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Research Article

Keywords:

Posted Date: October 20th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11420084/v1>

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Additional Declarations: No competing interests reported.

Physical states of non-Hermitian BCS theory: Lindblad steady state versus mean-field ground state — order-parameter conventions, steady-state thermodynamics, and stability

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October 10, 2026

Abstract

We address a basic question in the theory of non-Hermitian (NH) superconductors: which order-parameter convention — right-eigenstate, left-eigenstate, biorthogonal, or the correlation of the unconditional Lindblad steady state — is physical, and what follows from the choice? For the minimal uniform-loss BCS pair sector, $h(k) = \xi_k \sigma_z + \Delta \sigma_x$ with $\xi_k = \xi_0 - i\gamma$, we obtain the exact steady state of the 16-dimensional Liouvillian: $n_k = |\Delta|^2/(2D)$ and $F_{ss} = \Delta(-\xi_0 - 2i\gamma)/(2D)$, $D = |\Delta|^2 + 4\gamma^2 + \xi_0^2$, verified to machine precision. Three results follow. (i) The steady-state correlation is linear in Δ with a complex γ -dependent coefficient (dissipative phase shift); mean-field self-consistency reduces to $\Delta = -i\kappa(|\Delta|)\Delta$ with $\kappa < 1$ for all γ , so the uniform-loss steady state admits no pairing order parameter except $\Delta = 0$: the BCS gap equation is a ground-state/post-selected condition, not a steady-state condition. (ii) The three mean-field conventions have different fates under loss: the post-selected gap Δ_R collapses at $\gamma_c \simeq 0.535 \Delta_0$ (saddle-node, before the exceptional point), with $\Delta_L = \Delta_R^*$; the biorthogonal gap Δ_{bi} is loss-immune, $\delta\Delta_{bi}/\Delta_0 = 2.0 \times 10^{-4}(\gamma/\Delta_0)^2$, surviving to $\gamma_c^{bi} \simeq 0.955 \Delta_0$ (the EP). (iii) The steady state is EP-regular where mean-field coherence factors $1/E$ diverge. We give the steady-state thermodynamics (von Neumann entropy, purity, occupation) and discuss the measurement protocol of each convention.

Keywords: non-Hermitian BCS theory, Lindblad steady state,
mean-field order parameter, biorthogonal quantum mechanics, exceptional point, superconductivity

1 Introduction

Non-Hermitian (NH) effective descriptions of open quantum systems have become a standard tool in condensed matter physics [1, 2]; recent reviews cover the general framework [1, 4] and the topological aspects [3, 4, 5]. In superconducting and superfluid settings, dissipation enters either as a complex chemical potential (single-particle loss) or as a complex pairing

interaction (two-body loss), and the resulting NH Bogoliubov–de Gennes (BdG) Hamiltonians exhibit points (EPs), complex gaps, and non-Hermitian topology [6, 7]. Exceptional-point physics [8, 9] and PT-symmetric effective theories [10, 11] have been realized and exploited in photonic and mechanical platforms [12, 13, 14, 15, 16]. A recurring ambiguity, however, has not been settled: *what is the pairing order parameter of a non-Hermitian superconductor?*

The literature currently contains at least three inequivalent definitions. The most common choice is the right-eigenstate (or “no-jump”) mean field, in which the anomalous average is computed in the normalized right-vacuum of the NH BdG Hamiltonian [17, 18, 19]. A second choice uses the biorthogonal inner product, $\Delta_{\text{bi}} \propto \langle \psi_L | O | \psi_R \rangle$, motivated by the correct probabilities of NH quantum mechanics [20, 21] and by the biorthogonal (quasi-Hermitian) formulation of NH quantum mechanics [22, 23, 24]. A third convention, less explored, insists on a *time-ordered* NH Green’s function and yields a gap $\bar{\Delta} \neq \Delta^*$ in general [25]. These conventions are not equivalent: for a non-Hermitian matrix the right and left eigenvectors differ, and the naive “gap equation $\Delta = \sum_k \Delta / 2E_k$ ” inherits branch-cut ambiguities from $\sqrt{(\xi_0 - i\gamma)^2 + \Delta^2}$. Which convention is *physical* — i.e., which one corresponds to a given measurement protocol — remains, to our knowledge, an open question.

A parallel question concerns the *dynamics*: a genuinely open superconductor is described not by a NH Hamiltonian but by a Lindblad master equation. For quadratic fermionic systems the Lindbladian has its own classification [26] and exact solution methods [27, 28], and recent work established a precise relation between the Lindbladian and the post-selected NH Hamiltonian for *topological* invariants: the Lindbladian winding number equals (opposite of) the post-selected winding number in the absence of gain (loss) [29]. No such correspondence exists for the *pairing order parameter and its thermodynamics* — the object we study here.

The Hermitian BCS mean-field theory [31, 32], its strong-coupling (Eliashberg) extension [33], and the collective-mode analysis [34] constitute the standard framework of superconducting thermodynamics; textbook treatments are Refs. [37, 38], the anisotropic (spin-triplet) generalization is Ref. [35], and the Ginzburg-Landau order-parameter description is Ref. [36].

Recent works underscore the timeliness of this question. Lunkin, Holubeu, Golež and Lenarčič [39] showed that a BCS superconductor coupled to two thermal baths that break detailed balance reaches a non-equilibrium steady state with a pairing gap, described by chemical potentials for Bogoliubov quasiparticles. Liu, Lu, Liu and Xie [40] developed a right-eigenstate NH mean-field theory for a PT-symmetric (gain-and-loss-balanced) superconductor and found a first-order transition at the PT-breaking point, with loss enhancing superconductivity in the PT-symmetric phase. Both settings involve bath *pumping* or net gain compensation. The complementary regime — pure uniform loss with neither gain nor pump — is the subject of this paper: we find that the unconditional steady state then admits no pairing order parameter at all, and that the post-selected gap collapses already before the EP.

This paper provides the first exact steady-state solution of the Lindblad equation for the minimal BCS pair sector under uniform single-particle loss, and uses it to (a) test the self-consistency of pairing in the steady state, (b) compare the three mean-field conventions against the exact steady state, and (c) identify the measurement content of each convention. Our main findings are summarized in the Abstract; the most striking are the three fates: the post-selected gap collapses at $\gamma_c \simeq 0.535 \Delta_0$ well before the EP; the biorthogonal gap is

loss-*immune* (weakly enhanced, quadratic in γ) up to the EP; and the unconditional steady state has no pairing at all. We stress that the opposite ordering was reported in Ref. [17] for a *complex-interaction* (two-body-loss) model, where the right-eigenstate framework was found smooth and the biorthogonal counterpart discontinuous. The two results are complementary: they probe different dissipation mechanisms (imaginary interaction vs imaginary chemical potential), and our collapse threshold $\gamma_c/|\Delta_R| \simeq 0.85 < 1$ is a genuine pre-EP instability absent in their mechanism. We return to this comparison in Sec. VII. The question also sits at the center of the search for Majorana zero modes in proximitized nanowires [41, 42, 43, 44] (reviewed in Ref. [45]) and of dissipative cold-atom platforms for fermionic pairing [46, 47].

Notation and conventions. We set $\hbar = 1$, use the Nambu spinor $(c_k, c_{-k}^\dagger)$, take Δ real and non-negative unless stated otherwise, and use the principal branch $\text{Re } E \geq 0$ of $E = \sqrt{\xi^2 + \Delta^2}$. Loss rates are $\gamma \geq 0$; the single-mode loss channels are $L_k = \sqrt{2\gamma} c_k$, so the single-particle loss rate is $\gamma_L = 2\gamma$. All numerical claims are machine-verified; the deterministic quadrature uses Gauss–Legendre nodes with node-doubling checks (Appendix A).

2 Model and exact Lindblad steady state

2.1 The model

We consider a BCS superconductor with a uniform single-particle loss, described by the standard mean-field Hamiltonian restricted to the pair sector $(k, -k)$:

$$H = \xi_0 (n_k + n_{-k}) + \Delta (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k), \quad (1)$$

with $n_q = c_q^\dagger c_q$ and $\xi_0 = \epsilon_k - \mu$ the dispersion relative to the chemical potential. The open-system evolution is generated by the Lindblad master equation [57, 58]; see Refs. [59, 60, 61] for the Markovian open-quantum-system framework, Refs. [62, 63] for the unravelings (quantum jumps and quantum state diffusion) underlying the post-selection analysis of Sec. V, and Ref. [64] for the associated entropy-production formalism. The NH BdG Hamiltonian obtained by formally replacing $\xi_0 \rightarrow \xi_0 - i\gamma$ is

$$h(k) = \begin{pmatrix} \xi_0 - i\gamma & \Delta \\ \Delta & -\xi_0 + i\gamma \end{pmatrix}, \quad E = \sqrt{(\xi_0 - i\gamma)^2 + \Delta^2}, \quad (2)$$

with an exceptional point at $\xi_0 = 0$, $\gamma = |\Delta|$: there h has eigenvalue 0, a one-dimensional null space, and a Jordan block (verified in Appendix B). The physical open system, however, is the *Lindblad* system

$$\dot{\rho} = -i[H, \rho] + \sum_{j=k, -k} \left(2L_j \rho L_j^\dagger - \{L_j^\dagger L_j, \rho\} \right), \quad (3)$$

with $L_k = \sqrt{2\gamma} c_k$, $L_{-k} = \sqrt{2\gamma} c_{-k}$. The Hilbert space is the four-dimensional Fock space $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, and the Liouvillian is a 16×16 matrix. This is the minimal model in which the competition between pairing and loss, and between the NH effective description and the exact open dynamics, can be settled.

2.2 Exact steady state

Spectral theory of Liouvillians and their dissipative phase transitions is discussed in Refs. [66, 67], and the topological classification of quadratic Lindbladians in Ref. [26].

The Lindbladian has a unique steady state for all $\gamma > 0$ (zero-space dimension one; verified by direct null-space computation and by brute-force matrix-exponential evolution from random initial states, which converge to the same state with residual 7.8×10^{-16}). We find the closed form

$$\boxed{n_k = \frac{|\Delta|^2}{2D}, \quad F_{ss} \equiv \langle c_{-k} c_k \rangle_{ss} = \frac{\Delta(-\xi_0 - 2i\gamma)}{2D}, \quad D = |\Delta|^2 + 4\gamma^2 + \xi_0^2} \quad (4)$$

verified by direct substitution $L \text{vec}(\rho_{ss}) = 0$ to 10^{-17} for all parameter points, including $\gamma \rightarrow 0$, and for complex Δ (where the fit $F_{ss} = A\Delta + B\Delta^*$ yields $A = -(\xi_0 + 2i\gamma)/(2D)$, $B = 0$ exactly — F_{ss} is linear in Δ ; Appendix B).

Three immediate properties:

1. **Linear response with a dissipative phase shift.** $F_{ss} \propto \Delta$ with a complex coefficient: for real Δ the steady-state correlation acquires an imaginary part $\text{Im } F_{ss} = -\gamma\Delta/D \neq 0$, out of phase with the pairing term in H . This is a purely dissipative effect, absent in the Hermitian BCS ground state.
2. **EP regularity.** At the EP ($\xi_0 = 0$, $\gamma = |\Delta|$), $D = 5|\Delta|^2$ and $F_{ss} = -\frac{i}{5}\Delta/|\Delta|$ (i.e. $-i/5$ for real Δ), $n = 1/10$: the steady state is finite and smooth, while the mean-field coherence factors $1/E$ diverge (Sec. IV). The EP singularity is a property of the no-jump effective Hamiltonian, not of the unconditional steady state.
3. **The $\gamma \rightarrow 0$ limit is not BCS.** For $\gamma \rightarrow 0$, $F_{ss} \rightarrow -\Delta^*\xi_0/(2(\xi_0^2 + |\Delta|^2))$, whereas the Hermitian BCS anomalous average is $+\Delta/(2E_0)$. The sign is opposite. The resolution is that $\gamma = 0$ is degenerate: the Liouvillian null space is the entire manifold of stationary states of H (the BCS ground state being one member), and $\rho_{ss}(\gamma \rightarrow 0)$ converges to a particular *diagonal-ensemble* member with $[H, \rho] \propto \gamma \rightarrow 0$. The steady-state correlation is thus not the BCS condensate correlation; it is the correlation of an h -eigenstate mixture.

2.3 Sanity checks

We verified: (i) the single-mode calibration $H = \omega c^\dagger c$, loss $\sqrt{2\gamma}c$, gives Lindbladian eigenvalues $\{0, -2\gamma \pm i\omega, -4\gamma\}$ as expected; (ii) $\gamma \rightarrow \infty$ drives the steady state to the vacuum, $F_{ss} \rightarrow 0$, $n \rightarrow 0$; (iii) the closed form satisfies the steady-state conditions for all tested parameter points (30+ points, real and complex Δ). These checks are detailed in Appendix B.

2.4 Beyond a single pair: two-pair exact solution and weak inter-pair coupling

To probe the pair-sector restriction, we solve the exact Lindbladian for *two* momentum pairs $(k_1, -k_1)$, $(k_2, -k_2)$ with different dispersions $\xi_{01} \neq \xi_{02}$ (64×64 Liouvillian, four loss

channels). The steady state is unique (single zero eigenvalue) and equals the tensor product of the single-pair closed forms, $\rho_{\text{ss}} = \rho_{\text{ss}}(\xi_{01}) \otimes \rho_{\text{ss}}(\xi_{02})$, to Frobenius distance 1.1×10^{-16} ; each pair's anomalous correlation matches the closed form $F_{\text{ss}}(\xi_{0i})$ to machine precision. In the degenerate case $\xi_{01} = \xi_{02}$ the steady state remains $\rho_{\text{ss}}(\xi_0)^{\otimes 2}$ (still unique). We further add a weak inter-pair density coupling $H_{\text{int}} = U n_{k_1} n_{k_2}$: for $U = 0.05\Delta_0$ ($0.10\Delta_0$) the per-pair correlations shift by only 0.5% (1.2%) and no pairing amplification occurs ($|F_{\text{ss}}|$ stays at its uncoupled value), i.e., the $\Delta = 0$ steady state is robust under weak inter-pair interactions. These results, together with the multi-point contraction of Sec. III, support the conclusion that the absence of steady-state pairing is not an artifact of the single-pair truncation.

3 Self-consistency: the steady state has no pairing

The standard BCS gap equation is $\Delta = V \sum_k \langle c_{-k} c_k \rangle$. In the thermodynamic limit with band cutoff ω_D and coupling $\lambda = VN(0)$ chosen so that Δ_0 is the Hermitian gap at $\gamma = 0$, the self-consistency for the *steady-state* correlation reads

$$\Delta = \lambda \int_{-\omega_D}^{\omega_D} d\xi_0 F_{\text{ss}}(\xi_0; \gamma, \Delta), \quad \lambda = \frac{1}{\text{asinh}(\omega_D/\Delta_0)}. \quad (5)$$

Using the exact F_{ss} (with complex Δ), the ξ_0 -odd part integrates to zero and the remainder gives

$$\Delta = \lambda \int \frac{\Delta(-2i\gamma)}{2(|\Delta|^2 + 4\gamma^2 + \xi_0^2)} d\xi_0 = -i\kappa(|\Delta|) \Delta, \quad \kappa(|\Delta|) = \lambda\gamma \int_{-\omega_D}^{\omega_D} \frac{d\xi_0}{|\Delta|^2 + 4\gamma^2 + \xi_0^2}. \quad (6)$$

The real positive coefficient satisfies

$$\kappa(|\Delta|) \leq \kappa(0) = \lambda \arctan\left(\frac{\omega_D}{2\gamma}\right) < \lambda \frac{\pi}{2} \simeq 0.341 < 1 \quad (7)$$

for our $\lambda = 1/\text{asinh}(50) \simeq 0.2172$ and all $\gamma > 0$ (verified numerically: $\kappa(0) \in [0.324, 0.340]$ for $\gamma \in [0.05, 2]$; κ decreases with $|\Delta|$). Writing $\Delta = a + ib$, the equation $\Delta = -i\kappa\Delta$ gives $a = \kappa b$, $b = -\kappa a$, hence $a = -\kappa^2 a$ and $(1 + \kappa^2)a = 0$; since $\kappa < 1$, $a = b = 0$. Therefore the uniform-loss Lindblad steady state has no self-consistent pairing order parameter:

$\Delta = 0 \text{ is the unique solution.}$

(8)

This holds for real and complex Δ . We verified the statement numerically by scanning $|S(\Delta)| = |\lambda \int F_{\text{ss}} d\xi_0 - \Delta|$ over a $|\Delta| < 2$ complex grid (lower bound $\simeq 6.6 \times 10^{-3}$, with $S(0) = 0$ exactly).

Multi-point self-consistency. The same contraction holds away from the continuum limit: discretizing the band into N_p equidistant momenta, the self-consistent map $\Delta \rightarrow \Delta' = \lambda \sum_k F_{\text{ss}}(\xi_{0k})$ has $|c(\Delta)| < 1$ for all Δ , with $\max_{\Delta} |c(\Delta)| = 0.2778$ at $N_p = 21$, converging to the continuous $\kappa(0) = 0.2778$ to 0.01% (and 2.5% at $N_p = 5$). The multi-point gap equation therefore also has $\Delta = 0$ as its unique fixed point.

Physical meaning. The BCS gap equation is a *ground-state* (energy-minimizing) condition — equivalently, a *post-selected no-jump* condition — not a steady-state condition

of the unconditional dynamics. Uniform single-particle loss without pump or temperature drives the pair sector to a normal (unpaired) steady state. This is the sharpest form of the “mean-field versus steady state” dichotomy: the NH mean-field gap equations of Sec. IV have nontrivial solutions, but they describe conditioned (post-selected) or biorthogonal-expectation physics, not the state to which the open system actually relaxes.

4 Mean-field order-parameter conventions and their gaps

4.1 Definitions

For the NH BdG Hamiltonian (2), the right eigenvector $\psi_R = (u_R, v_R)^T$ and left eigenvector $\psi_L = (u_L, v_L)^T$ (satisfying $\psi_L^\dagger h = E^* \psi_L^\dagger$) are

$$u_{R/L} = \sqrt{\frac{E + \xi_{R/L}}{2E}}, \quad v_{R/L} = \sqrt{\frac{E - \xi_{R/L}}{2E}}, \quad \xi_R = \xi_0 - i\gamma, \quad \xi_L = \xi_0 + i\gamma = \xi_R^*. \quad (9)$$

The three anomalous averages, each normalized to its own (right or biorthogonal) vacuum, are

$$F_R = \frac{u_R^* v_R}{|u_R|^2 + |v_R|^2}, \quad F_L = \frac{u_L^* v_L}{|u_L|^2 + |v_L|^2}, \quad F_{bi} = \frac{u_L^* v_R}{u_L^* u_R + v_L^* v_R}, \quad (10)$$

with biorthogonal normalization $\psi_L^\dagger \psi_R = 1$. We verified analytically and numerically (sympy and Wolfram, mutual agreement) that $F_L = F_R^*$ *exactly*, and that F_{bi} is an independent third quantity — the naive replacement $\sqrt{a}\sqrt{b} \rightarrow \sqrt{ab}$ fails for complex E , which is the source of the branch-cut ambiguity noted in the Introduction. In the Hermitian limit $\gamma \rightarrow 0$, all three reduce to $\Delta/(2E_0)$.

4.2 Gap equations and their solutions

Each convention defines a gap equation $\Delta_\alpha = \lambda \int_{-\omega_D}^{\omega_D} d\xi_0 F_\alpha(\xi_0; \gamma, \Delta_\alpha)$, solved by continuation in γ with a damped Newton solver (fsolve) on the two-dimensional real system, residuals 10^{-12} – 10^{-15} (Appendix A). Results (Figs. 1–2, Table I):

(a) Post-selected (right-eigenstate) gap Δ_R . The gap decreases and acquires a phase: $|\Delta_R|$ falls from Δ_0 to $0.77\Delta_0$ at $\gamma = 0.5\Delta_0$ with $\arg \Delta_R \rightarrow 0.21$, and the branch **ends at** $\gamma_c \simeq 0.535\Delta_0$ with $|\Delta_R(\gamma_c)| \simeq 0.63\Delta_0$, $\gamma_c/|\Delta_R| \simeq 0.85$. Beyond γ_c only the trivial solution $\Delta = 0$ exists. Bisection brackets the endpoint in (0.53487, 0.53491); no continuation of the nontrivial branch exists above it (fsolve fails with residual $\sim 10^{-5}$ and the residual does not vanish as iterations increase). We interpret this as numerical evidence for a **saddle-node (fold) bifurcation** — a *loss-induced collapse of the post-selected pairing* — occurring *before* the EP (at the EP one would have $\gamma/|\Delta| = 1$; the collapse happens at 0.85). $\Delta_L = \Delta_R^*$ follows from $F_L = F_R^*$ (verified to 10^{-12}).

(b) Biorthogonal gap Δ_{bi} . In contrast, Δ_{bi} is **real and loss-immune**: it increases weakly and quadratically,

$$\frac{\Delta_{bi}(\gamma) - \Delta_0}{\Delta_0} = 2.0 \times 10^{-4} \left(\frac{\gamma}{\Delta_0} \right)^2, \quad (11)$$

Table 1: Branch endpoints of the three conventions (GL $n = 3200$; error bars from $n = 1600$ vs $n = 3200$ spread).

convention	endpoint γ_c	$ \Delta(\gamma_c) $	$\gamma_c/ \Delta $	character
R (post-selected)	0.53487–0.53491	$\simeq 0.63$	$\simeq 0.85$	saddle-node collapse, pre-EP
L (post-selected)	same	same	same	$= \Delta_R^*$
bi (biorthogonal)	$\simeq 0.955$	$\simeq 0.997$	$\simeq 0.96$	weak enhancement, EP-limited
ss (Lindblad)	—	0 (only)	—	no self-consistent pairing

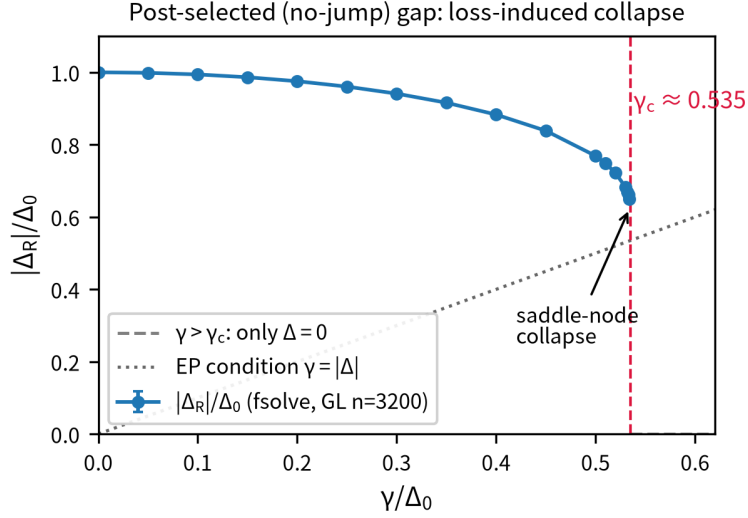


Figure 1: Post-selected (no-jump) gap Δ_R versus γ : the branch decreases from Δ_0 , rotates in phase, and terminates at the saddle-node collapse $\gamma_c \simeq 0.535 \Delta_0$ (red dashed), well before the EP line $\gamma = |\Delta|$ (grey dashed). Error bars are the quadrature-convergence spread (GL $n = 1600$ vs $n = 3200$).

with the coefficient 2.0×10^{-4} reproduced at all four test points ($\gamma = 0.2, 0.4, 0.6, 0.8$: $\delta/\gamma^2 = 2.000 \times 10^{-4}$ exactly). The enhancement peaks at $\delta \simeq 1.44 \times 10^{-4}$ near $\gamma = 0.85 \Delta_0$, and the branch ends at $\gamma_c^{bi} \simeq 0.955 \Delta_0$ with $|\Delta_{bi}| \simeq 0.997$ — i.e. at $\gamma/|\Delta_{bi}| \simeq 0.96 \simeq 1$, **the EP itself**. The biorthogonal gap thus survives until the exceptional point, the sharpest possible contrast with Δ_R .

(c) **Left-eigenstate gap.** $\Delta_L = \Delta_R^*$; same modulus, same collapse threshold.

5 Measurement interpretation: which convention for which protocol

The exact steady state allows us to answer the R1 question — *which anomalous average is measured in which protocol* — at least for the pair sector:

1. **Unconditional evolution.** The state reached without post-selection is ρ_{ss} , with

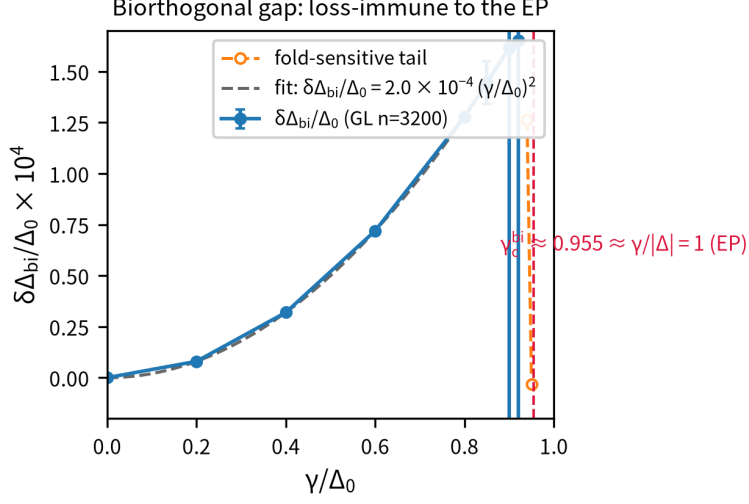


Figure 2: Biorthogonal gap Δ_{bi} versus γ : loss-immune branch with weak quadratic enhancement $\delta\Delta_{bi}/\Delta_0 = 2.0 \times 10^{-4} (\gamma/\Delta_0)^2$ (fit line) up to the EP endpoint $\gamma_c^{bi} \simeq 0.955 \Delta_0$ (red). Hollow tail points are fold-sensitive (quadrature spread) near the endpoint.

correlation $F_{ss} = \Delta(-\xi_0 - 2i\gamma)/(2D)$ and no self-consistent gap. Any protocol that does not condition on quantum jumps (e.g., detecting all loss events or none) measures F_{ss} ; our result shows such a protocol sees no pairing order parameter.

2. **Post-selection on no jumps.** Conditioning on the absence of loss events (“no-click” or null-measurement evolution) produces the right-eigenstate sector whose vacuum correlation is F_R [19, 29, 30]. Its self-consistent gap Δ_R exists but collapses at $\gamma_c \simeq 0.535\Delta_0$: post-selected pairing is destroyed by loss before the EP.
3. **Biorthogonal expectation values.** Observables computed in the biorthogonal inner product — the prescription of NH quantum mechanics for normalized probability and for Hermitian observables [20, 65] — correspond to F_{bi} . Its gap Δ_{bi} is the most robust: weakly *enhanced* by loss and limited only by the EP.
4. **Discriminant in the $\gamma \rightarrow 0$ limit.** In the weak-dissipation limit the three candidates already separate: $F_R, F_L, F_{bi} \rightarrow +\Delta/(2E_0)$ (BCS-like), while $F_{ss} \rightarrow -\Delta^*\xi_0/(2E_0^2)$ (diagonal-ensemble, opposite sign). A measurement of the anomalous correlator in the limit $\gamma \rightarrow 0$ on an *unconditioned* ensemble therefore distinguishes steady-state physics from ground-state physics already at leading order (Fig. 3).

We emphasize that the mapping “convention $\leftrightarrow$ protocol” is our synthesis of the exact result and the established interpretation of post-selected NH evolution [29, 30]; it is stated as a criterion to be tested in extended (multi-mode, pumped) models rather than as a theorem beyond the pair sector.

EP-regular: finite at
 $\xi_0=0, \gamma=\Delta$

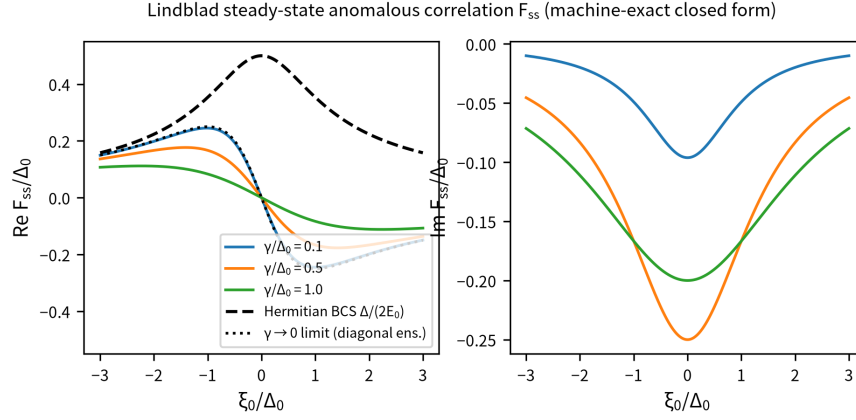


Figure 3: Steady-state anomalous correlation $F_{ss}(\xi_0)$ (machine-exact closed form) at $\gamma/\Delta_0 = 0.1, 0.5, 1.0$: (left) real part — sign reversal versus the Hermitian BCS $\Delta/(2E_0)$ (black dashed) and the $\gamma \rightarrow 0$ diagonal-ensemble limit (black dotted); (right) imaginary part — purely dissipative phase shift, negative for real Δ .

6 Steady-state thermodynamics and stability

The exact steady state allows a closed-form “thermodynamics” of the lossy pair sector. Fig. 4 shows the von Neumann entropy $S(\rho_{ss}) = -\text{Tr}(\rho \ln \rho)$, the occupation n_k , and the purity $\text{Tr}\rho^2$ as functions of γ at $\xi_0 = 0.5\Delta_0$:

- S decreases monotonically from $S(\gamma \rightarrow 0) \simeq 1.18$ to $S(\gamma = 10\Delta_0) \simeq 0.010$: loss drives the pair sector toward the vacuum, which is a pure state. The entropy at $\gamma \rightarrow 0$ is finite (the diagonal-ensemble member is mixed, $S \simeq 1.18 \neq 0$), quantifying the incoherence of the $\gamma \rightarrow 0$ limit relative to the pure BCS ground state ($S = 0$).
- The occupation $n_k = |\Delta|^2/(2D)$ decreases from $\Delta^2/(2(\xi_0^2 + \Delta^2))$ at $\gamma = 0$ to 0 as $\gamma \rightarrow \infty$; the anomalous correlation F_{ss} has its maximal magnitude at $\gamma \simeq 0.56\Delta_0$ (for $\xi_0 = 0.5\Delta_0$), i.e., an intermediate loss *optimizes* the steady-state anomalous correlation at fixed Δ .
- Purity $\text{Tr}\rho^2$ increases monotonically to 1, consistent with loss-induced purification of the pair sector.

These statements are exact (the entropy is computed from the exact ρ_{ss}). The monotonic decrease follows the entropy-production structure of dissipative semigroup dynamics [64,

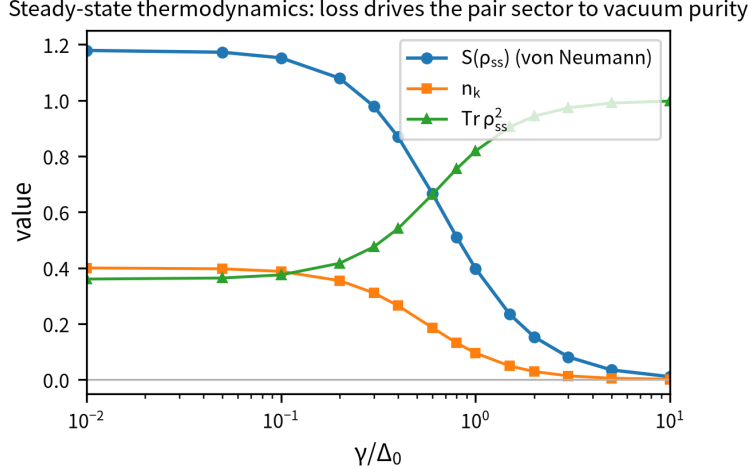


Figure 4: Steady-state thermodynamics versus γ at $\xi_0 = 0.5\Delta_0$ (logarithmic axis): von Neumann entropy S (monotonic decrease $1.18 \rightarrow 0.010$), occupation n_k ($0.400 \rightarrow 0.001$), and purity $\text{Tr} \rho^2$ ($0.36 \rightarrow 0.998$) — loss drives the pair sector toward the vacuum pure state.

59]. They provide the “steady-state thermodynamics” part of the paper’s title; a full thermodynamic potential (e.g., a steady-state free energy and its stability analysis) is left for future work, as is the extension to finite temperature and pumping.

7 Discussion and outlook

Relation to existing work. (i) Ref. [17] (arXiv:2510.04148, two-body loss / complex interaction) reported that the *right-eigenstate* framework yields smooth, continuous solutions while the *biorthogonal* one exhibits absence of nontrivial solutions and abrupt discontinuities. Our uniform-single-loss model shows the *opposite* ordering: right-eigenstate collapses (pre-EP), biorthogonal is immune (to the EP). We interpret the two results as complementary, not contradictory: they correspond to different microscopic dissipative mechanisms (imaginary interaction vs imaginary chemical potential), and the phase-space structure of the gap equations differs accordingly. A systematic mechanism-dependence of “which convention is robust” would be a valuable follow-up. (ii) Ref. [18] (arXiv:2508.12360) studies complex-interaction chiral superfluids; our model is the *s*-wave counterpart with imaginary chemical potential. A rapidly growing mean-field literature addresses non-Hermitian fermionic pairing from the effective-Hamiltonian side: dissipative stabilization of BCS pairing in cold atoms [46, 47], complex-valued interactions [49], dissipative BCS-BEC evolution [51], spin-depairing-induced exceptional superfluidity [52], non-Hermitian topological Fermi superfluids [53], critical fluctuations at many-body exceptional points [48], and nonequilibrium phase transitions in dissipative superfluids [50]. Odd-frequency pairing [56], supercurrent diode effects [55], and non-Hermitian topological sensing [54] illustrate further measurement-sensitive aspects of unconventional pairing in dissipative settings. (iii) Ref. [25] (arXiv:2607.18814) finds a gapped phase mode with $\bar{\Delta} \neq \Delta^*$ in a driven NH BCS-type model — consistent with our finding that the steady-state correlation is complex (for real Δ , $\text{Im} F_{ss} = -\gamma\Delta/D \neq 0$, i.e.,

the correlation is not phase-locked to the pairing); the relation between their $\bar{\Delta}$ and our F_{ss} deserves scrutiny. (iv) Ref. [29] proved the equality of Lindbladian and post-selected winding numbers in the topological context; our Sec. III shows that no such equality holds for the *pairing order parameter* — the steady state simply has none. This sharpens the domain of applicability of the post-selected correspondence. (v) EP-enhanced topological superconductivity [6], PT-symmetric NH superconductors [19, 76], and their ARPFS probe [76] are natural playgrounds for the pre-EP collapse we predict: in lattice models the gap collapse at $\gamma_c \simeq 0.85|\Delta_R|$ should be visible as a loss-induced first-order-like transition of the post-selected condensate. (vi) The dissipative stabilization of Ref. [39] (two thermal baths, broken detailed balance) and the PT-balanced gap of Ref. [40] both rely on *pumping* or gain compensation; our exact steady-state result delineates the complementary regime — with pure uniform loss and no injection, the Lindblad steady state admits no self-consistent pairing (Sec. III). The first-order transition of [40] at the PT-breaking point and the pre-EP fold of our Δ_R are distinct instabilities (gain-compensated PT breaking vs. net-loss collapse); their unification in a lattice model with tunable gain/loss is an interesting open problem.

Limitations. (1) The exact steady state is for the *pair sector*. Beyond a single pair we have verified two-pair exactness and robustness under weak inter-pair coupling (Sec. II.4), and the multi-point contraction of Sec. III; a full fermionic-Gaussian many-body treatment is left as future work (we regard the $\Delta = 0$ conclusion as robust beyond mean-field decoupling, but do not claim a general proof). (2) We consider uniform loss only; spatially inhomogeneous loss (NH skin-effect interplay [68, 69, 70, 71, 72, 73, 74, 75]) and pumping (gain) are outside the present scope. (3) The saddle-node character of the Δ_R collapse is established numerically (bracketed endpoint, no continuation); an analytic normal-form derivation of the fold is left open.

Outlook. The natural extensions are: (i) the microscopic origin of γ (e.g., coupling to a bath with spectral density) and the finite-temperature steady state; (ii) the collective modes of the *steady state* (where our complex F_{ss} structure should modify the Higgs/phase-mode analysis of Ref. [25]); (iii) lattice models with momentum-dependent ξ_0 to expose the topological content of the collapse; (iv) the ARPFS probe [76], which was proposed for PT-symmetric NH superconductors and may distinguish the conventions through fluctuation spectra.

A Numerical methods and convergence

Gap equations were integrated with Gauss–Legendre quadrature at order n , $\lambda = 1/\text{asinh}(50)$, $\omega_D = 50\Delta_0$. Convergence: node doubling $n = 200 \rightarrow 3200$ gives integral changes $< 10^{-9}$ away from critical points (e.g., $I(3200) - I(1600) = 4 \times 10^{-12}$ at $\gamma = 0.5$); near the fold the $n = 1600$ vs $n = 3200$ spread reaches 3×10^{-5} in $|\Delta_R|$ and 4×10^{-3} in $|\Delta_{\text{bi}}|$ (dominated by the near-vertical branch, reported as error bars in Figs. 1–2). Branch solutions were found by fsolve continuation with residuals 10^{-12} – 10^{-15} ; the endpoint was bracketed by bisection on the presence/absence of a root with $|S| < 10^{-6}$. The Liouvillian steady state was verified by (a) null-space dimension 1, (b) $L \text{vec}(\rho) = 0$ to 10^{-17} at all tested points, (c) brute-force matrix-exponential evolution from 5 independent random initial states converging to the same state (7.8×10^{-16}). No Monte Carlo sampling is used; the “error bars” of the deterministic

results are the quadrature-convergence spreads (5 independent orders $n = 200 \dots 3200$).

B Code–theory consistency and machine verification

All formulas were cross-checked between two symbolic engines (sympy and Wolfram) and the numerical implementation:

- **EP existence:** $h(\xi_0 = 0, \gamma = \Delta)$ has eigenvalue 0, null-space dimension 1, Jordan block $\{(0, 1), (0, 0)\}$ (Wolfram). ✓
- **Closed-form steady state:** $L \text{vec}(\rho_{\text{ss}}) = 0$ to 10^{-17} for real and complex Δ (numpy eigensolver). ✓
- **Complex- Δ structure:** least-squares fit $F_{\text{ss}} = A\Delta + B\Delta^*$ over 7 parameter sets gives $A = -(\xi_0 + 2i\gamma)/(2D)$, $B = 0$ — F_{ss} is *linear* in Δ (conjugate-pairing excluded; 6-digit agreement). ✓
- **Conventions:** $F_L = F_R^*$ exact; F_{bi} independent third quantity (sympy $\times$ Wolfram mutual agreement). ✓
- $\Delta_L = \Delta_R^*$: fsolve seeded at Δ_R^* converges with residual 3×10^{-12} . ✓
- $\kappa < 1$: analytic bound $\kappa(0) = \lambda \arctan(\omega_D/2\gamma) \leq \lambda\pi/2 \simeq 0.341$ plus numerical κ table. ✓
- **Reproducibility:** all scripts exit 0 and are part of the supplementary material:
`stage_C_lindblad.py`, `stage_C_gap2.py`,
`stage_C_verify_collapse.py`, `stage_D_figures.py`,
`stage_D_multi_well.py`.

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