

# External and Internal Impedance for Robot Teams Sharing a Load: Exact Decoupling and Decay Certificates

Mehdi Heydari Shahna

`mehdi.heydarishahna@tuni.fi`

Tampere University

Jouni Mattila

Tampere University

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# External and Internal Impedance for Robot Teams Sharing a Load: Exact Decoupling and Decay Certificates

Mehdi Heydari Shahna  and Jouni Mattila 

**Abstract**—The six-degree-of-freedom dual-arm impedance framework combines external object impedance with internal end-effector impedance under perfect inner-loop tracking. This letter extends it to graph-compiled teams of  $N$  robots with mobile bases, floating bases in fixed support modes, and redundant actuation. A model-based inner loop fits requested actuator rates to admissible motion using the rate-gain metric and feeds forward the corresponding effort. With an exact model and a measured environment wrench, an error-power identity accounts for tracking errors, and the internal wrench separates into the command, opposing loads from inconsistent requests, and a decaying motion-error term. Internal efforts do not drive motion, even with inertial model errors, when motion requests and environment wrenches are independent of the internal command. With an exact model and a compliant environment satisfying a stated stability condition, all errors converge locally exponentially at an explicit rate. Reference displacements produce internal commands without motion bias exactly when they lie in a computable subspace; a run-time condition certifies grasp and support feasibility. Simulations verify the compiled-model results. With contact physics, four humanoids carry a 63 kg aluminium girder within their rated payload. A static capacity estimate matches all twelve mass-sweep outcomes; comparable palm-load changes produce 0.2 mm girder motion with internal commands versus 1.9 mm with displaced hand references.

**Index Terms**—Cooperative manipulation, impedance control, internal forces, robot teams, Lyapunov methods.

## I. INTRODUCTION

Robots carrying a shared object must control both its motion and the internal wrenches that load it without moving it [1], [2]. Impedance control [3] shapes both responses: object impedance governs contact with the environment [4], and end-effector impedance governs internal loading [5]. Caccavale *et al.* [6] combined them with geometrically consistent six-degree-of-freedom (6-DOF) stiffness [7]. Their outer loops generate object and end-effector references, which industrial joint PID loops track. Experiments with two arms showed limited external and internal wrenches, prevention of slipping, and nearly independent action of the two impedances.

The analysis in [6] assumes perfect inner-loop tracking and establishes independence of the internal impedance at steady state. Rigidly grasping arms generate uncommanded internal wrenches when their references violate the closed-chain constraints. The measured-wrench decomposition also depends on the grasp-matrix pseudoinverse [2], [8], [9]. Exact-model control decouples closed-chain motion and internal

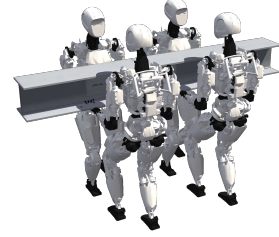


Fig. 1. Four Apollo humanoids carry a 63 kg aluminium girder (16 kg per robot) with the proposed two-loop impedance (MuJoCo, contact physics; all robots in single support).

force [1]. Heck *et al.* [10] combine both impedances with inverse dynamics for nonredundant arms and prove local asymptotic stability without a rate. Other approaches use force-sensor-free control [11], graph rigidity [12], adaptation [13], projection of contradicting forces [14], passive decomposition [15], or internal-force impedance with virtual decomposition control (VDC) for planar dual arms [16]. Exponential stability under torque constraints is established for single manipulators [17]. For the two-impedance scheme of [6], two questions remain: how to certify an explicit exponential rate for  $N$  robots with mobile or floating bases and redundant actuation while accounting for inner-loop errors, and how inconsistent references affect the internal impedance.

This letter extends the scheme of [6] to teams compiled from mechanism graphs [18], including humanoids carrying shared loads [19]. The inner loop follows model-based VDC [20]–[23] and modular control of heavy-duty manipulators and actuators under input constraints [24], [25]. It fits composite (required) velocities [26] using the rate-gain metric and feeds forward the corresponding effort. The feedforward effort combines the inertial model with the measured environment wrench. Reference [27] develops the fit (3), the identity of Theorem 1(a), and the rate of Theorem 2(a) without internal commands or inconsistent requests; it also synthesizes the gains robot by robot. These ingredients are proved here for completeness. The new results are:

- 1) *Internal-wrench structure.* For an exact model,  $\lambda_I = \psi + \Psi K w_\perp + \Theta s$ : the command, opposing loads from inconsistent requests (*antagonism*), and a term linear in the inner-loop error. Motion decoupling is a known plant property [14]; it holds even with inertial model errors when motion requests and environment wrenches are independent of the internal command (Theorem 1).
- 2) *Certified realization.* For an exact model and a stable

This work was supported by the Post Docs in Companies (PoDoCo) program, Finland. Authors are with the Faculty of Engineering and Natural Sciences, Tampere University, Tampere, Finland. (Corresponding author e-mail: mehdi.heydarishahna@tuni.fi). Code, simulation results, and a demonstration video are available at <https://github.com/Mehdi-Heydari-Shahna/Team-Impedance-Control>.

target interaction, inner-loop, payload, posture, internal-wrench and compliant-frame errors converge locally exponentially at any rate below  $\min\{\kappa_s, \kappa_\xi, \kappa_z, \kappa_T\}$ . The rate follows from the gains and models, without time-scale separation (Theorem 2).

- 3) *Reference commands and feasibility.* Reference displacements produce internal commands without motion bias exactly when they lie in a computable subspace. Otherwise, they bias motion and may create antagonism. For an exact model and convergent antagonism, internal stiffness  $K_z$  reduces its steady-state effect by  $K_z(K_z + I)^{-1}$ ; integral action rejects it. A run-time inequality certifies grasp and support feasibility (Proposition 1, Corollary 1).

The construction task in Fig. 1 demonstrates the scheme: four humanoids walk to an aluminium girder, pick it up, carry it and unload it in contact physics (Section V). We compare internal commands with the reference displacements of [6].

*Notation.*  $\mathcal{R}(X)$ ,  $\mathcal{N}(X)$ : range and null space of  $X$ ;  $X \succ 0$  ( $\succeq 0$ ): symmetric positive (semi)definite;  $X^+$ : Moore–Penrose inverse;  $\|x\|_M := (x^\top M x)^{1/2}$ ;  $\lambda_{\min}(X, Y)$ : smallest generalized eigenvalue of  $(X, Y)$ ,  $Y \succ 0$ , and  $\lambda_{\min}(X) := \lambda_{\min}(X, I)$ ;  $S(\cdot)$ : cross-product matrix,  $(\cdot)^\vee$  its inverse;  $\log$ : principal logarithm of  $SO(3)$ ;  $D^+$ : upper right Dini derivative.

## II. TEAM MODEL AND PROBLEM

### A. Compiled team model

Consider  $N$  robots holding a rigid payload through  $N_g \geq N$  rigid grasps (e.g., two hands per humanoid). The team compiler [18], [27] joins their mechanism graphs through the payload. With all grasps closed, it returns the constraint Jacobian  $A(q) \in \mathbb{R}^{r \times n}$  for grasps and other ideal connections (wheels, ground and feet), a tangent lift  $E(q) \in \mathbb{R}^{n \times d}$  whose columns span  $\mathcal{N}(A)$ , the actuator map  $B \in \mathbb{R}^{n \times m}$ , and the rigid-body model

$$M\dot{v} + C(q, v)v + g + D_v v = Bf + A^\top \lambda + J_o^\top h, \quad Av = 0, \quad (1)$$

where  $v$  is the generalized velocity,  $M(q) \succ 0$ ,  $\dot{M} - 2C$  is skew-symmetric,  $D_v \succeq 0$  is damping,  $f \in \mathbb{R}^m$  contains actuator efforts, and  $\lambda \in \mathbb{R}^r$  contains constraint multipliers. The environment applies wrench  $h$  to the payload, with moment about its origin  $p$ . The payload twist  $\nu := (\dot{p}, \omega) = J_o(q)v$  has world-frame components. Admissible velocities are  $v = Eu$ ,  $u \in \mathbb{R}^d$ , and  $L(q) := B^\top E$  maps independent speeds to actuator rates. We choose  $u = (\nu, u_b)$ , where  $u_b = \dot{q}_b$  contains rates of the remaining independent coordinates  $q_b$  (omnidirectional bases, lifts, posture). The compiled interfaces admit this choice of speeds [18].

The unactuated payload contacts the robots only through the grasps. Let  $\lambda_g \in \mathbb{R}^{6N_g}$  contain the grasp wrenches (rows of  $\lambda$ ) in the tool frames. The payload rows of (1) give its Newton–Euler equations  $M_o \dot{\nu} + c_o = G\lambda_g + h$ , where  $c_o$  contains velocity and gravity terms and the grasp matrix  $G \in \mathbb{R}^{6 \times 6N_g}$  has rank 6.  $G$  depends on payload pose only

through the net-wrench frame, so  $\mathcal{N}(G)$  is constant while the grasps remain closed.

### B. Load distribution and internal wrench

Motion fixes the net wrench  $G\lambda_g$ ; its distribution among robots is a choice. We define a weighted load distribution and its complementary internal projector,

$$D_G := W^{-1}G^\top(GW^{-1}G^\top)^{-1}, \quad P_I := I - D_G G, \quad (2)$$

with a constant block-diagonal  $W \succ 0$  (moments scaled by a characteristic length). Then  $GD_G = I$ ,  $P_I^\top = P_I$ ,  $\mathcal{R}(P_I) = \mathcal{N}(G)$ ,  $\lambda_g = D_G G\lambda_g + P_I \lambda_g$ , and  $D_G G$  and  $P_I$  are constant while the grasps are closed.  $\lambda_I := P_I \lambda_g$  is the *internal wrench* relative to the declared distribution; whether  $D_G G\lambda_g$  itself squeezes the payload depends on  $W$  [8]. The columns of  $V_I$  form a basis of  $\mathcal{N}(G)$ , with left inverse  $V_I^+$ .

### C. Problem statement

*Problem.* For (1) with  $N \geq 2$ , design the efforts in two loops such that (i) the payload realizes the external impedance of [6] toward the environment, (ii)  $\lambda_I$  converges to a desired  $\lambda_d \in \mathcal{N}(G)$  with the dynamics of an internal impedance in wrench coordinates, and (iii) all tracking and interaction errors, including inner-loop errors, converge exponentially at an explicit rate.

**Assumption 1** (Regular mode). *On a compact operating set  $\mathcal{O}$  of plant and controller states,  $A$  has full row rank, the partition  $u = (\nu, u_b)$  is valid (its dependent block of  $A$  is invertible),  $L$  has full column rank (the actuators can generate effort in every admissible direction), and the map  $\Psi$  of Lemma 1 is onto  $\mathcal{N}(G)$ .*

**Lemma 1** (Effort split, cf. [14]). *Let  $K \succ 0$  be diagonal. On  $\mathcal{O}$ : (i)  $\mathbb{R}^m = \mathcal{R}(KL) \oplus \mathcal{N}(L^\top)$ . (ii) For  $f \in \mathcal{N}(L^\top)$  there is a unique  $\mu(f)$  with  $A^\top \mu(f) = Bf$ , namely  $\mu(f) = (AA^\top)^{-1}ABf$ , and its grasp rows satisfy  $G\mu_g(f) = 0$ ; we set  $\Psi f := -\mu_g(f) \in \mathcal{N}(G)$ . (iii) Adding  $f \in \mathcal{N}(L^\top)$  to the effort in (1) leaves  $\dot{v}$  unchanged and changes  $\lambda$  by  $-\mu(f)$ , hence  $\lambda_g$  by  $\Psi f$ , whatever  $M$ ,  $C$ ,  $g$  and  $h$ .*

*Proof.* (i) If  $KLx \in \mathcal{N}(L^\top)$ , then  $L^\top KLx = 0$  and  $x = 0$ ; the dimensions add up to  $m$ . (ii)  $L^\top f = E^\top Bf = 0$  gives  $Bf \in \mathcal{N}(E^\top) = \mathcal{R}(A^\top)$ , and  $A^\top$  is injective. The payload rows of  $A^\top \mu = Bf$  read  $G\mu_g = 0$ . (iii)  $(\dot{v}, \lambda - \mu(f))$  solves (1) with  $f$  added, and the solution  $(\dot{v}, \lambda)$  is unique since  $M \succ 0$  and  $A$  has full row rank.  $\square$

Let  $\tilde{\Psi} := -[(AA^\top)^{-1}AB]_g$  (grasp rows), so that  $\Psi = \tilde{\Psi}$  on  $\mathcal{N}(L^\top)$ , and let  $\Pi_L := I - L(L^\top L)^{-1}L^\top$ . Under Assumption 1,  $\Psi^+ := (\tilde{\Psi}\Pi_L)^+$  is smooth on  $\mathcal{O}$  (constant rank), maps into  $\mathcal{N}(L^\top)$ , and satisfies  $\Psi\Psi^+\psi = \psi$  for  $\psi \in \mathcal{N}(G)$ :  $\Psi^+\psi$  is the minimum-norm effort in  $\mathcal{N}(L^\top)$  that produces grasp loading  $\psi$ . Any smooth right inverse of  $\Psi$  from  $\mathcal{N}(G)$  to  $\mathcal{N}(L^\top)$  can replace  $\Psi^+$  below; if each robot can hold any grasp wrenches with its own actuators against its own connections, a block-diagonal one exists, so internal efforts are computed robot by robot. For fixed-base six-joint arms with one grasp each,  $\dim \mathcal{N}(L^\top) = 6N - 6 = \dim \mathcal{N}(G)$  and  $\Psi$  is invertible.

### III. TWO-LOOP TEAM IMPEDANCE

#### A. Inner loop: fitted reference and required effort

The outer loops request actuator rates  $w \in \mathbb{R}^m$ , which may violate the closed-chain constraints. We fit the closest admissible motion in the metric of the rate gains  $K = \text{diag}(k_i) \succ 0$  [27],

$$u_r := (L^\top K L)^{-1} L^\top K w, \quad w_\perp := w - L u_r, \quad (3)$$

so that  $L^\top K w_\perp = 0$ , and  $v_r := E u_r$ . The required effort is a pair  $(f_r, \lambda_r)$  with

$$B f_r + A^\top \lambda_r = \tau, \quad P_I \lambda_{r,g} = 0, \quad (4)$$

where  $\tau := M \dot{v}_r + C(q, v) v_r + g + D_v v_r - J_o^\top h$ , and  $\dot{v}_r := E \dot{u}_r + \dot{E} u_r$  is evaluated along the measured motion. Such a pair exists, and its grasp component  $\lambda_{r,g}$  is unique. Set  $\bar{f} := K L (L^\top K L)^{-1} E^\top \tau$ . Then  $\tau - B \bar{f} = A^\top \bar{\lambda}$  for some  $\bar{\lambda}$ , and  $f_r := \bar{f} - \Psi^+ P_I \bar{\lambda}_g$ ,  $\lambda_r := \bar{\lambda} + \mu(\Psi^+ P_I \bar{\lambda}_g)$  solve (4). Two solutions differ by an  $f \in \mathcal{N}(L^\top)$  with  $\Psi f \in \mathcal{N}(G) \cap \mathcal{N}(P_I) = \{0\}$ ; this difference loads only the other connections. The inner loop is

$$f = f_r - K(B^\top v - w) + \Psi^+ \psi, \quad (5)$$

with an internal command  $\psi \in \mathcal{N}(G)$ .

*Remark 1* (VDC and [6]). In VDC terms, the requests are required actuator velocities,  $(v_r, f_r)$  are the team's required velocity and force, and the eliminated constraint term in Theorem 1(a) sums the virtual power flows at the connections [20], [22]. With requests  $w = \dot{a}_r + K^{-1} K_a (a_r - a)$ , diagonal  $K_a \succ 0$ , from references  $a_r$  of the actuator positions  $a$ , (5) is a joint PD loop with model-based feedforward  $f_r$ , whereas the PID loops of [6] use no model.

#### B. External impedance and payload requests

Let  $x_d = (p_d, R_d) (\Sigma_d)$  be the desired pose, of class  $C^2$ . The compliant frame  $x_c = (p_c, R_c) (\Sigma_c)$ , with  $\nu_c := (\dot{p}_c, \omega_c)$ , obeys the impedance [6, (20)] in the form of Remark 4

$$M_c \Delta \dot{\nu} + D_c \Delta \nu + h_\Delta(x_d, x_c) = -h, \quad \Delta \nu := \nu_d - \nu_c, \quad (6)$$

The desired twist, expressed at  $p_c$ , is  $\nu_d := (\dot{p}_d + S(\omega_d)(p_c - p_d), \omega_d)$ , with  $M_c, D_c \succ 0$ . The elastic wrench  $h_\Delta = (f_\Delta, \mu_\Delta)$  uses the geometrically consistent stiffness [6, (14)–(17)] with  $\Sigma_r = \Sigma_c$  and moment about  $p_c$ . Its components are  $f_\Delta = K'_p \Delta p$  and  $\mu_\Delta = \frac{1}{2} S(\Delta p) R_c K_p R_c^\top \Delta p + \mu_{\Delta o}$ , where  $\Delta p := p_d - p_c$  and  $K'_p = \frac{1}{2} (R_d K_p R_d^\top + R_c K_p R_c^\top)$ . The orientation term is  $\mu_{\Delta o} = 2 R_c E_\epsilon^\top K_o \epsilon$ , with  $E_\epsilon := \eta I - S(\epsilon)$ . Here  $(\eta, \epsilon)$ ,  $\eta \geq 0$ , is the unit quaternion of  $R_c^\top R_d = (\eta^2 - \epsilon^\top \epsilon) I + 2 \epsilon \epsilon^\top + 2 \eta S(\epsilon)$  [7], and  $K_p, K_o \succ 0$ . The potentials are  $V_p = \frac{1}{2} \Delta p^\top K'_p \Delta p$  and  $V_o = 2 \epsilon^\top K_o \epsilon$ .

Angular velocities have world-frame components ( $\dot{R}_c = S(\omega_c) R_c$ ,  $\dot{R}_d = S(\omega_d) R_d$ ), so  $\dot{\epsilon} = \frac{1}{2} E_c R_c^\top (\omega_d - \omega_c)$  and  $\dot{V}_o = \mu_{\Delta o}^\top (\omega_d - \omega_c)$  for any  $K_o \succ 0$ . Since  $h$  is measured (Assumption 2),  $M_c$  can be chosen freely; [6, (23)] instead ties it to payload inertia to use wrist sensing alone.

Define the pose error  $\xi_o := (p_c - p, \varphi)$ ,  $\varphi := \log(R_c R^\top)^\vee$ ,  $\|\varphi\| < \pi$ . The payload tracks  $x_c$ , while the remaining

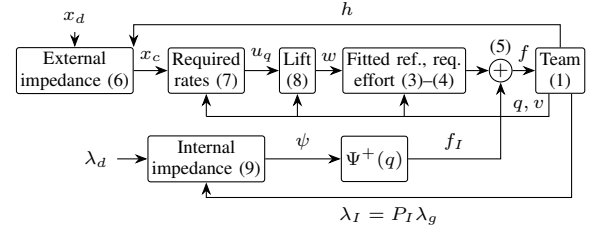


Fig. 2. Two-loop impedance: motion path (top), internal path (bottom). Internal efforts  $f_I = \Psi^+ \psi \in \mathcal{N}(L^\top)$  do not drive admissible motion under the conditions of Theorem 1(d).

independent coordinates track their  $C^2$  references  $q_{b,d}$  (using logarithms for orientations). With  $\Gamma_o := \text{diag}(\gamma_p I_3, \gamma_o I_3)$ , choose the required rates

$$\nu_q := \nu_c + \Gamma_o \xi_o, \quad u_{b,q} := \dot{q}_{b,d} + \Gamma_b (q_{b,d} - q_b). \quad (7)$$

The compiled lift propagates them to every actuator,

$$w := L(q) u_q, \quad u_q := (\nu_q, u_{b,q}). \quad (8)$$

These requests are consistent: (3) gives  $u_r = u_q$  and  $w_\perp = 0$ . We write  $\xi := (\xi_o, q_{b,d} - q_b)$  and  $\Gamma := \text{diag}(\Gamma_o, \Gamma_b) \succ 0$ .

#### C. Internal impedance

The internal command responds to the measured internal-wrench error through mass–spring–damper dynamics in wrench coordinates,

$$\psi = \lambda_d + V_I z, \quad M_z \ddot{z} + D_z \dot{z} + K_z z = -V_I^+ (\lambda_I - \lambda_d), \quad (9)$$

with symmetric  $M_z, D_z \succ 0$ ,  $K_z \succeq 0$ , and  $\lambda_I = P_I \lambda_g$  measured by the wrist sensors;  $K_z = 0$  is integral action on the internal wrench. Fig. 2 shows the scheme.

### IV. MAIN RESULTS

**Theorem 1** (Exact structure). *Let Assumption 1 hold and apply (5) with  $C^1$  requests  $w$  and an internal command  $\psi \in \mathcal{N}(G)$ . Let  $s := v - v_r$ . Parts (a)–(c) require exact evaluation of (4); part (d) also allows inertial model error. Along solutions of (1) in  $\mathcal{O}$ :*

- (a)  $\frac{d}{dt} \frac{1}{2} s^\top M s = -s^\top (B K B^\top + D_v) s$ ;
- (b)  $\lambda - \lambda_r + \mu(K w_\perp + \Psi^+ \psi) = \Xi s$ , where

$$\Xi := (A M^{-1} A^\top)^{-1} (A M^{-1} (C + D_v + B K B^\top) - \dot{A});$$

in particular,  $\lambda_I = \psi + \Psi K w_\perp + \Theta s$  with  $\Theta := P_I \Xi_g$ , where  $\Xi_g$  denotes the grasp rows of  $\Xi$ ;

- (c)  $G \lambda_g = G \lambda_{r,g} + G \Xi_g s$ , independently of  $\psi$  and  $w_\perp$ ;
- (d) if  $w$  and  $h$ , as functions of time and states, do not depend on  $\psi$ , neither does the motion  $(q, v)$ , even when (4) is evaluated with an inexact inertial model.

*Proof.* Subtracting (4) from (1) with (5) gives, with  $f_\perp := K w_\perp + \Psi^+ \psi$ ,

$$M \dot{s} + (C + D_v + B K B^\top) s = B f_\perp + A^\top (\lambda - \lambda_r). \quad (10)$$

Both  $K w_\perp$  and  $\Psi^+ \psi$  lie in  $\mathcal{N}(L^\top)$ , so  $B f_\perp = A^\top \mu$  with  $\mu := \mu(f_\perp)$  by Lemma 1. (a)  $s = E(u - u_r) \in \mathcal{N}(A)$ ; multiplying (10) by  $s^\top$  removes the constraint term, and the skew-symmetry of  $M - 2C$  gives (a). (b) Differentiating  $A s =$

0 gives  $A\dot{s} = -\dot{A}s$ ; multiplying (10) by  $AM^{-1}$  gives the first claim. Its grasp rows read  $\lambda_g = \lambda_{r,g} + \Psi K w_\perp + \psi + \Xi_g s$ .  $P_I$  annihilates  $\lambda_{r,g}$  by (4) and is the identity on  $\mathcal{N}(G)$ . (c)  $G$  annihilates  $\mathcal{N}(G)$ . (d) Premultiplying (1) by  $E^\top$  eliminates  $\lambda$ :  $E^\top(M\dot{v} + Cv + g + D_v v - J_o^\top h) = L^\top f = L^\top f_r - L^\top K B^\top s$ , since  $L^\top$  annihilates  $K w_\perp$  and  $\Psi^+ \psi$ . With  $v = Eu$ ,  $\dot{v} = E\dot{u} + \dot{E}u$ , this determines  $\dot{u}$  from  $(q, v, w, \dot{w}, h)$ , for any model in (4).  $\square$

**Remark 2** (Meaning). Part (a) quantifies inner-loop tracking error. Part (b) separates the internal wrench into commanded loading, antagonism from inconsistent requests [14], and tracking-error loading. Parts (c)–(d) establish motion–internal-effort decoupling throughout the motion.

**Assumption 2** (Environment and target interaction). *The measured environment wrench  $h = h_e(x, \nu)$  depends on the payload state, with  $\|h_e(x, \nu) - h_e(x_c, \nu_c)\| \leq \ell(\|\xi_o\| + \|\nu - \nu_c\|)$  on  $\mathcal{O}$ . The target interaction is (6) evaluated at the compliant-frame state. With right side  $-h_e(x_c, \nu_c) + \delta$ , it has a solution  $(x_c^*, \nu_c^*)(t)$  for  $\delta = 0$  and an exponential-stability certificate  $V_T(t, \chi)$ ,  $\chi := (p_c - p_c^*, \log(R_c R_c^{*\top})^\vee, \nu_c - \nu_c^*)$ , with  $c_1 \|\chi\|^2 \leq V_T \leq c_2 \|\chi\|^2$  and  $\dot{V}_T \leq -2\kappa_T V_T + c_3 \|\chi\| \|\delta\|$  on  $\mathcal{O}$ ,  $c_1, c_2, c_3, \kappa_T > 0$ .*

**Theorem 2** (Certified convergence). *Let Assumptions 1–2 hold, apply (3)–(9) with the exact model, and let the solution remain in  $\mathcal{O}$  with  $\|\varphi\| < \pi$ . Let*

$$\kappa_s := \min_{\mathcal{O}} \lambda_{\min}(E^\top(BKB^\top + D_v)E, E^\top ME),$$

$$\bar{j} := \max_{\mathcal{O}} [\lambda_{\min}(E^\top ME)]^{-\frac{1}{2}}, \quad \bar{\vartheta} := \max_{\mathcal{O}} \|V_I^+ \Theta M^{-\frac{1}{2}}\|,$$

$\kappa_\xi := \lambda_{\min}(\Gamma)$ , and let  $\kappa_z > 0$  be the decay rate (negative of the spectral abscissa) of  $M_z \ddot{z} + D_z \dot{z} + (K_z + I)z = 0$ . Then:

- (a)  $\|s(t)\|_M \leq e^{-\kappa_s t} \|s(0)\|_M$ ;
- (b)  $\|\xi(t)\| \leq e^{-\kappa_\xi t} \|\xi(0)\| + \bar{j} \beta(t) \|s(0)\|_M$ , with  $\beta(t) := (e^{-\kappa_s t} - e^{-\kappa_\xi t}) / (\kappa_\xi - \kappa_s)$  if  $\kappa_\xi = \kappa_s$ ;
- (c) the internal-wrench error  $e_I := V_I^+ (\lambda_I - \lambda_d)$  and  $(z, \dot{z})$  decay exponentially with any rate below  $\min\{\kappa_z, \kappa_s\}$ ;
- (d)  $\chi \rightarrow 0$  with any rate below  $\min\{\kappa_s, \kappa_\xi, \kappa_T\}$ .

Hence the closed loop converges exponentially, with any rate below  $\min\{\kappa_s, \kappa_\xi, \kappa_z, \kappa_T\}$ , to the target interaction with  $\lambda_I = \lambda_d$ , and (a), (b) and (d) do not depend on (9).

*Proof.* (a) With  $s = E\sigma$ ,  $\sigma := u - u_r$ , Theorem 1(a) gives  $\frac{d}{dt} \|s\|_M^2 = -2\sigma^\top E^\top (BKB^\top + D_v) E \sigma \leq -2\kappa_s \|s\|_M^2$ . (b) Since  $u_r = u_q$ ,  $u = u_q + \sigma$  with  $\|\sigma\| \leq \bar{j} \|s\|_M$ . For the translation,  $\dot{p}_c - \dot{p} = -\gamma_p(p_c - p) - \sigma_p$ . For the rotation,  $\tilde{R} := R_c R^\top$  obeys  $\dot{\tilde{R}} = S(\omega_c - \tilde{R}\omega) \tilde{R}$ , so  $\dot{\varphi} = J_I^{-1}(\varphi)(\omega_c - \tilde{R}\omega)$  with the left Jacobian  $J_I$  of  $SO(3)$ ; since  $\varphi^\top S(\varphi) = 0$  gives  $\varphi^\top J_I^{-1}(\varphi) = \varphi^\top$  and  $\tilde{R}^\top \varphi = \varphi$ ,  $\frac{d}{dt} \frac{1}{2} \|\varphi\|^2 = \varphi^\top (\omega_c - \omega) = -\gamma_o \|\varphi\|^2 - \varphi^\top \sigma_\omega$ . Finally  $\frac{d}{dt} (q_{b,d} - q_b) = -\Gamma_b(q_{b,d} - q_b) - \sigma_b$ , with orientation blocks treated as  $\varphi$ . Hence  $D^+ \|\xi\| \leq -\kappa_\xi \|\xi\| + \bar{j} \|s\|_M$ , and the comparison lemma [28] with (a) gives (b). (c) With  $w_\perp = 0$ , Theorem 1(b) gives  $e_I = z + \vartheta$ ,  $\vartheta := V_I^+ \Theta s$ ,  $\|\vartheta\| \leq \bar{\vartheta} \|s\|_M$ , and (9) becomes  $M_z \ddot{z} + D_z \dot{z} + (K_z + I)z = -\vartheta$ : an exponentially stable linear system driven

by an input that decays with rate  $\kappa_s$ . Variation of constants gives (c). (d) With  $\delta := h_e(x_c, \nu_c) - h_e(x, \nu)$ , (6) is the target interaction driven by  $\delta$ ; since  $\nu - \nu_c = \Gamma_o \xi_o + \sigma_\nu$ ,  $\|\delta\| \leq \ell(1 + \|\Gamma\|) \|\xi\| + \ell \bar{j} \|s\|_M$ , which decays by (a)–(b). Then  $D^+ \sqrt{V_T} \leq -\kappa_T \sqrt{V_T} + c_3 \|\delta\| / (2\sqrt{c_1})$ , and the comparison lemma gives (d). Theorem 1(d) gives the last claim.  $\square$

**Remark 3** (Invariance, robustness, and the internal impedance). The bounds of Theorem 2 are linear in the initial errors  $(s, \xi, z, \dot{z}, \chi)(0)$ . If the nominal motion  $(s = 0, \xi = 0, z = \dot{z} = 0, \chi = 0)$  stays a distance  $\varrho > 0$  inside  $\mathcal{O}$ , sufficiently small initial errors keep solutions in  $\mathcal{O}$ , establishing local exponential stability.

An inertial model error adds  $\tilde{\tau} := \hat{\tau} - \tau$  ( $\hat{\tau}$ : model-computed effort) to the right side of (10); then, while the solution stays in  $\mathcal{O}$ ,  $\|s(t)\|_M \leq e^{-\kappa_s t} \|s(0)\|_M + \kappa_s^{-1} \sup_{t' \leq t} \|M_u^{-1/2} E^\top \tilde{\tau}(t')\|$ ,  $M_u := E^\top M E$ , and Theorem 2(b)–(d) become input-to-state stability (ISS) bounds in  $s$  and  $\tilde{\tau}$ . At rest with constant references and exact  $D_v$ , the requests settle at  $u_q = -(L^\top K L + E^\top D_v E)^{-1} E^\top \tilde{\tau}$  and bias  $\xi$  by  $\Gamma^{-1} u_q$ .

With consistent requests and an exact model,  $\psi \equiv \lambda_d$  already gives  $\lambda_I \rightarrow \lambda_d$  with rate  $\kappa_s$ . Internal feedback (9) attenuates convergent antagonism (Proposition 1(iv)); with  $K_z = 0$ , it rejects this antagonism and convergent model-error terms, which add  $-P_I[(AM^{-1}A^\top)^{-1}AM^{-1}\tilde{\tau}]_g$  to  $\lambda_I$ .

**Remark 4** (The stiffness of [6]). With the definitions stated before [6, (15)], the power identity there holds for  $K_p'' = -\frac{1}{2} S(p_d - p_r) R_d K_p R_d^\top$  (opposite sign) and for the twist  $(\frac{d}{dt}[R_r^\top(p_d - p_r)], R_r^\top(\omega_d - \omega_r))$ , which then also replaces  $R_r^\top(\dot{p}_d - \dot{p}_r)$  in [6, (20)]; with the moment about  $p_c$ ,  $\dot{V}_p + \dot{V}_o = \Delta v^\top h_\Delta$  for any  $K_p, K_o \succ 0$ , the twists of (6) and all motions of  $\Sigma_d$  and  $\Sigma_c$ . Hence, for constant  $x_d$  and  $h_e = (-K_e(p - p_e) - D_e \dot{p}, 0)$ ,  $K_e, D_e \succeq 0$ , the target interaction is a mechanical system with potential  $V_p + V_o + \frac{1}{2} \|p_c - p_e\|_{K_e}^2$  and damping  $D_c + \text{diag}(D_e, 0) \succ 0$ , and Assumption 2 holds locally at a nondegenerate minimum of the potential, with any  $\kappa_T$  below the decay rate of the linearization.

**Proposition 1** (Reference-level commands and inconsistent requests). *Let  $w = L(q)u_q + w_a$ , where the  $C^1$  request perturbation  $w_a$  need not satisfy the constraints. The motion bias and internal loading are:*

- (i)  $u_r = u_q + \delta u_a$ ,  $\delta u_a := (L^\top K L)^{-1} L^\top K w_a$ ; Theorem 1 holds, Theorem 2(b) has  $\int_0^t e^{-\kappa_\xi(t-t')} \|\delta u_a(t')\| dt'$  added, and (c)–(d) hold as ISS bounds;
- (ii)  $\lambda_I = \psi + \alpha + \Theta s$  with the antagonism  $\alpha := \Psi K(w_a - L \delta u_a)$ ;
- (iii)  $w_a = K^{-1} \Psi^+ \psi'$ ,  $\psi' \in \mathcal{N}(G)$ , gives  $\delta u_a = 0$  and  $\alpha = \psi'$ , equivalent to (5) with  $\psi + \psi'$ . For requests from actuator references as in Remark 1, a displacement  $\delta a$  of  $a_r$  in the stiffness term adds  $w_a = K^{-1} K_a \delta a$  and, by the linearity of (3),  $\delta u_a = (L^\top K L)^{-1} L^\top K_a \delta a$  and  $\alpha = \Psi(K_a \delta a - K L \delta u_a)$ : it issues the internal command  $\Psi K_a \delta a$  without biasing the motion if and only if  $\delta a(t) \in K_a^{-1} \mathcal{N}(L^\top(q(t)))$  for all  $t$ ;

(iv) under (9) and  $\alpha(t) \rightarrow \alpha_\infty$ ,  $e_I \rightarrow K_z(K_z + I)^{-1}V_I^+\alpha_\infty$ : integral action ( $K_z = 0$ ) rejects the antagonism, positive internal stiffness reduces it by  $K_z(K_z + I)^{-1}$ , and  $\psi \equiv \lambda_d$  gives  $e_I \rightarrow V_I^+\alpha_\infty$ . The part of  $Kw_\perp$  in  $\mathcal{N}(\Psi)$  loads the other connections and is not rejected.

*Proof.* (i) follows from the linearity of (3) and from the proof of Theorem 2(b) with  $u - u_q = \sigma + \delta u_a$ . (ii)  $w_\perp = w_a - L\delta u_a$ ; apply Theorem 1(b). (iii)  $L^\top K K^{-1}\Psi^+\psi' = 0$  and  $\Psi\Psi^+\psi' = \psi'$ . For  $w_a = K^{-1}K_a\delta a$ ,  $\delta u_a = 0$  if and only if  $L^\top K_a\delta a = 0$ , since  $L^\top K L$  is invertible. (iv) Substituting (ii) into (9) gives  $M_z\ddot{z} + D_z\dot{z} + (K_z + I)z = -V_I^+\alpha - \vartheta$ , so  $z \rightarrow -(K_z + I)^{-1}V_I^+\alpha_\infty$  while  $\vartheta \rightarrow 0$ , and  $e_I = z + V_I^+\alpha + \vartheta$ .  $\square$

**Remark 5** (The scheme of [6] revisited). The internal impedance of [6] displaces end-effector references. With PD-type joint loops and rigid grasps, these displacements bias motion unless they lie in  $K_a^{-1}\mathcal{N}(L^\top)$  (Proposition 1(iii)); for the arms of [6],  $u = \nu$ , so the payload moves. Here  $\lambda_d$  commands the squeeze needed for friction grasps, and Corollary 1 checks feasibility. Feeding joint loops with  $Lu_r$  instead of  $w$  removes antagonism, as in the projection of [14].

**Corollary 1** (Grasp and support feasibility). *Under Theorem 2,  $\lambda = \lambda^* - \mu(\Psi^+V_I z) + \Xi s$  with the computable  $\lambda^* := \lambda_r - \mu(\Psi^+\lambda_d)$ ; on the grasp rows,  $\lambda_g^* = \lambda_{r,g} + \lambda_d$  and  $-\mu_g(\Psi^+V_I z) = V_I z$ . For a sufficient feasibility condition, let  $\mathcal{C}_k$  be a closed set of admissible wrenches of connection  $k$  (e.g., the contact-wrench cone of a grasp or foot, or a unilateral wheel load),  $\lambda_k^*(t) \in \mathcal{C}_k$ , and  $\rho_k(t)$  its distance from the boundary of  $\mathcal{C}_k$  (moments scaled by a length). If  $\|\mu(\Psi^+V_I z(t))\|_k + \bar{\zeta}_k\|s(t)\|_M < \rho_k(t)$  for all  $t$ , with  $\bar{\zeta}_k := \max_{\mathcal{O}}\|\Xi_k M^{-1/2}\|$ , then  $\lambda_k(t) \in \mathcal{C}_k$  for all  $t$ ; by Theorem 2, the left side converges to zero exponentially.*

*Proof.* Theorem 1(b) with  $w_\perp = 0$ ,  $\psi = \lambda_d + V_I z$ , the linearity of  $\mu$ , and  $-\mu_g(\Psi^+\psi) = \Psi\Psi^+\psi = \psi$ .  $\square$

**Remark 6** (Scope). The certificates apply to continuous-time dynamics within each fixed compiled mode, with unsaturated actuators, ideal bilateral connections (including grounded feet and omnidirectional wheels), and a smooth environment. With kinematic errors,  $\hat{\Psi}^+\psi \in \mathcal{N}(\hat{L}^\top)$  drives motion through  $L^\top \hat{\Psi}^+\psi = (L - \hat{L})^\top \hat{\Psi}^+\psi$ , entering as  $E^\top \tilde{\tau}$  in Remark 3. Sampling likewise adds a bounded input. Under a zero-order hold,  $\Psi^+(q_k)\psi_k$  leaves  $\mathcal{N}(L^\top)$  between samples, so Theorem 1(d) holds up to terms of the order of the sampling period.

## V. APPLICATION

Four Apptronik Apollo humanoids support a 2.4 m aluminium I-beam with an HEA 280 cross-section (Fig. 1). Each places both palms beneath the bottom flange without gripping ( $N_g = 8$ ). Each Menagerie model [29] weighs 80.9 kg and has 32 torque-limited actuators. The girder weighs 63 kg;

TABLE I  
SIMULATION AND CONTROLLER PARAMETERS (SI UNITS)

|                                            |                                                                                            |
|--------------------------------------------|--------------------------------------------------------------------------------------------|
| Plant                                      | MuJoCo, 1 ms step, implicit-fast, elliptic cones                                           |
| Friction: sole, palm, pads                 | 0.8, 0.7, 0.7 (wood)/0.9 (rubber)                                                          |
| Contact time constants                     | soles 8, palms 5, pads 7/12 ms (per contact)                                               |
| Robot; girder                              | 80.9 kg, limits 10.6–494 N m; 2.4 m, 63.1 kg                                               |
| Control; QP; $h$ filter                    | 500 Hz ( $-KB^\top v$ continuous); 50 Hz; 60 rad/s                                         |
| $k_i$ ; $\gamma_p = \gamma_o$ ; $\Gamma_b$ | 1–240 N m s/rad; 4; 4I                                                                     |
| $M_c$ ; $D_c$                              | diag(150I <sub>3</sub> , 20I <sub>3</sub> ); diag(1900I <sub>3</sub> , 270I <sub>3</sub> ) |
| $K_p$ ; $K_o$ ; $M_z$ ; $D_z$ ; $K_z$      | 6000I <sub>3</sub> ; 900I <sub>3</sub> ; 0.03I; 0.35I; 0                                   |
| QP margins; weights                        | 70 % of friction/CoP range, $\geq 30/25$ N; 1/10 (sole/palm)                               |
| QP limits; filter                          | full cones, 85 % of $f_i^{\max}$ (moments/0.1 m); 80 ms                                    |
| Gait                                       | 0.25 m steps, 1 s swing, $\approx 10$ s cycle, synchronous                                 |

TABLE II  
CHECKS ON THE COMPILED MODEL

| Result       | Quantity                                                           | Value                     |
|--------------|--------------------------------------------------------------------|---------------------------|
| Thm. 1(d)    | $\max  \Delta q $ (m, rad) due to $\psi$ , exact/inexact           | $(11/4.5) \cdot 10^{-14}$ |
| Remark 6     | same, zero-order hold $T = 0.5/4$ ms, 1 s                          | $(1.0/8.2) \cdot 10^{-5}$ |
| Thm. 2       | decay rate ( $s^{-1}$ ), 2–3 s: $\ s\ _M$ ; $\xi$ , $e_I$ , $\chi$ | 3.071; 2.97–3.01          |
| Cor. 1       | trajectory feasibility check satisfied from                        | 1.6 s                     |
| Prop. 1(iv)  | final $\ e_I\ $ , $\psi \equiv \lambda_d / K_z = I / K_z = 0$      | 100/50/0 N                |
| Remark 3     | same, +20 % payload model, $\psi \equiv \lambda_d / K_z = 0$       | 9.9/0 N                   |
| Prop. 1(iii) | girder deviation in 2.6 s, $\delta a$ frozen / projected           | 146/<10 <sup>-10</sup> mm |

Apollo’s reported lifting capacity is 25 kg<sup>1</sup>.

The internal-wrench space has dimension  $\dim \mathcal{N}(G) = 42$ . In double support (both feet grounded),  $\dim \mathcal{N}(L^\top) = 66$  and  $\Psi$  maps onto  $\mathcal{N}(G)$ : 24 internal-effort directions load only the feet. In single support (one foot grounded),  $\dim \mathcal{N}(L^\top) = 42$  and  $\Psi$  is invertible, so no internal-effort direction loads only the feet.  $W$  scales moments by 0.3 m;  $V_I$  is orthonormal in  $(N, N \text{ m})$ .

MuJoCo [30] includes soft frictional contacts, contact switching, joint friction, torque limits and sampling (Table I). Palm, sole and trestle-pad wrenches provide ideal force/torque measurements. On hardware, the pad wrench  $h$  would be estimated from wrist sensors and the payload model [6, (22), (23)].

We test the theory using (1) as the ideal plant (Table II, Fig. 3). The team moves the girder by 6 cm, 5 cm and 5°, then returns in 2.6 s. Increasing  $\max \|\lambda_I\|$  from 67 to 566 N leaves motion unchanged to machine precision, even with link-inertia perturbations up to  $\pm 30\%$  and a +20 % payload-mass increase.

With nonzero control errors and a compliant support (stiffness  $2 \cdot 10^4$  N/m, damping 300 N s/m),  $\kappa_s = 3.07 \text{ s}^{-1}$  governs the slowest predicted decay of  $\|s\|_M$ ,  $\xi$ ,  $e_I$  and  $\chi$  (Fig. 3(c)). It is below  $\kappa_\xi = 4$ ,  $\kappa_z = 5$  and  $\kappa_T = 6 \text{ s}^{-1}$ . The reported  $\kappa_s$  and  $\bar{\zeta}_k$  are evaluated a posteriori over visited states. With these trajectory-based values, the condition of Corollary 1 is satisfied for all 16 contact wrenches from 1.6 s; observed cone violations end at 0.25 s.

For Proposition 1(iii),  $\delta a = K_a^{-1}\Psi^+(q(0))\psi'$  ( $K_a = K$ ) gives a 150 N squeeze per palm at  $t = 0$ . Held constant during motion, it leaves the configuration-dependent space  $K_a^{-1}\mathcal{N}(L^\top(q))$ , causing a 146 mm girder deviation within 2.6 s. Updating it to  $K_a^{-1}\Psi^+(q(t))\psi'$  removes the deviation.

<sup>1</sup>M. Oitzman, “Apptronik unveils Apollo humanoid robot,” *The Robot Report*, Aug. 23, 2023.

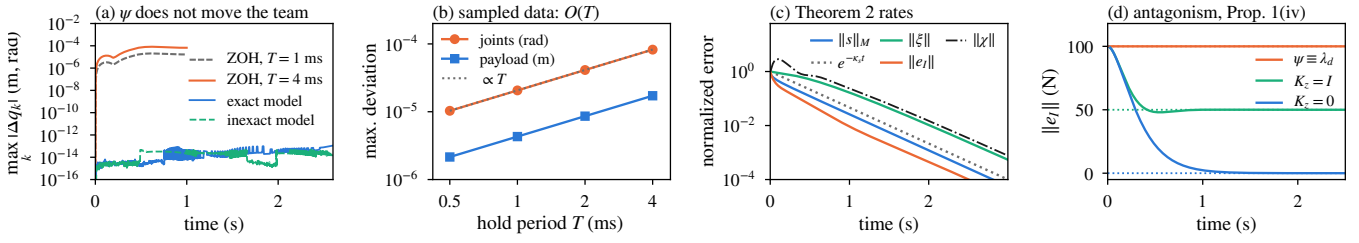


Fig. 3. Compiled-model verification. (a) Motion difference from an internal command (zero-order hold: 1 s runs). (b) Largest deviation versus hold period. (c) Normalized errors of Theorem 2 and the trajectory-based decay estimate. (d) Internal-wrench error under constant antagonism; dotted: predicted final values.

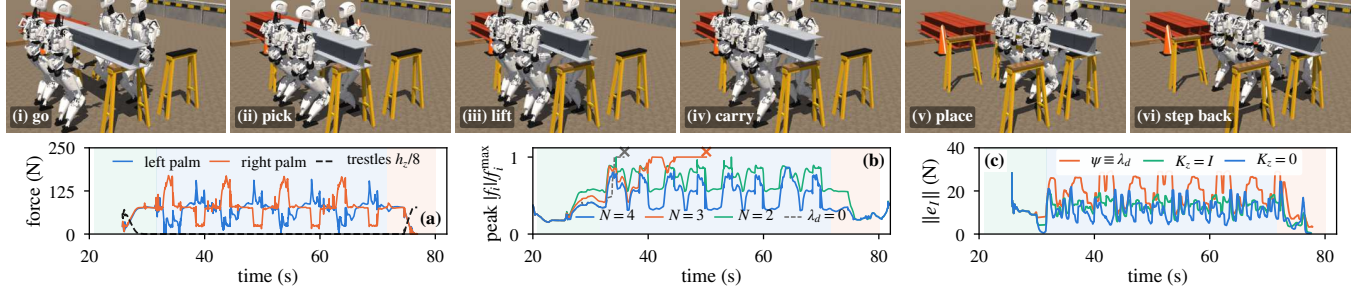


Fig. 4. Mission with  $N = 4$  and  $K_z = 0$ ; top: snapshots. (a) Palm forces of robot 1 and trestle force per palm ( $h_z/8$ ); shaded phases: pick and lift, carry, place and unload. (b) Peak actuator utilization ( $f_i^{\max}$ : torque limit);  $\times$ : girder lost. (c) Internal-wrench error with kinematic errors (0.5 s moving median).

The team walks 0.5 m to the girder on two trestles and places its palms beneath the flange. Moving  $p_d$  from  $p - K_p^{-1}h_f$  to  $p$  transfers the load to the palms;  $h_f$  is the force component of  $h$ . The team lifts the girder 0.1 m, carries it 1 m in four gait cycles, sets it down, unloads it and steps back (Figs. 4 and 5). Contact events switch the compiled mode; grasp changes reset  $z$  to zero.

A 50 Hz quadratic program (QP) selects  $\lambda_d$  and efforts in  $\mathcal{N}(L^\top) \cap \mathcal{N}(\Psi)$ . Its soft constraints target specified palm and sole friction and center-of-pressure (CoP) margins for  $\lambda^*$  from Corollary 1, and feedforward effort below 85 % of actuator limits; violations are penalized. Palm margins receive ten times the sole-margin weight because palm loads are low. At the pick posture, full contact cones and a 5 N minimum normal force define  $U^*$ , the minimum achievable peak actuator utilization and a static capacity estimate. Removing left-foot contacts gives single support. At saturation,  $e_I$  in (9) is set to zero for anti-windup.

With  $N = 4$ , the mission takes 91 s. The girder stays within 7.0 mm of the compliant frame (Fig. 5(c)), with largest deviations during lift and set-down. During carry-cycle weight shifts, it advances 0.25 m with up to 5.5 mm lag; leg joints move up to 0.6 rad (Fig. 5(d)). In single support, the stance foot carries the robot's full load. The internal channel shifts that robot's girder load toward its stance-side palm, which carries 79–86 % on average (99th percentile: 93 %, Fig. 5(e)). Palm forces differ by up to 160 N (149 N for robot 1, Fig. 4(a)). No actuator saturates: peak utilization is 0.87 (Fig. 4(b)). Wrists and ankles are closest to their limits; knee utilization remains below 33 % at the 99th percentile (Fig. 6(b)).

Both  $N = 4$  and  $N = 2$  carry the 63 kg girder ( $U^* = 0.47$  and 0.58 in single support; about 16 and 32 kg per robot,

TABLE III  
MISSION OUTCOMES IN MUJoCo ( $N = 4$ ,  $K_z = 0$  UNLESS STATED)

| 63 kg girder                                                                                                                                                                     | result                                                             | $\ e_I\ $ rms/median | $\max \ p_c - p\ $ |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------|----------------------|--------------------|
| $K_z = 0$                                                                                                                                                                        | done                                                               | 16/8.3               | 7.0                |
| $K_z = I$                                                                                                                                                                        | done                                                               | 17/12                | 7.1                |
| $\psi \equiv \lambda_d$                                                                                                                                                          | done                                                               | 23/18                | 7.0                |
| $\lambda_d = 0$                                                                                                                                                                  | lost, 36 s                                                         | —                    | —                  |
| $N = 3$ ; $N = 2$                                                                                                                                                                | lost, 50 s; done                                                   | —; 11/4.0            | —; 7.0             |
| noise 5/0.5; 15/1.5                                                                                                                                                              | done                                                               | 16/8.4; 16/9.3       | 7.1                |
| kin. error <sup>†</sup>                                                                                                                                                          | done                                                               | 16/10; 18/13; 24/20  | 6.9–7.0            |
| Mass sweep beyond the rating (kg): $U^*$ in single support; other runs done                                                                                                      |                                                                    |                      |                    |
| $N = 4$                                                                                                                                                                          | 183: 0.72, 240: 0.91, 260: 0.98, 280: 1.04 (lost 38 s)             |                      |                    |
| $N = 3$                                                                                                                                                                          | 120: 0.78, 140: 0.89, 160: 0.99, 183: 1.11 (lost 35 s)             |                      |                    |
| $N = 2$                                                                                                                                                                          | 100: 0.81, 120: 0.94, 140: 1.06 (lost 38 s), 183: 1.34 (lost 32 s) |                      |                    |
| Girder model (kg), $K_z = I$ : $(p_c - p)_z$ , meas./pred.;                                                                                                                      |                                                                    |                      |                    |
| 70: −6.3/−6.4, 66: −2.7/−2.7, 60: 2.6/2.8, 55: 6.7/7.4; 50, 40: lost (35 s)                                                                                                      |                                                                    |                      |                    |
| $K_z = 0$ : 60, 55 done; lost: 70, 66 (50, 62 s), 50, 40 (32, 34 s), 30 (29 s)                                                                                                   |                                                                    |                      |                    |
| $U^* = 0.25/0.47$ in double/single support; $\ e_I\ $ in N, $p_c - p$ in mm; lost: 0.15 m off the compliant frame or a fall; noise: std (N/Nm), white, 500 Hz, $e_I$ noise-free; |                                                                    |                      |                    |
| <sup>†</sup> link offsets, 1 mm std; $K_z = 0$ ; $K_z = I$ ; $\psi \equiv \lambda_d$ .                                                                                           |                                                                    |                      |                    |

respectively, the latter above rating; Table III, Fig. 6(a)). With a zero internal-wrench target ( $\lambda_d = 0$ ), the specified distribution is infeasible in single support, and the girder is lost in the first gait cycle (36 s).

The  $N = 3$  team ( $U^* = 0.55$ ), the only layout without point symmetry (two robots on one side), loses the girder in the second cycle. Palms slip along the light girder by 4 cm in the first cycle and over 10 cm before loss, while wrist-roll actuators saturate. The estimate  $U^*$  characterizes static actuator capacity with maintained grasps, while the contact simulation resolves palm slip. With 120–160 kg girders, increased palm loading allows  $N = 3$  to finish.

A twelve-run simulated mass sweep evaluates the capacity estimate beyond the rated payload at unchanged cross-section. Every run with single-support  $U^* < 1$  (up to 0.99) completes the mission; every run with  $U^* > 1$  loses the girder in the first

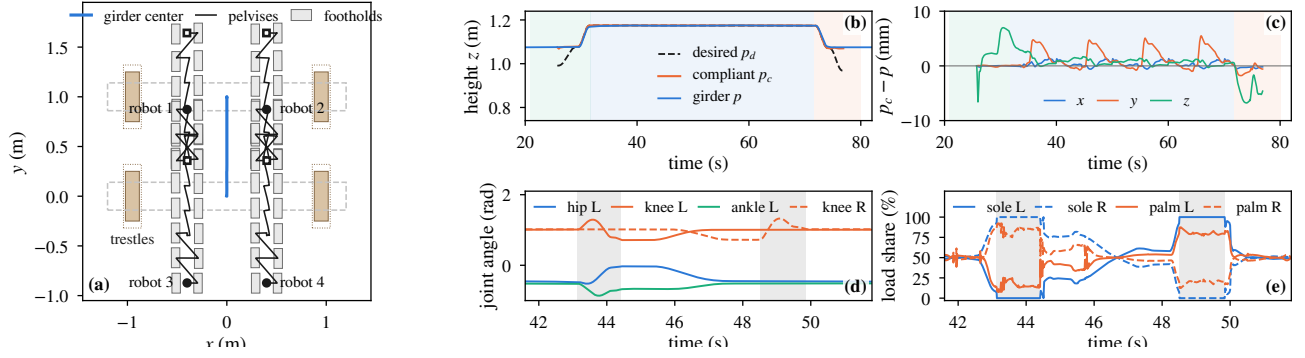


Fig. 5. Trajectories with  $N = 4$  and  $K_z = 0$ . (a) Top view: girder center, pelvises (● start, □ end), footholds and trestles; dashed: girder at pick-up and set-down. (b) Girder height. (c) Girder position error. (d) Leg joints of robot 1 over one carry cycle; gray: single support. (e) Load shares of the soles and palms of robot 1.

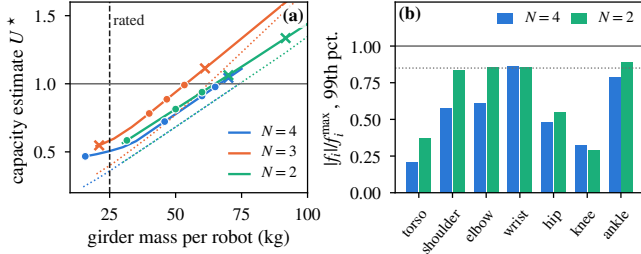


Fig. 6. (a) Capacity estimate  $U^*$  versus girder mass per robot (solid/dotted: single/double support), outcomes (● carried, × lost) and published payload (dashed). (b) Actuator utilization (99th percentile, load transfer to set-down) by joint group for the 63 kg girder; dotted: 85 % feedforward bound.

gait cycle. Successful runs near the boundary ( $U^* > 0.93$ ) include actuator saturation during 20–40 % of lifted phases and deviations up to 22 mm. The estimate therefore separates successful and unsuccessful missions in this sweep, while the measured deviations quantify tracking performance near the actuator-capacity boundary.

With  $K_z = I$ , model masses of 55–70 kg (−13 to +11 %) bias girder height during carry-cycle weight shifts: underestimation places it below the compliant frame, and overestimation above. The bias agrees within 10 % with the at-rest prediction of Remark 3: 0.92 mm per kg of model error (Table III, Fig. 7(d)). With a 50 kg model (−21 %), feedback supplies the missing modeled weight outside the QP; wrist-roll actuators (10.6 N m) saturate in the first step, and the girder is lost.

With  $K_z = 0$ , underestimates down to 55 kg permit carrying, but overestimates of 3 and 7 kg cause loss. Palm loads average 8.5 N below plan; palms assigned the 25 N minimum unload in single support. With the 70 kg model, a palm falls below 5 N during 31 % of the first two gait cycles, versus 0.6 % with the exact model. Integrating errors that unilateral contacts cannot remove causes  $z$  to wind up;  $K_z = I$  keeps  $z$  bounded. Measured  $G\lambda_g$  allows payload mass estimation after lift-off.

Link-offset errors (1 mm standard deviation; about 4 mm at palms and soles) make requests inconsistent with the true chain. They introduce  $w_a = (\hat{L} - L)u_q$  in Proposition 1 and additional model-error terms (Remark 6). All three variants finish within 7.0 mm. Median  $\|e_I\|$  is 10, 13 and 20 N for

$K_z = 0$ ,  $K_z = I$  and  $\psi \equiv \lambda_d$ , respectively (Fig. 4(c)): 1.3–2.3 N above runs without kinematic errors, ordered as predicted by Proposition 1(iv).

White noise of 5 N/0.5 N m or 15 N/1.5 N m on all force/torque measurements leaves true  $\|e_I\|_{\text{rms}}$  at 16 N (measured: 27 and 68 N) and girder deviation within 7.1 mm. A horizontal 100 N push enters measured  $h$ , displacing the compliant frame by  $K_p^{-1} \cdot 100 \text{ N} = 16.7 \text{ mm}$  (Fig. 7(a)). With joint friction, the measured displacement is 14.2 mm and the residual offset is 2.6 mm. With zero joint friction, the displacement matches the 16.7 mm impedance prediction and the girder returns to its initial position.

During a 4 cm motion in double support, with  $z$  frozen, an internal command shifts 25 N of normal force between palms (about one-third of their load), causing at most 0.20 mm girder deviation (Fig. 7(b)). Since all palm normals point upward, the command redistributes load rather than squeezing the girder.

We emulate the reference-displacement approach of [6] within the proposed inner loop. A hand-Jacobian pseudoinverse converts palm-normal hand displacements into actuator-reference displacements (Remark 1,  $K_a \propto K$ ). Palm-force shifts are similar (Fig. 7(c)), but the girder moves up to 1.9 mm (0.7 mm while held) and the pelvises up to 5.7 mm, as Proposition 1(iii) predicts. Projecting onto  $K_a^{-1}\mathcal{N}(L^\top)$  reproduces the internal command exactly. Without joint friction, the internal command moves the girder less than 1  $\mu\text{m}$  and the pelvises 0.06 mm; hand-reference displacement still moves the pelvises 5.8 mm.

With internal impedance active, a modulated command of up to 45 N per palm causes less than 0.01 mm girder motion without joint friction and up to 0.9 mm with it, as joints stick at the motion’s end. Commands of up to 90 N per palm exceed their load: four palms remain at CoP limits for most of the motion, and girder deviation reaches 6.5 mm without actuator saturation. The simulations use control updates at 500 Hz in simulation time. The centralized Python implementation has median wall-clock computation times of 9.0 ms per control step and 3.4 ms per QP solve.

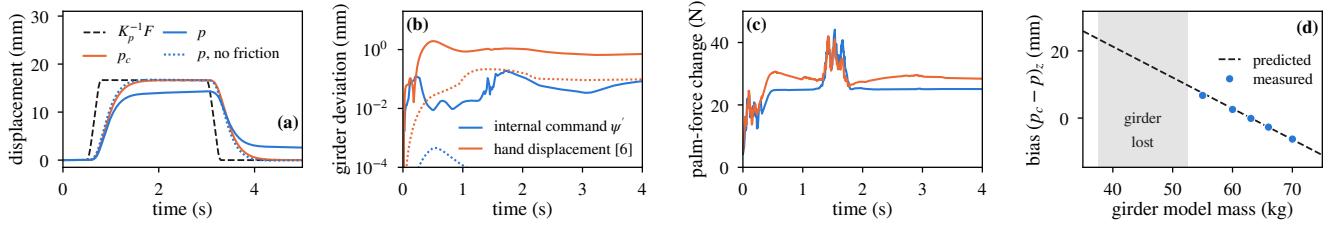


Fig. 7. Double-support tests after lift-off; dotted: without joint friction. (a) Displacement under a horizontal 100 N push  $F$  along the carry direction. (b), (c) Girder deviation and palm-force change (50 ms average) from a 25 N palm-load shift during a 4 cm motion, using an internal command or hand-reference displacements as in [6];  $z$  frozen. (d) Bias versus model mass during carry-cycle weight shifts,  $K_z = I$ ; shaded: girder lost (40 and 50 kg models).

## VI. CONCLUSION

This letter extends [6] to robot teams with an exact internal-wrench identity, bias-free reference commands and antagonism rejection. For an exact model and a stable target interaction, the analysis establishes explicit local exponential convergence rates within each fixed compiled mode. In MuJoCo contact simulations, four humanoids carried a 63 kg girder within their rated payload, with a maximum tracking deviation of 7.0 mm from the compliant frame. Comparable palm-load shifts produced 0.2 mm girder motion with internal commands versus 1.9 mm with hand-reference displacements under the same inner loop. A static capacity estimate matched all twelve mass-sweep outcomes. Together, the analysis and simulations connect internal-load regulation, motion tracking and contact-feasibility assessment for cooperative robot teams.

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