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Noise-robust distinguishability of collective coherence via Jensen–Shannon divergence in a cavity

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Abstract:

Identifying collective behavior from experimentally accessible observables remains a central challenge in driven-dissipative many-body systems. Here, we demonstrate that a local scaling exponent extracted from steady-state photon emission reveals collective critical behavior obscured in the photon-number magnitude itself. In a bad-cavity superradiant system, this response develops a crossover that sharpens with system size as collective coherence emerges, benchmarked against the underlying collective-emission fraction. Finite-size scaling and permutation-invariant quantum simulations support this critical interpretation, revealing a coincident minimum in the Liouvillian gap. Jensen–Shannon divergence further shows that the enhanced distinguishability of the scaling response remains robust against a finite Gaussian noise floor. Together, these results establish local photon-number scaling as an experimentally accessible probe of collective coherence and dissipative criticality in cavity QED and open quantum many-body systems.

Introduction:

A central challenge in non-equilibrium many-body physics is to identify collective organization from experimentally accessible observables. Strong collective correlations among emitters in open quantum systems articulate various emergent phenomena, including dissipative phase transitions, collective synchronization, and superradiant emission. Quantum dissipative criticality has therefore long been a focus of profound interest, from the early foundation of the Lipkin-Meshkov-Glick model^{1,2} and photon blockade in nonlinear oscillators³ to recent realizations in ultracold atomic ensembles⁴. Second-order dissipative phase transitions (DPTs) have been extensively studied in one- and two-photon driven Kerr resonators^{5,6} and are relevant to many-body quantum physics^{7,8}, condensed matter physics^{9,10}, and quantum optics^{11,12}. Their critical behavior is commonly characterized through order-parameter fluctuations¹³⁻¹⁹, Liouvillian gap analysis²⁰, finite-size scaling²¹, and susceptibilities^{11,22}. However, experimentally accessible signals do not necessarily provide direct information on the microscopic origin of the measured response. In particular, when independent and correlated processes contribute simultaneously to the same observable, its magnitude alone can obscure the collective organization developing near the critical regime²³⁻³².

Bad-cavity superradiance provides a natural setting in which this problem becomes directly visible in the emitted light³³⁻³⁶. Since Dicke's formulation, superradiance has distinguished independent spontaneous emission from many-emitter radiation organized by collective coherence^{37,38}. Cavity-QED realizations place this distinction in a driven-dissipative setting, where the cavity mediates collective dynamics and records them through the output field^{39,40}. In the bad-cavity limit, the optical field decays rapidly while the atomic ensemble stores the collective phase, making photon number a natural observable for steady-state superradiance^{41,42}. The diagnostic

question is therefore not simply whether the photon number becomes large, but whether the finite steady-state emission is governed by independent emitters or by a collective branch. After the lossy cavity mode is eliminated, the photon signal contains both an independent-emitter contribution and a collective-coherence contribution. One measured photon number can consequently contain a near-linear atom-number response from independent emission together with a near-quadratic response associated with collective emission. Across a finite driven-dissipative crossover, both contributions can coexist in the same parameter window, allowing the photon-number magnitude or its conventional normalization to vary smoothly without directly revealing how the emitted signal scales with system size.

This branch-composite nature suggests that additional information can be extracted by examining the response of the same observable to atom number. Photon-number response scaling changes the diagnostic from the magnitude measured at one fixed system size to the atom-number dependence of the emitted light⁴³⁻⁴⁵. When independent emission dominates, the photon-number response remains close to linear scaling, whereas the development of collective coherence drives the response toward near-quadratic scaling. Correlation-level observables provide the microscopic branch benchmark for this crossover. Collective spin correlations, the collective dipole, and collective branch fractions directly encode the contribution of collective coherence to the steady-state emission. However, reconstructing these quantities generally requires model-dependent access to microscopic correlations, whereas bad-cavity experiments naturally record emitted photons^{42,46,47}. The useful measurement route should therefore retain sensitivity to the collective branch while remaining constructed from the output observable available in experiment. In this sense, the collective fraction characterizes the buildup of the collective contribution, while photon-number response scaling probes how the same emitted signal changes with atom number. Scaling

of correlated quantum observables has played an important role in other quantum systems, including entanglement scaling near quantum phase transitions⁴⁸, yet steady-state photon-number response scaling remains largely unexplored as an experimentally accessible probe of dissipative criticality and collective coherence.

Here, we address an important new means to explore collective coherence induced by photon-number collective response scaling near a steady-state second-order dissipative superradiant phase transition in a coupled bad cavity. The photon collective response changes from near-linear to near-quadratic atom-number scaling across the critical regime as collective coherence develops. The collective response susceptibility exhibits an enhanced finite-size dependence, and the response obeys finite-size scaling with an effective critical exponent and collapses for different system sizes. We further compare the photon-number collective response with the normalized photon-number magnitude and collective fraction on matched finite-system cuts, thereby benchmarking the scaling response against the underlying collective-emission contribution while retaining photon number as the experimentally accessible observable. Jensen-Shannon distinguishability is used to characterize local response distributions near the critical boundary under the stated noise and parameter-separation assumptions⁴⁹⁻⁵². Finally, permutation-invariant quantum simulations reveal that the photon collective response crossover and a pronounced minimum of the Liouvillian gap occur in the same critical regime. Together, these results establish photon-number collective response scaling as an experimentally accessible route to probe collective coherence and non-equilibrium dissipative criticality in cavity QED, quantum sensing and open quantum many-body systems.

Results:

Cavity-mediated collective response of Photon-number

Bad-cavity photon number is a branch-composite signal because the emitted field combines an independent-emitter contribution with a collective-coherence contribution. After the adiabatic elimination of the cavity field, the reduced master equation contains the atoms-only terms with a more specific incoherent pumping term. By solving the steady-state density matrix ρ_{ss} , we extract the localized static size-dependent scaling metric q_N . The collective response scaling q_N quantifies the collective response of the intracavity photon number to variations in system size N by varying the local control parameters at steady-state. In order to quantify the collective response at the interface of the critical boundary/point, we measure the partial derivative of the logarithm of the mean intracavity photons with respect to the partial derivative of the logarithm of system size as a function of control parameters as

$$\begin{aligned} q_N(N, g, w) &\equiv \frac{\partial \log(\langle n \rangle_{ss})}{\partial \log(N)} \\ &= \frac{\partial \log[\Gamma N \rho_{ee} + \Gamma N(N-1) |\beta|^2]_{ss}}{\partial \log(N)} \end{aligned} \quad (1)$$

Here, N is the atom number, g is the atom-cavity coupling, w is the incoherent pump rate, $\langle n \rangle_{ss}$ is the steady-state photon number, $\Gamma = 4g^2/k$ is the collective decay rate, ρ_{ee} is the single-emitter excited-state population, and β denotes the collective coherence or pair-correlation amplitude. The branch-sensitive scaling q_N reveals the quantum critical behavior due to quantum fluctuations in the encoded branch-composite signal. This corresponds to enhanced collective coherence due to quantum dissipative criticality encoded in q_N . At the critical point, the atoms develop collective dipole and establish a strong synchronized state due to collective atom-light excitation. These

synchronized atomic dipoles construct a non-equilibrium steady-state superradiant phase at the quantum critical point (QCP), characterized by a vanishing Liouvillian gap closest to zero, as shown in Figure 1a. The suppression of the spectral gap represents the relaxation time of atoms from an uncorrelated state to gain a steady-state collectively dipole-synchronized state at the quantum critical boundary $g_c(w)$. In zero collective decay $\Gamma = 0$, due to uncorrelated atoms, it is in the normal phase below the critical boundary $g_c(w)$. It undergoes a superradiant phase transition for correlated atoms develop collective coherence above the critical boundary at a fixed incoherent pumping rate. The finite-size atomic ensemble in the bad cavity-limit is sufficient to establish a quantum dissipative critical region and to allow us to examine the signatures of CSD in the range $g_c(w) < g \ll k$.

Equation (1) summarizes the mechanism of Figure 1b, where one photon-number signal contains both an N -like independent-emitter branch and an $N(N - 1)$ -like collective-emission branch, while q_N configures the change in photon number when N is varied across critical coupling. The collective response q_N turns the composite photon number into a finite-window branch readout. If independent spontaneous emission dominates, adding emitters increases the photon number nearly linearly and q_N approaches one, whereas if collective coherence dominates, the pair term controls the response and the slope moves toward $q_N = 2$. Figure 1b represents this readout as a local slope of photon number versus atom number, so the accessible photon-number measurement becomes sensitive to branch content without replacing photon measurement with a microscopic correlation observable.

Dissipative phase transition *via* photon collective response q_N

Collective response q_N reveals an unexplored hidden universal feature of dissipative criticality in the underlying superradiant phase transition. The cavity mode couples to q_N and quantum critical fluctuations of the collective response scaling q_N significantly affect the collective response of mean intracavity photons. The mean intracavity photon as emission intensity $\langle n \rangle = \frac{4g^2}{k^2} \langle J_- J_+ \rangle_{ss}$ is coupled to an atomic coherence given by $\langle J_- \rangle_{ss} = N \langle \sigma^- \rangle_{ss}$. The net steady-state solution of collective coherence from mean-field equation (Method 9b) allows the extraction of a concise expression for the critical boundary $g_c(w)$, which is described by two solutions for ρ_{eg}

$$\begin{aligned} \rho_{eg} = 0 : \quad & \frac{4g^2}{k^2} \cdot N \cdot \rho_{ee} & g < g_c(w) \\ \rho_{eg} \neq 0 : \quad & \frac{4g^2}{k^2} \cdot N^2 \cdot |\rho_{eg}|^2 & g > g_c(w), \end{aligned} \quad (2)$$

$$g_c(w) = \sqrt{\frac{k(w + \Gamma + \gamma_{eg})\gamma_{\perp}}{2N(w - \Gamma - \gamma_{eg})}}$$

where $\gamma_{\perp} = \frac{w + \Gamma + \gamma_{eg}}{2} + 2\gamma$ is pure dephasing rate. At the critical boundary $g_c(w)$ [black dotted line], q_N spikes at $q_N \approx 1.5$ and creates a ridge to separates regions with near-linear scaling from regions approaching quadratic scaling, depicted in phase diagram Figure 2a. For $\rho_{eg} = 0$, atoms behave independently and exhibit incoherent emission for $\langle n \rangle \propto N\rho_{ee}$ in $g < g_c(w)$. At $g < g_c$, the required population inversion is $m < 0$ and the phase transition at steady-state is a normal phase (marked by the light-cyan region in Figure 2a). At this stage, the atoms in the atomic ensemble individually emit the photons and spontaneous emission dominate over collective emission. The collective cooperativity factor $C_N = \frac{4g^2 N}{k\gamma_{eg}}$ which is less than 1 at $g < g_c$ and not

high enough to overcome the incoherent pumping w and spontaneous decay γ_{eg} to produce a clear N^2 signature. In zero cavity field $a = 0$ and for non-zero collective coherence $\rho_{eg} \neq 0$, the system undergoes the superradiant phase transition at steady-state across the critical atom-cavity coupling $g = g_c$ at fixed incoherent pumping rate w . The deep superradiant phase transition (the red region) emerges at $g > g_c(w)$ and the atoms develop strong collective dipole. The population inversion for steady-state superradiance is $m_N > 0$, requiring $w > \Gamma + \gamma_{eg}$. The cooperativity factor becomes $C_N > 1$ and the atoms have a macroscopic dipole moment $|\rho_{eg}| > 0$. The aforementioned results are consistent with the thermodynamic limit in mean-field for a large number of atoms or $N \rightarrow \infty$, and $|\rho_{eg}|^2 \sim O(1)$ as the collective correlated state remains stable so the steady state collective mean intracavity photon number is $\langle n \rangle_{SR} \propto N^2$ and scales quadratically $q_N \approx 2.0$ right after the critical boundary $g_c(w)$.

q_N metric indicates the breakdown of mean-field theory in the strong coupling regime under $g_c(w) < g \ll k$, indicated by the light-blue region above the royal-blue region in Figure 2a. A finite-size numerical simulation is required to investigate the full dynamics of quantum criticality to reveal a clear signature of the dissipative superradiant phase transition. The strong atom-cavity coupling reaches to $g \geq 10^{-2}$, the collective decay rate $\Gamma = 4g^2/k$ becomes massive and surpass the required condition $w > \Gamma + \gamma_{eg}$. Finite-size quantum fluctuations at the critical boundary breakdown the deep superradiant state into a dissipative superradiant phase. The intracavity mean photon number for different finite atoms reveals the collective synchronization in Figure 2b. The peaks are high for large system size at $g = 10^{-2}$ and breakdown for further increase in coupling strength. The intracavity mean-photon number shows a prominent boost near the critical coupling and indicates a progressively sharper peak for larger number of atoms N .

To quantify the quantum dissipative criticality, we measure the collective response susceptibility $\chi_{q_N} = \frac{\partial q_N}{\partial g}$ which exhibits finite-size divergence. Figure 2c shows the coupling sweeps susceptibility in the finite-size limit with an increase in the system size. At system sizes above $N = 30$, size-dependent peak height and narrow width indicate the finite-size transition near the predicted critical coupling. This steady increment in system size indicates the enhancement in critical divergence. This critical divergence exhibits the sharpness of the peak with an increase in the system size. This evolution in finite-size divergence can be illustrated as quantum dissipative criticality encoded in the static-size-dependent observable q_N . These elucidations specify that the bad-cavity superradiant transition evolves continuously toward a genuine critical point in the finite-size limit, where the susceptibility is expected to diverge. Moreover, q_N obeys the finite-size scaling and collapse for all system sizes by equation 3

$$q_N(g, N) = F[g - g_c]N^{1/\nu} \quad (3)$$

analogous to Binder-comulant scaling, which validates the dissipative criticality to be identified directly from the steady-state photon collective response without measurement of conventional order parameters, shown in Figure 2d. F and ν are the universal scaling function and the effective correlation exponent, respectively. The collapse of the numerical results for different system sizes indicates that $q_N(g, N)$ is completely absorbed into the scaling variable $(g - g_c)N^{1/\nu}$ onto a single peak using a numerically extracted exponent $\frac{1}{\nu} = 0.89$. At the critical point, the pronounced peak reveals that the crossover from $q_N \approx 1$ to $q_N \leq 2$ becomes the sharpest and away from the critical boundary, convergence to the same asymptotic behavior confirms a negligible finite-size effect outside the critical window.

Collective response q_N comparison with collective fraction and normalized photon-number observables

Collective response scaling q_N as an order parameter plays a significant role in establishing the criteria of physical markers of collective emission rather than a numerical scaling label. The collective emission intensity $\langle J_+ J_- \rangle = N \frac{1 + \langle \sigma_j^z \rangle}{2} + N(N - 1) |\beta|^2$ only emphasises the total emission, where the first term is the individual emission by uncorrelated atoms and the second term exhibits the collective correlation contribution. The conventional order parameter $\Phi_{SR} = \frac{\langle n \rangle}{N^2}$ reports a photon number scale per quadratic normalization, yet it can vary smoothly through a dissipative crossover region where independent and collective terms coexist. This limitation is not a failure of normalization, but a consequence of applying one assumed N^2 -scale to a photon number that contains both N -like and N^2 -like terms. In a finite crossover, changes in ρ_{ee} and β can change the magnitude of Φ_{SR} without identifying which branch controls the atom-number dependence of the emitted field. The logarithmic response q_N configure the same photon-number signal changes when N is varied at fixed controls. Figure 3 compares this normalized photon-number view and the collective-fraction branch reference under matched control conditions. This comparison makes q_N a branch-selective physical marker of whether the emitted field follows an additive or collective atom-number response, rather than only a photon-number magnitude on a chosen normalization scale. The microscopic branch reference is supplied by the collective fraction as

$$F_{coll} = \frac{N(N-1)|\beta|^2}{N \frac{1 + \langle \sigma_j^z \rangle}{2} + N(N-1)|\beta|^2} \quad (4)$$

F_{coll} defines the branch meaning, while q_N provides the photon-number response. On matched cuts, q_N follows the independent-to-collective branch change represented by F_{coll} , while remaining a response constructed from photon number over atom number. When $F_{coll} \rightarrow 0$, the collective fraction corresponds to the normal phase due to uncorrelated individual alignment and in the superradiant phase, $F_{coll} \rightarrow 1$ due to collective dipole synchronization and dominance of collective coherence. F_{coll} supplies the microscopic branch reference, but it changes as a smooth fraction through the dissipative crossover. Φ_{SR} is also a magnitude normalized by an assumed quadratic scale, so it can mix changes in brightness with changes in branch content. The response, therefore converts the same photon-number data into branch-selective information about the atom-number law.

Collective coherence *via* Jensen-Shannon Divergence

To test whether the enhanced distinguishability of q_N persists in the presence of measurement noise, we compare the local response distributions across the critical boundary using the Jensen-Shannon divergence. We model the measurement uncertainty using Gaussian distributions with a finite noise floor. Under this noise, the conventional observables become strongly overlapping near the boundary, whereas the scaling response remains clearly distinguishable in Figure 4 (Method IV). We extract an exact, coordinate-invariant marker for the dissipative critical boundary g_c . JSD quantifies the overlap between two probability distributions, with values between $JSD = 0$

(overlap) and $\text{JSD} = 1$ (no overlap). These probability distributions $P(x) = \mathcal{N}(\sigma^2, \mu_-)$ and $Q(x) = \mathcal{N}(\sigma^2, \mu_+)$, are functions of the background noise floor σ^2 and observable parametric point $\mu_{\pm}$ across the critical boundary $g_c(w)$. Here, the background noise floor can be treated as single value, i.e, $\sigma_{\pm} \approx \sigma$ due to the symmetric nature of JSD and the observable parameter point is defined as $\mu_{\pm} = \text{observable}(g_{\pm})$. The dissipative phase transitions are characterized *via* infinitesimal variation ϵ below and above the critical boundary $g_c(w)$ by the parameter point $g_- = g_c(w) - \epsilon$ and $g_+ = g_c(w) + \epsilon$, respectively. The values across the infinitesimal boundary g_- and g_+ for traditional observables Φ_{SR} and F_{coll} are $\mu_{\pm} = \Phi_{SR}(g_{\pm})$ and $\mu_{\pm} = F_{coll}(g_{\pm})$, respectively. In the bad cavity regime, Φ_{SR} and F_{coll} vary smoothly across the critical boundary $g_c(w)$ due to continuous regulation of atomic polarization by emitted light intensity. The local variance of Φ_{SR} and F_{coll} scales quadratically under infinitesimal parameter divergence ϵ as $\epsilon^2 = 10^{-10}$, but their information-geometric sensitivity vanished by background noise floor ($\sigma = 0.015$), shifting their statistical distinguishability to zero ($\text{JSD} \rightarrow 0$), as shown in Figure 4a,c & d. Conversely, in the thermodynamic mean-field limit (see Methods) q_N configures the scaling re-indexing from linear to quadratic by a rapid step-function shift which allows the scaling sensitivity to escape noise floor suppression and mark an invariant information-theoretic singular peak at unity ($\text{JSD} \rightarrow 1.0$) at critical boundary g_c , as shown in Figure 4a. Collective response scaling q_N tracks how the total emission intensity scales as a function of system size. The performance evaluation of q_N at the mean-value difference after applying the limit is $\mu_+ - \mu_- = 2.0 - 1.0 = 1.0$, which shows that mean values jump abruptly to 1.0 bit on JSD divergence and that the difference is far larger than the noise variance σ in the bad cavity limit. The exact Shannon entropy limits of the JSD is $M(x) = \frac{1}{2} (P(x) + Q(x))$. So JSD for q_N is $\lim_{\substack{\epsilon \rightarrow 0 \\ N \rightarrow \infty}} \text{JSD}_{q_N}(P \parallel Q) \rightarrow 1.0 \text{ bit}$

which is depicted by green peak. The divergence reveals the significant signature of collective coherence. The information theoretic aspects establish the measurement of collective coherence in dissipative phase transition via structural shift from local uncorrelated emission to global correlated emission.

Dissipative phase transition and Liouvillian gap analysis

The non-equilibrium dissipative phase transition of collective response q_N is demonstrated by the emission intensity of intracavity mean photon derived by the steady-state master equation in the Born-Markov approximation $\partial \rho_{ss} = \mathcal{L} \rho_{ss}$. The observable in the Dicke basis $\langle J_+ J_- \rangle_{ss}$ computes the steady-state collective coherence. The total emission rate in the Dicke limit for $N+1$ symmetric states in the m -excitation subspace, $\left| \frac{N}{2}, m \right\rangle$ state in the symmetric Dicke ladder with collective

symmetric states $J = \frac{N}{2}$ and m -excitation is $\langle n \rangle = \frac{4g^2}{k^2} \left[\left(\frac{N}{2} + \left(m - \frac{N}{2} \right) \right) \left(\frac{N}{2} - \left(m - \frac{N}{2} \right) + 1 \right) \right]$.

Figure 5a shows a numerically simulated phase transition plot for collective response. From the quantum master equation, Liouvillian criticality based on system size $N = 40$ can be computed by collective sensitivity scaling by

$$q_N = \frac{\partial \log(\langle n \rangle)}{\partial \log(N)} = \frac{\partial \log \left(\frac{4g^2}{k^2} \langle J_+ J_- \rangle_{ss} \right)}{\partial \log(N)} \quad (5)$$

q_N indicates the phase transition from the normal phase into the superradiant phase at the critical point in the parameter space. The Dicke burst-like band indicated by the red arrow at the critical boundary (marked by yellow dotted line) exhibits the collective emission by correlated atoms in the atomic ensemble and $q_N \rightarrow 2$. The strong atom-cavity coupling reaches to $g \geq 10^{-2}$, the collective decay rate $\Gamma = 4g^2/k$ becomes massive and surpasses the required condition $w > \Gamma +$

γ_{eg} . Finite-size quantum fluctuations in critical boundary breakdown the deep superradiant state into dissipative superradiant phase.

The Dicke-basis Liouvillian is solved for the finite steady-state, and the emitted photon signal is reconstructed from $\langle J_+ J_- \rangle_{ss} = N \frac{1 + \langle \sigma_j^z \rangle}{2} + N(N-1)|\beta|^2$, and the same local log-log derivative is applied across atom number, shown in Figure 5b. The Liouvillian of the density matrix at the steady-state can be written as $\mathcal{L}\rho_{ss} = 0$ or simply $\frac{d\rho_{ss}}{dt} \approx 0$. The Liouvillian spectrum corresponds to n eigenvalues that satisfy the $\mathcal{L}\rho_n = \lambda_n \rho_n$ and the density matrix can be written as $\rho(t) = \rho_{ss} + \sum_{n>0} c_n e^{\lambda_n t} r_n$. The $\lambda_n = -\nu + i\omega$ at steady-state describes one of the eigenvalues that must correspond to $\lambda_0 = 0$ for the eigenvector $\mathcal{L}$. The ν and ω are the decay rate and energy oscillation, respectively. At steady-state, eigenvector $\mathcal{L}$ with zero eigenvalue corresponds to Liouvillian eigenstate with zero decay rate i.e., eigenstate with $\nu = 0$ and all other eigenvalues satisfy $Re(\lambda_n) < 0$. The Liouvillian spectral gap is $\delta_L = -Re(\lambda_1)$ and for the eigenvalues according to $\delta_n < \delta_{n+1}$ with steady-state solutions, where λ_1 is the closest non-zero eigenvalue. Figure 5b exhibit obvious signature of quantum dissipative criticality encoded in photon-number collective response q_N by approaching of the Liouvillian gap δ_L and the peak of collective response occurring at the same quantum critical point near $g \approx 10^{-2}$. This coincidence represents the enhance collective coherence induced by photon-number collective response scaling q_N emerges at same quantum critical point due to quantum fluctuations (shown in schematic of Figure 1a). It is not only that the transition from individual emission to collective dipole synchronization near critical boundary, but also showing the vanishing of the spectral gap closest to zero. The spectral dip of δ_L reveals the second-order phase transition of the normal phase and the deep superradiant phase in the dissipative phase transition. The time scale over which the collective

response q_N prolongs and approaches to $q_N = 1.99$ (orange line in Figure 5b) depends upon the relaxation rate towards the non-equilibrium steady-state.

As a result, the system's relaxation time into the steady state is determined by the smallest nonzero eigenvalue and $\tau = 1/\delta_L$ is the relaxation time, shown in Figure 5c-d. A small system size $N = 10$ (black curve) endures a 5-fold amplification in relaxation decay rate, whereas a large system size $N = 40$ (green curve) show maximum peak by 18-fold factor, yielding two-orders-of-magnitude dynamic point. The larger system size shows massive spike at critical point and show clear signature of prolonged relaxation time of system to approach steady-state. In the normal phase $g < g_c(w)$, the atoms behave as uncorrelated and the Liouvillian spectrum owns a finite gap $\delta_L > 0$. This leads to a generic exponential decay of an observable. In contrast, the gap forms a prominent minimum $\delta_L \approx 0$ as it approaches the critical coupling $g_c(w)$ and it gradually shrinks with increasing system size. As the system size increases, the gap shrinks and shows a narrow dip, exhibited by the green curve for $N = 40$. All the curves approach the same value of coupling strength (depicted in Figure 5 c-d). This reveals that the spectral gap is not an energy gap of a Hamiltonian but the real part of the eigen-values $\delta_L = -\text{Re}(\lambda_1)$, prevailing the slow mode of the dissipative open quantum system's density matrix towards the steady-state density matrix ρ_{ss} .

Discussion:

Experimental Feasibility

The dissipative superradiant phase transition studied here can be realised using ultra narrow transitions of strontium and ytterbium atoms trapped within optical lossy resonators. These

platforms can fall into the adiabatic elimination to access the dissipative phase transition via a dissipative quantum critical point and its verification by Liouvillian gap analysis. Crucially, the Jensen-Shannon divergence may be directly generated from empirically recorded photon-count distributions, offering an information-theoretic probe of dissipative criticality without requiring full quantum-state tomography or access to tiny atomic observables.

Summary and Outlook

In summary, our approach combines analytical, statistical and numerical methods to characterize the non-equilibrium steady-state phase transition through the critical boundary via the collective response q_N in bad cavity regime. Collective response scaling provides a finite-window way to read bad-cavity superradiance from the observable that experiments naturally record. The emitted field contains both independent-emitter and collective-emission contributions, so photon number at one atom number can remain ambiguous even when the system is bright. The scaling response q_N changes the question by reading whether the photon number grows as an additive emitter signal or as a collective-coherence signal. In the considered finite steady-state setting, the scaling response tracks the branch change represented by F_{coll} while preserving photon number as the measured observable. At standard mean-field analysis, q_N exhibits the non-equilibrium dissipative steady-state phase transition. The role of scaling in bad-cavity superradiance is therefore set by the relation between microscopic branch references and an accessible response of the emitted light. F_{coll} , $\langle J_+ J_- \rangle_{ss}$, and related correlations define the collective-coherence contribution in the model. q_N supplies a finite-window output response that can be compared with that reference on matched cuts. This separation

keeps the interpretation physical because the response is strongest when it exposes the independent-to-collective change already present in the branch-composite output.

The role of collective coherence at a superradiant phase transition is not captured by statistical mechanics. We measure quantum dissipative criticality with quantum information by exploring the collective coherence resource of a system close to its quantum critical point via Jensen-Shannon divergence (JSD). Collective coherence shows scaling behavior near the critical boundary via probability distribution of the photon-collective response in JSD distinguishability. Jensen-Shannon divergence distinguishability reveals that photon-collective response scaling separates and characterizes the collective coherence more clearly than normalized photon-number magnitude and collective fraction.

q_N navigates the standard diagnostic approach to mark its physical importance both analytically and statistically validated by a quantum approach to locate the boundaries, rather than just a scaling index. Our results exhibit that steady-state superradiant regimes can be identified through the collective response scaling, q_N rather than through traditional order parameters and collective fraction via quantum master equation. Liouvillian gap analysis validates quantum dissipative criticality encoded by photon-number response scaling. Our quantum simulations reveal that the photon collective response crossover and a pronounced minimum of the Liouvillian gap occur in the same critical regime. More generally, our findings show that crucial fluctuations in open quantum systems can be controlled using cavity-mediated synchronization. Using experimentally accessible cavity-QED architectures, new avenues for studying non-equilibrium universality, time-independent dynamical critical phenomena, and information-theoretic aspects of driven-dissipative many-body physics are made possible by the ability to access collective coherence and quantum dissipative criticality through photon-counting observables.

Figures

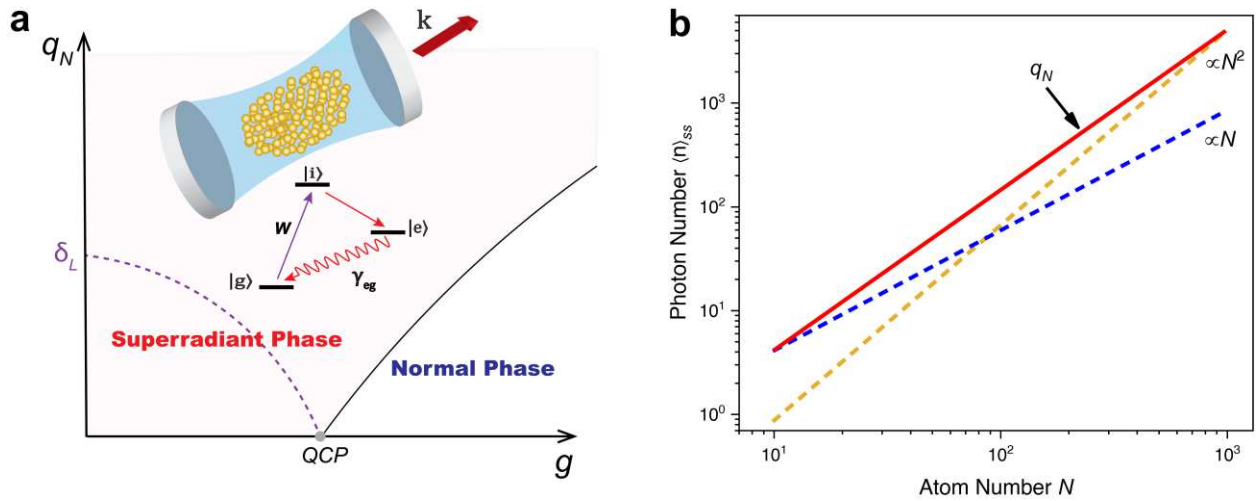


Figure 1: Schematic phase diagram of bad cavity-coupled atomic ensemble and photon collective response (a) An ensemble of three-level atoms is enclosed in a single-mode optical cavity in the bad-cavity regime ($\kappa \gg g, \gamma_{eg}$). Photon-number response q_N induces a steady-state dissipative superradiant phase transition at the quantum critical point (QCP) due to atom-cavity coupling strength. The Liouvillian gap also approaches near closet to zero in same critical regime. (b) Collective response q_N is characterized as the slope of the steady-state mean intracavity photon number as a function of atom number N . The response exponent q_N is branch composite scaling, separating N and N^2 response.

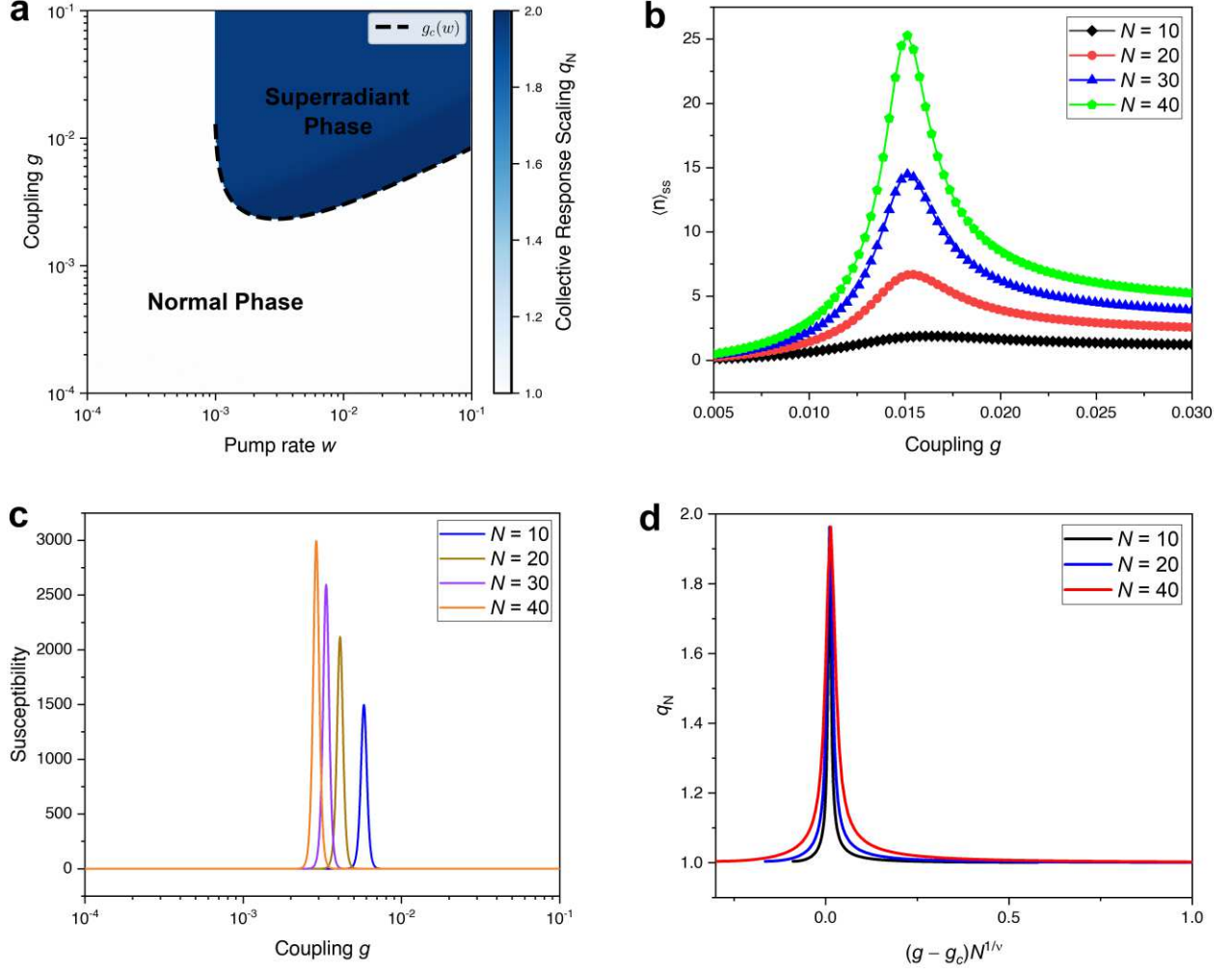


Figure 2: Dissipative criticality encoded in q_N via finite-size scaling analysis. (a) The steady-state dissipative phase transitions as a function of atom-cavity g coupling and incoherent pumping rate w via photon collective response q_N . The collective response q_N quantifies the logarithmic response of the intracavity photon number to variations in system size N by varying the rate of incoherent pumping w . Normal Phase and Superradiant Phase are shown by light-cyan and royal-blue regions in the plot, respectively. (b) The analytically measured q_N exhibits a sharp steps from the normal phase at $q_N = 1$ for coupling $g < g_c(w)$ to superradiant phase at $q_{SR} = 2$ for $g > g_c(w)$, marked by green and magenta line, respectively. (c) The coupling sweeps susceptibility $\chi_{q_N} = \frac{\partial q_N}{\partial g}$ which exhibits finite-size divergence with an increase in the system size. (d) The collapse of the numerical results for different system sizes indicates that $q_N(g, N)$ is completely absorbed into scaling variable $(g - g_c)N^{1/\nu}$ onto a single peak using a numerically extracted exponent $\frac{1}{\nu} = 0.89$.

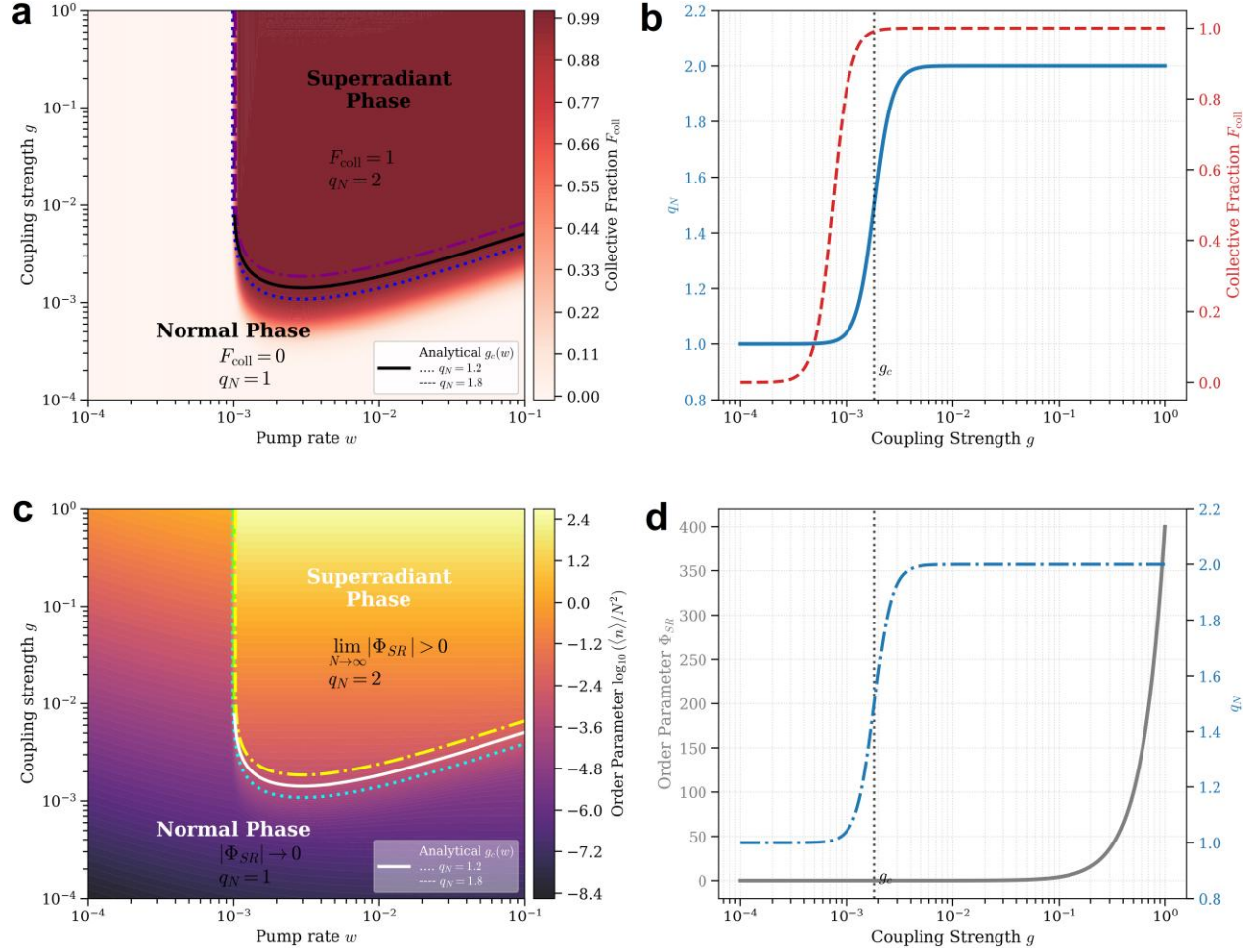


Figure 3: q_N as physical marker for collective coherence (a) The obscure boundary of the phase transition from normal to the superradiant phase for collective fraction F_{coll} comprehends the crossover region while q_N conceives the exact physical marker of collective emission. (b) The sharp step-over of q_N (depicted by blue line) versus the gradual increase of F_{coll} (red dotted line) corresponds to physical reliability on fixed w with swept g . (c-d) The comparison of q_N with the order parameter $\Phi_{SR} = \frac{\langle n \rangle}{N^2}$ on the same cut.

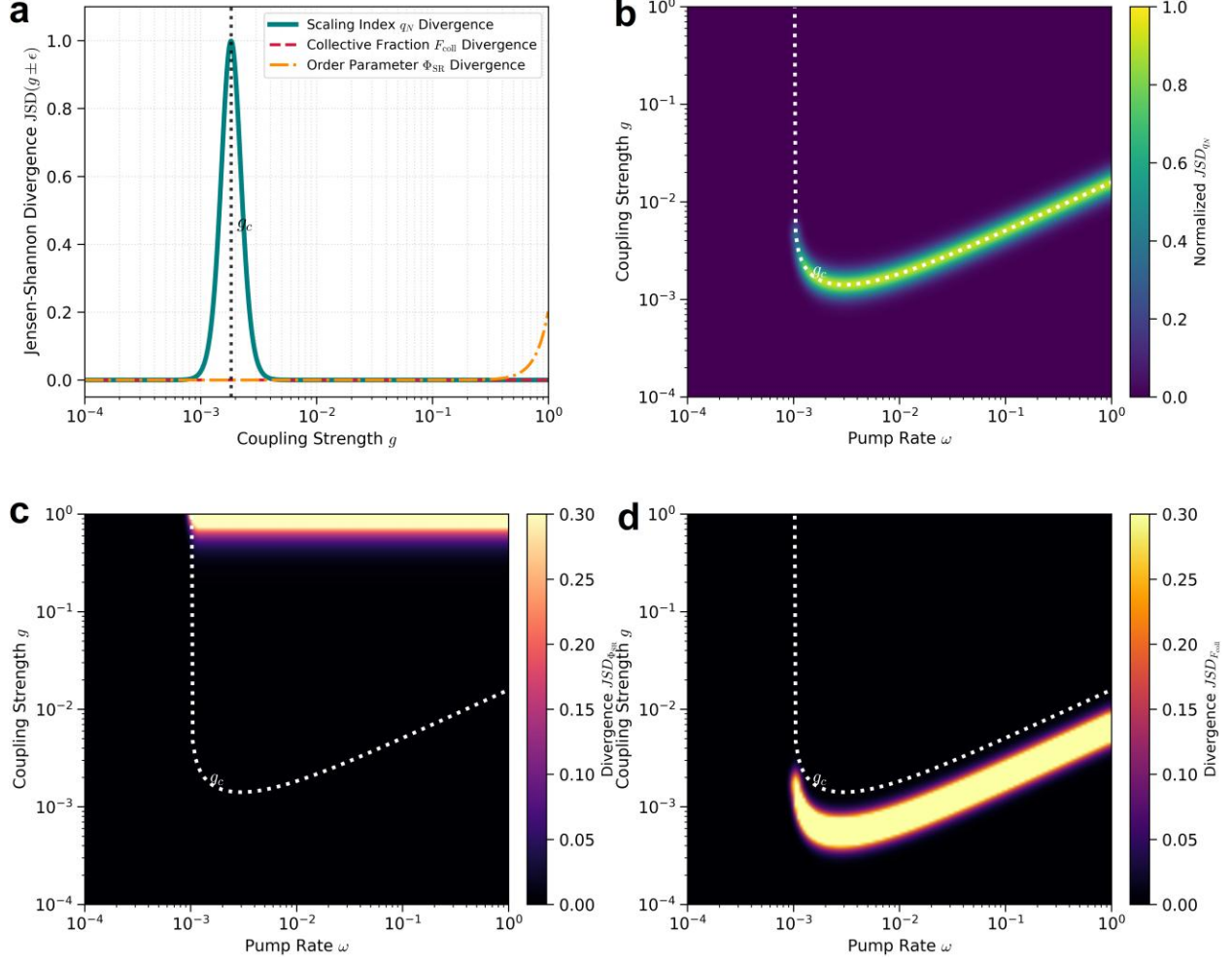


Figure 4: Collective coherence by performance of q_N via Jensen-Shannon Divergence. (a) The analytical description of F_{coll} and Φ_{SR} on $JSD \rightarrow 0$ reveals them to be statistically indistinguishable across $g_c(\omega)$. The q_N approaches the absolute maximum theoretical limit of information divergence across $g_c(\omega)$ as depicted by $JSD \rightarrow 1.0$ (b) JSD_{q_N} reveals a high-intensity ridge that accurately follows the superradiant phase boundary. (c-d) $JSD_{\Phi_{SR}}$ and $JSD_{F_{coll}}$ exhibit background noise accumulated band above and below the critical boundary g_c on JSD divergence, respectively.

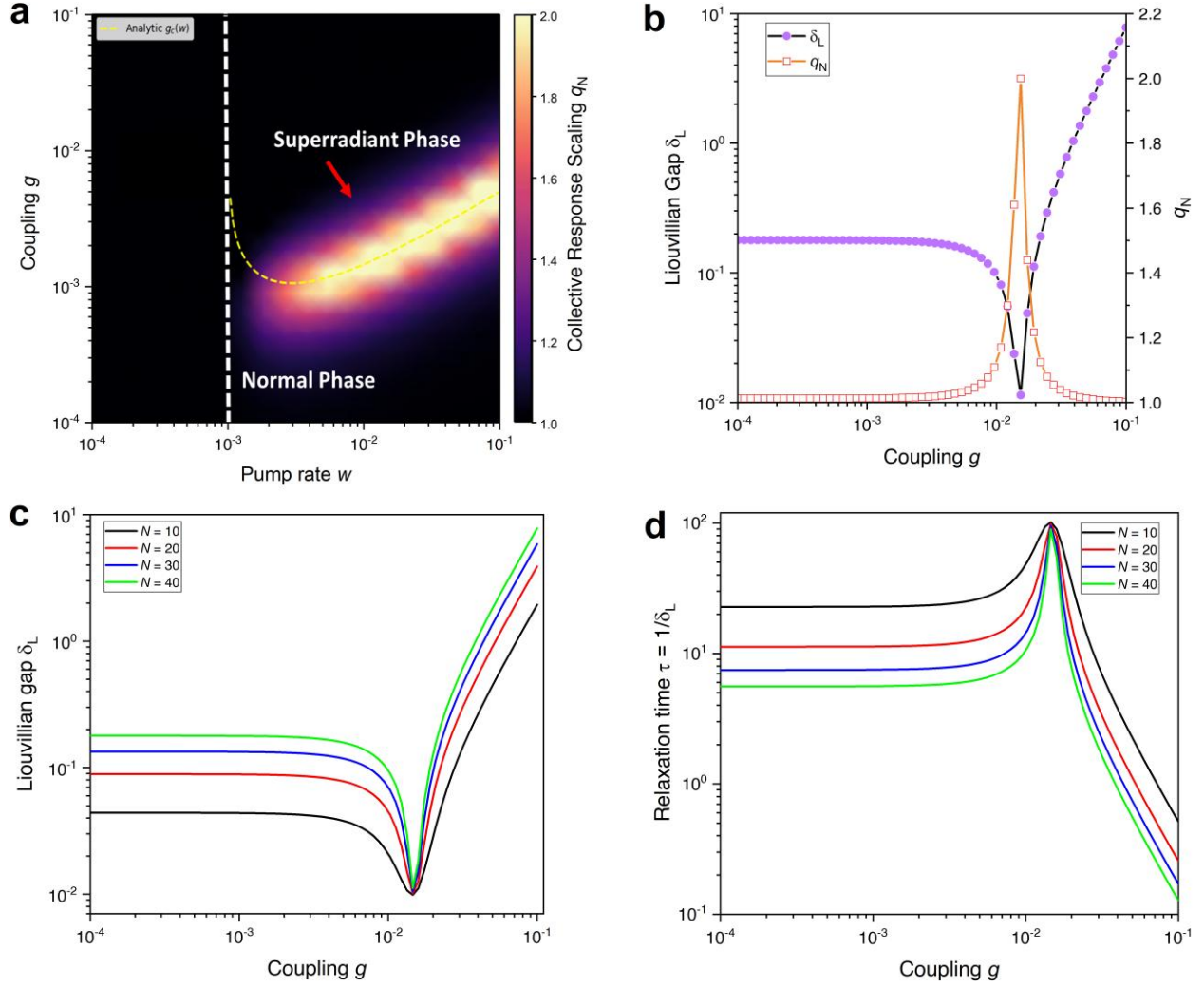


Figure 5: Dissipative superradiant phase transition and Liouvillian gap analysis by PIQS (a) We conceive the permutational symmetry of identical atoms using the Permutational Invariant Quantum Solver (PiQS) to indicate the phase transition from the normal phase to superddiant phase allowing upwards of 40 identical atoms in a homogeneous ensemble via q_N . The yellow dotted line shows the analytical results. **(b)** q_N and δ_L reveal the dissipative criticality at the same critical point. **(c-d)** δ_L for all system sizes converges to a single critical point. The system with large number of atoms $N = 40$ shows a narrow shrink (green curve) and a long dip as compared to other system sizes. Similarly, $\tau = 1/\delta_L$ reveals the critical slowing down for different system sizes.

Data availability

Data is available from the corresponding author upon request.

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Competing Interests

The authors declare no competing interests.

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Methods

I. Bad-cavity Model

Figure 1a shows the model as a single-mode cavity coupled to a homogeneous atomic ensemble of three-level atoms leading to steady-state superradiance. The atoms are configured with atomic ground state $|g\rangle$ and optical excited state $|e\rangle$ with intermediate state $|i\rangle$ to maintain populations via pumping. The atoms in ground state $|g\rangle$ are incoherently pumped with a pumping rate w from $|g\rangle$ to $|i\rangle$ for narrow atomic transitions from $|i\rangle$ to $|e\rangle$ and $|e\rangle$ to $|g\rangle$ via the fast decay rate γ_{ie} and spontaneous decay rate γ_{eg} along with the collective decay rate $N\Gamma$, respectively. The cavity is in the optical bad-cavity limit $k \gg g$, and is considered cryogenically optimized, having less thermal noise with no Brownian motion of atoms, sustained long coherence time, and suppressed inhomogeneous broadening. The incoherent pumping and effective decay are balanced to achieve phase transitions in steady-state superradiance. In the bad cavity limit ($\kappa \gg g\sqrt{N}$), a strongly damped cavity mode can be adiabatically eliminated, yielding an effective collective dissipative term $\Gamma L[J_-]$ for the atomic degrees of freedom. After the adiabatic elimination of the cavity field and $|i\rangle$, the reduced form of the quantum master equation contains only dissipators (Supplementary I). The full dynamics of the quantum system is

$$\dot{\rho} = \Gamma L[J_-] + \gamma_{eg} \sum_j L[\sigma_j^-] + w \sum_j L[\sigma_j^+] + \gamma \sum_j L[\sigma_j^z] \quad (6)$$

where $\Gamma = \frac{4g^2}{k}$ is the collective correlated emission, γ_{eg} is individual or spontaneous emission, w is the incoherent pumping rate, and γ is the pure dephasing rate. The Pauli operators for j atoms, the lowering, raising and collective operators are $\sigma_{ge}^j \equiv \sigma_j^-$, $\sigma_{eg}^j \equiv \sigma_j^+$ and $J_- =$

$\sum_{j=1}^N \sigma_j^-$, respectively. At steady-state $\dot{\rho} = 0$, the dynamics of the system can be evaluated by configuring the atomic coherence J_- and mean-field Bloch equations for the cavity field a .

II. Branch decomposition and response exponent

The photon-number analysis uses a fixed output-normalization convention within each matched comparison. The decomposition follows from expanding the collective correlation (Supplementary Section),

$$\begin{aligned} \langle J_+ J_- \rangle &= \sum_{j=1}^N \sum_{i=1}^N \langle \sigma_i^+ \sigma_j^- \rangle \\ &= N \frac{1 + \langle \sigma_j^z \rangle}{2} + N(N-1) |\beta|^2 \end{aligned} \quad (7)$$

For identical emitters, the diagonal part gives the independent contribution and the off-diagonal part gives the pair-coherence contribution, so that the bad-cavity output takes the form

$$\langle n \rangle_{ss} \propto \Gamma N \rho_{ee} + \Gamma N(N-1) |\beta|^2 \quad (8)$$

Here, ρ_{ee} is the single-emitter excited-state population and β denotes the collective coherence or pair-correlation amplitude. The common prefactor, output convention, and control parameters are held fixed within each matched cut, so the logarithmic derivative tests the atom-number dependence of the same observable rather than a change of normalization. The response is evaluated as

$$q_N(N, g, w) = \frac{\partial \log(\langle n \rangle_{ss})}{\partial \log(N)} \quad (9)$$

In the calculations, this derivative is implemented as a finite-window local slope over the available atom numbers, using either finite differences or a local fit of $\mathbf{log}(\langle \mathbf{n} \rangle_{ss})$ against $\partial \mathbf{log}(N)$. The same convention applies to both an additive emitter response and a collective pair response.

III. Critical Boundary via Mean-field approximation

The phase transition diagram from normal to superradiant phase is simulated by the quantum master equation and mean-field approximation for the correlation. The coupled differential equations for our system are

$$\langle \dot{a} \rangle = -\frac{k}{2} \langle a \rangle - i \sum_{i=1}^N g_i \langle \sigma_i^- \rangle \quad (10a)$$

$$\langle \dot{J}_- \rangle = -\gamma_{\perp} \langle J_- \rangle + 2ig \langle a \rangle \langle J^z \rangle \quad (10b)$$

$$\langle \dot{\sigma}_j^z \rangle = w(1 - \langle \sigma_j^z \rangle) - \gamma_{eg}(1 + \langle \sigma_j^z \rangle) - 2ig(\langle a^\dagger \rangle \langle \sigma^- \rangle - a \langle \sigma^+ \rangle) \quad (10c)$$

The steady-state solution of these coupled equations allows the extraction of a concise expression for the critical boundary condition from self-consistent equation which is

$$\left[1 - \frac{4g^2 Nm}{k\gamma_{\perp}} \right] \rho_{eg} = 0 \quad (11)$$

where $m = (\rho_{ee} - \rho_{gg})/2$ is the inversion variable used in the source derivation. The solution $\rho_{eg} = 0$ configures incoherent emission. The non-zero coherence branch $\rho_{eg} \neq 0$ reveals collective emission. The solution requires $m > 0$, which is a valid condition for population inversion. The population inversion for steady-state superradiance is $m_N > 0$, requiring $w > (\Gamma + \gamma_{eg})$. This equation generates the critical boundary point based on atom-cavity coupling g_c ,

and $m_N = \frac{w - \Gamma - \gamma_{eg}}{2(w + \Gamma + \gamma_{eg})}$, above equation can be written as

$$g_c = \sqrt{\frac{k(w + \Gamma + \gamma_{eg})\gamma_{\perp}}{2N(w - \Gamma - \gamma_{eg})}} \quad (12)$$

IV. Distinguishability of q_N via Jensen-Shannon Divergence

The more rigorous statistical approach conceived for analytical measurements is delimited for q_N employing Jensen-Shannon Divergence.⁴⁹⁻⁵² JSD configures the measure of the overlap of two probability distributions within JSD statistical range between JSD = 0 (overlap) and JSD = 1 (no overlap). The indistinguishability of F_{coll} and Φ_{SR} near the boundary is due to the overlapping of sample values just below and above the critical boundary $g_c(w)$. The infinitesimal variations ϵ across the phase transition below and above the critical boundary $g_c(w)$ are represented by parameter points $g_- = g_c(w) - \epsilon$ for the normal phase and $g_+ = g_c(w) + \epsilon$ for the superradiant phase, respectively. The Jensen-Shannon Divergence can be defined as

$$\text{JSD}(P \parallel Q) = \frac{1}{2}D_{KL}(P \parallel M) + \frac{1}{2}D_{KL}(Q \parallel M) \quad (13)$$

Where D_{KL} is the Kullback-Leibler divergence and $M = \frac{1}{2}(P + Q)$ is the average distribution. We further bound the measurement distributions at g_- and g_+ along with a variance σ^2 (any measured metric x form a continuous distribution for its mean value $\langle x \rangle$) as Gaussian distributions $P(x) = \mathcal{N}(\sigma_-^2, \mu_-)$ and $Q(x) = \mathcal{N}(\sigma_+^2, \mu_+)$. Since the Jensen-Shannon divergence is symmetric and $\sigma_{\pm} \approx \sigma$, the JSD can be analytically estimated via the Taylor expansion of the KL divergence

$$\text{JSD}(P \parallel Q) \approx \frac{\log_2 e}{8} \frac{(\mu_+ - \mu_-)^2}{\sigma^2} \quad (14)$$

The values across the infinitesimal boundary g_- and g_+ for the order parameter $\Phi_{SR} = \frac{\langle n \rangle}{N^2}$ are $\mu_- = \Phi_{SR}(g_-)$ and $\mu_+ = \Phi_{SR}(g_+)$, respectively. The difference using the first-order derivative for Φ_{SR} as a continuous function measures the $\mu_+ - \mu_- = \frac{\partial \Phi_{SR}}{\partial g} \Big|_{g_c(w)} \cdot (2\epsilon)$. Substituting this into above JSD approximation

$$\text{JSD}(P \parallel Q) \approx \frac{\log_2 e \cdot 4\epsilon^2}{8\sigma^2} \left(\frac{\partial \Phi_{SR}}{\partial g} \Big|_{g_c(w)} \right)^2 \quad (15)$$

Taking the limit $\epsilon \rightarrow 0$,

$$\lim_{\epsilon \rightarrow 0} \text{JSD}_{\Phi_{SR}}(P \parallel Q) = 0 \quad (16)$$

Similarly, the collective fraction F_{coll} performance at the mean values across the infinitesimal boundary is $\mu_- = F_{coll}(g_-)$ and $\mu_+ = F_{coll}(g_+)$, respectively. The JSD approximation in terms of F_{coll} is

$$\text{JSD}(P \parallel Q) \approx \frac{\log_2 e \cdot \epsilon^2}{2\sigma^2} \left(\frac{\partial F_{coll}}{\partial g} \Big|_{g_c(w)} \right)^2$$

Taking the limit $\epsilon \rightarrow 0$,

$$\lim_{\epsilon \rightarrow 0} \text{JSD}_{F_{coll}}(P \parallel Q) = 0 \quad (17)$$

q_N tracks how the total emission intensity scales as function of system size. The mean values of q_N are

$$\mu_- = \lim_{\epsilon \rightarrow 0} q_N(g_c(w) - \epsilon) = 1.0$$

$$\mu_+ = \lim_{\epsilon \rightarrow 0} q_N(g_c(w) + \epsilon) = 2.0$$

Then, $\mu_+ - \mu_- = 2.0 - 1.0 = 1.0$, which shows that mean values jump abruptly and is far larger than the noise variance σ in bad cavity limit. The exact Shannon entropy limits of the JSD is

$$M(x) = \frac{1}{2} (P(x) + Q(x))$$

So JSD for q_N under thermodynamic limit,

$$\lim_{\substack{\epsilon \rightarrow 0 \\ N \rightarrow \infty}} \text{JSD}_{q_N}(P \parallel Q) = 1.0 \text{ bit} \quad (18)$$

The analytical description of F_{coll} and Φ_{SR} on $\text{JSD} \rightarrow 0$ reveals the phase boundary as a continuous blur and phases across $g_c(w)$ to be statistically indistinguishable. The q_N approaches the absolute maximum theoretical limit of information divergence across $g_c(w)$ as depicted by $\text{JSD} \rightarrow 1.0$. The phase transition configures as a structural shift from local uncorrelated emission to global dipole synchronization.

V. Numerical simulation of Dissipative phase transition *via* PIQS

The analytical results are verified by quantum simulation using Permutational Invariant Quantum Solver (PIQS), which are shown in Figure 6. PIQS is an open-source QuTiP library in Python that is a prominent quantum simulation tool for addressing steady-state superradiance in driven dissipative systems. We solve the quantum master equation for modelling the full dynamics of our large atomic ensemble containing 40 identical atoms using PIQS. In a bad cavity limit, we adiabatically eliminated the cavity mode, and the coherent exchange between atoms in the ensemble and cavity photons has been absorbed into an effective dissipator. Thus, in the bad-cavity limit, the cavity field k is proportional to the collective Lindblad term Γ (Supplementary Section I). Here, $\Gamma = (4g^2)/k$ is the cavity-mediated collective correlated emission rate. The

indistinguishability between atoms has been set by an effective spectral bandwidth due to collective decay, Γ . The cavity field populations from the interaction Hamiltonian can be written as

$$H = \hbar g(a^\dagger J_- + a J_+)$$

$$\langle a^\dagger a \rangle = \frac{\Gamma}{k} \langle J_+ J_- \rangle = \sum_{i \neq 1}^N \langle \sigma_j^- \sigma_j^+ \rangle + \sum_{i=1}^N \langle \sigma_j^- \sigma_j^+ \rangle \quad (19)$$

The first term describes the sum of collective emission from correlated atoms, and the second term represents the individual emission leading to spontaneous emission from uncorrelated atoms. We simulate the quantum master equation considering the correlated atoms from the above equation for identical atoms within the bad-cavity limit by setting up a symmetric subspace of the Dicke system. Our simulation used the power method in the robust steady state solver for the Dicke basis Liouvillian. The Liouvillian of the density matrix at the steady-state can be written as $\mathcal{L}\rho_{ss} = 0$ or simply

$$\frac{d\rho_{ss}}{dt} \approx 0$$

The Liouvillian spectrum corresponds to n eigenvalues that satisfy the $\mathcal{L}\rho_n = \lambda_n \rho_n$. The $\lambda_n = -\nu + i\omega$ at steady-state describes one of the eigenvalues that must correspond to $\lambda_0 = 0$ for the eigenvector $\mathcal{L}$. The ν and ω are the decay rate and energy oscillation, respectively. At steady-state, eigenvector $\mathcal{L}$ with zero eigenvalue corresponds to Liouvillian eigenstate with zero decay rate i.e., eigenstate with $\nu = 0$. The observable in the Dicke basis $\langle J_+ J_- \rangle_{ss}$ computes the steady-state collective coherence. The total emission rate in the Dicke limit for $N+1$ symmetric states in the m -excitation subspace, $\left| \frac{N}{2}, m \right\rangle$ state in the symmetric Dicke ladder from the above equation is

$$\begin{aligned}
\langle n \rangle &= \frac{4g^2}{k^2} \langle J_+ J_- \rangle_{ss} = \frac{4g^2}{k^2} [J_+ J_- |J, M\rangle] \\
&= \frac{4g^2}{k^2} [(J + M)(J - M + 1) |J, M\rangle]
\end{aligned} \tag{20}$$

For collective symmetric states $J = \frac{N}{2}$ and m-excitation

$$\langle n \rangle = \frac{4g^2}{k^2} \left[\left(\frac{N}{2} + \left(m - \frac{N}{2} \right) \right) \left(\frac{N}{2} - \left(m - \frac{N}{2} \right) + 1 \right) \right] \tag{21}$$

Figure 5a shows a numerically simulated phase transition plot for collective sensitivity scaling. From the quantum master equation, Liouvillian criticality based on system size $N = 40$ can be computed using collective response scaling as

$$q_N = \frac{\partial \log(\langle n \rangle)}{\partial \log(N)} = \frac{\partial \log \left(\frac{4g^2}{k^2} \langle J_+ J_- \rangle_{ss} \right)}{\partial \log(N)} \tag{22}$$

The phase transition map for q_N indicated the phase transition from the normal phase into the superradiant phase at the critical point. The burst-like band indicated by the red arrow at the critical boundary exhibits the collective emission by correlated atoms in the atomic ensemble.