Online Resource 1

Article title: Satellites of Broucke's prograde family R in the equal-mass planar three-body problem: periodic orbits at the rotation numbers $1/n$ in two reversing-symmetry classes
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Procedure details (Online Resource 1)

A.1 Branch points, continuation and the zero loop

The n-fold repeat is solved with $q = 8$, 16, 24, 32 or 40 pieces, tried in this order, or with a number given by `-pieces`; it is accepted when the residual is below the repeat tolerance, 1×10^{-9} by default or 1×10^{-8} with `-repeat-tol=1e-8`. The singular values of $[J \,|\, \partial F/\partial\Lambda]$ (87×88 for $n = 5$, 6 and 8) are computed with a one-sided Jacobi method; the first is zero because the matrix has one more column than rows. The quadratic bifurcation equation is fitted at 36 directions of the null space, and its two real roots are the two directions. In the continuation the tangent is the null vector of $[J \,|\, \partial F/\partial\Lambda]$, oriented like the previous one. A step is accepted only if the corrector converges, the tangent turns by less than 0.35 rad and the orbit at the new point can be integrated. The step is halved after a failure, down to 1×10^{-7}, and grows by a factor 1.3 after a success, up to the smaller of 0.03 and 6 % of the path length so far (but not below 3×10^{-4}); in the first step the normal of the constraint plane has the tangent of the other curve removed. A continuation stops at a zero of θ, a closest approach below 0.03 (in the start units), a collapse of the step, a failed integration, or 700 accepted points. The zero is located when θ changes sign between two accepted points (the earlier $|\theta|$ below 0.05 turn) by bisection along the chord, with tolerance 1×10^{-11} and at most 60 bisections; the points at 1/2, 1/4, 3/4, 1/8 and 7/8 of the chord are tried in this order if the corrector fails. The continuation of each curve stops at the first zero of θ; the second zeros of the curves of $n = 5$ and 6 (A.11) were found with the first version of the tool (`CHANGELOG.md`, version 0.2.15).

From Section 2.3 (the continuation near the branch point).

With small steps near the branch point, where the invariants (T^*, L^*, θ) of the satellite curve and of the repeat curve differ by only a few 1×10^{-4}.

From Section 2.3 (the zero of the rotation).

The zero of θ is located by bisection along the chord, each new point being corrected back onto the curve.

From Section 2.3 (the test for repeats).

Each zero is tested for repeats: return after T/n, up to a rotation, n up to 200.

A.2 Refinement

The refinement uses a Gragg–Bulirsch–Stoer integrator in multiple precision with a working precision of 40 digits and a step tolerance of 1×10^{-26}. Newton's method uses a finite-difference Jacobian in u_1, u_2 and λ, and the derivative with respect to t from the end state; the iteration stops when the residual is below 1×10^{-24}. Since 2026-10-10 the closure is measured on the values rounded to 30 digits, and not on the unrounded solution.

A.3 Declared changes of the method, and the repaired zero loop

For n up to 26 and for $n = 28$ the continuation was run with the default settings. For $n = 27$ and $n \geq 29$ (except $n = 39$) the n-fold repeat did not close to the residual 1×10^{-9}, because the system is singular at the branch point and the residual stalls near $3.5 \times 10^{-11} n$. Four changes of the method were then made, each written into the rules file before it was run: the accepted residual of the repeat was relaxed to 1×10^{-8}; the repeat was solved with 16 pieces of the multiple shooting instead of 8; a point of the continuation whose orbit cannot be integrated is rejected like a failed step; and the caps on the number of integration steps were raised by a factor 20 (numbers in `src/perp.js` and `src/shooting.js`). The files for $n = 29$ to 56 were made with `-repeat-tol=1e-8 -pieces=16`, the file for $n = 27$ with `-repeat-tol=1e-8`, and the others with the defaults; for $n = 29$ to 36 the zeros are reached in 358 to 386 steps. These are changes of the method, not of the orbits; the end orbits are checked independently of the repeat tolerance by their own closure, the 30-digit refinement and REBOUND. For the relation B the bounds were written in `rules-n21-up.txt` (2026-10-10, 0.05 for $n \geq 21$), `rules-relation-tests.txt` (2026-10-11, 0.005, after the zeros for $n = 21$ to 36 were known) and `rules-test1-test2.txt` (2026-10-11, 0.005 before the runs for $n = 37$ to 58).

An error in the zero loop was found and repaired. Until version 0.2.23 the loop stopped for good when the corrector failed at a midpoint. This left four zeros ($n = 24$, 26, 27 and 28) with a rotation angle of up to 5.0×10^{-5} turns, and 15 of the 20 zeros for $n = 37$ to 56 (all of $n = 42$ to 56) with a rotation angle between 2.6×10^{-9} and 2.4×10^{-5} turns; a rotation of 5×10^{-5} turns moves L^* by about 1×10^{-3} and T^* by about 4×10^{-4}. The repaired tool gives identical files for $n = 13$ and 21; for $n = 26$ and 42 it differs only in the zero, now at 7.4×10^{-12} and 1.9×10^{-12} turns. The stored files were not changed. The 19 affected files were regenerated with the repaired tool into new files in `data/branch-points-regenerated/`: they equal the stored files except for the zeros, in the same sense for all 19, with the rotation left at the zero below 1×10^{-10} for 7, below 1×10^{-8} for 12 and at most 2.7×10^{-8} for all (against up to 5.0×10^{-5} before); 11 of the 19 ($n = 24$, 27, 28, 44, 45, 48 and 50 to 54) still stop above 1×10^{-9} turns and 12 above 1×10^{-10} ($n = 55$, at 4.2×10^{-10}, is the twelfth). The files and the run logs do not record why the loop stopped there (all five points of the chord failed, or 60 steps without reaching 1×10^{-11}), and the label "zero of theta" in these files does not mean that the tolerance 1×10^{-11} was reached (the field `turns` has the value), so the 30-digit orbits remain the values to use. The 30-digit orbits were refined from the old zeros and have $\theta = 0$, so the numbers in this paper do not depend on the stored zeros.

A.4 The stability computation in multiple precision

The monodromy matrix of the full period $2t$ is built from the flow in multiple precision (Gragg–Bulirsch–Stoer) by central differences in six directions of the subspace with zero centre of mass and momentum, each with the steps h and $2h$ and combined by Richardson extrapolation; the plain central difference is kept, and its difference from the extrapolated value estimates the differencing error. The six directions are the vector field $\mathbf{f}$, the rotation generator $\mathbf{g}$, and four vectors $\mathbf{b}_1$ to $\mathbf{b}_4$ completing them to a basis of W, the subspace on which the energy and the angular momentum are stationary (their gradients are computed analytically). W is invariant if the orbit closes, and the coefficients of the image of each basis vector along the two vectors that complete the basis of the whole space (the invariance defect) measure how far it is. The coefficients along $\mathbf{b}_1$ to $\mathbf{b}_4$ form the nontrivial 4×4 matrix N of eq. (9), whose determinant is 1 and whose characteristic polynomial is palindromic if the map is symplectic; both are reported. Tier A uses the 30-digit orbit (working precision 50 digits, tolerance 1×10^{-36}, $h = 1 \times 10^{-10}$), tier B the orbit refined to 40 digits (60 digits, 1×10^{-46}, $h = 1 \times 10^{-12}$). For the nine cases ($n = 5$ to 12 and the double-length orbit for $n = 8$) all four moduli equal 1 within 3.6×10^{-24} (A) and 2.6×10^{-35} (B), the invariance defect is at most 3×10^{-23} (A) and 5.4×10^{-34} (B), and the angles of the two tiers differ by at most 1.1×10^{-24}. The same method, with the isosceles start, gives the multipliers of the class I end orbits. For the class

C orbits of $n = 13$ to 20 and the class I end orbits of $n = 5$ to 20 only tier A was computed; the pair of the class C orbits of $n = 13$ to 20 on the unit circle has modulus 1 within 7×10^{-25}, and for the class I orbits of $n = 13$ to 20 the determinant of the 4×4 matrix is 1 within 7×10^{-24} and the invariance defect is at most 4.0×10^{-23}.

A.5 The class I search and its rules

The start is the isosceles state (5) of Section 2.1 with the unknowns p, w and t and the continuation parameter η. The symmetric residual at the end of the half orbit has three components, with $\mathbf{e}$ the unit vector from the centre of mass to body 0 and $\mathbf{e}_\perp$ the vector $\mathbf{e}$ turned by 90°: $(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{e}$, $\mathbf{v}_0 \cdot \mathbf{e}$ and $(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{e}_\perp$; the rotation per relative period is twice the angle of $\mathbf{e}$ at the end. The code reproduces the 242 stored members of R from their isosceles states (T^* and L^* to 6.4×10^{-10} relative, the rotation to 3.3×10^{-10} turns, the multipliers to 4×10^{-10}). The rules for $n = 5$ were written before the run, with predictions. The comparison of the two curves written down before the run for $n = 5$ was undecided by the rule written for it: the curves were to be called the same if every compared point lay within a tolerance (relative, in T^* and L^*, 2×10^{-6} or more where the stored polyline is coarse) and different if at least 5 points lay farther than 1×10^{-4} (or 10 times the tolerance, if larger), and anything in between undecided; 21 and 28 points (the two senses) lay outside the tolerance, none lay farther than 1×10^{-4} (the largest distance was 8.7×10^{-5}), and the threshold was not changed afterwards; the three facts of Section 3.3 (symmetry class, multiplier structure, differences far above the closures) were added after the data were seen. The rules for $n = 6$ to 12 were written before the runs and fix the criterion: an end orbit counts as different from the class C orbit of the same n if its symmetry class differs (an isosceles state and no state fixed by S, against the reverse) and its multipliers differ or its T^* or L^* differ by more than 100 times the closure accuracy; it is established only if its 30-digit closure is below 1×10^{-20} and a REBOUND integration closes it to below 1×10^{-9} after one period. All predictions written for $n = 6$ to 12 were met. For $n = 13$ to 20 the rules were written before the runs with the same criterion, with the addition that stability alone does not count, because both orbits may be unstable: an end orbit counts as different from the class C orbit of the same n only if its symmetry class differs and its T^* or L^* differ by more than 100 times the larger closure or the largest multipliers differ by more than 1×10^{-6} relative. Of the predictions written for $n = 13$ to 20 three failed (both senses reaching $\theta = 0$ at $n = 15$, no CPC row within 5×10^{-3} at $n = 17$, and the time of seven of the eight class I jobs) and the others were met. The comparison with the atlas follows `tools/compare-atlas-all.py` on the list of the 415 entries (key, T^*, L^*, syzygies) of 2026-10-09 and, for the CPC tables, the T^* and L^* of `data/cpc-converted.json`.

A.6 Commands and files

All code and data are available at the repository `github.com/Utshob-k/orbit-hunter`, tag `v0.2.33-corrected` (commit ac1b1d7), version DOI 10.5281/zenodo.23453312. Each table and figure is produced by a script listed in `docs/reproducing.md`: Tables 2, 3 and 4 by `tools/tables.mjs` (with the arguments 1 and 2; Table 5 is made from `node tools/class-i-summary.mjs 5 20`), Table S1 by `tools/relation-b-table.mjs`, Figures 2 to 4 by `tools/figures.mjs` (with the arguments 1, 2 and 3) and Figure 1 by `tools/figures.mjs 4`, the supplement by `tools/supplement.mjs`, the summary of the class I runs by `tools/class-i-summary.mjs`, the checks of the relation of Hénon by `tools/henon-integral-check.mjs`, `tools/henon-slope-check.mjs`, `tools/henon-sign-check.mjs` and `tools/henon-dip-check.mjs`, the class I search by `tools/branch-point-iso.mjs`, `tools/iso-refine.py`, `tools/iso-rebound.py` and `tools/iso-monodromy-mp.py`, and the multiple-precision stability of class C by `tools/monodromy-mp.py`. The short checks of Sections 2.1, 3.1 and 3.2 are `tools/half-period-check.mjs`, `tools/`

Table S1: The largest $|e|$ in T^* of the three predictions, by range of n.

n	no correction	first order (Hénon)	relation B
5 to 12	0.0583	0.0037	0.0478
13 to 20	0.0343	0.0053	0.0013
21 to 36	0.0182	0.0063	0.0031
37 to 56	0.0067	0.0066	0.0039

`isosceles-check.mjs`, `tools/two-senses-check.mjs`, `tools/long-run-rebound.py` and `tools/results-numbers.mjs`, and `tools/trace-checks.mjs` for the numbers of the trace quoted in Section 2.1. The multipliers along the curves, the signs of the largest multipliers and the check of the reduction (Sections 2.5, 3.2 and 3.3) are made by the tools listed in A.13.

A.7 The relation B in detail

This appendix gives the definitions and the statistics behind Section 4.2. Three predictions of the T^* of the end orbit are compared, with the error $e = T^*(\text{end orbit}) - \text{prediction}$. (i) No correction: nT^*_{member} ($\delta = 0$). (ii) The first-order prediction from eq. (10) without a free parameter: $nT^*_{\text{member}} - \pi L^*_{\text{member}}\varepsilon_n$, which assumes that the slope πL^* of the branch point holds along the whole curve. (iii) The relation B, an empirical relation written down before the runs for $n \geq 21$: $T^*(n) = nT^*_R(n) - (a + b/n)$, with $T^*_R(n)$ the T^* of the member read from the trace by interpolation, and $a = -0.0033397$, $b = 0.463315$ the least-squares values on $n = 8$ to 20 (13 values).

The first-order prediction is the best of the three for $n \leq 12$ (error 0.0037 at most) and overestimates δ_n by a factor that grows from 1.00 at $n = 5$ to 4.8 at $n = 56$. B is better than no correction for every n from 13 to 36, where δ_n is large, and from $n = 48$ on no correction is better than B (from $n = 49$ with the exact member; B is better than no correction for $n = 6$ to 47). The bound 0.05 was written before the runs for $n = 21$ to 36, and the bound 0.005 only before the runs for $n = 37$ to 56; neither separates B from no correction, and the error of B, about 3×10^{-3}, does not shrink with n. B is kept here as an empirical remark and not as a result.

The bounds written before the runs do not separate B from no correction. The bound 0.05 was written on 2026-10-10 before the runs for $n = 21$ to 36, and the bound 0.005 on 2026-10-11 before the runs for $n = 37$ to 56 only; a pass at 0.005 for $n = 21$ to 36 is not a pre-registered test. With no correction at all, e is inside 0.05 for all 16 values of $n = 21$ to 36 (the largest 0.0182, at $n = 21$) and inside 0.005 for 15 of the 20 values of $n = 37$ to 56 (16 with the interpolated member that B uses); so the bound 0.05 does not separate B from no correction, and the bound 0.005 separates them only for 5 of the 20 values of $n = 37$ to 56. Relative to δ_n, the bound 0.05 is 2.7 times $\delta_{21} = 0.0182$ and 7.0 times $\delta_{36} = 0.0071$, and the bound 0.005 is 0.75 times $\delta_{37} = 0.0067$ and 2.9 times $\delta_{56} = 0.0017$ (δ_n with the exact member); with the interpolated member δ_n is 0.0174, 0.0064, 0.0062 and 0.0011 at these four values of n, and the ratios are 2.9, 7.8, 0.8 and 4.5. A bound that is a few times δ_n cannot test a correction of the size of δ_n. At $n = 56$ the fitted δ of B is 0.0049, against a measured 0.0011 with the interpolated member (a factor of about 4) and 0.0017 with the exact member (a factor of about 3), so B passes the bound 0.005 there only because δ_n itself has fallen below it. B carries information where δ_n is large: its error is 0.0013 at $n = 13$ to 20 against 0.0343 without correction, and it is better than no correction for every n from 13 to 36; the first-order prediction is the best of the three for $n \leq 12$ (error 0.0037 at most) and overestimates δ_n by a factor that grows from 1.00 at $n = 5$ to 4.8 at $n = 56$; from $n = 48$ on no correction is better than B. B also did not meet the criteria written for its robustness tests (the largest error in the trailing windows is 5.18×10^{-3} for a bound of 0.005), its error does not go to zero (0.0029 to 0.0039 from $n = 37$ on), it fails for $n = 5$ and 6 (errors 0.0478 and 0.0174), and with the same $\delta(n)$ it holds for none of the ten end orbits with $j > 1$, where the rotation excess is larger. The error of B, about 3×10^{-3} (between 0.0011 and

0.0039 for $n = 21$ to 56), does not shrink with n. As a check with exact members, B refitted on the exact members of $n = 8$ to 20 ($a = -0.0035851, b = 0.482212$) has an error between 0.0011 and 0.0033 for $n = 21$ to 56, so the error of about 3×10^{-3} is not an artefact of the interpolation of the member (the interpolated $nT^*_R(n)$ is below the exact nT^*_{member} by 2.0×10^{-5} to 2.4×10^{-3}). B is kept here as an empirical remark and not as a result.

A.8 Secondary numbers of the sequence, the stability and the trace (from Sections 3.1, 3.2 and 3.4)

The orbits for $n = 5$, 6 and 8 are the orbits with $T^* = 17.847$ and 19.769 and, for $n = 8$, the shortest orbit of the orbit with $T^* = 47.065$, with the same syzygy sequences as the other orbits: $(01)^n$ with $2n$ syzygies and no repeat in the return test. The closest approach between two bodies, multiplied by $|E|$ to remove the scale, falls at every step from 0.7127 at $n = 5$ to 0.4365 at $n = 56$.

The orbits are stored with 30 digits. For $n = 7$ and 9 to 56 the closure of the full period is between 1.1×10^{-25} and 7.1×10^{-23}; all 49 are "reliable" at the first level, and REBOUND closes their double-precision start values to between 1.6×10^{-14} and 1.5×10^{-12}. For $n = 5$, 6 and 8 the closures are 1.1×10^{-26}, 1.1×10^{-26} and 1.8×10^{-24} in separate files, and refining them changed T^* by at most 2.0×10^{-11}.

L^* grows with positive steps that shrink from 0.02291 ($n = 5$ to 6) to 0.00032 ($n = 55$ to 56). The member of R with $\nu = 1/n$ has almost the same T^*/n as the end orbit of its satellite curve, which is lower by the relative amount 2.34×10^{-3} at $n = 5$, 1.05×10^{-3} at $n = 13$, 4.35×10^{-4} at $n = 20$, 9.82×10^{-5} at $n = 36$ and 1.63×10^{-5} at $n = 56$; the end orbit has a lower L^* than its member for every n, by 0.0219 at $n = 5$ and 1.3660 at $n = 56$.

For the stability of $n = 5$ to 12 in multiple precision, the angles of the two precisions agree to 1.1×10^{-24} turns, the two angles of an orbit differ (the smallest difference is 0.099, at $n = 5$), and the invariance of the subspace on which the trivial multipliers were removed holds to 5×10^{-34}. In the notation of the main text the two angles are α_r and α_t.

The last 32 points of the stored trace of R do not move (steps of 1×10^{-8} to 1×10^{-9}).

A.9 Secondary numbers of the class I search (from Section 3.3)

For every $n = 5$ to 12 the member with $\nu = 1/n$ is found in the isosceles description with $|\nu - 1/n|$ below 7×10^{-12}. The matrix $[J \,|\, \partial F/\partial \Lambda]$ has a two-dimensional null space (the second smallest singular value is between 2.3×10^{-13} and 5.3×10^{-12}, the third between 7.5×10^{-4} and 4.8×10^{-3}), so that the bifurcation equation has at most two roots; it has two real roots, and the angle between them rises from 59.4° at $n = 5$ to 63.2° at $n = 12$ (55.9° and 56.5° at $n = 5$ and 6 in class C). The curve of the non-repeat root runs to a zero of θ in both senses, with path lengths between 1.3 and 7.9, and L^* minus that of the member is between -0.023 ($n = 5$) and -0.28 ($n = 12$) at their ends.

For $n = 13$ to 20 the member with $\nu = 1/n$ is found in the same way, and the matrix $[J \,|\, \partial F/\partial \Lambda]$ has a two-dimensional null space (the second smallest singular value is between 3.4×10^{-12} and 1.1×10^{-11}, the third between 2.5×10^{-4} and 7.3×10^{-4}); the bifurcation equation has two real roots, and the angle between them rises from 63.5° at $n = 13$ to 65.2° at $n = 20$. For $n = 13$ and 14 the curve of the non-repeat root runs to a zero of θ in both senses; for $n = 15$ to 20 it does so in one sense only (sense -1 for $n = 15$ and 16, sense 1 for $n = 17$ to 20), the other sense stopping by a collapse of the step. L^* minus that of the member is -0.313 ($n = 13$) to -0.544 ($n = 20$) at the end.

In the comparison of the two curves for $n = 5$, 21 and 28 points of the two senses are outside the

tolerance, and none is above the threshold. The symmetry class of the end orbit of class I is that it has an isosceles state (fixed by S') and no state fixed by S, while the class C orbit has a state fixed by S and no isosceles state with any body as apex.

An independent integration with REBOUND (IAS15, default settings) closes the start values of the class I orbits to between 8×10^{-15} and 3×10^{-13} after one period ($n = 5$ to 20); after twenty periods the distance is 4×10^{-13} to 3.7×10^{-12} for $n = 5$ to 12 and 4.2×10^{-8} at $n = 13$, 7.1×10^{-5}, 2.2×10^{-3}, 1.5×10^{-2}, then 1.8, 1.5, 1.3 and 0.68 for $n = 17$ to 20, as expected for unstable orbits. The angle α_t of the transverse pair is above that of the class C orbit by 0.0004 at $n = 5$, 0.0025 at $n = 6$, 0.0047, 0.0071, 0.0100, 0.0141, 0.0212 and 0.0489 at $n = 12$. Closer to the branch point than a path length of 0.23 ($n = 5$) the distance of the resonant pair from $+1$ is below 1×10^{-6} in the earlier run behind this sentence; in the run of A.13 the resonant pair is classified from the path length 0.76 ($n = 5$, class I) on. For $n = 13$ to 20 the class I orbit has no pair on the unit circle.

Each end orbit of class I is established by a criterion fixed before the runs for $n = 6$ to 20 (for $n = 5$ it was added after the data were seen, A.5): the closure of the 30-digit orbit is below 4×10^{-25} for $n = 5$ to 12 and below 3.6×10^{-24} for $n = 13$ to 20, an independent integration with REBOUND (IAS15) closes the start values after one period (below 3×10^{-13} for $n = 13$ to 20) and, for $n = 5$ to 12, after twenty periods, and the syzygy sequence is $(01)^n$. For $n = 13$ to 20 the distance after twenty periods is not part of the criterion: it grows with the multiplier, to the size of the orbit from $n = 17$, as it does for any unstable orbit.

A.10 Checks of the sign and the size of dT^* (from Section 4.1)

In turns the relation is $dT^* = \pi L^* d(\Phi/2\pi)$, with Φ in radians. The angular momentum of the 54 entries of `data/satellite-candidates.json` (the 49 orbits of the sequence for $n = 7$ and 9 to 56, the two second zeros of $n = 5$ and 6, and three orbits with $j > 1$) is positive, so eq. (35) gives $dT/d\Phi > 0$; in the stored satellite curves dT^* and $d\rho$ have opposite signs for all 11,497 pairs of neighbouring points with $|d\rho| \geq 1 \times 10^{-6}$, and $dT^*/(-\pi L^* d\rho)$ has the median 1.0000. The trace of R has the sign of Φ: T^* increases with its rotation in all 101 stable steps.

A.11 Where the orbits are, start formats and reproducing (from Section 7.2)

- $n = 7$ **and 9 to 56:** `data/satellite-candidates.json` holds 54 orbits, 49 of them the orbits of Table 2 (30 digits); the label of an entry says which orbit it is, and its entries for $n = 5$ and 6 are other orbits (second zeros of the same curves). Each entry has u_1, u_2, lam, t, the residual, the closure, the largest multiplier and the closure tier.

- $n = 5$**, 6 and 8:** `data/refined/T17.847.json`, `T19.769.json` and `T47.065.json` ($n = 8$: the shortest orbit of the orbit with $T^* = 47.065$), with 30 digits.

- **Branch points and curves, the trace of R, the rules:** `data/branch-points/<n>-upper.json` (member, singular values, directions, and the points of both senses with 10 significant digits), `data/r-family-trace.json` (242 steps), `docs/rules/`.

- **Class I and the multipliers:** `data/class-i/` holds, for $n = 5$ to 20, `curves/` (both senses of the curve from the branch point), `refined/` (the 30-digit end orbit), `stability/` (the multipliers, tier A), `rebound/` and `check/`, and `partner-orbits.json` (the 16 end orbits); `data/stability-mp/class-c/` holds the multipliers of class C ($n = 5$ to 12 in tiers A and B, $n = 13$ to 20 in tier A); `data/supplement-initial-conditions.json` has the start values of

the 52 orbits of class C and the 16 of class I; `node tools/class-i-summary.mjs` prints the summary of the class I runs from these files.

- **Multipliers along the curves and the figure of orbits:** `data/curve-multipliers/` holds the multipliers along the 53 curves of A.13 (`class-c/` and `class-i/`, one file per curve and sense with the points and the coefficients a and b of the characteristic polynomial of the 4×4 block, 10 significant digits, and the zero; `summary.json`); `data/largest-multipliers-n13-56.json` holds the largest multipliers of the class C end orbits for $n = 13$ to 56; `figures/end-orbits.svg` is Figure 1.

With $G = m = 1$ the bodies of class C start on the x axis at $-1 - \lambda/3$, $\lambda - \lambda/3$ and $1 - \lambda/3$ with the velocities $(0,\ u_1)$, $(0,\ u_2)$ and $(0,\ -u_1 - u_2)$; the orbits of class I start at the isosceles state of Section 2.1. The relative period is $2t$, which is the full period for the orbits of the tables. The values in the files are strings with 30 digits; T^* and L^* are computed from them by `perpInfo` in `src/perp.js`. Node 18 or later is needed; the Python scripts need `mpmath` and, for the REBOUND check, `rebound` and `numpy`. `npm test` runs eleven test files (about 10 seconds; 46 checks pass at this tag). The commands are listed in `docs/reproducing.md` (Section A.6).

From Section 3.4 (the files of the trace).

The points beyond the stored trace are in `data/r-family-trace-extended.json`; the stored trace `data/r-family-trace.json` is unchanged.

From Section 7.2 (the checks and the files of the multipliers).

The short checks quoted in Sections 2.1, 3.1 and 3.2 are in `tools/` (`half-period-check.mjs`, `isosceles-check.mjs`, `two-senses-check.mjs`, `long-run-rebound.py` and `results-numbers.mjs`) and are listed in `docs/reproducing.md`; the numbers of Sections 2.1, 3.2 and 3.3 added in version 0.2.32 are printed by `trace-checks.mjs`, `largest-multipliers.mjs` and `curve-multiplier-types.mjs`. The multipliers along the 53 curves are in `data/curve-multipliers/` and the largest multipliers of the class C end orbits for $n = 13$ to 56 in `data/largest-multipliers-n13-56.json`.

A.12 Pre-registration, failed predictions and run times (from Sections 2.6, 3.3 and 6.2)

The rules of each run are in `docs/rules/`; they were written before the run, with the clock reading of the machine, and a later change was declared in the same file before it was run (the files are listed in A.3, A.5 and A.7). The comparison of the two curves for $n = 5$ was undecided by the rule written for it (the rule and the numbers are in A.5), and the three facts of Section 3.3 on which "different" rests were added after the data were seen (A.5). The predictions that failed are named in A.5 (class I, $n = 13$ to 20: both senses reaching $\theta = 0$ at $n = 15$, a CPC row within 5×10^{-3} at $n = 17$, and the time of seven of the eight jobs) and in A.7 (the relation B: the bounds 0.05 and 0.005 do not separate it from no correction, and it did not meet the criteria written for its robustness tests). The 19 regenerated zero files took 54 to 119 minutes each (A.3). The run of the multipliers along the curves (A.13) had its rules written before the run, two declared changes and four failed predictions, named in A.13; its 79 jobs took 27 minutes.

A.13 The multipliers along the satellite curves (from Sections 2.5, 3.2 and 3.3)

Curves and code. The multipliers were computed at every accepted point of the continuations of 53 curves: class C, $n = 5$ to 20, 27 curves (both senses for $n = 5$ to 15, one sense for $n = 16$ to 20), and class I, $n = 5$ to 20, 26 curves (both senses for $n = 5$ to 14, one sense for $n = 15$ to 20); these are the senses that reach a zero of θ in the stored files. The curves were followed again with the code and the settings of the stored runs (`tools/branch-point-mult.mjs` for class C, with one sense per run and the full start and the multipliers stored at every point; `tools/branch-point-iso.mjs` for class I, as released), and every retraced curve equals the stored curve within the tolerances written before the run (1×10^{-7} in s, T^* and L^*, 1×10^{-9} turn in the rotation; 53 of 53).

Reduction. At each point the monodromy matrix of the full period, rotated back by θ, is reduced to the 4×4 matrix N of eq. (9) in double precision (`tools/reduced-multipliers.mjs`, for class I `tools/reduced-multipliers-iso.mjs`): the quotient by $\mathrm{span}\{\mathbf{f}, \mathbf{g}\}$ on W removes the trivial multipliers exactly, so no eigenvalue is picked as the one "closest to 1" (in double precision the four trivial eigenvalues are spread by about 1×10^{-4}, and such a choice would give wrong values near the branch point). The characteristic polynomial of N is $x^4 - ax^3 + bx^2 - ax + 1$, and the files store a and b at every point; a reciprocal pair μ, $1/\mu$ has $y = \mu + 1/\mu$, the roots of $y^2 - ay + (b - 2) = 0$.

Reference checks. (a) At the 30-digit starts of the 32 end orbits (class C and class I, $n = 5$ to 20) the reduction gives the four y values of the multiple-precision tier A with a largest relative difference of 6.3×10^{-8} (limit 1×10^{-6}; 32 of 32). (b) At the zero of each of the 53 curves the double precision values agree with tier A to 4.1×10^{-8} at most in class C and to 2.0×10^{-5} in class I (the largest at $n = 14$, sense -1, whose zero was left at 4.8×10^{-8} turn; limit 1×10^{-3}). Where both senses reach $\theta = 0$ the two zeros give the same four y values, inside the limit of (b).

Classification. A pair is elliptic if both moduli are within 1×10^{-6} of 1, real positive or real negative if both eigenvalues are real and the moduli differ from 1 by more than 1×10^{-6}, and complex otherwise. A pair is classified at a point only where $|y - 2| \geq 1 \times 10^{-6}$ and $|y + 2| \geq 1 \times 10^{-6}$: the noise of y near the double multiplier $+1$ is about 1×10^{-7}, and the eigenvalues then scatter by about 1×10^{-4}, so that the type cannot be read there in double precision. A change of type counts only if the new type holds for at least 3 consecutive classified points; a shorter run at the end of a curve is listed as an end run and not counted (the only one is the transverse pair of the class I curve of $n = 12$, sense -1, one real negative point past the zero). A change is given as a bracket between the last point of the old type and the first point of the new type, in L^* and in the rotation, and no change was refined; the brackets of single curves are 0.0009 to 0.0052 wide in L^*. The zero (the end orbit) is inserted as a point of the sequence, between the last point before it and the first point past it, which is the last point stored. The first point of a curve, the n-fold repeat itself, has no multipliers.

Tracking. At the first point with multipliers the resonant pair is the root y nearer to 2; at every later point the assignment of the two roots to the two pairs is the one with the smallest total change of y, and a step is ambiguous if the other assignment costs less than three times as much. The tracking by continuity is unambiguous at every step; the only ambiguity is at the very first classified point of the 10 curves with $n \geq 16$ (the other root of y is within 1×10^{-2} of 2 at the branch point there), and it does not matter: at the zero of every one of the 53 curves the tracked resonant pair is the pair nearest $+1$, by a factor of at least 200 in $|y - 2|$, the ratio of the distances from 2 of the other root y and of the tracked one (smallest 201: class C, $n = 6$, sense -1). No complex quadruplet occurs on any curve.

Where the resonant pair is classified. The resonant pair is first classified at a path length between 0.76 and 4.9 over the 53 curves (0.76 to 0.94 for $n = 5$, at $L^* = 2.5865$), the transverse pair from the first point after the branch point. Nothing is known of the type of the resonant pair between the branch point and that point. For $n = 5$ the classified part of the curves is the last part

before the end orbit (L^* from 2.5865 to 2.5721, against 2.5940 at the branch point), so a hyperbolic stretch of the class C curve near the branch point, as $n = 6$ shows from its first classified point, can be neither excluded nor confirmed.

Rules, declarations and failed predictions. The rules of the run, with the predictions, were written before any script of the run was run (machine clock 14:02:42 on 2026-10-18). Two changes were declared in the same file before they took effect: (1) after the smoke tests, a pair is classified where $|y - 2|$ and $|y + 2|$ are at least 1×10^{-6} (instead of all eigenvalues at least 1×10^{-4} from ± 1), and check (b) is made at the zero and not at the last stored point, which is past the zero; (2) after a look at two curves, the zero is inserted in the sequence and runs shorter than 3 points at the end are listed. Four predictions failed: that the resonant pair of class C stays elliptic along the whole curve, that the resonant pair of class I stays real positive along the whole curve, that the resonant pair is classified from a path length below 0.05, and that the panel for $n = 56$ of Fig. 1 can be read without a zoom. The others held (retraces equal the stored curves, the reference checks (a) and (b), the transverse pair for $n \leq 12$ on the unit circle and for $n = 13$ to 20 with one change, no complex quadruplet, unambiguous tracking, `npm test`).

What this does not show. It does not show why the resonant pair changes type where it does, or whether the transverse pair reaches -1 at a special point of the curve (nothing special was looked for); it does not cover the lower rotation number, the curves that stop by a collapse of the step, orbits outside the two symmetry classes, the part of a curve beyond its zero, or the multipliers of the curves for $n \geq 21$. The multipliers along the curves are in double precision (tier A only at the end orbits).

The largest multipliers for $n = 13$ to 56. The reduction applied to the 30-digit start of each class C end orbit, rounded to double precision, gives a largest multiplier that is real negative for 44 of 44 values of n ($n = 13$ to 56). The moduli agree with the stored double precision values to 1.2×10^{-7} at most ($n = 14$) and to 8.3×10^{-8} at most for $n = 21$ to 56, and with tier A ($n \leq 20$) to 2.4×10^{-8}; for $n = 21$ to 56 they run from 8.5188 to 53.1762, one eigenvalue outside the unit circle and its inverse.

Table S2: The type of the resonant and of the transverse pair along the satellite curves of both classes, $n = 5$ to 20, with the bracket in L^* of each change of type (both senses together; "A $\to$ B at x to y": the last point of type A and the first point of type B lie at these L^*), the L^* of the branch point and the L^* of the end orbits (class C / class I). The resonant pair is classified only from a path length of 0.76 to 4.9 on; the transverse pair from the first point after the branch point.

n	class C: resonant pair	class C: transverse pair	class I: resonant pair	class I: transverse pair	L^* of the branch point	L^* of the end orbits (C / I)
5	elliptic, no change	elliptic, no change	real positive, no change	elliptic, no change	2.5940	2.5721 / 2.5721
6	real positive $\to$ elliptic at 2.6242 to 2.6264	elliptic, no change	elliptic $\to$ real positive at 2.6233 to 2.6259	elliptic, no change	2.6495	2.5950 / 2.5948
7	real positive $\to$ elliptic at 2.6460 to 2.6478	elliptic, no change	elliptic $\to$ real positive at 2.6448 to 2.6473	elliptic, no change	2.7025	2.6124 / 2.6118
8	real positive $\to$ elliptic at 2.6628 to 2.6654	elliptic, no change	elliptic $\to$ real positive at 2.6628 to 2.6652	elliptic, no change	2.7531	2.6259 / 2.6251
9	real positive $\to$ elliptic at 2.6763 to 2.6792	elliptic, no change	elliptic $\to$ real positive at 2.6763 to 2.6801	elliptic, no change	2.8015	2.6367 / 2.6357
10	real positive $\to$ elliptic at 2.6881 to 2.6917	elliptic, no change	elliptic $\to$ real positive at 2.6862 to 2.6894	elliptic, no change	2.8478	2.6456 / 2.6443
11	real positive $\to$ elliptic at 2.6953 to 2.6989	elliptic, no change	elliptic $\to$ real positive at 2.6962 to 2.6996	elliptic, no change	2.8922	2.6530 / 2.6515
12	real positive $\to$ elliptic at 2.7030 to 2.7069	elliptic, no change	elliptic $\to$ real positive at 2.7048 to 2.7083	elliptic, no change	2.9349	2.6593 / 2.6576
13	real positive $\to$ elliptic at 2.7114 to 2.7155	elliptic $\to$ real negative at 2.6661 to 2.6702	elliptic $\to$ real positive at 2.7121 to 2.7167	elliptic $\to$ real negative at 2.6683 to 2.6717	2.9761	2.6647 / 2.6628
14	real positive $\to$ elliptic at 2.7177 to 2.7222	elliptic $\to$ real negative at 2.6758 to 2.6810	elliptic $\to$ real positive at 2.7181 to 2.7227	elliptic $\to$ real negative at 2.6808 to 2.6847	3.0158	2.6694 / 2.6673
15	real positive $\to$ elliptic at 2.7243 to 2.7280	elliptic $\to$ real negative at 2.6902 to 2.6935	elliptic $\to$ real positive at 2.7218 to 2.7270	elliptic $\to$ real negative at 2.6917 to 2.6966	3.0542	2.6735 / 2.6712
16	real positive $\to$ elliptic at 2.7294 to 2.7335	elliptic $\to$ real negative at 2.6987 to 2.7024	elliptic $\to$ real positive at 2.7272 to 2.7316	elliptic $\to$ real negative at 2.7002 to 2.7047	3.0914	2.6771 / 2.6747

n	class C: resonant pair	class C: transverse pair	class I: resonant pair	class I: transverse pair	L^* of the branch point	L^* of the end orbits (C / I)
17	real positive → elliptic at 2.7330 to 2.7373	elliptic → real negative at 2.7081 to 2.7122	elliptic → real positive at 2.7316 to 2.7358	elliptic → real negative at 2.7105 to 2.7147	3.1275	2.6803 / 2.6778
18	real positive → elliptic at 2.7351 to 2.7395	elliptic → real negative at 2.7179 to 2.7221	elliptic → real positive at 2.7336 to 2.7378	elliptic → real negative at 2.7212 to 2.7253	3.1625	2.6832 / 2.6806
19	real positive → elliptic at 2.7387 to 2.7432	elliptic → real negative at 2.7255 to 2.7299	elliptic → real positive at 2.7384 to 2.7426	elliptic → real negative at 2.7302 to 2.7343	3.1966	2.6858 / 2.6831
20	real positive → elliptic at 2.7415 to 2.7461	elliptic → real negative at 2.7325 to 2.7370	elliptic → real positive at 2.7408 to 2.7449	elliptic → real negative at 2.7367 to 2.7408	3.2297	2.6882 / 2.6853

From Section 3.2 (where the transverse pair passes through -1).

The passage lies in class C between $L^* = 2.6661$ and 2.6702 at $n = 13$ (the end orbit is at 2.6647) and between 2.7325 and 2.7370 at $n = 20$ (the end orbit is at 2.6882), in class I at similar values; the brackets of all n are in Table S2. The L^* of the passage through -1 rises with n faster than that of the end orbit, and the bracket lies above the end orbit by at least 0.0014 ($n = 13$) to 0.044 ($n = 20$) in L^* in class C and by at least 0.0055 to 0.051 in class I.

From Section 3.3 (where along the curve the resonant pair changes type).

The change lies at 0.42 ($n = 6$) to 0.90 ($n = 20$) of the way from the branch point to the end orbit, measured in L^*: the fraction is $f = (L^*_{\text{branch}} - L^*)/(L^*_{\text{branch}} - L^*_{\text{end}})$, computed at the two edges of the bracket of the change in Table S2, with the L^* of the branch point and of the end orbit of the same class given in that table (4 decimals; the L^* of the end orbits is also in Tables 2, 3 and 4). The fraction rises with n; the centres of the brackets give 0.44 at $n = 6$ and 0.90 at $n = 20$.

From Section 3.3 (the brackets of the change of the resonant pair).

The change takes place at nearly the same L^* in the two classes (for $n = 12$ between 2.7030 and 2.7069 in class C and between 2.7048 and 2.7083 in class I; all brackets are in Table S2, and the centres of the two brackets of one n differ by 0.0001 to 0.0021 in L^*).

A.14 Symmetries, closure and multipliers: details moved from Sections 2.1 to 2.5

From Section 2.1 (reversing symmetries).

Devaney also cites De Vogelaere's classification of symmetric periodic points by the parity of the period (Prop. 4.3) and shows that a symmetric periodic orbit of a suitably non-degenerate kind lies on a smooth one-parameter family of symmetric periodic orbits (Prop. 7.3).

From Section 2.1 (the isosceles symmetry of R).

Over the 242 stored points the half period state, written again as a perpendicular start, equals the original start to 7×10^{-8} at most in the start parameters (u_1, u_2, λ, t); the deviation of the state at t itself from the exchanged and rotated start state is at most 3.81×10^{-6} in the state space, larger because it contains the departure from the collinear and perpendicular form, which is largest at the first points of the trace (closest approach 0.0295 to 0.0352 at the first six points) and which the rewriting removes. All 242 points are distinct points of one curve, and the last 32 lie within 1×10^{-8} to 1×10^{-9} of each other (Online Resource 1, A.8).

Γ is a symmetry of the equations, so $z(s+t) = \Gamma z(s)$ for all s. The map $S' = \Gamma S$ is another reversing symmetry and an involution: S commutes with the exchange and turns the rotation by φ into one by $-\varphi$, so $S\Gamma S = \Gamma^{-1}$. If $z(-s) = Sz(s)$, then $S'z(s) = \Gamma z(-s) = z(t - s)$, so the states $z(t/2)$ and $z(3t/2)$, at a quarter and at three quarters of the relative period $2t$, are symmetric states of S'. They are isosceles: $\Gamma Sz = z$ says that body 0 is on the line ℓ through the centre of mass at the angle $\varphi/2$ to the x axis (the rotation by φ after the reflection in the x axis is the reflection in ℓ), with its velocity perpendicular to ℓ, and that body 2 is the mirror image of body 1 in ℓ, with $\mathbf{v}_2$ the negative of the mirror image of $\mathbf{v}_1$.

On the 242 stored points this isosceles test, with body 0 as the apex, holds at a quarter and at three quarters of the relative period with errors at most 1.2×10^{-8} in the positions and 2.5×10^{-9} in the velocities. The control, which tests body 1 as the apex (bodies 0 and 2 as the mirrored pair), gives errors of 2.3×10^{-4} and more; body 2 as the apex was not tested.

From Section 2.3 (the second null vector).

At a branch point the matrix $[J \,|\, \partial F/\partial \Lambda]$ has a second null vector: for $n = 5$, 6 and 8 the second smallest singular values are 3.4×10^{-12}, 3.4×10^{-12} and 6.9×10^{-13}, against 5.5×10^{-3}, 3.8×10^{-3} and 2.2×10^{-3} for the next.

From Section 2.4 (the closure at three levels).

The closure is checked at three levels. (1) In double precision with two integrators: the largest difference of a component after one period, best rotation removed, is below 1×10^{-8} ("reliable"; orbits that do not meet this are called "weak"). (2) With REBOUND (IAS15; Rein and Liu 2012, Rein and Spiegel 2015), an independent integrator, from the start values rounded to double precision. (3) In 30 digits, integrating the full period from the values as stored.

From Section 2.5 (the multipliers in multiple precision).

For $n = 5$ to 12 the matrix is computed again in multiple precision, with the trivial multipliers removed by reduction and not by picking the four eigenvalues closest to 1, which in double precision are spread by about 1×10^{-4} because the time shift and the rotation make Jordan blocks. The matrix N is built by central differences of the flow with Richardson extrapolation, for the 30-digit orbits and for the orbits refined to 40 digits, which gives an error estimate (Online Resource 1, A.4); the class C orbits of $n = 13$ to 20 and the end orbits of class I ($n = 5$ to 20) are treated in the same way, with the 30-digit orbit only.

From Section 2.2 (the monodromy matrix along the trace of R).

At each step the relative monodromy matrix is computed from the variational equations over one relative period, rotated back by θ and restricted to the 8-dimensional part with zero centre of mass and momentum; the four eigenvalues closest to 1 count as trivial.

A.15 Details of the results: moved from Sections 3.1 to 3.4 and 4.3

From Section 3.1 (the two senses; the change of T^* and L^* along the curve).

That the two end states are the two symmetric points of one orbit was checked on the double-precision end states of the continuation: the end state of one sense, integrated over half its relative period and written again as a perpendicular start (rotation, scale and the order of the bodies allowed for), equals the end state of the other sense, with a largest difference of 1.9×10^{-9} in (u_1, u_2, λ) and 1.7×10^{-9} relative in t over $n = 5$ to 15 and both directions, while the other variant of the transformation, the half turn, differs by more than 1.1. The two senses are therefore linked by the flow over half a period (Section 2.1).

This fits Hénon's relation (Section 4.1): along the curve the rotation changes only from $-\varepsilon_n$ to 0, by at most 7.0×10^{-3} turns, so T^* changes little (by at most 0.058), while the relation does not limit the change of L^*, which is up to 1.4.

The curves of $n = 5$ and 6 reach $\theta = 0$ a second time further on, at $T^* = 17.7208$ and 19.6472 (`data/satellite-candidates.json`); these two orbits are not part of the sequence, and the continuation of the other curves stops at the first zero (Online Resource 1, A.1).

From Section 3.2 (long runs with REBOUND, and the brackets of the transverse pair in class I).

In double precision, REBOUND runs of 1000 and 10,000 periods keep the orbits within 1.1×10^{-8} of the start state, with an energy and angular momentum drift below 9×10^{-14}; a perturbation (each component of the start state multiplied by $1 + 1 \times 10^{-8}\xi$, ξ a standard normal number, so that the six components that are zero in the start state are not perturbed) grows linearly in the number of periods (the log–log slope is 0.99 to 1.01), the signature of a frequency shift and not of an instability, and no exponential growth appears in 10,000 periods.

In class I the transverse pair changes from elliptic to real negative between 2.6683 and 2.6717 at $n = 13$ (end orbit 2.6628) and between 2.7367 and 2.7408 at $n = 20$ (end orbit 2.6853).

The change takes place at nearly the same L^* in the two classes, for $n = 12$ between 2.7030 and 2.7069 in class C and between 2.7048 and 2.7083 in class I, for $n = 6$ between 2.6242 and 2.6264 and between 2.6233 and 2.6259, and for $n = 20$ between 2.7415 and 2.7461 and between 2.7408 and 2.7449; the centres of the two brackets of one n differ by 0.0001 to 0.0021 in L^*, and the brackets of single curves are 0.0009 to 0.0052 wide.

From Section 3.3 (the two zeros of $n = 14$, and the differences of T^* and L^* between the two end orbits).

For $n = 14$ the two zeros differ in T^* by 4×10^{-7}, which is πL^* times the rotation of 4.8×10^{-8} turns left at the zero of sense -1 (eq. (10); that of sense 1 is 2.6×10^{-12}): the zero of sense -1 is less converged, on the same curve.

T^* of the class I orbit is above that of the class C orbit by 3.0×10^{-7} at $n = 5$, rising to 1.1×10^{-5} at $n = 12$ (relative 1.7×10^{-8} to 3.6×10^{-7}), and its L^* is below by 6.2×10^{-5} at $n = 5$, rising to 1.7×10^{-3} at $n = 12$ (relative 2.4×10^{-5} to 6.5×10^{-4}); the values for each n are in Table 3.

T^* of the class I orbit is above that of the class C orbit by 1.18×10^{-5} at $n = 13$, rising to 1.49×10^{-5} at $n = 20$ (relative 3.6×10^{-7} and 3.3×10^{-7}), and its L^* is below by 1.92×10^{-3} at $n = 13$ and 2.88×10^{-3} at $n = 20$ (relative 7.2×10^{-4} and 1.07×10^{-3}); all values are in Table 4.

From Section 3.4 (the end of the stored trace); from Section 4.3 (the extrema of the dip).

The step rule of the tracing code rejects a half period t below 0.3 of the t at which the piece of the trace started, and the last stored point has $t = 0.29999$ of that value; the end of the stored trace is a limit of the tool and not a property of the family.

Because $dT^* = -(L^*/2)d\theta$, T^* decreases where θ increases and increases where θ decreases, so the lowest θ of the dip is the highest T^*, a coincidence of extrema that the relation of Hénon implies; on 51 of the 52 curves they are at the same stored point, and on the 52nd ($n = 7$, dip 2.9×10^{-11} turns) at neighbouring points.

A.16 The relation of Hénon: derivation and slope check (moved from Section 4.1)

How δ_n arises follows from a relation of Hénon (1977). For a family of periodic solutions of a system with an axis of symmetry, $\partial T/\partial A = -\partial\Phi/\partial E$, with T the period, A the angular momentum, Φ the angle through which the system has rotated after one period and E the energy (his eq. 30). For the N-body problem he combines this with the mechanical similarity of Newtonian orbits and obtains, along a family of fixed energy, $dT/d\Phi = -A/(2E)$ (his eq. 35). The relation is used here at fixed energy and with scale-free variables. For $E < 0$ it reads $dT/d\Phi = A/(2|E|)$, and with $T^* = T|E|^{3/2}$ and $L^* = |A||E|^{1/2}$, and $A > 0$, $dT^* = |E|^{3/2}\,dT = (A|E|^{1/2}/2)\,d\Phi = (L^*/2)\,d\Phi$. The similarity of Newtonian orbits (lengths scaled by c, times by $c^{3/2}$, the energy by $1/c$, the angular momentum by $c^{1/2}$) leaves T^*, L^* and Φ unchanged, so any family can be brought to one energy, and $dT^* = (L^*/2)d\Phi$ holds along a family whatever the energies of its members. Φ is Hénon's rotation angle; the rotation angle θ of this paper has the opposite sign, because it is the angle of the rotation that takes the end state of one period back onto the start state, so $\theta = -\Phi$ and $dT^* = -\frac{L^*}{2}\,d\theta = -\pi L^*\,d\rho$, with $\rho = \theta/2\pi$ the rotation in turns (eq. (10) of the article).

At the branch point both the satellite curve and the repeat curve have the same L^*, so both have the slope πL^* in absolute value; the slope of the repeat curve, read from the R family alone, agrees with the local slope for 50 of the 52 values of n. The two that do not are $n = 52$ and $n = 55$, where

the slope of the repeat curve is above the local slope of the satellite curve by 3.7 and 9.2 percent, more than the tolerance of the check (the larger of 3 percent and the estimated uncertainty of either slope). Both are at the end of the range, where the local slope of the satellite curve is least well determined (uncertainty 0.38 and 1.03, against slopes of about 12.6 and 12.7).

A.17 The comparison with the atlas and the orbits of the thesis (moved from Sections 5.2 and 5.3)

The comparison used the local copy of 2026-10-09 of all 415 equal-mass entries with non-zero angular momentum (the catalogue was not downloaded again): an entry matches an orbit when L^* agrees to 1×10^{-5} relative, T^* to 1×10^{-5} relative where one of the two is n times the other (n up to 12), and the number of syzygies is the same.

The 16 class I end orbits ($n = 5$ to 20) were compared in the same way with the 415 entries, among them the 14 satellites, and with the 99 satellites of the CPC tables (T^* and L^* only, with repeats up to 12): none matches at 1×10^{-5}, and the nearest entries in (T^*, L^*) are far (for $n = 13$ to 20 the larger of the two relative differences is 6×10^{-2} or more); no atlas entry lies within 5×10^{-3} with the same number of syzygies except, for $n = 7$, the entry R7 1/1, which is the class C orbit and lies 2.1×10^{-4} in L^* from the class I orbit. No CPC row lies within 5×10^{-3} except, for $n = 17$, the rows $N = 90$ and 91 ($k = 42$), which lie within 5×10^{-3} in (T^*, L^*) at twice T^*: this is a coincidence of two numbers, because those rows have θ not 0 and the test does not use the number of syzygies.

The orbits were converted like the rows of the CPC tables, closed again and tested, with the following result. The orbit $k = 39$ is a rotating member (θ not 0) of the satellite curve of the member of R with the lower $\nu = 1/39$; it has no repeat, its word is $(01)^{39}$, and the member has a second null vector (second smallest singular value 4.3×10^{-12} against 8.1×10^{-5}); the orbit with $\theta = 0$ on this curve was not computed.

A.18 Sources, search and repositories (moved from Sections 1.4 and 6.3)

In the two preliminary searches of Janković, Dmitrašinović and Šuvakov (2020), described in Section 1.4 of the article, L is the angular momentum for $G = m = 1$ with the size of the configuration fixed and the energy left free (their Section 3, Eq. 5), so the two values are not values of L^* and cannot be converted to L^* without the energy of each orbit; their scale-invariant angular momentum, for equal masses, is $L|E|^{1/2}$ (their Section 2), the L^* used here.

The sources checked were the Three Body Orbits atlas (415 equal-mass entries with $L \neq 0$, 2026-10-09), the archived copies of 2019 of the pages of the Šuvakov gallery (57 satellite pages, all of them rows of the CPC tables) and the tables of Davoust and Broucke (1982), of Janković, Dmitrašinović and Šuvakov (2020) (99 satellites) and of Li et al. (2025) (18 rows); the statements of Janković and Dmitrašinović (2016), Janković, Dmitrašinović and Šuvakov (2020) and Li and Liao (2020, arXiv:2008.13550) on the direct branch were read (Sections 1.3, 1.4 and 5.4).

The public listing of github.com/sjtu-liao/three-body (viewed on 17 October 2026) holds pictures of orbits with equal masses and zero angular momentum, of orbits with unequal masses and zero angular momentum and of free-fall orbits, initial conditions of three-dimensional orbits, the five papers of the group, and in `roadmap/Data` four data files of BHH satellites with various masses; whether these files hold equal-mass orbits with non-zero angular momentum was not examined.

A literature search of 2026-10-11 used OpenAlex, Semantic Scholar, Crossref and arXiv, 15 keyword queries, 269 distinct works that cite Broucke (1975), Davoust and Broucke (1982), Hénon (1976),

Janković and Dmitrašinović (2016) or Janković, Dmitrašinović and Šuvakov (2020), and the reference list of the 2025 review of Li and Liao (Li and Liao 2020, arXiv:2008.13550, had no citing works in OpenAlex or Semantic Scholar).

A.19 Further details moved from Sections 2.2, 2.3, 3.3, 3.4, 4.2, 6.1 and 6.2

From Section 3.3 (the pre-registration of the comparison; the distances of the pairs in Table 5).

For $n = 5$ the comparison of the two curves that was written down before the run was undecided, and these three facts were added after the data were seen; for $n = 6$ to 20 the criterion was written before the runs and does not use the curves near the branch point, and it is met for $n = 6$ to 12 and for $n = 13$ to 20 (Online Resource 1, A.5 and A.12).

For $n \leq 12$ the transverse pair is at the distance $1/2 - \alpha_t$ from -1: 0.397 at $n = 5$ and 0.098 at $n = 12$ in class C, 0.397 and 0.049 in class I. The resonant pair is at the angle α_r (0.0039 to 0.0095) from $+1$ in class C, and at $\mu_r - 1$ (0.025 to 0.062) in class I.

From Section 3.4 (the extension of the trace of R).

The trace was continued from the last stored point with the same code, in two runs. The first ended at $\lambda = 0.9496$, where the step rule of the tool (a half period below 0.3 of the one at which the run started is rejected) was reached again (the tool's stop message says "lost the family", but the family was not lost); the second was started from its last point with a new reference half period, continues the family smoothly (see below) and ended at $L^* = 7.25$ ($\lambda = 0.970$, $T^* = 1.646$) by the closest-approach stop of the tool (closest approach 0.030). The runs hold 109 stored points, of which 78 are distinct to 5 decimals in (λ, L^*): the first run stored its last point again after it had stalled. The first point of the second run is one regular step beyond the last distinct point of the first: the steps in λ, L^*, T^* and the rotation across the join (9.2×10^{-4}, 4.7×10^{-2}, -2.5×10^{-3} and -1.4×10^{-4}) are within 8 percent of the last regular steps of the first run before the stall (9.7×10^{-4}, 4.7×10^{-2}, -2.7×10^{-3} and -1.5×10^{-4}). All the points are stable. Both rotation numbers fall monotonically, from 0.0170 to 0.0023, and each $1/n$ for $n = 59$ to 439 is crossed once by each, as a bracket between consecutive points; with the stored trace the rotation numbers $1/n$ are crossed for every n from 3 to 439.

From Section 6.1 (the lower rotation number and the rotation numbers j/n).

For the lower rotation number, the satellite curves for $n = 13$ to 20 never reached $\theta = 0$ (16 continuations, all stopped by a limit); for $n \leq 12$, 23 of the 40 satellite continuations were stopped by a limit, among them the lower branch for $n = 4$, 6 and 8; the lower curves for $n \geq 21$ were not run.

Satellites with rotation number j/n and $j > 1$ were followed from 21 members: 33 of the 42 continuations reach $\theta = 0$, and 12 of those 33 zeros have a weak or failed closure (Section 2.4); they are not part of the sequence.

From Section 6.2 (the hierarchical structure of the members at large n).

In the start state the distance of bodies 1 and 2 is $1 - \lambda = 0.115$ at $n = 56$, against 1.885 from body 0 to body 1, a ratio of 0.061, and 0.30 at $n = 5$.

The rotation of the member per relative period is 0.017869 turns at $n = 56$, against $1/56 = 0.017857$.

From Section 2.2 and Section 2.3 (finding the member; the bifurcation equation).

A member with $\nu_{\text{up}} = 1/n$ is found by bracketing the crossing on two stable steps and bisecting

until $|\nu - 1/n| < 1 \times 10^{-11}$, closing the member by multiple shooting (residual below 1×10^{-11}).

A second null vector is necessary for a second branch, and a quadratic bifurcation equation on the null space then says whether one exists: it has two real directions, one the tangent of the repeat curve (computed independently from the repeats of the neighbouring members, to six digits) and one the satellite.

From Section 4.2 (the three predictions of the T^* of the end orbit).

Three predictions of the T^* of the end orbit were compared: no correction, the first-order prediction from eq. (10) without a free parameter, and an empirical relation B, written down before the runs for $n \geq 21$, with two constants fitted on $n = 8$ to 20. The first-order prediction is the best of the three for $n \leq 12$ (error 0.0037 at most). B is better than no correction for $n = 6$ to 47, but the bounds written before the runs (0.05 for $n = 21$ to 36, 0.005 for $n = 37$ to 56) do not separate it from no correction, that is B is not discriminated from the null model, and from $n = 48$ ($n = 49$ with the exact member) B is worse than no correction; the error of B, about 3×10^{-3}, does not shrink with n.

From Section 6.2 (the precision of the multipliers).

The multipliers are computed in multiple precision with error estimates from two precisions (tiers A and B) for the class C orbits of $n = 5$ to 12, in multiple precision in one tier (the 30-digit orbit) for the class C orbits of $n = 13$ to 20 and for the 16 end orbits of class I ($n = 5$ to 20), and in double precision for the other orbits (for $n = 21$ to 56 the largest multiplier and its sign, by the reduction of Section 2.5).

The branch points for large n rest on singular values (at $n = 29$ the smallest are 2×10^{-12} and 2.6×10^{-5}) with a relaxed repeat tolerance; the end orbits are checked independently of it by their own closure, the 30-digit refinement and REBOUND (Online Resource 1, A.3).

From Section 6.1 (the cases $n = 57$ and 58).

With 16 pieces of the multiple shooting both senses of both curves stopped at the step limit after a path length of 3.9 to 8.0, and a further change for these two cases (24 and 32 pieces) was declared in the rules file before it was run; its results are not part of this paper.