended the rank-two result to arbitrary finite local dimensions, proving that every rank-two bipartite entangled state is projectively steerable in at least one direction, and in both directions when $m = n$. They also derived the rank condition $(r-1)(n-1) \leq m-1$ guaranteeing projective steering from Alice to Bob [22].

Despite this progress, two issues remain. First, the rank boundary between entanglement and steering has yet to be fully determined. Second, the continuous-measurement criterion needs to be converted into an explicit finite steering inequality. This would allow experimental certification with finitely many settings, as is possible with existing linear steering inequalities [23–25].

We prove that every bipartite entangled state with $r \leq \max\{m, n\}$ is steerable in both directions using finitely

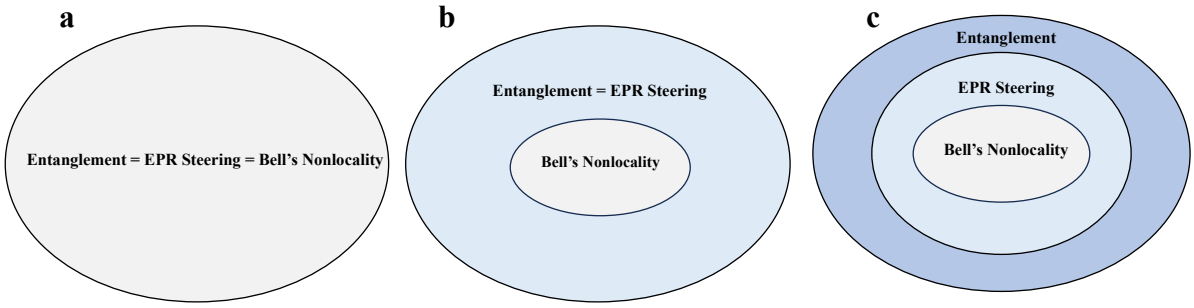

FIG. 1. Rank bound for bipartite quantum correlations. (a) Entanglement, steering, and Bell nonlocality coincide for pure states. (b) For $r \leq \max\{m, n\}$, entanglement and two-way projective steering coincide. (c) At every rank above the bound, there exist entangled states that are unsteerable in both directions under arbitrary POVMs.

many projective measurements. The bound is sharp: for every $m, n \geq 2$, each higher rank admits entangled states that are unsteerable in both directions under arbitrary positive-operator-valued measurements (POVMs). This extends the previous rank criterion [22] to a broader rank range and guarantees two-way steering throughout. We also convert the continuous-measurement criterion of the boundary geometry study [21] into a finite steering inequality. The corresponding experimental tests require only two fixed measurement bases on the trusted side, while the other party uses projective measurements or a fixed interferometer with controlled attenuation.

This paper is organized as follows. We first construct the finite steering criterion and then prove the low-rank theorem and its sharpness. The experimental implementation is presented next, followed by a final discussion of the physical implications and robustness against noise.

Theorem 1. *For an $m \times n$ system, every rank-r entangled state is two-way steerable using finite sets of projective measurements if*

$$r \leq \max\{m, n\}. \tag{1}$$

If $r > \max\{m, n\}$, there exists a rank-r entangled state that is two-way unsteerable under arbitrary POVMs.

Every rank-two entangled state is therefore two-way projectively steerable in arbitrary finite local dimensions. The rank bound is illustrated in Fig. 1.

Finite steering criterion.—For Alice's measurement setting j and outcome a, Bob's unnormalized conditional state is denoted by $\sigma_{a|j}$. A local hidden-state (LHS) model requires

$$\sigma_{a|j} = \mathrm{Tr}_A[(M_{a|j} \otimes \mathbb{I})\rho] = \int_\Lambda p(a|j, \lambda)\tau_\lambda \, d\mu(\lambda). \tag{2}$$

Here $M_{a|j}$ is Alice's measurement effect. The hidden states τ_λ are normalized, and μ is a probability distribution on the hidden-variable space Λ. The response function $p(a|j, \lambda)$ gives the probability that Alice reports outcome a for setting j and hidden variable λ. As Alice changes her measurement setting, $p(a|j, \lambda)$ may change, but the hidden states τ_λ and their distribution μ must remain fixed.

Choose two orthogonal Bob states $|\phi\rangle, |\beta\rangle$. Consider a family of Bob's unnormalized conditional states σ_t, generated by Alice's local measurements and parametrized by t. Denote their coherence by $b_t = \langle\phi|\sigma_t|\beta\rangle$ and population by $d_t = \langle\beta|\sigma_t|\beta\rangle$. Suppose that, for constants $\ell, c > 0$ and sufficiently small $t > 0$,

$$|b_t| \geq \ell t, \qquad d_t \leq ct^2. \tag{3}$$

We will show that we can select finitely many conditional states from this family and the corresponding Alice measurements for which any LHS explanation requires a total hidden-state weight $\int_\Lambda d\mu(\lambda) > 1$. This contradicts the normalization of μ.

Write $q_\lambda = \langle\beta|\tau_\lambda|\beta\rangle$ and let $f_t(\lambda)$ be Alice's response probability for the selected outcome. Positivity gives $|\langle\phi|\tau_\lambda|\beta\rangle| \leq \sqrt{q_\lambda}$. For $0 \leq u < v$, define $I = \{\lambda : u^2 < q_\lambda \leq v^2\}$. For $q_\lambda \leq u^2$, we have $\sqrt{q_\lambda} \leq u$; for $q_\lambda > v^2$, we have $\sqrt{q_\lambda} \leq q_\lambda/v$. The triangle inequality and positivity therefore give

$$\begin{aligned} |b_t| &\leq \int_{q_\lambda \leq u^2} f_t\sqrt{q_\lambda}\, d\mu + \int_I f_t\sqrt{q_\lambda}\, d\mu + \int_{q_\lambda > v^2} f_t\sqrt{q_\lambda}\, d\mu \\ &\leq u + \int_I f_t\sqrt{q_\lambda}\, d\mu + \frac{d_t}{v}. \end{aligned}$$

Here we use $b_t = \int_\Lambda f_t\langle\phi|\tau_\lambda|\beta\rangle\, d\mu$ and $f_t \leq 1$. Rearranging and applying Cauchy–Schwarz to the integral gives

$$\begin{aligned} \left(|b_t| - u - \frac{d_t}{v}\right)_+ &\leq \int_I f_t\sqrt{q_\lambda}\, d\mu \\ &\leq \sqrt{\int_I f_t q_\lambda\, d\mu}\,\sqrt{\int_I f_t\, d\mu} \\ &\leq \sqrt{d_t\, \mu(I)}. \end{aligned} \tag{4}$$

Here $(x)_+ = \max\{x, 0\}$, and the last step uses $f_t \leq 1$ and $d_t = \int_\Lambda f_t q_\lambda\, d\mu$.

For $N > 4c/\ell^2$, take $t_{j+1} = \ell^2 t_j/(16c)$, with t_1 small enough that Eq. (3) holds for all $j = 1, \ldots, N$. Set $u_j = \ell t_j/4$ and $v_j = 4ct_j/\ell$, and denote the corresponding intervals by I_j. Since $v_{j+1} = u_j$, these intervals are disjoint. Equation (3) gives $|b_{t_j}| - u_j - d_{t_j}/v_j \geq \ell t_j/2$, so Eq. (4) yields

$$\mu(I_j) \geq \frac{\ell^2}{4c}.$$

An LHS model would therefore require

$$1 = \mu(\Lambda) \geq \sum_{j=1}^{N} \mu(I_j) \geq N\frac{\ell^2}{4c} > 1.$$

This contradicts the normalization of μ.

We now give a useful block criterion. For a bipartite state ρ, suppose there exist orthonormal pairs $|\alpha_0\rangle, |\alpha_1\rangle$ on Alice's side and $|\phi\rangle, |\beta\rangle$ on Bob's side such that

$$\langle \alpha_0, \beta|\rho|\alpha_0, \beta\rangle = 0, \qquad g := \langle \alpha_0, \phi|\rho|\alpha_1, \beta\rangle \neq 0. \tag{5}$$

Then ρ is steerable in both directions using finite sets of projective measurements. This also recovers the criterion of Ref. [21].

By positivity of ρ, the first condition implies $\rho|\alpha_0, \beta\rangle = 0$. Adjusting the phases of the local states, we may take $g > 0$. Let Alice measure

$$|\xi_t\rangle = \frac{|\alpha_0\rangle + t|\alpha_1\rangle}{\sqrt{1+t^2}}.$$

Writing $D = \langle \alpha_1|\rho|\alpha_1\rangle$, direct calculation gives Bob's conditional coherence and population:

$$b_t = \frac{tg + t^2\langle\phi|D|\beta\rangle}{1+t^2}, \qquad d_t = \frac{t^2\langle\beta|D|\beta\rangle}{1+t^2}. \tag{6}$$

Since $g > 0$, Eq. (6) gives $\mathrm{Re}\, b_t \geq \ell t$ and $d_t \leq ct^2$ for sufficiently small $t > 0$, so the finite criterion proves Alice-to-Bob steering. The reverse direction follows by symmetry under exchange of the parties.

This condition also implies a negative partial transpose. In the chosen local basis, the block of the partial transpose on Bob corresponding to $|\alpha_0, \beta\rangle$ and $|\alpha_1, \phi\rangle$ is

$$\begin{pmatrix} 0 & g \\ g & \langle\phi|D|\phi\rangle \end{pmatrix}, \qquad \det = -g^2 < 0. \tag{7}$$

Proof of theorem.—We first show that, within the rank bound of Eq. (1), a state admitting no local states satisfying Eq. (5) is separable.

Relabel the parties so that $n \geq m$, hence $r \leq n$. Decompose $\rho = \sum_{j=1}^{r} |\psi_j\rangle\langle\psi_j|$ into linearly independent, unnormalized vectors. For an Alice vector $|x\rangle$, let $V(x)$ be the $n \times r$ matrix whose jth column is $(\langle x| \otimes \mathbb{I})|\psi_j\rangle$. As $|x\rangle$ ranges over Alice's space, these matrices form a complex linear space. Their columns collectively span Bob's effective space, since $\rho_B = \sum_s V(\chi_s)V(\chi_s)^\dagger$ for any orthonormal Alice basis $\{|\chi_s\rangle\}$.

For orthogonal Alice states, write $V = V(x)$ and $W = V(y)$. The zero-probability condition is $V^\dagger|\beta\rangle = 0$, and the cross coherence is $\langle\phi|VW^\dagger|\beta\rangle$. The vector $VW^\dagger|\beta\rangle$ is then orthogonal to $|\beta\rangle$. If nonzero, its normalized direction gives a state $|\phi\rangle$ satisfying Eq. (5). The absence of this structure therefore requires

$$V^\dagger|\beta\rangle = 0 \quad \Longrightarrow \quad VW^\dagger|\beta\rangle = 0. \tag{8}$$

For nonorthogonal Alice states, subtracting the component of $|y\rangle$ parallel to $|x\rangle$ changes W by a multiple of V. This leaves $VW^\dagger|\beta\rangle$ unchanged when $V^\dagger|\beta\rangle = 0$. Thus Eq. (8) holds for every pair V, W in the matrix family.

Choose a member V_0 of maximal rank k. Changing Bob's basis and unitarily recombining the pure components puts its only nonzero entries into an invertible upper-left $k \times k$ block. Equation (8) forces the lower-left block of every W to vanish. Writing W_{22} for the lower-right block of W, column elimination gives $\mathrm{rank}(V_0 + \epsilon W) = k + \mathrm{rank}\, W_{22}$ for sufficiently small nonzero ϵ. Since $V_0 + \epsilon W$ belongs to the same family, maximality forces $W_{22} = 0$. All columns therefore lie in a common k-dimensional subspace. They span Bob's space, so $k = n$; together with $k \leq r \leq n$, this gives $r = n$ and an invertible V_0.

The family $\widetilde{V}(x) = V_0^{-1}V(x)$ contains $\mathbb{I}$ and still satisfies Eq. (8), with the kernel vector replaced by $V_0^\dagger|\beta\rangle$. If $\widetilde{V}^\dagger|e\rangle = \bar{\lambda}|e\rangle$, applying the condition to $\widetilde{V} - \lambda\mathbb{I}$ and $\mathbb{I}$ gives $\widetilde{V}|e\rangle = \lambda|e\rangle$. The line through $|e\rangle$ and its orthogonal complement are therefore invariant under $\widetilde{V}$. Repeating on the orthogonal complement gives an orthonormal eigenbasis, so every member is normal.

Every complex linear combination is also normal. Expanding this identity for $\widetilde{V} + \widetilde{W}$ and $\widetilde{V} + i\widetilde{W}$ gives $\widetilde{V}\widetilde{W}^\dagger = \widetilde{W}^\dagger\widetilde{V}$. The matrices therefore preserve one another's eigenspaces. Diagonalizing within these invariant subspaces gives a common orthonormal eigenbasis $|e_j\rangle$.

Write $\widetilde{V}(x)|e_j\rangle = \lambda_j(x)|e_j\rangle$. Each $\lambda_j(x)$ is linear in $\langle x|$, so there is an Alice vector $|a_j\rangle$ such that $\lambda_j(x) = \langle x|a_j\rangle$. Let U be the unitary matrix whose columns are $|e_j\rangle$, and define the recombined components by $|\psi'_j\rangle = \sum_k U_{kj}|\psi_k\rangle$. By the definition of $V(x)$,

$$(\langle x| \otimes \mathbb{I})|\psi'_j\rangle = V(x)|e_j\rangle = \langle x|a_j\rangle V_0|e_j\rangle.$$

Expanding $|\psi'_j\rangle$ in the Alice basis $\{|\chi_s\rangle\}$ therefore gives

$$|\psi'_j\rangle = \sum_s \langle\chi_s|a_j\rangle|\chi_s\rangle \otimes V_0|e_j\rangle = |a_j\rangle \otimes V_0|e_j\rangle.$$

Since U is unitary, the recombination leaves ρ unchanged, yielding

$$\rho = \sum_{j=1}^{n} |\psi'_j\rangle\langle\psi'_j| = \sum_{j=1}^{n} |a_j\rangle\langle a_j| \otimes \left(V_0|e_j\rangle\langle e_j|V_0^\dagger\right). \tag{9}$$

Thus ρ is separable. An entangled state within the rank bound must therefore admit the structure in Eq. (5), which gives finite projective steering in both directions.

For sharpness, fix $2 \le m \le n$ and any $n < r \le mn$. Consider the rank-three two-qubit state

$$\omega = \frac{51}{100}|\Phi^+\rangle\langle\Phi^+| + \frac{49}{200}\big(|01\rangle\langle 01| + |10\rangle\langle 10|\big), \qquad (10)$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. This Bell-diagonal state has correlation matrix $T = \mathrm{diag}(51/100, -51/100, 1/50)$. The POVM LHS model of Ref. [26] applies when $R_T \ge 6/5$. Here Cauchy–Schwarz gives

$$R_T = \frac{2\pi}{\int_{S^2} \|T\vec{u}\|\, d\Omega} \ge \frac{\sqrt{3}}{2\sqrt{\mathrm{Tr}(T^2)}} > \frac{6}{5}, \qquad (11)$$

since $\mathrm{Tr}(T^2) = 2603/5000 < 25/48$. Swap symmetry gives unsteerability in both directions.

Embed ω into the first two levels of each party and set $i_j = \min\{j, m-1\}$ for $2 \le j < n$. Choose $r-n-1$ distinct vectors $|v_\ell\rangle$ from the $mn-n-1$ computational basis vectors other than $|01\rangle$, $|10\rangle$, $|11\rangle$ and the $|i_j, j\rangle$, and define

$$\rho_r = \frac{\omega + \sum_{j=2}^{n-1}|i_j, j\rangle\langle i_j, j| + \sum_{\ell=1}^{r-n-1}|v_\ell\rangle\langle v_\ell|}{r-2}. \qquad (12)$$

All product components are distinct computational basis vectors orthogonal to $|11\rangle$, whereas $|\Phi^+\rangle$ has a nonzero $|11\rangle$ component. The r pure components are therefore linearly independent, giving rank exactly r. Together with ω, the vectors $|i_j, j\rangle$ populate all local levels, giving local dimensions exactly m, n.

The added projectors leave the $|01\rangle, |10\rangle$ block of the partial transpose unchanged, so

$$\rho_r^{T_B}|\Psi^-\rangle = -\frac{|\Psi^-\rangle}{100(r-2)}, \qquad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \qquad (13)$$

Thus ρ_r is entangled. Embedding preserves the POVM LHS models of ω, and mixing with product states preserves unsteerability in both directions. This constructs a POVM-unsteerable entangled state at every rank above the bound and completes the proof of Theorem 1.

Experimental tests and examples.—The quantities in the finite criterion can be measured using two fixed bases on Bob's side: the basis Z containing $|\phi\rangle, |\beta\rangle$, and the interference basis X containing $(|\phi\rangle \pm |\beta\rangle)/\sqrt{2}$, with outcomes $s = \pm 1$. Label Alice's outcomes by $a = 0, \pm 1$. The joint probabilities give the coherence signal c_j and population p_j:

$$c_j = \tfrac{1}{2}\sum_{a,s=\pm 1} as P_j(a, s|X),$$
$$p_j = \sum_{a=\pm 1} P_j(a, \beta|Z).$$

For the projective measurements constructed above, label $|\xi_{t_j}\rangle$ by $+1$ and all other outcomes by 0. Then $c_j = \mathrm{Re}\, b_{t_j}$ and $p_j = d_{t_j}$. Equation (4) gives the finite steering inequality

$$\mathcal{S} = \sum_{j=1}^{N} \frac{\big(|c_j| - u_j - p_j/v_j\big)_+^2}{p_j} \le 1, \qquad (14)$$

where $0 \le u_j < v_j$ and the intervals $(u_j, v_j]$ are disjoint. The summand is taken as zero when $p_j = 0$, and $p_j/v_j = 0$ when $v_j = \infty$. The inequality also holds when both outcomes ± 1 are used, since the absolute difference of their response probabilities cannot exceed their sum. The projective settings and interval endpoints constructed above violate Eq. (14).

Experimentally, Alice's measurement amplitude can be varied by controlled attenuation. Attenuating mode α_1 by an amplitude factor t and combining it with mode α_0 in a fixed balanced interferometer gives the measurement effects

$$M_{\pm|t} = \tfrac{1}{2}(|\alpha_0\rangle \pm t|\alpha_1\rangle)(\langle\alpha_0| \pm t\langle\alpha_1|), \qquad 0 < t \le 1.$$

The effect $M_{0|t} = \mathbb{I} - M_{+|t} - M_{-|t}$ accounts for all remaining outcomes, including loss. Under the conditions in Eq. (5), the measured quantities satisfy

$$c_j = t_j g, \qquad p_j = t_j^2 \langle\beta|D|\beta\rangle$$

exactly. For exponential attenuation, equal increments in attenuation time generate the geometric sequence of amplitudes used in the proof.

For example, consider the rank-two state

$$\rho = \tfrac{1}{6}\big[(2|00\rangle + |11\rangle)(2\langle 00| + \langle 11|) + |10\rangle\langle 10|\big].$$

Using the computational modes on both sides, with $|\phi\rangle = |0\rangle$ and $|\beta\rangle = |1\rangle$, gives $c_j = t_j/3$ and $p_j = t_j^2/6$. Take two attenuation settings $t_1 = 1$, $t_2 = 1/150$, and intervals $(1/30, \infty)$ and $(0, 1/30]$. Substitution into Eq. (14) gives

$$\mathcal{S} = \frac{27}{50} + \frac{27}{50} = \frac{27}{25} > 1.$$

Discussion.—We fully determine the rank boundary between bipartite entanglement and steering. For any $m, n \ge 2$, every entangled state of rank at most $\max\{m, n\}$ is two-way projectively steerable, while counterexamples exist at every admissible rank above the bound. This boundary determines the minimum number of pure-state components needed for classical random mixing to separate entanglement from steering: $\max\{m, n\} + 1$. The universal connection between entanglement and steering for pure states thus extends to a range of mixed-state ranks that grows with local dimension. This reveals the fine structure of the hierarchy of bipartite quantum nonlocality.

The proof converts the continuous-measurement criterion from the boundary geometry study [21] into a finite-measurement steering inequality based on coherence and measurement probabilities. The first-order coherence and second-order population identified in that work allow finitely many appropriately chosen settings to exclude any LHS explanation. The low-rank argument further guarantees that every entangled state within the rank bound satisfies the block condition underlying this mechanism. The trusted party needs only two fixed measurement bases, while the other party can vary the measurement amplitude through controlled attenuation in a fixed interferometer, allowing a direct experimental test of the mechanism.

Beyond the rank bound, the block condition in Eq. (5) may still hold. How to study the states satisfying this condition systematically through algebraic methods and obtain a complete canonical form, thereby clarifying the structure of this family of boundary states, remains an open problem.

Finite measurements also extend the result to noisy states. Once the constructed measurement settings and interval endpoints are fixed, $\mathcal{S}$ is continuous near the original state, so a strict violation of Eq. (14) survives small perturbations. Applying this argument separately to both directions gives a neighborhood of two-way-steerable states around the original state, including full-rank states obtained by adding a small amount of the maximally mixed state. Effective measurements can therefore first be identified in the low-rank limit; even after noise removes the exact rank deficiency, the observed inequality violation can still certify steering. How much noise this method can tolerate for different states and noise models, and how this tolerance can be optimized by choosing measurement settings and interval endpoints, remain open questions.

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