

A Quadratic Resource-Allocation Reduction for Portfolio Selection in the ε -Contaminated Binomial Model

Federica Ricca

Università degli Studi di Roma La Sapienza <https://orcid.org/0000-0002-7925-7911>

Andrea Scozzari

andrea.scozzari@unicusano.it

Università degli Studi Niccolò Cusano: Università degli Studi Niccolò Cusano <https://orcid.org/0000-0003-3038-3957>

Research Article

Keywords: Portfolio optimization, Uncertainty modeling, Ambiguity, quadratic knapsack problem, ε -contamination

Posted Date: October 20th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11432694/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.
[Read Full License](#)

Additional Declarations: The authors declare no competing interests.

A Quadratic Resource-Allocation Reduction for Portfolio Selection in the ε -Contaminated Binomial Model

Federica Ricca

Dept. Methods and Models for Economics, Territory, and Finance

University of Rome La Sapienza

Via del Castro Laurenziano, 9, Rome, Italy

`federica.ricca@uniroma1.it`

ORCID: 0000-0002-7925-7911

Andrea Scozzari*

Dept. of Economic, Psychological, Communication, Educational,

and Motor Sciences, University Niccolò Cusano

Via Don Carlo Gnocchi 3, Rome, Italy

`andrea.scozzari@unicusano.it`

ORCID: 0000-0003-3038-3957

Abstract

We study the reduced portfolio selection problem in the ε -contaminated binomial model of Petturiti and Vantaggi. We give an explicit finite-state construction of a least favourable probability measure through a strictly convex continuous quadratic knapsack problem over the contamination class. This construction specializes the least favourable measure framework for robust utility maximization and connects it to established resource-allocation algorithms. The auxiliary optimizer is independent of both the utility function and the initial budget. For increasing, strictly concave utility functions satisfying regularity and boundary conditions, optimal terminal wealth is reconstructed through inverse marginal utility and a scalar budget equation. For constant relative risk aversion (CRRA) utility, the budget equation is solved explicitly for every parameter $\gamma > 0$, including logarithmic utility when $\gamma = 1$.

Given the probability vectors, the auxiliary problem can be solved in $O(n \log n)$ time by sorting and scanning, or in $O(n)$ time using median-search algorithms, where $n = T + 1$. For every fixed $\gamma > 0$, CRRA wealth reconstruction also takes $O(n)$ operations. This procedure avoids the exponential active-set enumeration used in Algorithm 1 of Petturiti and Vantaggi. We also present an affine-constrained wealth formulation and illustrative numerical comparisons for $\gamma = 2$.

Keywords: Portfolio optimization, Uncertainty modeling, Ambiguity, quadratic knapsack problem, ε -contamination.

1 Introduction and relation to existing work

Petturiti and Vantaggi [11] study dynamic portfolio selection under ambiguity in a binomial market. Their Theorem 1(ii) establishes that optimal terminal wealth is a function of the terminal stock price. This allows a reduction from the 2^T price paths to the $T + 1$ terminal stock-price states. Building on this property, their Theorem 2 provides equivalent formulations

*Corresponding author.

as a family of linearly constrained concave subproblems and as a single compact formulation. We take these results as our starting point.

For CRRA utility, Theorem 3 of [11] characterizes optimal terminal wealth through $T + 1$ combinatorial subproblems. The resulting procedure is implemented in Algorithm 1 of [11] by exhaustive active-set enumeration. Its exponential enumeration cost limits its applicability to small horizons. Petturiti and Vantaggi also consider an equivalent nonlinear formulation, their model (19), which they solve numerically to address larger horizons.

In this paper, we give a direct quadratic resource-allocation construction for the reduced portfolio problem. The auxiliary probability vector is the unique optimizer of a continuous quadratic knapsack problem over the contamination class. It depends on the reference and risk-neutral distributions and on the contamination parameter, but not on the utility function or the initial budget.

For increasing, strictly concave utility functions satisfying the stated regularity and boundary conditions, we reconstruct optimal terminal wealth through inverse marginal utility and a scalar budget equation. For CRRA utility, including logarithmic utility, the budget equation is solved explicitly. The same auxiliary solution can therefore be reused across different utility functions and budgets.

The continuous quadratic knapsack problem is a classical special case of separable convex resource allocation with a single resource constraint and bound constraints.

Continuous separable resource allocation has a long history. Patriksson [9] surveys multiplier-search and pegging (that is, variable fixing) methods; Patriksson and Strömberg [10] provide algorithmic developments and computational comparisons. Breakpoint search, sorting, median selection, and variable fixing for continuous quadratic knapsack problems are studied in Kiwiel [6, 7, 8]. Related pegging methods are developed by Bretthauer and Shetty [4].

The interpretation of the auxiliary vector is grounded in the literature on least favourable measures. Schied [14] establishes that, when an appropriate least favourable measure exists, robust utility maximization reduces to expected-utility maximization under that measure. His framework includes inverse-marginal-utility reconstruction and a characterization through convex divergences. Bernard et al. [1] further develop the connections between least favourable measures, stochastic dominance, and robust cost-efficiency.

Our construction specializes this framework to the reduced finite-state contamination model. The contribution is the explicit identification of a continuous quadratic knapsack problem whose optimizer yields a utility-independent least favourable measure, together with the reconstruction of optimal terminal wealth and the resulting algorithmic complexity bounds for the CRRA family. We verify these properties directly in the finite-state setting. The general reduction through a least favourable measure and the resource-allocation algorithms used to solve the auxiliary problem are established results; the contribution lies in their explicit connection for the portfolio problem considered here.

Given the probability vectors, the auxiliary problem can be solved in $O(n \log n)$ time by sorting and scanning, or in $O(n)$ time using Kiwiel's established median-search algorithms. For every fixed CRRA parameter $\gamma > 0$, wealth reconstruction also requires $O(n)$ operations. Hence, optimal CRRA terminal wealth can be computed in $O(n)$ time.

In contrast, Algorithm 1 of [11] examines up to $n2^{n-1} = (T+1)2^T$ active-set configurations. The threshold construction avoids this enumeration and computes the same optimal terminal wealth throughout the CRRA family, including logarithmic utility.

We also introduce an affine-constrained formulation of the wealth optimization problem. In the special case $\gamma = 2$, we present preliminary computational results comparing this formulation with our implementation of formulation (19) from [11]. These experiments are restricted to the two nonlinear wealth optimization models and should therefore be viewed independently of the algorithmic complexity results obtained through the quadratic reformulation. Related applications of ε -contamination include randomized reinsurance with a fixed loss marginal, as

studied in [12]. Their focus is on optimal retention and the impact of ambiguity on the benefits of randomization, rather than on the resource-allocation construction considered here.

2 Reduced state space portfolio selection problem

The classical multi-period binomial market model (see, e.g., [5, 13]) considers a perfect (competitive and frictionless) market under no-arbitrage, where two basic securities are traded: a non-dividend-paying stock and a risk-free bond. We consider model (14) in [11], which seeks an optimal terminal wealth random variable within the set of all wealth outcomes V_T attainable from a given initial wealth V_0 . For a detailed description of the model, the reader is referred to formulation (14) in [11].

Let $T \in \mathbb{N}$, $T \geq 1$, denote the finite investment horizon, and set $n = T + 1$. Working on the reduced state space of terminal stock prices, represent terminal wealth by a vector $x = (x_k)_{k=1}^n \in \mathbb{R}_{++}^n$, where x_k is the wealth associated with terminal state k . Let $P = (P_k)_{k=1}^n$ and $q = (q_k)_{k=1}^n$ denote, respectively, the reference and risk-neutral probability distributions on this reduced state space. The reference distribution reflects the investor's beliefs about the stock-price evolution and can be calibrated using historical data, whereas the risk-neutral distribution is determined by the market parameters through the no-arbitrage condition [11]. Assume

$$P_k > 0, \quad q_k > 0, \quad \sum_k P_k = \sum_k q_k = 1, \quad B = (1 + r)^T V_0 > 0.$$

where r is the risk-free interest rate over each period. The feasible wealth set is

$$\mathcal{X} = \{x \in \mathbb{R}_{++}^n : q^\top x = B\}.$$

For $0 \leq \varepsilon < 1$, set $a_k = (1 - \varepsilon)P_k$ and define

$$\mathcal{W}_\varepsilon = \{w \in \mathbb{R}^n : w_k \geq a_k, \sum_k w_k = 1\}.$$

For $\varepsilon > 0$, the set $\mathcal{W}_\varepsilon$ coincides with the contamination class

$$\{(1 - \varepsilon)P + \varepsilon R : R \in \Delta_n\}, \quad \Delta_n = \left\{R \in \mathbb{R}_+^n : \sum_{k=1}^n R_k = 1\right\}.$$

Indeed, if $w \in \mathcal{W}_\varepsilon$, then $R_k = (w_k - a_k)/\varepsilon$ is nonnegative and

$$\sum_{k=1}^n R_k = \frac{1 - \sum_{k=1}^n a_k}{\varepsilon} = 1.$$

Conversely, every vector $w = a + \varepsilon R$ with $R \in \Delta_n$ belongs to $\mathcal{W}_\varepsilon$.

For any increasing utility function $U(\cdot)$, the worst-case expected utility is therefore

$$\begin{aligned} F(x) &= \min_{w \in \mathcal{W}_\varepsilon} \sum_{k=1}^n w_k U(x_k) \\ &= \sum_{k=1}^n a_k U(x_k) + \varepsilon \min_{R \in \Delta_n} \sum_{k=1}^n R_k U(x_k) \\ &= \sum_{k=1}^n a_k U(x_k) + \varepsilon \min_{1 \leq k \leq n} U(x_k) \\ &= \sum_{k=1}^n a_k U(x_k) + \varepsilon U\left(\min_{1 \leq k \leq n} x_k\right). \end{aligned} \tag{1}$$

The third equality follows because a convex combination of the values $U(x_k)$ is minimized by concentrating all probability on a state attaining their minimum. The last equality follows from the monotonicity of $U(\cdot)$. For $\varepsilon = 0$, the same identity holds directly, since $\mathcal{W}_0 = \{P\}$. Hence, the reduced problem is

$$\sup_{x \in \mathcal{X}} F(x).$$

We recall, in our notation, the reduced formulation established by Theorems 1 and 2 of [11]. The extreme distributions have coefficients

$$p_k^{(h)} = a_k + \varepsilon \mathbf{1}_{\{k=h\}}.$$

For fixed h , define $\mathcal{X}_h = \{x \in \mathcal{X} : x_h \leq x_k \ \forall k\}$. Then $\mathcal{X} = \bigcup_h \mathcal{X}_h$ and $F(x) = \sum_k p_k^{(h)} U(x_k)$ on $\mathcal{X}_h$. Then,

$$\sup_{x \in \mathcal{X}} F(x) = \max_h \sup_{x \in \mathcal{X}_h} \sum_k p_k^{(h)} U(x_k).$$

This explains the permutation notation in formulation (18) of [11]: only its first element matters, and each subproblem has $n - 1$ ordering inequalities. When the utility function is of CRRA type, we consider

$$U_\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad (\gamma > 0, \ \gamma \neq 1), \quad U_1(x) = \log x,$$

so that $U'_\gamma(x) = x^{-\gamma}$ and $U''_\gamma(x) = -\gamma x^{-\gamma-1} < 0$.

2.1 An affine-constrained formulation

For a strictly increasing utility function $U(\cdot)$, introduce an auxiliary scalar variable $m > 0$ representing a common lower bound on terminal wealth across all states. For $0 < \varepsilon < 1$, the reduced problem is equivalent to

$$\begin{aligned} \max_{x > 0, m > 0} \quad & \sum_k a_k U(x_k) + \varepsilon U(m) \\ \text{s.t.} \quad & q^\top x = B, \\ & m - x_k \leq 0, \quad k = 1, \dots, n. \end{aligned} \tag{2}$$

For any fixed feasible wealth vector x , the constraints imply $m \leq \min_k x_k$. Since $\varepsilon > 0$ and $U(\cdot)$ is strictly increasing, the objective is maximized with respect to m at $m = \min_k x_k$. Thus, at every optimal solution, m equals the minimum terminal wealth. For CRRA utility, we take $U = U_\gamma$. In particular, for $\gamma = 2$, minimizing the negative of the objective yields the equivalent formulation

$$\begin{aligned} \min_{x > 0, m > 0} \quad & G(x, m) = \sum_k \frac{a_k}{x_k} + \frac{\varepsilon}{m} \\ \text{s.t.} \quad & q^\top x = B, \\ & m - x_k \leq 0, \quad k = 1, \dots, n. \end{aligned} \tag{3}$$

Its Hessian is positive definite on the positive domain. Each ordering constraint involves only m and one wealth component. For comparison, using our notation, formulation (19) of [11] is

$$\begin{aligned} \max_{x > 0, b \in \mathbb{R}} \quad & b \\ \text{s.t.} \quad & q^\top x = B, \\ & b - \sum_k a_k U_\gamma(x_k) - \varepsilon U_\gamma(x_h) \leq 0, \quad h = 1, \dots, n. \end{aligned} \tag{4}$$

This is also a convex optimization problem, after changing the sign of its linear objective. Both formulations have $n + 1$ variables and n inequalities; their difference is structural rather than a reduction in variable or constraint counts.

3 The auxiliary quadratic allocation problem

This section develops the auxiliary quadratic construction and its application to wealth reconstruction. We first characterize the auxiliary optimizer and verify its least favourable measure property. We then give the finite-dimensional reconstruction for general utility and derive the explicit CRRA formula. Finally, we describe threshold algorithms and establish the computational complexity of the CRRA solution procedure.

Consider the (auxiliary) problem

$$\begin{aligned} \min_{w \in \mathbb{R}^n} \quad & J(w) = \frac{1}{2} \sum_k \frac{w_k^2}{q_k} \\ \text{s.t.} \quad & \sum_k w_k = 1, \\ & w_k \geq a_k, \quad k = 1, \dots, n. \end{aligned} \tag{5}$$

This problem has a nonempty compact feasible set and a strictly convex objective, hence a unique optimizer. The feasible set is exactly $\mathcal{W}_\varepsilon$.

Proposition 1. *Assume $0 < \varepsilon < 1$ and define*

$$\rho_k = \frac{a_k}{q_k}, \quad H(c) = \sum_k (cq_k - a_k)_+.$$

There is a unique solution $c > 0$ of $H(c) = \varepsilon$, and the unique optimizer of (5) is

$$w_k^\star = \max\{a_k, cq_k\}. \tag{6}$$

Proof. The function H is continuous and piecewise affine, is zero for $c \leq \min_k \rho_k$, is strictly increasing above that value, and diverges to infinity. Thus, since $\varepsilon > 0$, the equation $H(c) = \varepsilon$ has a unique solution, which satisfies $c > \min_k \rho_k$. The optimal vector satisfies $w^\star \geq a$ and

$$\sum_k w_k^\star = \sum_k a_k + H(c) = 1 - \varepsilon + \varepsilon = 1.$$

For the Lagrangian

$$J(w) + \eta(\sum_k w_k - 1) + \sum_k \nu_k(a_k - w_k),$$

choose $\eta = -c$ and $\nu_k = (\rho_k - c)_+$. Then $w_k^\star/q_k + \eta - \nu_k = 0$, $\nu_k \geq 0$, and $\nu_k(a_k - w_k^\star) = 0$. The KKT conditions are sufficient for this convex problem [3] and strict convexity establishes uniqueness of the solution. $\square$

The auxiliary optimizer also admits an interpretation as a least favourable probability measure. The following result establishes this property in terms of first-order stochastic dominance, following the definition of Bernard et al. [1].

Proposition 2. *Under the assumptions of Proposition 1, let $w^\star$ solve (5), and define $\ell_k^\star = w_k^\star/q_k$. Then, for every nondecreasing function $f : (0, \infty) \rightarrow \mathbb{R}$,*

$$\sum_k w_k^\star f(\ell_k^\star) \leq \sum_k w_k f(\ell_k^\star) \quad \text{for every } w \in \mathcal{W}_\varepsilon.$$

Thus, $w^\star$ is a least favourable measure with respect to first-order stochastic dominance in the sense of Bernard et al. [1].

Proof. Set $d_k = w_k^* - a_k = (cq_k - a_k)_+$. Then $d_k \geq 0$, $\sum_k d_k = \varepsilon$, and $d_k > 0$ implies $\ell_k^* = c$. Consequently,

$$\sum_k w_k^* f(\ell_k^*) = \sum_k a_k f(\ell_k^*) + \varepsilon f(c).$$

For any $w \in W_\varepsilon$, write $w = a + \varepsilon R$ with $R \in \Delta_n$. Since $\ell_k^* \geq c$ and f is nondecreasing,

$$\sum_k w_k f(\ell_k^*) = \sum_k a_k f(\ell_k^*) + \varepsilon \sum_k R_k f(\ell_k^*) \geq \sum_k a_k f(\ell_k^*) + \varepsilon f(c).$$

The claimed inequality follows. $\square$

Since both q and w are probability vectors,

$$\sum_k \frac{(w_k - q_k)^2}{q_k} = \sum_k \frac{w_k^2}{q_k} - 2 \sum_k w_k + \sum_k q_k = \sum_k \frac{w_k^2}{q_k} - 1.$$

Thus, problem (5) can also be interpreted as the projection of the risk-neutral probability vector q onto W_ε with respect to the weighted norm $\|v\|_{q^{-1}}^2 = \sum_k v_k^2 / q_k$. This projection is carried out in probability space, while the corresponding optimal terminal wealth can subsequently be reconstructed from w^* as established in the theorem presented in the next subsection.

Finally, we observe that Kiwiel [6, 7, 8] considers $\min_v \{\frac{1}{2} v^\top D v - \alpha^\top v : \beta^\top v = R, \ell \leq v \leq u\}$, with positive diagonal D . Thus, problem (5) is obtained by taking

$$D = \text{diag}(1/q_k), \quad \alpha = 0, \quad \beta = \mathbf{1}, \quad R = 1, \quad \ell = a, \quad u = \mathbf{1}.$$

The upper bounds are redundant because the components are positive and sum to one. Kiwiel's clipped multiplier formula, with multiplier $t = -c$, gives (6) at the optimum. The scalar threshold equation is then a special case of the classical quadratic knapsack multiplier equation.

3.1 Wealth reconstruction and the CRRA case

The following result specializes Schied's least favourable measure reduction [14] to the explicit finite-state construction obtained above. It records the scalar budget equation needed for general utility and provides the basis for the explicit CRRA reconstruction.

Theorem 3. Assume $0 < \varepsilon < 1$. Let $U : (0, \infty) \rightarrow \mathbb{R}$ be continuously differentiable, with a positive and strictly decreasing derivative satisfying

$$\lim_{x \downarrow 0} U'(x) = +\infty, \quad \lim_{x \uparrow \infty} U'(x) = 0,$$

and let $I = (U')^{-1}$.

Let w^* solve (5), and define

$$\ell_k^* = \frac{w_k^*}{q_k} = \max\{\rho_k, c\}.$$

There exists a unique $\lambda^* > 0$ satisfying

$$\sum_k q_k I\left(\frac{\lambda^*}{\ell_k^*}\right) = B. \tag{7}$$

The unique optimizer of (2) is

$$x_k^* = I\left(\frac{\lambda^*}{\ell_k^*}\right), \quad m^* = I\left(\frac{\lambda^*}{c}\right). \tag{8}$$

Moreover,

$$x_k^* = m^* \iff \rho_k \leq c.$$

The auxiliary optimizer w^* , the threshold c , and the minimum-wealth state set are independent of both the utility function U and the budget B .

Proof. The assumptions on U' imply that $I : (0, \infty) \rightarrow (0, \infty)$ is continuous and strictly decreasing, with

$$\lim_{t \downarrow 0} I(t) = +\infty, \quad \lim_{t \uparrow \infty} I(t) = 0.$$

Consequently, the function

$$\Phi(\lambda) = \sum_k q_k I\left(\frac{\lambda}{\ell_k^*}\right)$$

is continuous and strictly decreasing on $(0, \infty)$, and satisfies

$$\lim_{\lambda \downarrow 0} \Phi(\lambda) = +\infty, \quad \lim_{\lambda \uparrow \infty} \Phi(\lambda) = 0.$$

Thus, there exists a unique $\lambda^* > 0$ solving (7).

The proposed point (x^*, m^*) is strictly positive and satisfies the budget constraint. Since $\ell_k^* \geq c$ and I is strictly decreasing,

$$x_k^* = I\left(\frac{\lambda^*}{\ell_k^*}\right) \geq I\left(\frac{\lambda^*}{c}\right) = m^*.$$

Equality holds if and only if $\ell_k^* = c$, equivalently $\rho_k \leq c$. Since $H(c) = \varepsilon > 0$, at least one index satisfies $\rho_k < c$. Hence, $m^* = \min_k x_k^*$.

To verify optimality, minimize the negative objective in (2). The Lagrangian is

$$L(x, m, \lambda, \mu) = -\sum_k a_k U(x_k) - \varepsilon U(m) + \lambda(q^\top x - B) + \sum_k \mu_k(m - x_k).$$

The stationarity conditions are

$$-a_k U'(x_k) + \lambda q_k - \mu_k = 0, \quad -\varepsilon U'(m) + \sum_k \mu_k = 0.$$

Set

$$d_k = (cq_k - a_k)_+, \quad \mu_k^* = \frac{\lambda^*}{c} d_k.$$

Then $\mu_k^* \geq 0$ for every k . If $\rho_k > c$, then $d_k = 0$ and $\ell_k^* = \rho_k = a_k/q_k$, so

$$a_k U'(x_k^*) = a_k \frac{\lambda^*}{\ell_k^*} = \lambda^* q_k.$$

If $\rho_k \leq c$, then $x_k^* = m^*$ and $d_k = cq_k - a_k$, so

$$a_k U'(x_k^*) + \mu_k^* = a_k \frac{\lambda^*}{c} + \frac{\lambda^*}{c} (cq_k - a_k) = \lambda^* q_k.$$

Thus, the stationarity conditions with respect to x hold in both cases.

Furthermore,

$$\sum_k \mu_k^* = \frac{\lambda^*}{c} \sum_k d_k = \frac{\lambda^*}{c} \varepsilon = \varepsilon U'(m^*),$$

where we used $H(c) = \sum_k d_k = \varepsilon$. This proves stationarity with respect to m .

Complementary slackness follows because $\mu_k^* > 0$ implies $\rho_k < c$ and therefore $x_k^* = m^*$. The proposed point and multipliers satisfy all KKT conditions, which are sufficient for global optimality. Finally, U' strictly decreasing implies that U is strictly concave. Since $a_k > 0$ for every k and $\varepsilon > 0$, the objective in (2) is strictly concave in (x, m) , establishing uniqueness. $\square$

We observe that the reconstructed wealth vector and the auxiliary probability measure also satisfy a saddle-point property. Define

$$L(x, w) = \sum_{k=1}^n w_k U(x_k).$$

Then

$$L(x, w^*) \leq L(x^*, w^*) \leq L(x^*, w) \quad \text{for all } x \in X, w \in W_\varepsilon.$$

Indeed, the reconstruction formula gives

$$w_k^* U'(x_k^*) = \lambda^* q_k, \quad k = 1, \dots, n.$$

By concavity of U and the budget constraint,

$$\begin{aligned} L(x, w^*) - L(x^*, w^*) &\leq \sum_{k=1}^n w_k^* U'(x_k^*) (x_k - x_k^*) \\ &= \lambda^* q^\top (x - x^*) = 0. \end{aligned}$$

For the second inequality, write $w^* = a + d$, where $d_k = (cq_k - a_k)_+$ and $\sum_k d_k = \varepsilon$. Since $d_k > 0$ implies $x_k^* = m^*$, we have

$$L(x^*, w^*) = \sum_{k=1}^n a_k U(x_k^*) + \varepsilon U(m^*).$$

Every $w \in W_\varepsilon$ can be written as $w = a + \varepsilon R$ with $R \in \Delta_n$. Since $x_k^* \geq m^*$ and U is increasing,

$$L(x^*, w) \geq \sum_{k=1}^n a_k U(x_k^*) + \varepsilon U(m^*) = L(x^*, w^*).$$

Consequently,

$$\max_{x \in X} \min_{w \in W_\varepsilon} L(x, w) = L(x^*, w^*) = \min_{w \in W_\varepsilon} \max_{x \in X} L(x, w).$$

Thus, w^* is a worst-case measure for x^* , and x^* maximizes expected utility under w^* .

For general utility, the reconstruction requires the solution of the scalar budget equation (7). For CRRA utility, this equation can be solved explicitly.

Corollary 4. Assume $0 < \varepsilon < 1$ and let $U = U_\gamma$ with $\gamma > 0$, including logarithmic utility when $\gamma = 1$. Let w^* solve (5), and define

$$z_k = \left(\frac{w_k^*}{q_k} \right)^{1/\gamma} = \max\{\rho_k, c\}^{1/\gamma}, \quad D_\gamma = \sum_k q_k z_k.$$

The unique optimizer of (2) is

$$x_k^* = \frac{B z_k}{D_\gamma}, \quad m^* = \frac{B c^{1/\gamma}}{D_\gamma}. \quad (9)$$

Proof. For CRRA utility,

$$U'_\gamma(x) = x^{-\gamma}, \quad I(t) = t^{-1/\gamma}.$$

Equation (7) becomes

$$(\lambda^*)^{-1/\gamma} \sum_k q_k (\ell_k^*)^{1/\gamma} = (\lambda^*)^{-1/\gamma} D_\gamma = B.$$

Hence,

$$\lambda^* = \left(\frac{D_\gamma}{B} \right)^\gamma.$$

Substitution into (8) gives (9). □

Corollary 5. For $\gamma = 2$, let

$$D_2 = \sum_k q_k \sqrt{\frac{w_k^*}{q_k}}.$$

The minimum value of (3) is D_2^2/B .

Proof. Let

$$A = \{k : \rho_k \leq c\}, \quad z_k = \sqrt{\max\{\rho_k, c\}}.$$

The threshold equation gives

$$\sum_{k \in A} (cq_k - a_k) = \varepsilon,$$

and therefore

$$\sum_{k \in A} a_k + \varepsilon = c \sum_{k \in A} q_k.$$

Using $a_k = q_k \rho_k$, we obtain

$$\begin{aligned} \sum_k \frac{a_k}{z_k} + \frac{\varepsilon}{\sqrt{c}} &= \frac{\sum_{k \in A} a_k + \varepsilon}{\sqrt{c}} + \sum_{k \notin A} \frac{a_k}{\sqrt{\rho_k}} \\ &= \sqrt{c} \sum_{k \in A} q_k + \sum_{k \notin A} q_k \sqrt{\rho_k} \\ &= D_2. \end{aligned}$$

By (9),

$$G(x^*, m^*) = \frac{D_2}{B} \left(\sum_k \frac{a_k}{z_k} + \frac{\varepsilon}{\sqrt{c}} \right) = \frac{D_2^2}{B}.$$

□

Remark 1. For $\varepsilon = 0$, the contamination class reduces to $W_0 = \{P\}$. In the CRRA case, the unique optimal terminal wealth is

$$x_k^* = \frac{B(P_k/q_k)^{1/\gamma}}{\sum_j q_j (P_j/q_j)^{1/\gamma}}.$$

Thus, the solution is obtained directly from the standard CRRA formula.

At the other endpoint, $\varepsilon = 1$, the robust objective reduces to $U(\min_k x_k)$. For any strictly increasing utility function U , the unique optimal terminal wealth is $x_k^* = B$ for every k . Indeed,

$$\min_k x_k \leq \sum_k q_k x_k = B,$$

and equality is attained only by the constant wealth vector.

3.2 Computing the threshold and algorithmic scope

The threshold equation

$$H(c) = \sum_k (cq_k - a_k)_+ = \varepsilon$$

can be solved by specializing classical breakpoint-search methods for the continuous quadratic knapsack problem.

Sort the ratios $\rho_k = a_k/q_k$ so that

$$\rho_{(1)} \leq \dots \leq \rho_{(n)},$$

using the same permutation for a and q . Define

$$A_k = \sum_{i=1}^k a_{(i)}, \quad Q_k = \sum_{i=1}^k q_{(i)}, \quad \rho_{(n+1)} = +\infty.$$

On the interval $[\rho_{(k)}, \rho_{(k+1)}]$,

$$H(c) = cQ_k - A_k.$$

Thus, if the solution of $H(c) = \varepsilon$ lies in this interval, it is given by

$$c_k = \frac{\varepsilon + A_k}{Q_k}.$$

This gives the following sorting-and-scanning procedure.

Sorting-and-scanning procedure.

Input: positive probability vectors P, q and $0 < \varepsilon < 1$.

1. Set $a_k = (1 - \varepsilon)P_k$ and $\rho_k = a_k/q_k$ for $k = 1, \dots, n$.
2. Sort the ratios ρ_k in nondecreasing order, applying the same permutation to a and q .
3. Initialize $A = 0$ and $Q = 0$, and set $\rho_{(n+1)} = +\infty$.
4. For $k = 1, \dots, n$:
 - (a) Update $A \leftarrow A + a_{(k)}$ and $Q \leftarrow Q + q_{(k)}$.
 - (b) Set $c = (\varepsilon + A)/Q$.
 - (c) If $\rho_{(k)} \leq c \leq \rho_{(k+1)}$, stop the scan.
5. Return c and $w_k^* = \max\{a_k, cq_k\}$ in the original state ordering.

The piecewise-affine representation of H guarantees that the scan identifies the unique solution of $H(c) = \varepsilon$. If the solution coincides with a breakpoint, more than one interval may satisfy the test, without changing c or w^* . Sorting requires $O(n \log n)$ operations, while the scan and reconstruction of w^* require $O(n)$ operations. The procedure uses $O(n)$ storage.

Alternatively, Kiwiel's median-search algorithms [6, 7] solve the auxiliary quadratic problem in $O(n)$ time without sorting all breakpoints.

Corollary 6. *Given the probability vectors P and q , assume $0 < \varepsilon < 1$ and fix a CRRA parameter $\gamma > 0$. The unique optimal terminal wealth can be computed using $O(n \log n)$ operations by sorting and scanning, or $O(n)$ operations using Kiwiel's median-search algorithms.*

Proof. The auxiliary problem is a continuous quadratic knapsack problem with the coefficients specified above. Its optimizer is obtained with the stated complexity by the corresponding threshold method. Once w^* is available, computing z_k , D_γ , and all components of x^* in Corollary 4 requires $O(n)$ operations. $\square$

These bounds concern the computation of terminal wealth on the reduced state space, given the probability vectors. For general utility satisfying the assumptions of Theorem 3, the auxiliary quadratic problem and its complexity are unchanged. Wealth reconstruction additionally requires solving the scalar budget equation (7); its computational cost is separate from the threshold computation.

Compared with Algorithm 1 of [11], which examines up to $n2^{n-1} = (T+1)2^T$ active-set configurations, the threshold construction avoids combinatorial enumeration. For CRRA utility, it yields the same optimal terminal wealth with the complexity stated in Corollary 6. The auxiliary optimizer and threshold can be reused when the utility function or budget changes.

4 Some computational comparisons

4.1 Instances and implementation

The experiments compare the affine-constrained formulation (3) with our implementation of formulation (19) in [11], represented by (4) in our notation. They provide an illustrative comparison of the two nonlinear wealth formulations, rather than a computational assessment of the threshold algorithms in Section 3.2. Both models were implemented in AMPL and solved using BONMIN 1.8.9, with Cbc 2.10.8 and IPOPT 3.12.13. The experiments were run on a local Windows installation, calling Python 3.11 from MATLAB. BONMIN [2] was used for both continuous models, with IPOPT [15] as its nonlinear optimization component.

We reconstruct the META parameter family from [11] using

$$p = 0.5528, \quad u = 1.0292, \quad d = 0.9717, \quad r = 1.0508^{1/250} - 1, \quad q_{\text{up}} = \frac{1 + r - d}{u - d},$$

with $V_0 = 1000$, $\varepsilon = 0.02$, and $\gamma = 2$. For $j = 0, \dots, T$ (corresponding to index $k = j + 1$),

$$P_{j+1} = \binom{T}{j} p^j (1-p)^{T-j}, \quad q_{j+1} = \binom{T}{j} q_{\text{up}}^j (1-q_{\text{up}})^{T-j}.$$

The implementation evaluates log-probabilities and normalizes each vector to sum to one before conversion to ordinary floating-point coefficients. It records the number of coefficients lost to underflow. The recomputed $q_{\text{up}} \simeq 0.49562133495$ differs from the rounded value 0.4963 printed in [11]; the experiments must therefore be identified as reconstructed instances, not exact replications of the original numerical tables.

Both models use $y = x/B$, giving $q^\top y = 1$. Our affine-constrained formulation uses $s = m/B$ and objective $\sum_k a_k/y_k + \varepsilon/s$. The minimization representation of (4) uses objective t and constraints $\sum_k a_k/y_k + \varepsilon/y_h \leq t$. Initial values are $y_k = 1$, with $s = 1$ or $t = 1$. Both use a numerical lower bound 10^{-10} on positive variables. This bound is not part of the economic model; sensitivity to it remains to be investigated, although no returned wealth component was at the bound in the reported tests and the reference solution lies above it.

The recorded options are: *bonmin.algorithm=B-BB*, *tol=1e-9*, *max_iter=3000*, and *bonmin.time_limit=120*; AMPL presolve is disabled. Solution times are measured as the elapsed time of the AMPL `solve` call, including communication with the solver but excluding data and model construction. Each model runs in a separate Python process.

4.2 Reference and validation

We compute a numerical reference solution using the threshold characterization of Proposition 1 and the wealth-reconstruction formula of Corollary 4. Specifically, we approximate the unique solution of $H(c) = \varepsilon$ by 100 bisection steps on $[0, 2]$, obtaining c^{ref} . The normalized reference wealth vector is then reconstructed as

$$y_k^{\text{ref}} = \frac{\max\{\rho_k, c^{\text{ref}}\}^{1/\gamma}}{\sum_j q_j \max\{\rho_j, c^{\text{ref}}\}^{1/\gamma}},$$

with $\gamma = 2$ in the reported experiments. The implementation evaluates the likelihood ratios in logarithmic form.

Thus, the reference uses the proposed mathematical reduction, with bisection as the numerical method for computing the threshold. It is used to assess the accuracy of the solutions returned by BONMIN and is not used to initialize the solver.

To compare the two formulations using the same criterion, we evaluate the objective directly from the normalized terminal wealth vector $\hat{y}$ returned by the solver:

$$G(\hat{y}) = \sum_k \frac{a_k}{\hat{y}_k} + \frac{\varepsilon}{\min_k \hat{y}_k}.$$

The relative objective discrepancy with respect to the numerical reference solution is

$$E_{\text{obj}} = \frac{|G(\hat{y}) - G(y^{\text{ref}})|}{G(y^{\text{ref}})}.$$

We also compare the terminal wealth vectors using the maximum relative componentwise discrepancy and the probability-weighted absolute discrepancy:

$$E_{\infty} = \max_k \left| \frac{\hat{y}_k}{y_k^{\text{ref}}} - 1 \right|, \quad E_q = \sum_k q_k \left| \hat{y}_k - y_k^{\text{ref}} \right|.$$

We check the feasibility of each returned solution and assess its accuracy. A feasible solution passes objective validation if $E_{\text{obj}} \leq 10^{-6}$, and additionally passes statewise validation if $E_{\infty} \leq 10^{-3}$. Feasibility is checked using tolerances of 10^{-7} for the budget and inequality constraints and 10^{-9} for lower-bound violations.

4.3 Illustrative computational comparisons

In order to test formulations (3) and (4) on large instances, the experiment comprises ten repetitions for each of the two formulations at each horizon $T \in \{100, 125, 250, 500, 1000\}$, hence 100 attempted runs. These are repeated executions of five fixed instances, not 100 independently generated portfolio instances. The following tables present the results of the comparison.

T	(3) (s)	(19) (s)	Time Ratio	Objective validation
100	0.152075	0.324415	2.13	both
125	0.156105	0.469250	3.01	both
250	0.185675	2.584850	13.92	both
500	0.225620	65.483500	290.24 [†]	(19) only
1000	0.332255	—	—	neither validated

[†] The formulation does not meet the objective-accuracy criterion at this horizon.

Table 1: Median solve-call times (seconds) over ten runs per formulation and horizon. The ratio is the median time for formulation (19) of [11] divided by that for our formulation (3).

T	E_{obj} : (3) / (19)	E_q : (3) / (19)
100	$2.9049 \cdot 10^{-9}$ / $1.4797 \cdot 10^{-9}$	$1.5442 \cdot 10^{-8}$ / $8.5478 \cdot 10^{-9}$
125	$1.3534 \cdot 10^{-8}$ / $8.7397 \cdot 10^{-9}$	$7.4196 \cdot 10^{-8}$ / $4.7689 \cdot 10^{-8}$
250	$7.5825 \cdot 10^{-9}$ / $8.0044 \cdot 10^{-8}$	$3.3013 \cdot 10^{-8}$ / $3.1954 \cdot 10^{-7}$
500	$2.0954 \cdot 10^{-6}$ / $5.1463 \cdot 10^{-8}$	$4.4433 \cdot 10^{-6}$ / $1.5157 \cdot 10^{-7}$
1000	$9.5300 \cdot 10^{-5}$ / —	$9.5395 \cdot 10^{-5}$ / —

Table 2: Median relative objective discrepancies E_{obj} and probability-weighted wealth discrepancies E_q with respect to the numerical reference solution, over ten runs per formulation and horizon.

Both formulations satisfy the feasibility and objective-accuracy criteria in all repetitions for $T = 100, 125, 250$. At these horizons, the median solve-call time of formulation (19) is

approximately 2.13, 3.01, and 13.92 times that of formulation (3), respectively. At $T = 500$, formulation (3) is faster but fails the objective-accuracy criterion, whereas formulation (19) satisfies it. At $T = 1000$, formulation (3) returns feasible solutions that fail this criterion, while all runs of formulation (19) reach the external process timeout without returning a solution.

None of the returned solutions satisfies the prescribed componentwise accuracy criterion. Thus, a small objective discrepancy does not ensure accurate wealth reconstruction in every state. Moreover, at $T = 1000$, some reference probabilities underflow to zero, so the floating-point instances differ from the strictly positive probability model assumed in the analysis. The results therefore illustrate both the structural differences between the formulations and the numerical limitations of their implementation.

These comparisons concern only the two nonlinear wealth formulations and are separate from the complexity results in Section 3.2. The numerical reference is obtained through the proposed reduction, using bisection to compute the threshold. Its computation is not timed separately. Accordingly, the experiments do not assess the running times of the sorting-and-scanning procedure or Kiwiel’s median-search algorithms.

5 Conclusions

We have developed an explicit quadratic resource-allocation construction for the reduced portfolio selection problem in the ε -contaminated binomial model. The auxiliary optimizer is a least favourable probability measure and is independent of both the utility function and the initial budget. For increasing, strictly concave utility functions satisfying the stated regularity and boundary conditions, optimal terminal wealth is reconstructed through inverse marginal utility and a scalar budget equation. This construction specializes the general least favourable measure framework of Schied [14]. The emphasis is on its explicit finite-state realization and its connection to continuous quadratic knapsack algorithms. For CRRA utility, the budget equation is solved explicitly. Consequently, given the probability vectors and for every fixed $\gamma > 0$, optimal CRRA terminal wealth can be computed using $O(n \log n)$ operations by sorting and scanning, or $O(n)$ operations using Kiwiel’s established median-search algorithms. The resulting procedure replaces the active-set enumeration of Algorithm 1 in [11], which examines up to $n2^{n-1} = (T+1)2^T$ configurations, with a single threshold computation and explicit wealth reconstruction. The same auxiliary solution can be reused across different budgets and utility functions.

Further research may examine weaker assumptions on utility, identify additional utility families with explicit budget normalization, and analyze the numerical solution of the scalar budget equation when no closed-form reconstruction is available.

References

- [1] Bernard, C., Junike, G., Lux, T., Vanduffel, S. (2024). Cost-efficient payoffs under model ambiguity. *Finance and Stochastics*, 28(4), 965–997.
- [2] Bonami, P., Biegler, L. T., Conn, A. R., Cornuéjols, G., Grossmann, I. E., Laird, C. D., Lee, J., Lodi, A., Margot, F., Sawaya, N., Wächter, A. (2008). An algorithmic framework for convex mixed integer nonlinear programs. *Discrete Optimization*, 5(2), 186–204.
- [3] Boyd, S., Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press.
- [4] Bretthauer, K. M., Shetty, B. (2002). A pegging algorithm for the nonlinear resource allocation problem. *Computers & Operations Research*, 29(5), 505–527.

- [5] Černý, A. (2009). Mathematical techniques in finance: Tools for incomplete markets (2nd ed.). Princeton University Press.
- [6] Kiwiel, K. C. (2007). On linear-time algorithms for the continuous quadratic knapsack problem. *Journal of Optimization Theory and Applications*, 134, 549–554.
- [7] Kiwiel, K. C. (2008). Breakpoint searching algorithms for the continuous quadratic knapsack problem. *Mathematical Programming*, 112, 473–491.
- [8] Kiwiel, K. C. (2008). Variable fixing algorithms for the continuous quadratic knapsack problem. *Journal of Optimization Theory and Applications*, 136, 445–458.
- [9] Patriksson, M. (2008). A survey on the continuous nonlinear resource allocation problem. *European Journal of Operational Research*, 185(1), 1–46.
- [10] Patriksson, M., Strömberg, C. (2015). Algorithms for the continuous nonlinear resource allocation problem - New implementations and numerical studies. *European Journal of Operational Research*, 243(3), 703–722.
- [11] Petturiti, D., Vantaggi, B. (2024). The impact of ambiguity on dynamic portfolio selection in the epsilon-contaminated binomial market model. *European Journal of Operational Research*, 314(3), 1029–1039.
- [12] Petturiti, D., Stabile, G., Vantaggi, B. (2026). Optimal reinsurance in the epsilon-contaminated loss distribution preserving model. *Fuzzy Sets and Systems*, 110043.
- [13] Pliska, S. (1997). *Introduction to mathematical finance: Discrete time models*. Oxford: Blackwell Publishing.
- [14] Schied, A. (2005). Optimal investments for robust utility functionals in complete market models. *Mathematics of Operations Research*, 30(3), 750–764.
- [15] Wächter, A., Biegler, L. T. (2006). On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106, 25–57.