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The period function at a C^1 switching level: loss of smoothness and a critical period for a pendulum with unilateral elastic stops

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Abstract

We study the period function of planar conservative systems $\ddot{x} + W'(x) = 0$ whose potential is C^1 but piecewise analytic, the switching being caused by a unilateral elastic constraint that engages at $x = x_g$. If $W'(x_g) = d > 0$ and W'' jumps by J at x_g, we prove that the period satisfies $T(h) = \widetilde{T}(h) - \frac{4\sqrt{2}}{3} J d^{-3} (h - h_g)^{3/2} + O\big((h - h_g)^{5/2}\big)$ above the grazing energy h_g, where $\widetilde{T}$ is the period of the unswitched system. Hence T is C^1 but not C^2 at h_g, with $T'' \to -\operatorname{sign}(J)\,\infty$. The result is applied to a pendulum with two horizontal springs that act only beyond prescribed clearances. For this system we obtain: an exact solution by Jacobi and Weierstrass elliptic functions and a closed-form period; stiffness-independent grazing energies; strict monotonicity of the period in each stiffness; the rigid-stop limit with correction $\pi/(\cos\theta_g \sqrt{\kappa})$ per stop; and, for stiff stops, a critical period located at $\delta_* = T_0'(E_g)^2 d^6/(8n^2\kappa^2 \cos^4\theta_g)\,(1 + O(\kappa^{-1}))$ above grazing, so that the period function is not monotone. All statements are confirmed numerically with independent time integration.

Keywords: period function; critical period; piecewise-smooth systems; unilateral constraint; grazing; pendulum; elliptic functions.
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1 Introduction

For a planar system with a center, the *period function* assigns to each periodic orbit of the period annulus its minimal period. Its monotonicity and its critical points (critical periods) are classical subjects: criteria for monotonicity go back to Chicone [3] and Schaaf [10], bounds on the number of critical periods to Chow and Sanders [4], and the case of homogeneous nonlinearities was analysed by Gasull, Guillamon, Mañosa and Mañosas [7]. For analytic systems the period function is analytic on the interior of the period annulus, so the question is how many times it can turn.

Mechanical systems with clearances and one-sided supports lead to vector fields that are only piecewise smooth [1, 8, 5]. When the constraint is elastic the force is continuous, the potential is C^1, and a period annulus still exists; but the period function is no longer analytic, because orbits start touching the constraint at a *grazing energy*. Grazing is well known to organise the dynamics of forced nonsmooth systems through singular terms in the associated return maps [9, 6, 11]. The corresponding question for the conservative backbone—what exactly happens to the period function at grazing—does not appear to have been answered.

In this paper we answer it and show that grazing produces both a universal loss of smoothness and, for stiff constraints, a critical period. Our results are of two kinds.

A general result. Let W be C^1, analytic on each side of a switching point x_g with $W'(x_g) = d > 0$, and let J be the jump of W'' there. Theorem 2.3 shows that

$$T(h) = \widetilde{T}(h) - \frac{4\sqrt{2}}{3}\,\frac{J}{d^3}\,(h-h_g)^{3/2} + O\big((h-h_g)^{5/2}\big), \qquad h \downarrow h_g,$$

where $\widetilde{T}$ is the period of the system without switching, and that $T - \widetilde{T} = \delta^{1/2}G(\delta)$ with G analytic. Thus the period function is always C^1 and never C^2 at a grazing level with $J \neq 0$, and the sign of J fixes the sign of the infinite curvature.

A pendulum with unilateral elastic stops. We then study in detail a pendulum whose bob meets a horizontal spring on each side after crossing a clearance $d_\pm$. The geometric nonlinearity is kept exactly and the constitutive law is not regularised. We prove:

- an exact solution, with Jacobi functions in free flight and the Weierstrass function $\wp$ in contact, and a closed-form period built from incomplete elliptic integrals (Theorems 4.3 and 4.4);
- stiffness-independent grazing energies $E_{g,\pm} = 1 - \sqrt{1-d_\pm^2}$ and strict monotonicity of the period in each stiffness above them (Proposition 3.2, Theorem 5.1);
- convergence, as the stiffness $\kappa \to \infty$, to the period of the pendulum with rigid elastic-impact stops, with correction $\pi/(\cos\theta_g\sqrt{\kappa})$ per stop (Theorem 5.2);
- for stiff stops, the existence of a critical period (a strict local maximum of T) at an explicitly located energy $E_g + \delta_*$ with $\delta_* = O(\kappa^{-2})$, with its height (Theorem 6.2).

The last result shows that the period function of this nonsmooth pendulum is not monotone, although the free pendulum and its rigid-stop limit both have monotone period functions on the relevant energy ranges.

The paper is organised as follows. Section 2 contains the general result. Section 3 introduces the pendulum, its grazing energies and contact regimes. Section 4 gives the exact solution and period. Section 5 proves monotonicity in stiffness and the rigid-stop limit. Section 6 establishes the near-grazing expansions and the critical period. Section 7 reports the numerical confirmation, and Section 8 concludes.

2 The period function at a C^1 switching level

Consider

$$\ddot{x} + W'(x) = 0, \qquad H = \tfrac{1}{2}\dot{x}^2 + W(x), \tag{1}$$

on an interval $I \ni 0$ with the following hypotheses.

Hypothesis 2.1. (H1) $W \in C^1(I)$, $W(0) = 0$, and $xW'(x) > 0$ for $x \in I\setminus\{0\}$; hence the origin is a center whose period annulus contains the orbits with energies $0 < h < h_{\max}$.

(H2) There is $x_g \in I$, $x_g > 0$, and functions W_L, W_R analytic on a neighbourhood U of x_g such that $W = W_L$ on $U \cap \{x \le x_g\}$ and $W = W_R$ on $U \cap \{x \ge x_g\}$. Since $W \in C^1$, $W_L(x_g) = W_R(x_g) =: h_g$ and $W_L'(x_g) = W_R'(x_g) =: d > 0$. Set

$$J = W_R''(x_g) - W_L''(x_g).$$

Let $\widetilde{W}$ be the potential obtained by replacing W on $U \cap \{x \ge x_g\}$ by W_L, and let T and $\widetilde{T}$ denote the period functions of W and $\widetilde{W}$; they coincide for $h \le h_g$. For $h > h_g$ close to h_g write $\delta = h - h_g$, and let $X_R(u)$, $X_L(u)$ be the local inverses of W_R, W_L near x_g, i.e. $W_R(X_R(u)) = u$, $X_R(h_g) = x_g$, and similarly for L.

Lemma 2.2 (Excursion time)**.** *For $0 < \delta$ small, the time per period spent by the orbit of energy $h_g + \delta$ in $\{x > x_g\}$ is*

$$\mathcal{T}_R(\delta) = \sqrt{2}\int_0^{\delta} \frac{\gamma_R(v)\,dv}{\sqrt{\delta - v}} = \sqrt{2}\,\delta^{1/2}\int_0^1 \frac{\gamma_R(\delta t)}{\sqrt{1-t}}\,dt, \qquad \gamma_R(v) = \frac{1}{W_R'\big(X_R(h_g+v)\big)}, \tag{2}$$

and γ_R is analytic at $v = 0$ with $\gamma_R(0) = 1/d$, $\gamma_R'(0) = -W_R''(x_g)/d^3$. The same holds for $\widetilde{W}$ with R replaced by L.

Proof. By reversibility the time is twice the time from x_g to the turning point x_+, that is,

$$\mathcal{T}_R = 2\int_{x_g}^{x_+} \frac{dx}{\sqrt{2(h - W_R(x))}} = \sqrt{2}\int_{x_g}^{x_+} \frac{dx}{\sqrt{h - W_R(x)}}.$$

Since $W_R'(x_g) = d > 0$, $u = W_R(x)$ is an analytic change of variables near x_g, which gives (2); then put $v = \delta t$. Analyticity of γ_R follows from the analytic inverse function theorem, and $\gamma_R' = -W_R''/W_R'^3$ evaluated at x_g. □

Theorem 2.3 (Period function at a switching level)**.** *Under Hypothesis 2.1 there is an analytic function G near 0 with $G(0) = 0$ such that, for small $\delta = h - h_g > 0$,*

$$T(h) - \widetilde{T}(h) = \delta^{1/2}G(\delta) = -\frac{4\sqrt{2}}{3}\,\frac{J}{d^3}\,\delta^{3/2} + O(\delta^{5/2}). \tag{3}$$

Consequently, if $\widetilde{T}$ is C^2 near h_g, then T is C^1 near h_g, T is analytic on a right neighbourhood $(h_g, h_g + \epsilon)$ whenever $\widetilde{T}$ is, and

$$T''(h) = \widetilde{T}''(h) - \sqrt{2}\,\frac{J}{d^3}\,\delta^{-1/2} + O(\delta^{1/2}) \qquad (h \downarrow h_g). \tag{4}$$

In particular, if $J \neq 0$ then $T \notin C^2$ at h_g, and $T''(h) \to -\infty$ if $J > 0$, $T''(h) \to +\infty$ if $J < 0$.

Proof. The orbits of W and $\widetilde{W}$ with energy $h_g + \delta$ coincide in $\{x \le x_g\}$, so $T - \widetilde{T} = \mathcal{T}_R - \mathcal{T}_L$. By Lemma 2.2,

$$\mathcal{T}_R(\delta) - \mathcal{T}_L(\delta) = \sqrt{2}\,\delta^{1/2}\int_0^1 \frac{(\gamma_R - \gamma_L)(\delta t)}{\sqrt{1-t}}\,dt =: \delta^{1/2}G(\delta).$$

Expanding the analytic function $\gamma_R - \gamma_L = \sum_{j\ge 1} c_j v^j$ (note $c_0 = 0$) and integrating termwise, which is legitimate by uniform convergence on $[0,\delta]$, $G(\delta) = \sqrt{2}\sum_{j\ge1} c_j B_j \delta^j$ with $B_j = \int_0^1 t^j(1-t)^{-1/2}dt$, an analytic function. Since $c_1 = -(W_R'' - W_L'')(x_g)/d^3 = -J/d^3$ and $B_1 = 4/3$, (3) follows. Differentiating the convergent series termwise gives (4). □

Remark 2.4. (i) Nothing is assumed about the regularity of W away from U, other than (H1); the singular part of T at h_g is entirely local. (ii) The leading coefficient depends only on the slope d at the switching point and on the jump J. It is therefore insensitive to the global shape of the potential and gives a universal law for C^1 elastic contact. (iii) Several switching levels contribute additively. (iv) For rigid impacts the potential is not C^1, and the singular terms in grazing return maps are of order $\delta^{1/2}$ [9, 6]. The $\delta^{3/2}$ law here is the conservative, continuous-force counterpart.

3 A pendulum with two unilateral elastic stops

A point mass m on a massless rigid rod of length ℓ moves in a vertical plane; θ is measured from the downward vertical. A right spring of stiffness K_+ acts only when the horizontal coordinate $\ell\sin\theta$ exceeds g_+, and a left spring of stiffness K_- acts only when $\ell\sin\theta < -g_-$, with $0 < g_\pm < \ell$. The springs act horizontally (for instance through guided plungers), are massless, and there is no dissipation. With $\tau = \sqrt{g/\ell}\,t$, $d_\pm = g_\pm/\ell \in (0,1)$, $\kappa_\pm = K_\pm\ell/(mg) \geq 0$ and $[z]_+ = \max\{z, 0\}$, the dimensionless equation is

$$\ddot\theta + V'(\theta) = 0, \qquad V(\theta) = 1 - \cos\theta + \frac{\kappa_+}{2}\,[\sin\theta - d_+]_+^2 + \frac{\kappa_-}{2}\,[-\sin\theta - d_-]_+^2\,. \tag{5}$$

We work in the principal cell $|\theta| < \pi/2$, where the activation angles are

$$\theta_{g,+} = \arcsin d_+, \qquad \theta_{g,-} = -\arcsin d_-, \qquad c_\pm := \cos\theta_{g,\pm} = \sqrt{1 - d_\pm^2}.$$

On the contact branch $\sigma \in \{+1, -1\}$ the potential coincides with the entire function

$$V_\sigma(\theta) = 1 - \cos\theta + \frac{\kappa_\sigma}{2}\big(\sigma\sin\theta - d_\sigma\big)^2. \tag{6}$$

Proposition 3.1 (Well-posedness)**.** *$V \in C^1(\mathbb{R})$, V' is locally Lipschitz, solutions of (5) are unique, and $H = \frac{1}{2}\dot\theta^2 + V(\theta)$ is conserved. On $(-\pi/2, \pi/2)$, Hypothesis 2.1 holds at each activation angle with*

$$d = d_\sigma = |V'(\theta_{g,\sigma})|, \qquad J = J_\sigma = \kappa_\sigma c_\sigma^2,$$

the unswitched comparison system being the free pendulum.

Proof. $z \mapsto [z]_+^2$ is C^1 with derivative $2\,[z]_+$, so $V \in C^1$ with locally Lipschitz derivative; Picard–Lindelöf gives uniqueness and $\dot H = 0$. On $(0, \pi/2)$, $V'(\theta) = \sin\theta + \kappa_+\,[\sin\theta - d_+]_+\cos\theta > 0$, and symmetrically on the left, which is (H1). At $\theta_{g,+}$ we have $V' = \sin\theta_{g,+} = d_+$ and $V_+'' - (1-\cos)'' = \kappa_+\big[\cos^2\theta - (\sin\theta - d_+)\sin\theta\big]_{\theta_{g,+}} = \kappa_+ c_+^2$; the left side follows by $\theta \mapsto -\theta$. □

Put $E^* = \min\{V(-\pi/2), V(\pi/2)\} = 1 + \min_\sigma \kappa_\sigma(1 - d_\sigma)^2/2$. For $0 < E < E^*$ the orbit of energy E is a libration with turning points $\theta_-(E) < 0 < \theta_+(E)$; throughout, E is restricted to this range.

Proposition 3.2 (Grazing energies)**.** *The orbit of energy E reaches $\theta_{g,\sigma}$ if and only if $E \geq E_{g,\sigma} := 1 - c_\sigma = 1 - \sqrt{1 - d_\sigma^2}$. These energies do not depend on $\kappa_\pm$. For $E \leq \min_\sigma E_{g,\sigma}$ the motion is that of the free pendulum, with period $T_0(E) = 4K(k)$, $k = \sqrt{E/2}$.*

Proof. Both spring terms vanish at $\theta_{g,\sigma}$, so $V(\theta_{g,\sigma}) = 1 - c_\sigma$, and V is strictly monotone on each half-cell. The free period is classical [2]. □

Hence the orbits fall into three regimes: no contact for $E \leq \min_\sigma E_{g,\sigma}$, contact on one side only between the two grazing energies, and contact with both springs for $E > \max_\sigma E_{g,\sigma}$. For a prescribed admissible energy E_a, spring σ is never engaged if and only if $d_\sigma > \sqrt{2E_a - E_a^2}$.

4 Exact solution and exact period

Lemma 4.1 (Free flight)**.** *Let $0 < E < 2$, $k = \sqrt{E/2}$. While no spring is active, $\sin(\theta/2) = k\,\mathrm{sn}\big(u_0 \pm (\tau - \tau_0), k\big)$. If $E > E_{g,\sigma}$, the time per period spent between $\theta = 0$ and $\theta_{g,\sigma}$ is*

$$\mathcal{T}_{f,\sigma}(E) = 2F(\varphi_\sigma, k), \qquad \sin\varphi_\sigma = \sqrt{E_{g,\sigma}/E}. \tag{7}$$

Proof. With $\sin(\theta/2) = k\,\mathrm{sn}\,u$ the energy relation $\frac{1}{2}\dot\theta^2 + 2\sin^2(\theta/2) = 2k^2$ becomes $\dot u^2 = 1$ [2]. The time is $2\int_0^{|\theta_{g,\sigma}|} d\theta/\sqrt{2(E-1+\cos\theta)}$; the substitution $\sin(\theta/2) = k\sin\varphi$ gives (7), because $\sin^2(\theta_{g,\sigma}/2) = (1-c_\sigma)/2 = E_{g,\sigma}/2$. □

Lemma 4.2 (Quartic reduction)**.** *On the contact branch σ, $y = \tan(\theta/2)$ satisfies $\dot y^2 = P_\sigma(y) = \sum_{j=0}^4 A_j y^j$ with*

$$A_4 = \tfrac{E}{2} - 1 - \tfrac{\kappa_\sigma d_\sigma^2}{4}, \quad A_3 = A_1 = \sigma\kappa_\sigma d_\sigma, \quad A_2 = E - 1 - \kappa_\sigma\big(1 + \tfrac{d_\sigma^2}{2}\big), \quad A_0 = \tfrac{E}{2} - \tfrac{\kappa_\sigma d_\sigma^2}{4}. \tag{8}$$

Proof. With $\sin\theta = 2y/(1+y^2)$, $1-\cos\theta = 2y^2/(1+y^2)$ and $\dot\theta = 2\dot y/(1+y^2)$, the energy integral gives $\dot y^2 = \frac{1}{2}(1+y^2)^2(E - V_\sigma) = \frac{1}{2}\big[E(1+y^2)^2 - 2y^2(1+y^2) - \frac{\kappa_\sigma}{2}(d_\sigma y^2 - 2\sigma y + d_\sigma)^2\big]$; expanding gives (8). □

Fix $E \in (E_{g,\sigma}, E^*)$, let $r = \tan(\theta_\sigma(E)/2)$ correspond to the turning point on side σ, $p_j = P_\sigma^{(j)}(r)$, and

$$g_2 = \frac{p_2^2}{48} - \frac{p_1 p_3}{24}, \qquad g_3 = -\frac{p_2^3}{1728} + \frac{p_1 p_2 p_3}{576} - \frac{p_1^2 p_4}{384}, \tag{9}$$

the invariants of the quartic written through its Taylor coefficients at a root [12, §20.6].

Theorem 4.3 (Exact contact motion)**.** *(a) r is a simple root: $p_1 = -(1+r^2)V'(\theta_\sigma(E))$, so $\sigma p_1 < 0$. (b) Let τ_r be a time at which the orbit is at $\theta_\sigma(E)$, $y_{g,\sigma} = \tan(\theta_{g,\sigma}/2)$, $w_g = p_2/24 + p_1/(4(y_{g,\sigma} - r))$ and*

$$\Delta\tau_\sigma(E) = \int_{w_g}^{\infty} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}. \tag{10}$$

The integral converges and, for $|\tau - \tau_r| \le \Delta\tau_\sigma(E)$,

$$\theta(\tau) = 2\arctan\Big(r + \frac{p_1}{4\big[\wp(\tau - \tau_r; g_2, g_3) - p_2/24\big]}\Big). \tag{11}$$

(c) The contact time with spring σ per period is $2\Delta\tau_\sigma(E)$.

Proof. (a) $P_\sigma(y) = \frac{1}{2}(1+y^2)^2(E - V_\sigma(2\arctan y))$; differentiate at the root and use $d\theta/dy = 2/(1+y^2)$. (b) With $P_\sigma(r+x) = p_1 x + \frac{p_2}{2}x^2 + \frac{p_3}{6}x^3 + \frac{p_4}{24}x^4$ and $w = p_2/24 + p_1/(4(y-r))$, a direct computation gives $(dw/dy)^2 P_\sigma(y) = 4w^3 - g_2 w - g_3$. On the open contact arc $P_\sigma > 0$ and $p_1/(y-r) > 0$ by (a), so w is strictly monotone along the arc, tends to $+\infty$ at the turning point, and $\dot w^2 = 4w^3 - g_2 w - g_3 > 0$ on (w_g, ∞). Separation of variables gives $|\tau - \tau_r| = \int_{w(\tau)}^{\infty} ds/\sqrt{4s^3 - g_2 s - g_3}$, which converges at both ends. This integral is the inversion of $\wp$ on the real segment issuing from its pole [12, §20.22], and the cubic has no zero on (w_g, ∞); hence $w(\tau) = \wp(\tau - \tau_r)$, by evenness of $\wp$ on both sides of τ_r. Solving for y gives (11). (c) $w = w_g$ is the activation angle. □

Expressing $\wp$ through Jacobi functions (three real roots of the cubic, or one) shows that (10) is an incomplete elliptic integral of the first kind [2].

Theorem 4.4 (Exact period)**.** *For $0 < E < \min\{2, E^*\}$ and $k = \sqrt{E/2}$,*

$$T(E) = \sum_{\sigma=\pm1} T_\sigma(E), \qquad T_\sigma(E) = \begin{cases} 2K(k), & E \le E_{g,\sigma}, \\ 2F\big(\arcsin\sqrt{E_{g,\sigma}/E},\, k\big) + 2\Delta\tau_\sigma(E), & E > E_{g,\sigma}. \end{cases} \tag{12}$$

Proof. By reversibility, T is twice the time from θ_- to θ_+; split at $\theta = 0$ and apply Lemma 4.1 and Theorem 4.3(c). The pieces join with continuous $(\theta, \dot\theta)$ by uniqueness. □

5 Monotonicity in stiffness and the rigid-stop limit

For $E > E_{g,\sigma}$ write $\mathcal{T}_{c,\sigma}(E) = 2\Delta\tau_\sigma(E)$ and let $\Theta_\sigma(u;\kappa_\sigma)$ be the inverse of V_σ on the contact branch. As in Lemma 2.2,

$$\mathcal{T}_{c,\sigma}(E) = \sqrt{2}\int_{E_{g,\sigma}}^{E} \frac{du}{|V'(\Theta_\sigma(u;\kappa_\sigma))|\sqrt{E-u}}. \tag{13}$$

The change of variables is global on the branch because $|V'| \geq d_\sigma > 0$ there.

Theorem 5.1 (Monotonicity in stiffness)**.** *Fix $d_\pm$ and E. If $E \leq E_{g,\sigma}$, $T(E)$ does not depend on κ_σ. If $E > E_{g,\sigma}$, then $\mathcal{T}_{c,\sigma}(E)$, and hence $T(E)$, is strictly decreasing in κ_σ, on the range of κ_σ for which $E < E^*$.*

Proof. Take $\sigma = +1$ and write $\theta = \Theta_+(u;\kappa)$, $s = \sin\theta - d_+ > 0$, $c = \cos\theta > 0$. From $1 - \cos\theta + \kappa s^2/2 = u$, $\partial_\kappa\theta = -s^2/(2V')$, so

$$\partial_\kappa\big[V'(\Theta_+(u;\kappa))\big] = sc + V''\partial_\kappa\theta = \frac{s}{2V'}\big(2V'c - V''s\big),$$

with $V' = \sin\theta + \kappa sc$ and $V'' = c + \kappa(c^2 - s\sin\theta)$. A direct simplification gives

$$2V'c - V''s = c(\sin\theta + d_+) + \kappa s(c^2 + s\sin\theta) > 0.$$

Thus the integrand in (13) strictly decreases with κ for every $u \in (E_{g,+}, E]$, while (7) does not involve κ. □

Theorem 5.2 (Rigid-stop limit)**.** *Fix $E > E_{g,\sigma}$. As $\kappa_\sigma \to \infty$,*

$$\mathcal{T}_{c,\sigma}(E) = \frac{\pi}{c_\sigma\sqrt{\kappa_\sigma}} + o\big(\kappa_\sigma^{-1/2}\big). \tag{14}$$

In the symmetric case $d_\pm = d$, $\kappa_\pm = \kappa$, for $E > E_g$, $T(E) = T_\infty(E) + 2\pi/(c\sqrt{\kappa}) + o(\kappa^{-1/2})$, where $T_\infty(E) = 4F(\arcsin\sqrt{E_g/E}, k)$ is the period of the pendulum with perfectly elastic rigid stops at $\pm\theta_g$.

Proof. Take $\sigma = +1$, write κ, d, c_g for κ_+, d_+, c_+, $\theta = \Theta_+(u;\kappa)$, $s = \sin\theta - d$, and

$$\sqrt{\kappa}\,\mathcal{T}_{c,+} = \sqrt{2}\int_{E_g}^{E} \frac{h_\kappa(u)\,du}{\sqrt{E-u}}, \qquad h_\kappa(u) = \frac{\sqrt{\kappa}}{V'(\theta)}.$$

Pointwise limit. Since $u - E_g = (\cos\theta_g - \cos\theta) + \kappa s^2/2 \geq \kappa s^2/2$, $s \to 0$ and $\theta \to \theta_g$, whence $\kappa s^2/2 \to u - E_g$ and

$$h_\kappa(u) = \frac{\sqrt{\kappa}}{\sin\theta + \sqrt{\kappa}(\sqrt{\kappa}s)\cos\theta} \longrightarrow \frac{1}{c_g\sqrt{2(u-E_g)}}.$$

Domination. By the proof of Theorem 5.1, Θ_+ decreases in κ, so for $\kappa \geq \kappa_0$ the contact angles lie in $[\theta_g, \theta_M]$, $\theta_M = \Theta_+(E;\kappa_0) < \pi/2$. With $c_m = \cos\theta_M$, $L = \tan\theta_M$: $\cos\theta_g - \cos\theta \leq Ls$, so $\kappa(u-E_g) \leq L\kappa s + \frac{1}{2}\kappa^2 s^2$, while $V' \geq \max\{d, \kappa s c_m\}$. If $\kappa s \leq 1$, then $\kappa(u-E_g) \leq (L+\frac{1}{2})V'^2/d^2$; if $\kappa s > 1$, then $\kappa(u-E_g) \leq (L+\frac{1}{2})V'^2/c_m^2$. Hence $h_\kappa \leq M(u-E_g)^{-1/2}$, and $M[(u-E_g)(E-u)]^{-1/2}$ is integrable. By dominated convergence, $\sqrt{\kappa}\,\mathcal{T}_{c,+} \to c_g^{-1}\int_{E_g}^{E}[(u-E_g)(E-u)]^{-1/2}du = \pi/c_g$. □

Remark 5.3. $\pi/(c\sqrt{\kappa})$ is half the period of the linearised contact oscillation with tangent stiffness $V''(\theta_g^+) \approx \kappa c^2$. Numerically (Figure 5) the remainder in (14) is $O(\kappa^{-1})$. Both T_0 and T_∞ are monotone in E on the relevant ranges (increasing and decreasing, respectively), so the elastic system interpolates between two monotone period functions.

6 Near-grazing expansions and a critical period

Fix a side σ, write $d = d_\sigma$, $c = c_\sigma$, $\kappa = \kappa_\sigma$, $\theta_g = \theta_{g,\sigma}$, $E_g = E_{g,\sigma}$, $\delta = E - E_g$ and $a = c + \kappa c^2 = |V_\sigma''(\theta_g)|$.

Theorem 6.1 (Near-grazing expansions)**.** *For small $\delta > 0$,*

$$|\theta_\sigma(E) - \theta_g| = \frac{\delta}{d} - \frac{a}{2d^3}\delta^2 + O(\delta^3), \tag{15}$$

$$\mathcal{T}_{c,\sigma}(E) = \frac{2\sqrt{2}}{d}\,\delta^{1/2} - \frac{4\sqrt{2}\,a}{3d^3}\,\delta^{3/2} + O(\delta^{5/2}), \tag{16}$$

$$T(E) - T_0(E) = -\sum_{\sigma:\, E > E_{g,\sigma}} \frac{4\sqrt{2}\,\kappa_\sigma c_\sigma^2}{3d_\sigma^3}\,\delta_\sigma^{3/2} + O(\delta_\sigma^{5/2}). \tag{17}$$

In particular $T \in C^1(0, \min\{2, E^*\})$, T is analytic away from $E_{g,\pm}$, and $T''(E) \to -\infty$ as $E \downarrow E_{g,\sigma}$ whenever $\kappa_\sigma > 0$.

Proof. (15) is the reversion of $V_\sigma(\theta_g + x) - E_g = dx + \frac{a}{2}x^2 + O(x^3)$. (16) follows from Lemma 2.2 with $\gamma(0) = 1/d$, $\gamma'(0) = -a/d^3$ and $B_0 = 2$, $B_1 = 4/3$. (17) and the regularity statements are Theorem 2.3 with $J = \kappa_\sigma c_\sigma^2$ (Proposition 3.1); $T_0 = 4K(\sqrt{E/2})$ is analytic for $0 < E < 2$, and away from $E_{g,\pm}$ analyticity follows from (12). □

The leading terms of (15)–(16) are purely geometric; the stiffness enters at relative order δ, because the spring deformation vanishes at grazing.

Since $T_0'(E_g) > 0$ while $T'' \to -\infty$, the period just above grazing first increases and then may turn down. The next theorem shows that it does so for stiff stops, and locates the turning point.

Theorem 6.2 (Critical period)**.** *Fix $d_\pm$. Suppose either the configuration is symmetric ($d_\pm = d$, $\kappa_\pm = \kappa$; put $n = 2$), or only $\kappa_\sigma = \kappa$ varies and $E_{g,\sigma} < E_{g,-\sigma}$ (put $n = 1$). Let*

$$\delta_* = \frac{T_0'(E_g)^2\, d^6}{8\, n^2 \kappa^2 c^4}, \qquad T_0'(E) = \frac{E(k) - (1-k^2)K(k)}{k^2(1-k^2)} > 0,\ k^2 = \frac{E}{2}. \tag{18}$$

There are κ_0 and C such that for $\kappa \geq \kappa_0$:

1. *$T' > 0$ on $(0, E_g]$ and on $\big(E_g, E_g + (1 - C/\kappa)^2\delta_*\big)$, and $T' < 0$ on $\big(E_g + (1 + C/\kappa)^2\delta_*,\, E_g + 4\delta_*\big]$;*
2. *consequently T has a strict local maximum at some $E_{\max} = E_g + \delta_{\max}$ with $|\delta_{\max}/\delta_* - 1| \leq 3C/\kappa$;*
3. *$T(E_{\max}) - T(E_g) = \frac{1}{3}T_0'(E_g)\,\delta_*\big(1 + O(\kappa^{-1})\big)$.*

Proof. For $E \leq E_g$, $T = T_0$, which is increasing. For $0 < \delta \leq 4\delta_*$ (and κ_0 large enough that $E_g + 4\delta_*$ is below any other grazing energy), $T(E_g + \delta) = T_0(E_g + \delta) + nD(\delta)$, where, by (13) applied with stiffness κ and 0,

$$D(\delta) = \sqrt{2}\int_0^\delta \frac{\eta(v)\,dv}{\sqrt{\delta - v}}, \qquad \eta = \gamma_\kappa - \gamma_0, \quad \gamma_\kappa(v) = \frac{1}{V_\sigma'(\Theta_\sigma(E_g + v; \kappa))}.$$

Since $\eta(0) = 0$ and $\eta \in C^1$, integration by parts gives $D(\delta) = 2\sqrt{2}\int_0^\delta \eta'(v)(\delta - v)^{1/2}dv$ and therefore

$$D'(\delta) = \sqrt{2}\int_0^\delta \frac{\eta'(v)\,dv}{\sqrt{\delta - v}}. \tag{19}$$

Uniform estimate of η'. Take $\sigma = +1$. On the branch $V_\kappa' \geq \sin\theta \geq d$, so $V_\kappa(\theta_g + x) - E_g \geq dx$ and the contact angle satisfies $x = \Theta_+(E_g + v) - \theta_g \leq v/d \leq 4\delta_*/d = O(\kappa^{-2})$. With $s = \sin(\theta_g + x) - d = O(x)$,

$$V_\kappa' = \sin\theta + \kappa s\cos\theta = d + O(\kappa x) = d + O(\kappa^{-1}),$$
$$V_\kappa'' = \cos\theta + \kappa(\cos^2\theta - s\sin\theta) = a + O(\kappa x) = a + O(\kappa^{-1}),$$

uniformly for $0 \leq v \leq 4\delta_*$. Hence $\gamma_\kappa' = -V_\kappa''/V_\kappa'^3 = -a/d^3 + O(1)$ and $\gamma_0' = -c/d^3 + O(\kappa^{-2})$, so

$$\eta'(v) = -\frac{\kappa c^2}{d^3} + R(v), \qquad |R(v)| \leq C_1 \quad (0 \leq v \leq 4\delta_*).$$

Derivative of the period. Inserting into (19) and using $T_0'(E_g + \delta) = T_0'(E_g) + O(\delta)$,

$$T'(E_g + \delta) = T_0'(E_g) - q\,\delta^{1/2}\big(1 + \varepsilon(\delta)\big) + O(\kappa^{-2}), \qquad q = \frac{2\sqrt{2}\,n\kappa c^2}{d^3}, \quad |\varepsilon(\delta)| \leq \frac{C_2}{\kappa}.$$

By (18), $q\,\delta_*^{1/2} = T_0'(E_g)$. Writing $\delta = t^2\delta_*$ with $0 < t \leq 2$,

$$T'(E_g + t^2\delta_*) = T_0'(E_g)\big[1 - t(1 + \varepsilon)\big] + O(\kappa^{-2}),$$

which is positive for $t < 1 - C/\kappa$ and negative for $1 + C/\kappa < t \leq 2$, for suitable C and $\kappa \geq \kappa_0$. This is item 1.

For item 2, T is analytic on $(E_g, E_g + 4\delta_*]$ (Theorem 6.1), so its critical points there are isolated. T increases up to $E_g + (1 - C/\kappa)^2\delta_*$ and decreases after $E_g + (1 + C/\kappa)^2\delta_*$, so its maximum over $[E_g, E_g + 4\delta_*]$ is attained at an interior point $E_{\max}$ of the window between them, which is a strict local maximum. Since $(1 \pm C/\kappa)^2 = 1 \pm 2C/\kappa + O(\kappa^{-2})$, the bound on $\delta_{\max}$ follows.

For item 3, integrate: $T(E_g + \delta) - T(E_g) = T_0'(E_g)\delta - \frac{2}{3}q\,\delta^{3/2}(1 + O(\kappa^{-1})) + O(\kappa^{-2}\delta)$. At $\delta = \delta_{\max} = \delta_*(1 + O(\kappa^{-1}))$, $q\,\delta_{\max}^{3/2} = T_0'(E_g)\,\delta_*(1 + O(\kappa^{-1}))$, which gives $\frac{1}{3}T_0'(E_g)\delta_*(1 + O(\kappa^{-1}))$. □

Remark 6.3. (i) The critical period occurs at a distance $O(\kappa^{-2})$ above grazing and has height $O(\kappa^{-2})$; it is therefore invisible at the scale of Figure 1(a) but is clearly resolved in Figure 1(b). (ii) The argument uses only Theorem 2.3, the bound $V' \geq d$ on the branch, and the fact that the third derivative of V_σ at θ_g is $O(\kappa)$. It therefore extends to any family satisfying Hypothesis 2.1 with $J \to +\infty$ and $W_R''' = O(J)$ near x_g, provided $\widetilde{T}'(h_g) > 0$. (iii) For a softening switch ($J < 0$), Theorem 2.3 gives $T'' \to +\infty$ instead, and no critical period is created in this way when $\widetilde{T}'(h_g) > 0$.

7 Numerical confirmation

Turning points are computed by Brent's method on the cancellation-free form $V(\theta_g + x) - E_g = 2\sin(\theta_g + \frac{x}{2})\sin\frac{x}{2} + 2\kappa\cos^2(\theta_g + \frac{x}{2})\sin^2\frac{x}{2}$. Contact integrals use $\theta = \theta_\sigma(E) - x_* s^2$, which removes the endpoint singularity, together with the identity $V(b) - V(\theta) = 2\sin\frac{b-\theta}{2}\big[\sin\frac{b+\theta}{2} + \frac{\kappa}{2}\cos\frac{b+\theta}{2}(\sin b + \sin\theta - 2d)\big]$, which avoids subtractive cancellation; adaptive Gauss–Kronrod quadrature is used with relative tolerance 10^{-13}. Free-flight times use (7). As an independent check, (5) is integrated by an eighth-order Dormand–Prince method (relative tolerance 10^{-12}) and the period is located by event detection. The scripts are supplied as supplementary material.

General theorem. For the Duffing potential $W = x^2/2 + x^4/4 + \frac{J}{2}[x - 0.6]_+^2$ ($d = 0.816$), the ratio of the computed $T - \widetilde{T}$ to the prediction (3) is 0.999987, 0.998711, 0.987244 for $\delta = 10^{-6}, 10^{-4}, 10^{-3}$ when $J = 3$, and 0.999993, 0.999341, 0.993435 when $J = -0.5$ (softening switch).

Exact solution. Over 2000 random parameter samples the quartic (8) reproduces $\frac{1}{2}(1+y^2)^2(E-V_\sigma)$ to relative accuracy 2×10^{-14}, and the identity $(dw/dy)^2P_\sigma = 4w^3-g_2w-g_3$ holds to 6×10^{-13}. For $(E,d,\kappa)=(0.25,0.5,8)$, (10) gives 0.4707343065897 against 0.4707343065898 by direct quadrature, and (11) agrees with time integration on the whole contact arc to 3×10^{-14}.

Periods. Table 1 lists exact periods for $d_\pm = 0.5$ ($E_g = 1-\sqrt{3}/2 \approx 0.1339745962$). Each row decreases strictly in κ (Theorem 5.1), and below E_g the values equal $4K(\sqrt{E/2})$. For $d_+ = 0.35$, $d_- = 0.55$, $\kappa_+ = 8$, $\kappa_- = 3$ ($E_{g,+} \approx 0.0632503$, $E_{g,-} \approx 0.1648353$), the energies 0.05, 0.10, 0.25 give no, one-sided and two-sided contact (Figure 3); their periods 6.3230172231, 5.8491942069, 5.0815756188 agree with time integration to 2×10^{-11}.

Table 1: Exact periods for $d_\pm = 0.5$, $\kappa_\pm = \kappa$ (Theorem 4.4); last column: largest difference from direct time integration in the row.

E	$\kappa=0$	$\kappa=2$	$\kappa=5$	$\kappa=10$	max. diff.
0.10	6.364013815	6.364013815	6.364013815	6.364013815	1.4×10^{-13}
0.14	6.397692979	6.378492426	6.352322681	6.314378196	3.3×10^{-12}
0.20	6.449765395	6.099577672	5.781068301	5.472794334	8.5×10^{-12}
0.35	6.588936439	5.760390974	5.138388257	4.624796482	8.6×10^{-10}
0.60	6.855557793	5.715887688	4.834711284	4.150392775	2.1×10^{-11}
0.85	7.181066784	5.947631295	4.830531673	3.988166402	2.5×10^{-11}

Near-grazing expansions. For $d = 0.5$, $\kappa = 8$ (symmetric), Table 2 and Figure 4 compare the computations with Theorem 6.1. Leading-term errors decay like δ, two-term errors like δ^2, and $(T-T_0)$ divided by (17) tends to 1 linearly in δ.

Table 2: Near-grazing expansions, $d = 0.5$, $\kappa = 8$ (symmetric). $\mathcal{T}_c$ is the total contact time with both springs.

δ	pen./lead	pen. 2-term err.	$\mathcal{T}_c$/lead	$\mathcal{T}_c$ 2-term err.	$(T-T_0)/(17)$
10^{-7}	0.99999863	3.9×10^{-12}	0.99999817	6.3×10^{-12}	0.999996
10^{-6}	0.99998627	3.9×10^{-10}	0.99998169	6.3×10^{-10}	0.999962
10^{-5}	0.99986272	3.9×10^{-8}	0.99981697	6.3×10^{-8}	0.999615
10^{-4}	0.99863070	3.9×10^{-6}	0.99817530	6.3×10^{-6}	0.996166
10^{-3}	0.98664621	3.8×10^{-4}	0.98229286	6.1×10^{-4}	0.963022
10^{-2}	0.89182089	3.3×10^{-2}	0.86181835	5.2×10^{-2}	0.725628

Critical period. For $d = 0.5$ (symmetric, $n = 2$), $T_0'(E_g) = 0.8490193523$. Table 3 compares (18) with the maximiser of T found by bounded scalar optimisation. The relative discrepancy, 0.9%, 0.23%, 0.10% for $\kappa = 2, 5, 10$, decays like κ^{-1}, and the height agrees with $\frac{1}{3}T_0'(E_g)\delta_*$ to the same order. The prediction is already accurate at $\kappa = 2$, well outside the asymptotic regime required by the proof.

Rigid-stop limit. For $d = 0.5$, $T_\infty(0.2) = 3.8853668087$, $T_\infty(0.5) = 2.2024608173$, $T_\infty(0.85) = 1.6522670425$, and $\sqrt{\kappa}\,(T-T_\infty)$ at $\kappa = 10^6$ equals 7.2490, 7.2547, 7.2567, against $2\pi/c = 7.2552$ (Figure 5).

Table 3: The critical period above grazing, $d = 0.5$, symmetric.

κ	δ_* (18)	$\delta_{\max}$ numerical	$T(E_{\max}) - T(E_g)$	$\frac{1}{3}T_0'(E_g)\delta_*$
2	1.5643×10^{-4}	1.5786×10^{-4}	4.451×10^{-5}	4.427×10^{-5}
5	2.5029×10^{-5}	2.5087×10^{-5}	7.093×10^{-6}	7.083×10^{-6}
10	6.2572×10^{-6}	6.2632×10^{-6}	1.772×10^{-6}	1.771×10^{-6}

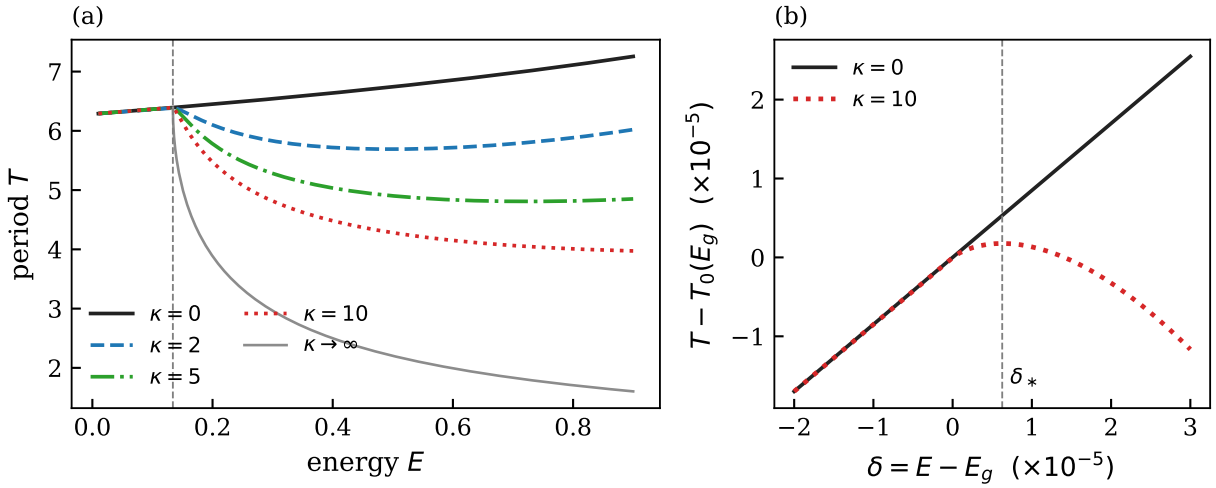


Figure 1: (a) Period function for $d_\pm = 0.5$ and several stiffnesses; the dashed line is E_g and the grey curve the rigid-stop limit T_∞. (b) Magnification at grazing for $\kappa = 10$: C^1 matching with the free pendulum and the critical period at δ_* given by (18).

8 Concluding remarks

At a switching level where a conservative system acquires extra stiffness through a C^1 contact, the period function has a universal $\delta^{3/2}$ singularity whose coefficient depends only on the local slope and the stiffness jump. It is C^1 but not C^2, and the sign of the infinite curvature is that of $-J$. For a pendulum with two unilateral elastic stops, this local result combines with an exact elliptic solution and global estimates into a complete picture of the period function. It is invariant below the stiffness-independent grazing energies, strictly decreasing in each stiffness above them, and convergent to the rigid elastic-impact period with an explicit $\kappa^{-1/2}$ correction. For stiff stops it has a critical period, which we located with relative error $O(\kappa^{-1})$.

Several questions remain open. One is the number of critical periods for moderate stiffness and asymmetric clearances. Another is the behaviour of the period function on the rotational orbits outside the principal cell. A third is the persistence of the critical period under weak damping and forcing, where it is expected to produce multiple resonant amplitudes near the onset of contact. The general Theorem 2.3 applies verbatim to other oscillators with elastic stops, for example Duffing and Helmholtz-type potentials.

Declarations

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Data and code availability. All numerical values and figures are generated from the formulas in this article by the Python scripts `model.py`, `verify.py`, `duffing_check.py` and

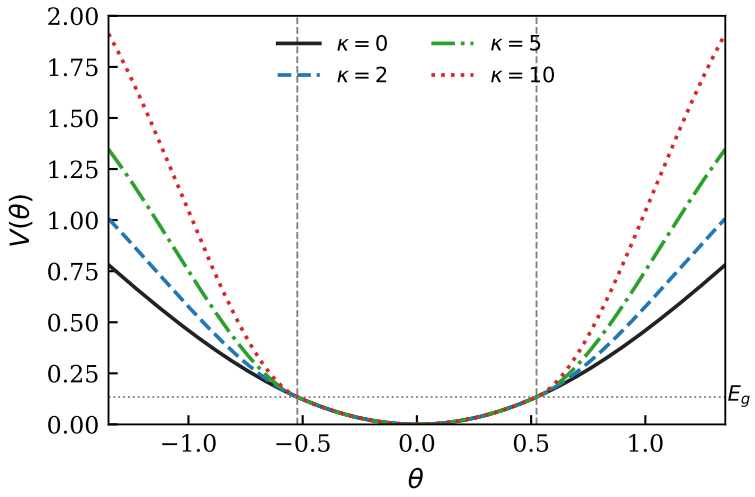


Figure 2: Potential for $d_{\pm} = 0.5$; dashed: activation angles; dotted: E_g. All stiffnesses coincide inside the clearance window.

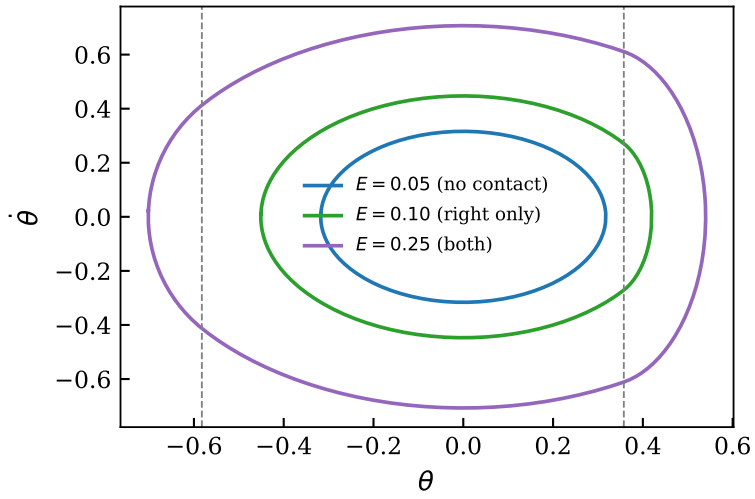


Figure 3: Energy contours for $d_+ = 0.35$, $d_- = 0.55$, $\kappa_+ = 8$, $\kappa_- = 3$: no contact, right-only contact and two-sided contact.

`make_figs.py`, supplied as supplementary material.

Author contributions. A. H. Salas: conceptualization, methodology, formal analysis, software, validation, writing.

Use of AI tools. An AI assistant (Claude, Anthropic) was used to help with language editing, restructuring of the manuscript and preparation of the verification scripts. All mathematical statements, proofs and numerical results were checked by the author, who takes full responsibility for the content of this article.

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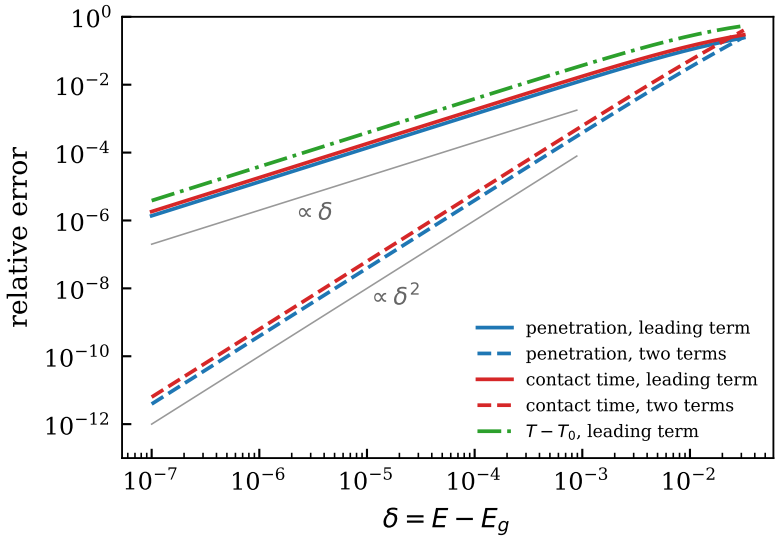


Figure 4: Relative errors of the expansions of Theorem 6.1, $d = 0.5$, $\kappa = 8$. Solid: leading terms (slope 1); dashed: two-term formulas (slope 2); dash-dotted: (17).

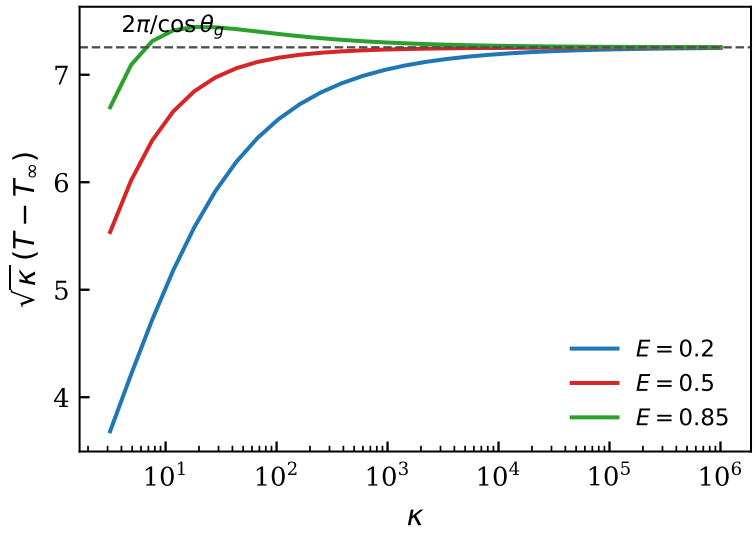


Figure 5: Convergence to the rigid-stop limit of Theorem 5.2 for $d = 0.5$.

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