

Online Resource 3

Supplementary Tables S1–S4 and Supplementary Figure S1.

Bounded Repair Reachability of Residual Critical Blocks in 3D Binary Masks under
SAFE-Gated Single-Voxel Flips

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Online Resource 3. Supplementary Tables S1–S4 and Supplementary Figure S1

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Manuscript: Bounded Repair Reachability of Residual Critical Blocks in 3D Binary Masks under SAFE-Gated Single-Voxel Flips.

Supplementary Table S1. Witness steps

(a) Main-text Figure 7: BraTS2021_00110, C00-F1, $L^* = 4$.

step	operation	voxel (z,y,x)	RAW	X_T	SAFE	Loc	FO-P	FID	Critl after	NRFR after
1	REM	(106,110,162)	1	1	✓	×	✓	×	1	1
2	REM	(106,110,161)	0	1	✓	×	✓	✓	1	1
3	ADD	(106,111,162)	0	0	✓	✓	✓	✓	0	1
4	ADD	(106,110,162)	1	1	✓	×	×	✓	0	0

(b) Figure S1: BraTS2021_00312, C00-F0, $L^* = 2$.

step	operation	voxel (z,y,x)	RAW	X_T	SAFE	Loc	FO-P	FID	Critl after	NRFR after
1	REM	(62,160,153)	1	1	✓	×	✓	×	1	1
2	ADD	(62,161,153)	0	0	✓	✓	✓	✓	0	1

(c) BraTS2021_00110, C00-F0, G26, $L^* = 2$ (139 states): the unrestricted F0 repair of the block of main-text Figure 7.

step	operation	voxel (z,y,x)	RAW	X_T	SAFE	Loc	FO-P	FID	Critl after	NRFR after
1	REM	(106,110,162)	1	1	✓	×	✓	×	1	1
2	ADD	(106,111,162)	0	0	✓	✓	✓	✓	0	1

Notes. The rule decisions SAFE, Loc, FO-P and FID are the recorded values of `PATHS.csv` in the production tables (Online Resource 4), evaluated in the state immediately before each move. Loc, FO-P and FID are recorded for comparison and are not rules of the systems that produced these witnesses (C00-F1, C00-F0). |Critl| is the number of critical blocks in the window, as stored by the production run. RAW, X_T and NRFR are computed from the stored witness states and the RAW mask by the re-aggregation script of Online Resource 4 when the witness bundles are supplied. $(\beta_0, \beta_1, \beta_2) = (1, 1, 3)$ (all steps before the 00110 witness, crop, closed-cube model; check of a necessary condition). In (c), the two voxels are step 1 (106,110,162) and step 3 (106,111,162) of the witness in (a). Step 1 removes RAW foreground (FID $\times$) and is therefore excluded under F2, and the final NRFR is 1. C01P-F0 finds the same path with the same number of states (139; identical rows in `PATHS.csv`), so flip-once does not exclude this F0 path.

Supplementary Table S2. Search outcomes of all systems at both horizons

Outcomes of the 680 target blocks per system and horizon ($680 \times 14 \times 2 = 19,040$ searches); entries are FOUND / EXH / DL / CAP. G14: R = 1, L = 4. G26: R = 2, L = 6. At G26, C00-F1 and C00-F2 rows that ended in CAP and were replaced by an earlier completed search (reuse log, 655 rows) carry that search's verdict; G14 has no replaced rows. Every system sums to 680 at both horizons.

System	G14	G26	of which inherited at G26 (FOUND / DL)
C00-F0	661 / 0 / 19 / 0	666 / 0 / 0 / 14	-
C00-F1	15 / 0 / 665 / 0	72 / 0 / 606 / 2	44 / 517
C00-F2	3 / 458 / 219 / 0	15 / 205 / 460 / 0	0 / 94
C01H-F0	588 / 0 / 92 / 0	632 / 0 / 7 / 41	-
C01H-F2	0 / 676 / 4 / 0	0 / 271 / 337 / 72	-
C01P-F0	588 / 0 / 92 / 0	636 / 0 / 4 / 40	-
C01P-F2	3 / 460 / 217 / 0	13 / 207 / 365 / 95	-
C10-F0	0 / 641 / 39 / 0	5 / 630 / 45 / 0	-
C10-F1	0 / 641 / 39 / 0	0 / 630 / 50 / 0	-
C10-F2	0 / 677 / 3 / 0	0 / 669 / 11 / 0	-
C11H-F0	0 / 680 / 0 / 0	0 / 680 / 0 / 0	-
C11H-F2	0 / 680 / 0 / 0	0 / 680 / 0 / 0	-
C11P-F0	0 / 642 / 38 / 0	0 / 631 / 49 / 0	-
C11P-F2	0 / 677 / 3 / 0	0 / 673 / 7 / 0	-

Supplementary Table S3. Search outcomes by cohort

Effective counts of Table 4 (G26) and Supplementary Table S2, split by cohort. C1: 614 blocks in 213 cases ($k = 2$ shellshell blocks that are shell-closed in the RAW mask). C2: 66 blocks in 39 cases (shell-closed in the final state of the first repair stage only). Entries are FOUND / EXH / DL / CAP. For F1, the C01P and C11P columns are the F2 searches (Section 5.3). For every cell, C1 + C2 equals the pooled count. Substituted rows are counted in the effective cells as in Table 4. Counts only are reported; no rate comparison between cohorts is made, because the two cohorts differ in size (614 and 66 blocks) and in how their blocks arise.

G26, F0

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	602 / 0 / 0 / 12	5 / 571 / 38 / 0	574 / 0 / 4 / 36	0 / 572 / 42 / 0
C2 (66)	64 / 0 / 0 / 2	0 / 59 / 7 / 0	62 / 0 / 0 / 4	0 / 59 / 7 / 0
pooled (680)	666 / 0 / 0 / 14	5 / 630 / 45 / 0	636 / 0 / 4 / 40	0 / 631 / 49 / 0

G26, F1

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	54 / 0 / 558 / 2	0 / 571 / 43 / 0	0 / 201 / 325 / 88	0 / 609 / 5 / 0
C2 (66)	18 / 0 / 48 / 0	0 / 59 / 7 / 0	13 / 6 / 40 / 7	0 / 64 / 2 / 0
pooled (680)	72 / 0 / 606 / 2	0 / 630 / 50 / 0	13 / 207 / 365 / 95	0 / 673 / 7 / 0

G26, F2

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	0 / 199 / 415 / 0	0 / 605 / 9 / 0	0 / 201 / 325 / 88	0 / 609 / 5 / 0
C2 (66)	15 / 6 / 45 / 0	0 / 64 / 2 / 0	13 / 6 / 40 / 7	0 / 64 / 2 / 0
pooled (680)	15 / 205 / 460 / 0	0 / 669 / 11 / 0	13 / 207 / 365 / 95	0 / 673 / 7 / 0

G14, F0

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	595 / 0 / 19 / 0	0 / 582 / 32 / 0	532 / 0 / 82 / 0	0 / 583 / 31 / 0
C2 (66)	66 / 0 / 0 / 0	0 / 59 / 7 / 0	56 / 0 / 10 / 0	0 / 59 / 7 / 0
pooled (680)	661 / 0 / 19 / 0	0 / 641 / 39 / 0	588 / 0 / 92 / 0	0 / 642 / 38 / 0

G14, F1

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	9 / 0 / 605 / 0	0 / 582 / 32 / 0	0 / 440 / 174 / 0	0 / 613 / 1 / 0
C2 (66)	6 / 0 / 60 / 0	0 / 59 / 7 / 0	3 / 20 / 43 / 0	0 / 64 / 2 / 0
pooled (680)	15 / 0 / 665 / 0	0 / 641 / 39 / 0	3 / 460 / 217 / 0	0 / 677 / 3 / 0

G14, F2

Cohort (blocks)	C00	C10	C01P	C11P
C1 (614)	0 / 439 / 175 / 0	0 / 613 / 1 / 0	0 / 440 / 174 / 0	0 / 613 / 1 / 0
C2 (66)	3 / 19 / 44 / 0	0 / 64 / 2 / 0	3 / 20 / 43 / 0	0 / 64 / 2 / 0
pooled (680)	3 / 458 / 219 / 0	0 / 677 / 3 / 0	3 / 460 / 217 / 0	0 / 677 / 3 / 0

Supplementary Figure S1. Control block BraTS2021_00312

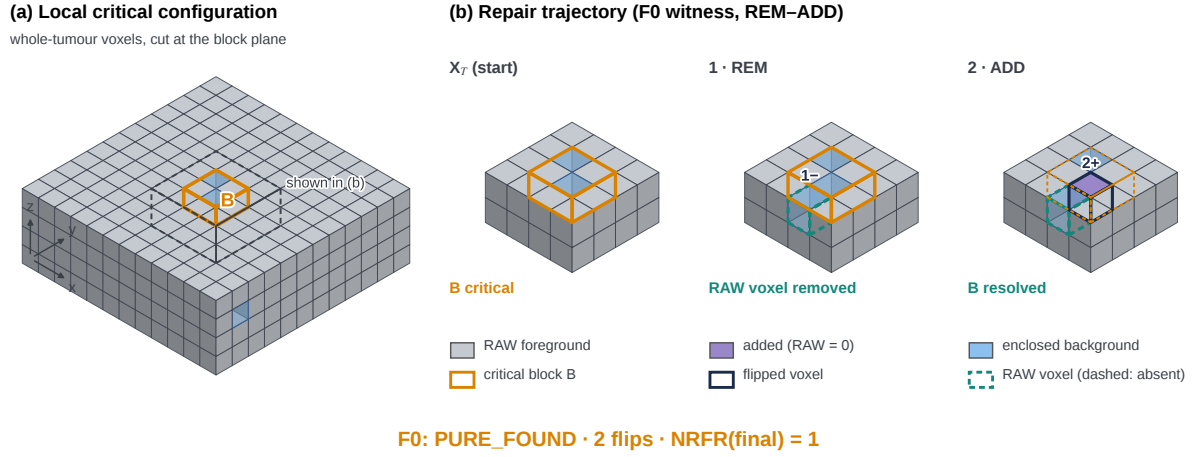


Figure S1. (a) Critical block B (orange frame) inside the tumour mask of BraTS2021_00312, same colour encoding as main-text Figure 7 (without its fill patterns). (b) The unrestricted repair (F0) removes a RAW foreground voxel (teal, dashed) and adds a voxel outside RAW; B is resolved after two flips, and the RAW voxel is not restored (NRFR(final) = 1). On the same block the fidelity-constrained searches do not return a repair within the horizon: F1 is depth-limited (at most 6 flips, 342,262 states; the search is not exhausted) and F2 is exhausted (no admissible move at X_T). Figure preparation used ChatGPT/OpenAI Codex assistance for code and layout.

Supplementary Table S4. Attachment of the filling voxel along the witness

For BraTS2021_00110, block k2xy@106, 111, 162, C00-F1 at G26, let $v = (106, 111, 162)$, with coordinates in (z, y, x) order. The states $Y_0 = X_T$, Y_1 and Y_2 are the initial state and the states after the first and second removals of the four-flip witness in Table S1(a). The removals act on $(106, 110, 162)$ and $(106, 110, 161)$, respectively. The target voxel v is background in all three states and is filled at step 3. We recompute $A_{Y_k}(v) = K(Y_k \setminus \{v\}) \cap C_v$ from the stored states.

State	Last operation	$\beta(A_{Y_k}(v))$	SAFE	(n_0, n_1, n_2)	(r_1, r_2)
$Y_0 = X_T$	initial	$(1, 0, 1)$	no	8,12,6	7,5
Y_1	REM 1	$(1, 0, 0)$	yes	8,12,5	7,5
Y_2	REM 2	$(1, 0, 0)$	yes	8,12,5	7,5

Here n_j is the number of attachment cells of dimension j , and r_j is the rank of the cellular boundary map ∂_j over $\mathbb{F}_2$. The Betti vector is obtained by

$$\beta_0 = n_0 - r_1, \quad \beta_1 = n_1 - r_1 - r_2, \quad \beta_2 = n_2 - r_2.$$

All three chain complexes satisfy $\partial_1 \partial_2 = 0$. The computation uses the closed contact cells from the 26-neighbourhood of v , including their faces. It agrees with the attachment computation used in the existing witness probe; no new search is performed. The filling voxel is already SAFE after REM 1 and remains SAFE after REM 2.

Relation to Table S1. The SAFE column of Table S1 tests the *voxel flipped in that row* in the state *immediately before that move*: v_i in Y_{i-1} for display steps $i = 1, \dots, 4$. Table S4 instead tests the *same fixed voxel* $v = (106, 111, 162)$ in each of Y_0, Y_1, Y_2 . Only its Y_2 row corresponds to the SAFE check for ADD 3 in Table S1(a). The SAFE checks for REM 1 and REM 2 in Table S1 concern different voxels and do not determine when the later filling voxel first becomes SAFE.

Table S4 provenance and reproducibility records. The original witness bundle is `FIGURE_DATA/shard_1/FIG/BraTS2021_00110__k2xy_106_111_162/C00-F1__G26.npz`, arrays `wit_states[0,:]`, `wit_states[1,:]` and `wit_states[2,:]` in the order of coords (180 voxels). The context is `ctx.npz`, with crop origin (96, 101, 152) and shape (21, 22, 22). The stored path is matched to `PRODUCTION/shard_1/PATHS.csv`, lines 211–214 (header is line 1); its steps 0–3 are display steps 1–4. Replay matches all five stored witness states exactly.

The non-image verification supplement `Online_Resource_4_S4_records_R4J.zip` contains `ATTACHMENT_RECALCULATION.json`, `ATTACHMENT_RECALCULATION.csv`, `ATTACHMENT_CELL_LEDGER.json`. The internal replay array `REPLAYED_FIRST_THREE_STATES.npz` is retained with the source witness for verification. The package `README` links these files to Table S4 and gives the reproduction command. It records SHA-256 digests of the context, witness bundle, path inputs, coordinate array and each state, with the byte encoding specified in the JSON. The boundary-rank calculation is identified by `code/close_evidence.py` (byte-identical in the records bundle as `close_evidence_FROZEN.py`), functions `cells_from_neighbours`, `columns` and `rank_f2`. Its SHA-256 is `957d7f51a7f6130f01a05779805402b497704ee6ac78647075350448b91592e2`. The existing probe is `MANUSCRIPT/PHASE_S/code/attach_along_witness.py::attach_betti`, SHA-256 `06bbc4253e33cab77ee854f5a3f3991f750df648a1ef5fb496ca441ca296e2ef`.

State	SHA-256 of the stored 180-entry occupancy array
Y0	28c660b4c34db09286b8e592903a132c727f04c3760a84da64f0e22408ec7ab6
Y1	f4ded876e53ebabbe9cd90f09a3f7e29681695bde257c3ce8d7f2cf0bef4fbff
Y2	f6beccf09260fd85e5b4a738fb793ca74b0ee80520c9ed7b5cb558fc4e894ab3