

Online Resource 1

Proof of homotopy preservation under SAFE flips.

Bounded Repair Reachability of Residual Critical Blocks in 3D Binary Masks under
SAFE-Gated Single-Voxel Flips

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Online Resource 1. SAFE single-voxel flips preserve the homotopy type of the closed-cube foreground complex

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This note proves that a flip passing the SAFE gate of Section 3.3 leaves the homotopy type of the closed-cube foreground complex $K(X)$ unchanged, and records the finite computations that accompany the proof. The statements concern $K(X)$ only. In the formulation by the cubical neighbourhood of a point, the statements are known from Niethammer, Kalies, Mischaikow and Tannenbaum [Niethammer06]; Lemma 3 and Corollary 4 below relate the attachment used here to that neighbourhood. The proofs are included so that the guarantee used in the main text is self-contained, and they are not offered as new theory.

1 Setting

For $q \in \mathbb{Z}^3$ let $C_q = q + [0,1]^3$ be the closed unit cube. The faces of the cubes C_q , which are products of intervals $[a,a]$ or $[a,a+1]$, form a cubical complex $\mathcal{C}$ of $\mathbb{R}^3$. For a finite $X \subset \mathbb{Z}^3$ let $K(X) = \bigcup_{q \in X} C_q$. This is a finite subcomplex of $\mathcal{C}$ (empty for $X = \emptyset$). For $v \in \mathbb{Z}^3$ write $C = C_v$, ∂C for the union of its 26 proper faces (8 vertices, 12 edges, 6 squares), and $N(v) = \{q \in \mathbb{Z}^3 : 0 < \|q - v\|_\infty \leq 1\}$ for the 26-neighbourhood. The attachment of v to X is

$$A_X(v) = K(X \setminus \{v\}) \cap C_v.$$

The flip of v maps X to $X \triangle \{v\}$; it is an addition if $v \notin X$ and a removal if $v \in X$. A flip is SAFE if the Betti numbers of $A_X(v)$ are $(\beta_0, \beta_1, \beta_2) = (1, 0, 0)$; by Proposition 1 below the choice of the coefficient field is immaterial.

For an offset $o = q - v \in \{-1, 0, 1\}^3 \setminus \{0\}$ let $f_v(o) = I_1 \times I_2 \times I_3$ with $I_i = \{v_i\}$ if $o_i = -1$, $I_i = \{v_i + 1\}$ if $o_i = +1$, and $I_i = [v_i, v_i + 1]$ if $o_i = 0$. The map $o \mapsto f_v(o)$ is a bijection from the 26 offsets onto the 26 proper faces of C_v .

Lemma 1. For finite $X \subset \mathbb{Z}^3$ and $v \in \mathbb{Z}^3$,

$$A_X(v) = \bigcup_{q \in X \cap N(v)} f_v(q - v).$$

Hence $A_X(v)$ is a subcomplex of ∂C that depends only on $X \cap N(v)$, is the same for X and $X \triangle \{v\}$, and every subcomplex of ∂C occurs for some X .

Proof. $K(X \setminus \{v\}) \cap C_v = \bigcup_{q \in X \setminus v} (C_q \cap C_v)$. For $q \neq v$ the cubes C_q and C_v intersect in a common face of both. The intersection is empty if $\|q - v\|_\infty \geq 2$ and equals $f_v(q - v)$ if $q \in N(v)$. Each $f_v(o)$ is a closed cell of ∂C together with all its faces, so the union is a subcomplex of ∂C . Conversely, a subcomplex A of ∂C is the union of its maximal faces $f_v(o_1), \dots, f_v(o_m)$, and $X = \{v + o_1, \dots, v + o_m\}$ gives $A_X(v) = A$. ■

2 Subcomplexes of the boundary of a cube

For a subcomplex A of ∂C let V, E, F be its numbers of vertices, edges and squares, $\chi(A) = V - E + F$, and $c(A)$ its number of connected components. A is *collapsible* if it can be reduced to a point by finitely many elementary collapses, that is, removals of a pair (e, Q) of a square Q and an edge e of Q that lies in no other

square of the current complex (a free edge), or of a pair (u, w) of an edge w and an endpoint u of w that lies in no other edge.

Lemma 2. Let $\emptyset \neq A \subseteq \partial C$ be a subcomplex with $A \neq \partial C$. Then A can be reduced by elementary collapses of the first kind to a graph G (a complex without squares). Consequently, for every coefficient field F ,

$$\beta_0(A; F) = c(A), \beta_1(A; F) = c(A) - \chi(A), \beta_2(A; F) = 0,$$

and $H_*(A; \mathbb{Z})$ is free.

Proof. We show that every subcomplex $A \neq \partial C$ that contains a square contains a square with a free edge. Because $A \neq \partial C$, some square s of ∂C is not in A (if all six squares were in A , all their faces would be as well, and $A = \partial C$). Each square of ∂C shares an edge with exactly four others and is disjoint from exactly one, the opposite square.

Case 1: a square $Q \in A$ shares an edge e with s . The edge e belongs to A as a face of Q . The only squares of ∂C containing e are Q and s , and $s \notin A$, so e is a free edge of Q .

Case 2: no square of A shares an edge with s . Then A contains none of the four squares adjacent to s , and since A contains a square, it contains the opposite square s' and no other square. Every edge of s' is shared with one of the four squares adjacent to s , none of which is in A , so all four edges of s' are free.

Let (e, Q) be a free pair, and $A' = A \setminus (\text{int } e \cup \text{int } Q)$. Because e lies in no other square of A , A' is a subcomplex, $s \notin A'$, and A' has one square fewer than A . Induction on the number of squares yields a graph G . Each step is a strong deformation retraction: Q deformation retracts onto the union of its three edges other than e , fixing them pointwise, and the retraction extends by the identity to A . Hence $A \simeq G$, and the steps preserve the number of components and χ (E and F each drop by one). For a finite graph, H_0 and H_1 are free of rank c and $E_G - V_G + c$, and $H_k = 0$ for $k \geq 2$; $E_G - V_G + c = c - \chi(G) = c - \chi(A)$. ■

Proposition 1 (decision rule). For a non-empty subcomplex A of ∂C the following are equivalent: (i) A is collapsible; (ii) A is contractible; (iii) $\beta(A; F) = (1, 0, 0)$ for some field F , equivalently for every field F ; (iv) $c(A) = 1$ and $\chi(A) = 1$. If $A = \emptyset$ then $\beta_0 = 0$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold for any complex, since collapses are homotopy equivalences and contractible spaces have the homology of a point. (iii) $\Rightarrow$ (iv): if $A = \partial C$ then $\beta_2 = 1$, so (iii) fails. Otherwise Lemma 2 gives $\beta_0 = c$ and $\beta_1 = c - \chi$, so $\beta_0 = 1$ and $\beta_1 = 0$ force $c = 1$ and $\chi = 1$. (iv) $\Rightarrow$ (i): ∂C has $\chi = 2$, so $A \neq \partial C$, and Lemma 2 gives a graph G with $c(G) = 1$ and $\chi(G) = 1$, a tree. A tree is reduced to a point by removing leaves, so A is collapsible. The Betti numbers in Lemma 2 do not depend on F , which gives the equivalence of "some field" and "every field". ■

In particular the SAFE condition does not depend on the coefficient field, and a flip is SAFE if and only if its attachment is non-empty, connected and has Euler characteristic 1. The lookup table of the searches is generated from exactly these quantities: the number of connected components of the 1-skeleton, the indicator that all six squares of the boundary are present, and the Euler characteristic, from which β_1 follows; no coefficient field enters that computation. The independent check of Section 4 computes the Betti numbers by boundary-matrix ranks over F_2 and over $\mathbb{Q}$.

3 A SAFE flip preserves the homotopy type of $K(X)$

Theorem 1. Let $X \subset \mathbb{Z}^3$ be finite, $v \in \mathbb{Z}^3$, $K = K(X \setminus \{v\})$, $K^+ = K(X \cup \{v\}) = K \cup C_v$ and $A = A_X(v) = K \cap C_v$. (a) If the flip of v is SAFE, then K is a strong deformation retract of K^+ . In particular the inclusion $K \hookrightarrow K^+$ is a homotopy equivalence, and $K(X)$ and $K(X \triangle \{v\})$ are homotopy equivalent. (b) If the flip of v is not SAFE, then the inclusion $K \hookrightarrow K^+$ is not an isomorphism on integral homology in some degree, and therefore not a homotopy equivalence.

Proof. (a) By Proposition 1, A is contractible, and $C = C_v$ is contractible, so the inclusion $A \hookrightarrow C$ is a homotopy equivalence. The pair (C, A) is a CW pair and therefore has the homotopy extension property [Hatcher02, Prop. 0.16]. By [Hatcher02, Cor. 0.20] A is a deformation retract of C : there is a homotopy H :

$C \times [0,1] \rightarrow C$ with $H_0 = \text{id}$, $H_1(C) \subseteq A$ and $H_t|_A = \text{id}$ for all t (the definition of a deformation retraction in [Hatcher02] fixes A pointwise). Define $F_t : K^+ \rightarrow K^+$ by $F_t|_K = \text{id}$ and $F_t|_C = H_t$. This is well defined because $K \cap C = A$ and H_t fixes A , and it is continuous by the pasting lemma, K and C being closed. Then $F_0 = \text{id}$, $F_1(K^+) \subseteq K$ and $F_t|_K = \text{id}$, so K is a strong deformation retract of K^+ . Since $\{K(X), K(X \triangle \{v\})\} = \{K, K^+\}$, the last claim follows.

- (b) If $K = \emptyset$ then $K^+ = C$ and $\emptyset \hookrightarrow C$ is not an equivalence. Let $K \neq \emptyset$. If $A = \emptyset$ then $K^+ = K \sqcup C$ and $H_0(K^+, K) \cong H_0(C) = \mathbb{Z} \neq 0$. If $A \neq \emptyset$, the pairs (K^+, K) and (C, A) are CW pairs, and the inclusion $C \subseteq K^+$ induces a homeomorphism $C/A \rightarrow K^+/K$ (a continuous bijection of compact Hausdorff spaces). Then by [Hatcher02, Prop. 2.22] $H_k(K^+, K) \cong \tilde{H}_k(K^+/K) \cong \tilde{H}_k(C/A) \cong H_k(C, A)$, and since C is contractible the reduced long exact sequence of (C, A) gives $H_k(C, A) \cong \tilde{H}_{k-1}(A; \mathbb{Z})$. When the flip is not SAFE, Proposition 1 shows that $\tilde{H}_j(A; F) \neq 0$ for some j and some field F ; as $H_*(A; \mathbb{Z})$ is free (Lemma 2 for $A \neq \partial C$, and $H_*(\partial C; \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}$), $\tilde{H}_j(A; \mathbb{Z}) \neq 0$. Thus $H_{j+1}(K^+, K; \mathbb{Z}) \neq 0$. If the inclusion induced isomorphisms in all degrees, the long exact sequence of the pair (K^+, K) would give $H_*(K^+, K) = 0$. Hence it does not. ■

Corollary 2. Let $X_0, X_1, \dots, X_n \subset \mathbb{Z}^3$ be finite with $X_{i+1} = X_i \triangle \{v_i\}$, where each flip of v_i is SAFE with respect to X_i . Then $K(X_0), \dots, K(X_n)$ are pairwise homotopy equivalent. Consecutive complexes are nested, and the smaller is a strong deformation retract of the larger; $K(X_0)$ and $K(X_n)$ are in general not nested. In particular $K(X_0)$ and $K(X_n)$ have the same number of connected components and isomorphic homology groups in every degree.

Proof. Apply Theorem 1(a) to each step and compose the homotopy equivalences $K(X_i) \rightarrow K(X_{i+1})$ (or their homotopy inverses). ■

Theorem 1(b) concerns the inclusion map. It does not assert that $K(X)$ and $K(X \triangle \{v\})$ fail to be homotopy equivalent by some other map when the flip is not SAFE.

Lemma 3 (cubical neighbourhood). For finite $X \subset \mathbb{Z}^3$ and $v \in \mathbb{Z}^3$ let $N_v = \bigcup_{q \in X \cap N(v)} C_q$ be the union of the closed cubes of the foreground voxels among the 26 neighbours of v . Then $N_v \cap C_v = A_X(v)$, and $A_X(v)$ is a strong deformation retract of N_v . In particular N_v and $A_X(v)$ are homotopy equivalent, and $H_*(N_v; G) \cong H_*(A_X(v); G)$ for every coefficient group G .

Proof. If $X \cap N(v) = \emptyset$ both sets are empty. Otherwise $N_v \cap C_v = \bigcup_{q \in X \cap N(v)} (C_q \cap C_v) = A_X(v)$ by the proof of Lemma 1. Let $\pi : \mathbb{R}^3 \rightarrow C_v$ be the coordinatewise clamp, which replaces the i -th coordinate x_i by $\min\{\max\{x_i, v_i\}, v_i + 1\}$. It is continuous and $\pi(x) = x$ for $x \in C_v$. For $q = v + o$ with $o \in \{-1, 0, 1\}^3 \setminus \{0\}$ write $C_q = J_1 \times J_2 \times J_3$ with $J_i = [v_i - 1, v_i], [v_i, v_i + 1]$ or $[v_i + 1, v_i + 2]$ according to $o_i = -1, 0$ or $+1$. The clamp maps J_i onto $\{v_i\}, [v_i, v_i + 1]$ or $\{v_i + 1\}$, which is the i -th factor I_i of $f_v(o)$, and $I_i \subseteq J_i$. Hence $\pi(C_q) = f_v(o) = C_q \cap C_v$. Put $H_t(x) = (1 - t)x + t\pi(x)$. For $x \in C_q$ every coordinate of $H_t(x)$ is a convex combination of two points of J_i , so $H_t(x) \in C_q \subseteq N_v$. Therefore H_t maps N_v into N_v for every $t \in [0, 1]$, $H_0 = \text{id}$, $H_1(N_v) = \bigcup \pi(C_q) = A_X(v)$, and $H_t(x) = x$ for $x \in A_X(v) \subseteq C_v$. ■

Corollary 4. For finite $X \subset \mathbb{Z}^3$ and $v \in \mathbb{Z}^3$ the following are equivalent: (i) the flip of v is SAFE; (ii) N_v has Betti numbers $(1, 0, 0)$ over some, equivalently every, field; (iii) N_v is acyclic, that is $\tilde{H}_k(N_v; \mathbb{Z}) = 0$ for all k and $N_v \neq \emptyset$; (iv) N_v is contractible; (v) the inclusion $K(X \setminus \{v\}) \hookrightarrow K(X \cup \{v\})$ induces isomorphisms in integral homology in all degrees; (vi) this inclusion is a homotopy equivalence.

Proof. By Lemma 3, N_v and $A_X(v)$ have the same homotopy type and the same homology with every coefficients. (i) $\Leftrightarrow$ (ii) is the definition of SAFE and Proposition 1. (i) $\Rightarrow$ (iv): $A_X(v)$ is contractible by Proposition 1. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii): a contractible space is acyclic, and acyclicity over $\mathbb{Z}$ gives Betti numbers $(1, 0, 0)$ over every field by the universal coefficient theorem. (i) $\Rightarrow$ (vi) is Theorem 1(a), (vi) $\Rightarrow$ (v) holds for any map, and (v) $\Rightarrow$ (i) is the contrapositive of Theorem 1(b). ■

For the neighbourhood $CN(p)$ of [Niethammer06], taken with $P = X \cup \{v\}$, $CN(v) = N_v$ and $CN(v) \cap C_v = A_X(v)$. Corollary 4 therefore states, in the closed-cube model, the equivalence of the homology-simple and homotopy-simple conditions with the acyclicity of the neighbourhood, which is the content of that paper

in dimension three. It is included so that the equivalence is proved for the objects used here, and it is not offered as new theory.

The domain of the image does not enter the statements. A finite domain Ω with background outside Ω is the case $X \subset \Omega \subset \mathbb{Z}^3$, and the attachment of any $v \in \mathbb{Z}^3$ is determined by the 26 values of X around v , with the value 0 outside Ω .

4 Finite computations

The 2^{26} occupancy patterns of $N(v)$ were enumerated and each pattern was mapped to its attachment by Lemma 1. The enumeration, carried out by a program written separately from the generator of the lookup table, gave the following.

Quantity	Value
Distinct attachments (subcomplexes of ∂C , including $\emptyset$)	15,935
Non-empty attachments	15,934
Attachments with $c = 1$ and $\chi = 1$ (SAFE)	2,062
Patterns with a SAFE attachment	25,985,144 of 67,108,864

For all 15,934 non-empty attachments the Betti numbers over $\mathbb{Q}$ and over F_2 agree, the integral homology computed by Smith normal form is torsion-free, and the conditions (i), (iii) and (iv) of Proposition 1 select the same 2,062 attachments, where (i) was evaluated by 4 random-order collapses per attachment and by a fixed-order greedy collapse in a separate program. Lemma 2 was also checked directly: for every attachment $A \neq \partial C$, collapses of square-edge pairs alone remove all squares. The lookup table used in the searches (SHA-256 59407fef2acb5b06059a13f19092047d172bbddd49bc85f811d822293b4c4830) agrees with condition (iv) on all 2^{26} patterns. These computations confirm Lemma 2 and Proposition 1 on the finite set of attachments that can occur; they are not used in the proofs above.

Programs: `s2_lemma_check.py` (fixed-order collapse and Betti numbers over $\mathbb{Q}$ and F_2 , SHA-256 fa4c9b20705deafa873e509a8f7cce39043a46c01906bed9fdf78215b0c5b4f5) and `safe_independent_check.py` (pattern-level enumeration, Smith normal form, random-order collapses and table comparison). SHA-256 of `safe_independent_check.py`: 961939b38f18c6283c86a6118c98c6b285948b338d3d86d762b3581ab384ad9c.

5 Scope

Theorem 1 and Corollary 2 state that SAFE flips preserve the homotopy type of the compact polyhedron $K(X) \subset \mathbb{R}^3$. They make no statement about the homotopy type of the complement of $K(X)$, about ambient isotopy, about images with more than one foreground label, or about the relation between the SAFE condition and the definitions of simple points in digital topology other than the cubical-neighbourhood formulation of Lemma 3 and Corollary 4; the topological-number definitions of simple points are not examined here.

References

- [Hatcher02] Hatcher A. Algebraic Topology. Cambridge University Press, 2002. Proposition 0.16 (CW pairs have the homotopy extension property), Corollary 0.20 (a subcomplex whose inclusion is a homotopy equivalence is a deformation retract), the definition of a deformation retraction in Chapter 0 (fixing A pointwise), Proposition 2.22 (good pairs) and the reduced long exact sequence of a pair in Section 2.1. Statement numbers refer to the author’s online version (pi.math.cornell.edu/hatcher/AT/ATpage.html).

[Niethammer06] Niethammer M, Kalies WD, Mischaikow K, Tannenbaum A. On the detection of simple points in higher dimensions using cubical homology. *IEEE Trans Image Process* 15(8):2462–2469 (2006). doi: 10.1109/TIP.2006.877309.