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Analysis of Cracked Micro-Polar Planes

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Abstract

The general three-dimensional quasi-static formulation of micro-polar solids is streamlined for the two-dimensional case, resulting in the decoupling of the governing equations for in-plane deformation from those of the anti-plane deformation. The solutions to the governing equations for screw and edge dislocations-disclinations in an infinite plane, intended for analyzing cracks via the distributed dislocation technique, are derived. The resultant stress and couple-stress tensors exhibit a Cauchy singular at the origin of the displacement discontinuity. An asymptotic analysis near the crack tip shows that while stress components are finite, the components of the couple-stress tensor demonstrate a square-root singularity. Utilizing the dislocation-disclination solutions, systems of Cauchy singular integral equations are derived to analyze an infinite plane weakened by a finite crack sustaining anti-plane or in-plane deformation. The systems contain equations with multiple Cauchy singular terms; thus, they are more general than those previously encountered in linear fracture problems based on local elasticity theory. To solve systems of integral equations and determine the densities of dislocations and disclinations on the crack surface, a generalization of an existing numerical procedure is required. For a cracked plane under far-field in-plane traction, crack deformation is such that mixed-mode I-II occurs. In addition, contrary to the local elasticity theory, crack opening occurs while far-field traction is compressive and closes when it is tensile. The components of stress and couple-stress tensors in a cracked plane for two planes comprised of different materials are determined, and the effects of material micro-polar constants are studied.

Keywords: Micro-polar cracked plane; Multiple Cauchy singular terms; Bounded stress tensor; Mixed mode deformation; Crack closing

Introduction

A principal axiom of classical continuum mechanics states that particles can sustain only forces. However, the translational degrees of freedom of a particle alone may not be adequate to describe the behavior of structures containing regions with severe stress gradients or composed of materials with significant microstructural effects, such as cellular materials. In the continuum micro-polar theory, a particle's motion consists of translation augmented by rotation, i.e., particles possess six degrees of freedom. Therefore, force and moment are defined independently, resulting in a more involved mathematical formulation. The foundations of the micro-polar theory of elasticity were elaborated on by several investigators, e.g., Eringen

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(1966), Ieşan (1969), Nowacki (1970), Dyszlewicz (2004), and Eremeyev et al. (2012). Nonetheless, among the myriad articles on fracture mechanics problems employing different non-local elasticity theories, only a few have addressed micro-polar materials. Atkinson and Leppington (1977) used the Wiener-Hopf technique to derive solutions for a semi-infinite crack in an infinite plane employing couple-stress and micro-polar theories. Paul and Sridharan (1981) analyzed a micro-polar plane weakened by a finite crack. The analyses of the cracked Cosserat-type and micro-polar planes were carried out, respectively, by Garajeu and Soós (2003) and Diegele et al. (2004). Shmoylova et al. (2007) utilized the boundary element method to derive a semi-analytic solution for stress concentration around a crack in a micro-polar plane subject to in-plane loading. In all the foregoing articles, plane strain conditions prevailed. Cracked micro-polar planes under anti-plane and in-plane deformations were studied by Li and Lee (2008, 2009). Yu et al. (2024) investigated stress and couple-stress fields at the crack tip in a micro-polar plane under anti-plane deformation. Employing the extended finite element method, Tuna et al. (2025) determined the basic mixed-mode parameters of fracture mechanics. In all the above-mentioned investigations, stress and couple-stress fields were considered square-root singular at a crack tip. Therefore, the behavior of the stress tensor in the micro-polar plane resembled that of the local elasticity theory.

In this article, we streamlined the three-dimensional formulation of the micro-polar elasticity theory to analyze two-dimensional problems. The simplification reveals that, akin to the local elasticity theory, in-plane and anti-plane deformations are decoupled and should be treated separately. Then, the solutions to a screw and an edge dislocation-disclination (hereinafter, referred to as dislocation for brevity) in a micro-polar plane are derived. An asymptotic analysis near the dislocation shows that both stress and couple-stress tensors are Cauchy singular. Furthermore, an asymptotic analysis of stress and couple-stress tensors in the vicinity of a crack tip results in bounded and square-root singular stress and couple-stress tensors, respectively. The boundedness of the stress tensor differs from that reported in the literature. However, in a special case, we may set a material micro-polar constant equal to zero in the asymptotic analysis, arriving at outcomes consistent with previous studies. The foregoing findings are integrated into the distributed dislocation technique to construct systems of Cauchy singular integral equations for a crack under anti-plane and in-plane deformations. The systems contain equations with multiple Cauchy singular terms, necessitating the generalization of a well-established numerical procedure. The numerical values of the densities of dislocations for two homogeneous cracked planes composed of different micro-polar materials subjected to far-field anti-plane and in-plane tractions are determined. These values are, then, utilized to compute the components of stress and couple stress tensors in the vicinity of a crack tip. The effects of the planes' material constants on the stress fields near a crack tip are studied.

Micro-Polar formulation

In micro-polar theory of elasticity, the strain tensor $\boldsymbol{E}$ and the micro-polar strain tensor $\boldsymbol{\Gamma}$ in terms of the displacement $U = (u_1, u_2, u_3)$ and micro-rotation $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ vectors are, Eringen (1966)

$$\boldsymbol{E} = \nabla \otimes U - \boldsymbol{\varepsilon} \cdot \Phi,$$
$$\boldsymbol{\Gamma} = \boldsymbol{\varepsilon}.(\nabla \otimes \Phi)^T \tag{1}$$

where, $\boldsymbol{\varepsilon}$ is the Levi-Civita permutation tensor, ∇ signifies the gradient differential operator, $\otimes$ represents the tensor product, and superscript T stands for the transpose of a matrix. In Eqs (1) and in the following, second and higher rank tensors are shown with bold-face letters. The equilibrium equations, ignoring body effects, are

$$\nabla \cdot \boldsymbol{\sigma} = 0,$$
$$\nabla \cdot \boldsymbol{m} + \boldsymbol{\varepsilon} : \boldsymbol{\sigma} = 0. \tag{2}$$

In the above equations, $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are the stress and couple-stress tensors, respectively. The constitutive equations for isotropic materials in the linear micro-polar elasticity read

$$\boldsymbol{\sigma} = \lambda(\nabla.U)\boldsymbol{I} + \mu(\nabla \otimes U)^T + (\mu + \kappa)\nabla \otimes U - \kappa\boldsymbol{\varepsilon} \cdot \Phi,$$
$$\boldsymbol{m} = \alpha(\nabla \cdot \Phi)\boldsymbol{I} + \beta(\nabla \otimes \Phi)^T + \gamma\nabla \otimes \Phi. \tag{3}$$

In Eqs (3), $\boldsymbol{I}$ is the unit matrix, λ, μ are the Lame and κ, α, β, γ are the micro-polar material constants. From the above equations we arrive at equilibrium equations in terms of displacement and micro-rotation vectors U and Φ, respectively, as

$$(\lambda + \mu)\nabla(\nabla \cdot U) + (\mu + \kappa)\nabla^2 U + \kappa(\nabla \times \Phi) = 0,$$
$$(\alpha + \beta)\nabla(\nabla \cdot \Phi) + \gamma\nabla^2\Phi + \kappa(\nabla \times U) - 2\kappa\Phi = 0. \tag{4}$$

The strain energy in a body with volume V reads as

$$W = \frac{1}{2}\int_V (\boldsymbol{\sigma} : \boldsymbol{E} + \boldsymbol{m} : \boldsymbol{\nabla\Phi})dV \tag{5}$$

The geometric and natural boundary conditions, in micro-polar elasticity, are

geometric	$U =$ known	natural	$t^n = n \cdot \boldsymbol{\sigma} =$ known
geometric	$\Phi =$ known	natural	$M^n = n \cdot \boldsymbol{m} =$ known

(6)

In Eqs (6), n is the unit outward vector perpendicular to the surface of the body, and t^n and M^n are, respectively, traction and micro-traction vectors. From Eqs (3), the natural boundary conditions (6) become

$$t^n = \lambda(\nabla.U)n + (2\mu + \kappa)(n.\nabla)U + \mu n \times \nabla \times U + \kappa n \times \Phi,$$
$$M^n = \alpha(\nabla.\Phi)n + (\beta + \gamma)(n.\nabla)\Phi + \gamma n \times \nabla \times \Phi. \tag{7}$$

We consider a three-dimensional micro-polar body infinitely extended in the z-direction of the Cartesian coordinates. The body is under external traction, which is independent of the variable z. Analogous to the plane-strain situations in local elasticity $\partial/\partial z = 0$, leading to displacement and micro-rotation vectors as

$$U = [u(x,y), v(x,y), w(x,y)], \qquad \Phi = \left[\varphi_x(x,y), \varphi_y(x,y), \varphi_z(x,y)\right]. \tag{8}$$

In view of Eqs (8), equilibrium equations (4) lead to two decoupled systems of equations involving $[u(x,y), v(x,y), \varphi_z(x,y)]$ for in-plane and $\left[w(x,y), \varphi_x(x,y), \varphi_y(x,y)\right]$ for anti-plane deformations of a micro-polar plane. From Eqs (4), for the in-plane deformation we have

$$\begin{aligned}
&(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+(\mu+\kappa)\nabla^2 u+\kappa\frac{\partial \varphi_z}{\partial y}=0,\\
&(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+(\mu+\kappa)\nabla^2 v-\kappa\frac{\partial \varphi_z}{\partial x}=0,\\
&\gamma\nabla^2\varphi_z+\kappa\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-2\kappa\varphi_z=0,
\end{aligned} \tag{9}$$

where, ∇^2 is the Laplace operator, in the Cartesian coordinates. For anti-plane deformation the governing equations read

$$\begin{aligned}
&(\mu+\kappa)\nabla^2 w+\kappa\left(\frac{\partial \varphi_y}{\partial x}-\frac{\partial \varphi_x}{\partial y}\right)=0,\\
&(\alpha+\beta+\gamma)\frac{\partial}{\partial x}\left(\frac{\partial \varphi_x}{\partial x}+\frac{\partial \varphi_y}{\partial y}\right)+\gamma\frac{\partial}{\partial y}\left(\frac{\partial \varphi_x}{\partial y}-\frac{\partial \varphi_y}{\partial x}\right)+\kappa\frac{\partial w}{\partial y}-2\kappa\varphi_x=0,\\
&(\alpha+\beta+\gamma)\frac{\partial}{\partial y}\left(\frac{\partial \varphi_x}{\partial x}+\frac{\partial \varphi_y}{\partial y}\right)+\gamma\frac{\partial}{\partial x}\left(\frac{\partial \varphi_y}{\partial x}-\frac{\partial \varphi_x}{\partial y}\right)-\kappa\frac{\partial w}{\partial x}-2\kappa\varphi_y=0.
\end{aligned} \tag{10}$$

Screw dislocation in micro-polar planes

We consider a screw dislocation situated at the origin of the Cartesian coordinate system, wherein on the half-line $x>0$, displacement and rotation vectors are discontinuous, $w(x,0^+)\neq w(x,0^-)$, $\varphi_i(x,0^+)\neq\varphi_i(x,0^+)$, $i\in\{x,y\}$. For the dislocation traction and micro-traction vectors t^n and M^n, respectively, are continuous. In general, screw dislocations may be represented as

$$\begin{aligned}
&\frac{\partial w(x,0^+)}{\partial x}-\frac{\partial w(x,0^-)}{\partial x}=B_z\delta(x),\\
&\frac{\partial \varphi_x(x,0^+)}{\partial x}-\frac{\partial \varphi_x(x,0^-)}{\partial x}=D_x\delta(x),\\
&\frac{\partial \varphi_y(x,0^+)}{\partial x}-\frac{\partial \varphi_y(x,0^-)}{\partial x}=D_y\delta(x),\\
&\sigma_{yz}(x,0^+)=\sigma_{yz}(x,0^-),\\
&m_{yi}(x,0^+)=m_{yi}(x,0^-), \qquad i\in\{x,y\}
\end{aligned} \tag{11}$$

where, $\delta(.)$ designates the Dirac delta function, B_z and $D_i, i\in\{x,y\}$ are the components of the dislocation Burgers vector in z-direction and disclination vector in the i_{th}-direction. Furthermore, making use of Eqs (3), and (6) the continuity of traction implies that

$$(\mu+\kappa)\frac{\partial w}{\partial y}(x,0^+)-\kappa\varphi_x(x,0^+)=(\mu+\kappa)\frac{\partial w}{\partial y}(x,0^-)-\kappa\varphi_x(x,0^-),$$
$$\beta\frac{\partial \varphi_y}{\partial x}(x,0^+)+\gamma\frac{\partial \varphi_x}{\partial y}(x,0^+)=\beta\frac{\partial \varphi_y}{\partial x}(x,0^-)+\gamma\frac{\partial \varphi_x}{\partial y}(x,0^-),$$
$$\alpha\frac{\partial \varphi_x}{\partial x}(x,0^+)+(\alpha+\beta+\gamma)\frac{\partial \varphi_y}{\partial y}(x,0^+)=\alpha\frac{\partial \varphi_x}{\partial x}(x,0^-)+(\alpha+\beta+\gamma)\frac{\partial \varphi_y}{\partial y}(x,0^-). \quad (12)$$

We take the complex Fourier transform, $\mathcal{F}$, over variable x of Eqs (10) and obtain

$$(\mu+\kappa)\frac{d^2W}{dy^2}-\kappa\frac{d\Phi_x}{dy}-\omega^2(\mu+\kappa)W-i\omega\kappa\Phi_y=0,$$
$$\gamma\frac{d^2\Phi_x}{dy^2}-i\omega(\alpha+\beta)\frac{d\Phi_y}{dy}+\kappa\frac{dW}{dy}-[(\alpha+\beta+\gamma)\omega^2+2\kappa]\Phi_x=0,$$
$$(\alpha+\beta+\gamma)\frac{d^2\Phi_y}{dy^2}-i\omega(\alpha+\beta)\frac{d\Phi_x}{dy}+i\omega\kappa W-(\gamma\omega^2+2\kappa)\Phi_y=0, \quad (13)$$

where, $i=\sqrt{-1}$. In Eqs (13), $[W,\Phi_x,\Phi_y]=\mathcal{F}\{[w,\varphi_x,\varphi_y];x\to\omega\}$. The Fourier transformations of Eqs (11) and (12) are

$$W(\omega,0^+)-W(\omega,0^-)=\frac{iB_z}{\omega},$$
$$\Phi_x(\omega,0^+)-\Phi_x(\omega,0^-)=\frac{iD_x}{\omega},$$
$$\Phi_y(\omega,0^+)-\Phi_y(\omega,0^-)=\frac{iD_y}{\omega},$$
$$(\mu+\kappa)\frac{dW}{dy}(\omega,0^+)-\kappa\Phi_x(\omega,0^+)=(\mu+\kappa)\frac{dW}{dy}(\omega,0^-)-\kappa\Phi_x(\omega,0^-),$$
$$\gamma\frac{d\Phi_x}{dy}(\omega,0^+)-i\omega\beta\Phi_y(\omega,0^+)=\gamma\frac{d\Phi_x}{dy}(\omega,0^-)-i\omega\beta\Phi_y(\omega,0^-),$$
$$(\alpha+\beta+\gamma)\frac{d\Phi_y}{dy}(\omega,0^+)-i\omega\alpha\Phi_x(\omega,0^+)=(\alpha+\beta+\gamma)\frac{d\Phi_y}{dy}(\omega,0^-)-i\omega\alpha\Phi_x(\omega,0^-). \quad (14)$$

The solution to Eqs (13) subject to conditions (14) is straightforward

$$W(\omega,y)=\frac{i}{2}B_z\,\mathrm{sgn}(y)\frac{1}{\omega}e^{-|\omega y|}+\frac{i}{2}D_x\left\{\frac{\beta+\gamma}{2\mu+\kappa}\mathrm{sgn}(\omega)e^{-|\omega y|}-\left[\frac{\omega(\beta+\gamma)}{s(2\mu+\kappa)}+\frac{\kappa}{\omega s(\mu+\kappa)}\right]e^{-s|y|}\right\}$$
$$+\frac{\beta+\gamma}{2(2\mu+\kappa)}D_y\,\mathrm{sgn}(y)\left(e^{-|\omega y|}-e^{-s|y|}\right),$$
$$\Phi_x(\omega,y)=-\frac{i}{4}B_z\,\mathrm{sgn}(\omega)\left[e^{-|\omega y|}-\frac{|\omega|}{t}e^{-t|y|}\right]$$
$$-\frac{i}{4}D_x\,\mathrm{sgn}(y)\left\{\frac{\beta+\gamma}{2\mu+\kappa}\omega e^{-|\omega y|}-2\left[\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\omega+\frac{1}{\omega}\right]e^{-s|y|}+\frac{\beta+\gamma}{\kappa}\omega e^{-t|y|}\right\}$$
$$-\frac{1}{4}D_y\left\{\frac{\beta+\gamma}{2\mu+\kappa}|\omega|e^{-|\omega y|}-\frac{2(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}se^{-s|y|}+\frac{1}{t}\left[\frac{(\beta+\gamma)\omega^2}{\kappa}+2\right]e^{-t|y|}\right\}, \quad (15)$$

$$\Phi_y(\omega,y) = -\frac{1}{4}B_z\,\mathrm{sgn}(y)\left(e^{-|\omega y|}-e^{-t|y|}\right)$$
$$-\frac{1}{4}D_x\left\{\frac{\beta+\gamma}{2\mu+\kappa}|\omega|e^{-|\omega y|}-\frac{2}{s}\left[\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\omega^2+1\right]e^{-s|y|}+\frac{\beta+\gamma}{\kappa}te^{-t|y|}\right\}$$
$$+\frac{i}{4}D_y\,\mathrm{sgn}(y)\left\{\frac{\beta+\gamma}{2\mu+\kappa}\omega e^{-|\omega y|}-\frac{2(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\omega e^{-s|y|}+\left[\frac{2}{\omega}+\frac{\beta+\gamma}{\kappa}\omega\right]e^{-t|y|}\right\}.$$

In Eqs (15), $s=\sqrt{\omega^2+\kappa_1^2}$, $t=\sqrt{\omega^2+\kappa_2^2}$, $\kappa_1=\sqrt{\kappa\,(2\mu+\kappa)/\gamma(\mu+\kappa)}$, $\kappa_2=\sqrt{2\kappa/(\alpha+\beta+\gamma)}$, and $\mathrm{sgn}(\cdot)$ signifies the sign function. Taking the inverse Fourier transformation of Eqs (15) results in

$$w(x,y)=\frac{B_z}{2\pi}\mathrm{sgn}(y)\int_0^\infty\frac{1}{\omega}e^{-\omega|y|}\sin(\omega x)\,d\omega$$
$$+\frac{(\beta+\gamma)D_x}{2\pi(2\mu+\kappa)}\int_0^\infty\left\{e^{-\omega|y|}-\left[\omega+\frac{\kappa(2\mu+\kappa)}{(\mu+\kappa)(\beta+\gamma)}\frac{1}{\omega}\right]\frac{1}{s}e^{-s|y|}\right\}\sin(\omega x)\,d\omega$$
$$+\frac{(\beta+\gamma)D_y}{2\pi(2\mu+\kappa)}\,\mathrm{sgn}(y)\int_0^\infty\left(e^{-\omega|y|}-e^{-s|y|}\right)\cos(\omega x)\,d\omega,$$

$$\varphi_x(x,y)=-\frac{B_z}{4\pi}\int_0^\infty\left[e^{-\omega|y|}-\frac{\omega}{t}e^{-t|y|}\right]\sin(\omega x)\,d\omega$$
$$-\frac{(\beta+\gamma)D_x}{4\pi(2\mu+\kappa)}\,\mathrm{sgn}(y)\int_0^\infty\left\{\omega e^{-\omega|y|}-2\left[\frac{\mu+\kappa}{\kappa}\omega+\frac{2\mu+\kappa}{\beta+\gamma}\frac{1}{\omega}\right]e^{-s|y|}\right.$$
$$\left.+\frac{2\mu+\kappa}{\kappa}\omega e^{-t|y|}\right\}\sin(\omega x)\,d\omega$$
$$-\frac{(\beta+\gamma)D_y}{4\pi(2\mu+\kappa)}\int_0^\infty\left\{\omega e^{-\omega|y|}-\frac{2(\mu+\kappa)}{\kappa}se^{-s|y|}\right.$$
$$\left.+\frac{2\mu+\kappa}{k}\left[\omega^2+\frac{2\kappa}{\beta+\gamma}\right]\frac{1}{t}e^{-t|y|}\right\}\cos(\omega x)\,d\omega,$$

$$\varphi_y(x,y)=-\frac{B_z}{4\pi}\,\mathrm{sgn}(y)\int_0^\infty\left(e^{-\omega|y|}-e^{-t|y|}\right)\cos(\omega x)\,d\omega$$
$$-\frac{(\beta+\gamma)D_x}{4\pi(2\mu+\kappa)}\int_0^\infty\left\{\omega e^{-\omega|y|}-2\left[\frac{\mu+\kappa}{\kappa}\omega^2+\frac{2\mu+\kappa}{\beta+\gamma}\right]\frac{1}{s}e^{-s|y|}\right.$$
$$\left.+\frac{2\mu+\kappa}{\kappa}te^{-t|y|}\right\}\cos(\omega x)\,d\omega$$
$$+\frac{(\beta+\gamma)D_y}{4\pi(2\mu+\kappa)}\,\mathrm{sgn}(y)\int_0^\infty\left\{\omega e^{-\omega|y|}-\frac{2(\mu+\kappa)}{\kappa}\omega e^{-s|y|}\right.$$
$$\left.+\left(\frac{2\mu+\kappa}{\beta+\gamma}\right)\left[\frac{2}{\omega}+\frac{\beta+\gamma}{\kappa}\omega\right]e^{-t|y|}\right\}\sin(\omega x)\,d\omega. \qquad (16)$$

To determine stress and couple-stress fields the spatial derivatives of displacement together with the components of the rotation vector are needed. However, in Eqs (16), we observe that the definite integrals are divergent and may not be evaluated numerically. To change the definite integrals to convergent ones, consider, e.g., the function $f(\omega)$, integrable in a finite interval $\Omega>\omega>0$, such that $\lim_{\omega\to\infty}f(\omega)=M(\omega)\neq 0$. Then, we may write

$$\int_0^\infty\{\sin(\omega x),\cos(\omega x)\}f(\omega)d\omega$$
$$=\int_0^\infty\{\sin(\omega x),\cos(\omega x)\}M(\omega)d\omega$$
$$+\int_0^\infty\{\sin(\omega x),\cos(\omega x)\}[f(\omega)-M(\omega)]d\omega. \qquad (17)$$

The first integral in the right-hand side should be determined analytically whereas the second integral may be obtained numerically. From Eqs (16) and (17), we obtain

$$\frac{\partial w}{\partial x} = -\frac{B_z}{2\pi}\frac{y}{x^2+y^2} + \frac{(\beta+\gamma)D_x}{2\pi(2\mu+\kappa)}\int_0^\infty \omega\left[\frac{\omega}{s}e^{-|y|s} - e^{-|y|\omega}\right]\cos(\omega x)\,d\omega$$
$$+ \frac{\kappa D_x}{2\pi(\mu+\kappa)}\int_0^\infty \frac{1}{s}e^{-|y|s}\cos(\omega x)\,d\omega$$
$$+ \operatorname{sgn}(y)\frac{(\beta+\gamma)D_y}{2\pi(2\mu+\kappa)}\int_0^\infty \omega\left(e^{-|y|s} - e^{-|y|\omega}\right)\sin(\omega x)\,d\omega,$$

$$\frac{\partial w}{\partial y} = -\frac{B_z}{2\pi}\frac{x}{x^2+y^2} - \operatorname{sgn}(y)\frac{(\beta+\gamma)D_x}{2\pi(2\mu+\kappa)}\int_0^\infty \omega\left[e^{-|y|s} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega$$
$$- \operatorname{sgn}(y)\frac{\kappa D_x}{2\pi(\mu+\kappa)}\int_0^\infty \frac{1}{\omega}e^{-|y|s}\sin(\omega x)\,d\omega$$
$$+ \frac{(\beta+\gamma)D_y}{2\pi(2\mu+\kappa)}\int_0^\infty \left[se^{-|y|s} - \omega e^{-|y|\omega}\right]\cos(\omega x)\,d\omega,$$

$$\varphi_x(x,y) = \frac{B_z}{4\pi}\int_0^\infty \left[\frac{\omega}{t}e^{-|y|t} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega$$
$$+ \operatorname{sgn}(y)\frac{D_x}{2\pi}\left\{\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\int_0^\infty \omega\left[e^{-|y|s} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega\right.$$
$$\left. + \int_0^\infty \frac{1}{\omega}e^{-|y|s}\sin(\omega x)\,d\omega - \frac{\beta+\gamma}{2\kappa}\int_0^\infty \omega\left[e^{-|y|t} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega\right\}$$
$$+ \frac{D_y}{2\pi}\left\{\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\int_0^\infty \left[se^{-|y|s} - \omega e^{-|y|\omega}\right]\cos(\omega x)\,d\omega\right.$$
$$\left. - \frac{\beta+\gamma}{2\kappa}\int_0^\infty \omega\left[\frac{\omega}{t}e^{-|y|t} - e^{-|y|\omega}\right]\cos(\omega x)\,d\omega + \int_0^\infty \frac{1}{t}e^{-|y|t}\cos(\omega x)\,d\omega\right\},$$

$$\varphi_y(x,y) = \operatorname{sgn}(y)\frac{B_z}{4\pi}\int_0^\infty \left[e^{-|y|t} - e^{-|y|\omega}\right]\cos(\omega x)\,d\omega$$
$$+ \frac{D_x}{2\pi}\left\{\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\int_0^\infty \omega\left[\frac{\omega}{s}e^{-|y|s} - e^{-|y|\omega}\right]\cos(\omega x)\,d\omega + \int_0^\infty \frac{1}{s}e^{-|y|s}\cos(\omega x)\,d\omega\right.$$
$$\left. - \frac{\beta+\gamma}{2\kappa}\int_0^\infty \left[te^{-|y|t} - \omega e^{-|y|\omega}\right]\cos(\omega x)\,d\omega\right\}$$
$$- \operatorname{sgn}(y)\frac{D_y}{2\pi}\left\{\frac{(\beta+\gamma)(\mu+\kappa)}{\kappa(2\mu+\kappa)}\int_0^\infty \omega\left[e^{-|y|s} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega\right.$$
$$\left. - \int_0^\infty \frac{1}{\omega}e^{-|y|t}\sin(\omega x)\,d\omega - \frac{\beta+\gamma}{2\kappa}\int_0^\infty \omega\left[e^{-|y|t} - e^{-|y|\omega}\right]\sin(\omega x)\,d\omega\right\}. \tag{18}$$

From Eqs (3), (16), and (18), the solutions of a screw dislocation located at (ξ,η), yield

$$\sigma_{ij}(x,y;\xi,\eta) = K_{ij}^z(x,y;\xi,\eta)B_z + K_{ij}^x(x,y;\xi,\eta)D_x + K_{ij}^y(x,y;\xi,\eta)D_y, \qquad ij \in \{xz,zx,yz,zy\}$$
$$m_{ij}(x,y;\xi,\eta) = K_{ij}^z(x,y;\xi,\eta)B_z + K_{ij}^x(x,y;\xi,\eta)D_x + K_{ij}^y(x,y;\xi,\eta)D_y, \qquad ij \in \{xx,yy,zz,xy,yx\} \tag{19}$$

The coefficients in Eqs (19) are

$$K_{xz}^z(x,y;\xi,\eta) = \frac{\mu+\kappa}{2\pi}\left(\frac{y-\eta}{r^2}\right) + \operatorname{sgn}(y-\eta)\frac{\kappa}{4\pi}\int_0^\infty \left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{xz}^x(x,y;\xi,\eta) = -\frac{\beta+\gamma}{4\pi}\int_0^\infty \left[te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{xz}^y(x,y;\xi,\eta) = \operatorname{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$+ \operatorname{sgn}(y-\eta)\frac{\kappa}{2\pi}\int_0^\infty \frac{1}{\omega}e^{-|y-\eta|t}\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{zx}^z(x,y;\xi,\eta) = \frac{\mu}{2\pi}\left(\frac{y-\eta}{r^2}\right) - \operatorname{sgn}(y-\eta)\frac{\kappa}{4\pi}\int_0^\infty \left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega, \tag{20}$$

$$K_{zx}^{x}(x,y;\xi,\eta) = -\frac{\beta+\gamma}{4\pi}\int_0^\infty \left[2\frac{\omega^2}{s}e^{-|y-\eta|s} - te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega$$
$$-\frac{\gamma\kappa_1^2}{2\pi}\int_0^\infty \frac{1}{s}e^{-|y-\eta|s}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{zx}^{y}(x,y;\xi,\eta) = \mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[2e^{-|y-\eta|s} - e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$-\mathrm{sgn}(y-\eta)\frac{\kappa}{2\pi}\int_0^\infty \frac{1}{\omega}e^{-|y-\eta|t}\sin[\omega(x-\xi)]d\omega,$$

$$K_{yz}^{z}(x,y;\xi,\eta) = -\frac{\mu+\kappa}{2\pi}\left(\frac{x-\xi}{r^2}\right) - \frac{\kappa}{4\pi}\int_0^\infty \left[\frac{\omega}{t}e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{yz}^{x}(x,y;\xi,\eta) = \mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{yz}^{y}(x,y;\xi,\eta) = \frac{\beta+\gamma}{4\pi}\int_0^\infty \left[te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega + \frac{\alpha\kappa_2^2}{4\pi}\int_0^\infty \frac{1}{t}e^{-|y-\eta|t}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{zy}^{z}(x,y;\xi,\eta) = -\frac{\mu}{2\pi}\left(\frac{x-\xi}{r^2}\right) + \frac{\kappa}{4\pi}\int_0^\infty \left[\frac{\omega}{t}e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{zy}^{x}(x,y;\xi,\eta) = \mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[2e^{-|y-\eta|s} - e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$+\mathrm{sgn}(y-\eta)\frac{\gamma\kappa_1^2}{2\pi}\int_0^\infty \frac{1}{\omega}e^{-|y-\eta|s}\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{zy}^{y}(x,y;\xi,\eta) = \frac{\beta+\gamma}{4\pi}\int_0^\infty \left[2se^{-|y-\eta|s} - te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega - \frac{\alpha\kappa_2^2}{4\pi}\int_0^\infty \frac{1}{t}e^{-|y-\eta|t}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{xx}^{z}(x,y;\xi,\eta) = \frac{\beta+\gamma}{4\pi}\int_0^\infty \left[te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega - \frac{\kappa}{2\pi}\int_0^\infty \frac{1}{t}e^{-|y-\eta|t}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{xx}^{x}(x,y;\xi,\eta) = \frac{(\beta+\gamma)(2\alpha+\beta+\gamma)}{2\pi(\alpha+\beta+\gamma)}\left(\frac{y-\eta}{r^2}\right)$$
$$+\mathrm{sgn}(y-\eta)\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega^2\left[\frac{2}{\gamma\kappa_1^2}e^{-|y-\eta|s} - \frac{1}{\kappa}e^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega$$
$$+\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{2\pi}\int_0^\infty \left[e^{-|y-\eta|s} + \frac{\alpha\kappa_2^2}{2\kappa}e^{-|y|t} - \frac{2\alpha+\beta+\gamma}{\alpha+\beta+\gamma}e^{-|y|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{xx}^{y}(x,y;\xi,\eta) = -\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega\left[\frac{2}{\gamma\kappa_1^2}se^{-|y-\eta|s} - \frac{1}{\kappa}te^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$-\frac{\alpha\kappa_2^2}{2\pi}\int_0^\infty \frac{1}{\omega t}e^{-|y-\eta|t}\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{yy}^{z}(x,y;\xi,\eta) = -\frac{\beta+\gamma}{4\pi}\int_0^\infty \left[te^{-|y-\eta|t} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega - \frac{\alpha\kappa_2^2}{4\pi}\int_0^\infty \frac{1}{t}e^{-|y-\eta|t}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{yy}^{x}(x,y;\xi,\eta) = -\mathrm{sgn}(y-\eta)\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega^2\left[\frac{2}{\gamma\kappa_1^2}e^{-|y-\eta|s} - \frac{1}{\kappa}e^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega$$
$$-\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{2\pi}\int_0^\infty \left[e^{-|y-\eta|s} - e^{-|y-\eta|t}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{yy}^{y}(x,y;\xi,\eta) = -\frac{(\beta+\gamma)(2\alpha+\beta+\gamma)}{2\pi(\alpha+\beta+\gamma)}\left(\frac{x-\xi}{r^2}\right) - \frac{\kappa}{\pi}\int_0^\infty \frac{1}{\omega t}e^{-|y-\eta|t}\sin[\omega(x-\xi)]\,d\omega$$
$$+\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega\left[\frac{2}{\gamma\kappa_1^2}se^{-|y-\eta|s} - \frac{1}{\kappa}te^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$-\frac{(\beta+\gamma)(2\alpha+\beta+\gamma)}{2\pi(\alpha+\beta+\gamma)}\int_0^\infty \left[\frac{\omega}{t}e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K_{zz}^{z}(x,y;\xi,\eta) = -\frac{\alpha\kappa_2^2}{4\pi}\int_0^\infty \frac{1}{t}e^{-|y-\eta|t}\cos[\omega(x-\xi)]\,d\omega,$$

$$K_{zz}^{x}(x,y;\xi,\eta) = \frac{\alpha(\beta+\gamma)}{2\pi(\alpha+\beta+\gamma)}\left\{\left(\frac{y-\eta}{r^2}\right) + \mathrm{sgn}(y-\eta)\int_0^\infty \left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$K_{zz}^{y}(x,y;\xi,\eta) = -\frac{\alpha(\beta+\gamma)}{2\pi(\alpha+\beta+\gamma)}\left\{\frac{x-\xi}{r^2} + \int_0^\infty \left[\frac{\omega}{t}e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega\right.$$
$$\left.+\frac{2\kappa}{\beta+\gamma}\int_0^\infty \frac{1}{\omega t}e^{-|y-\eta|t}\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$K^{z}_{xy}(x,y;\xi,\eta) = -\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K^{x}_{xy}(x,y;\xi,\eta) = -\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega\left[\frac{2}{\gamma\kappa_1^2}se^{-|y-\eta|s} - \frac{1}{\kappa}te^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$-\frac{\beta}{2\pi}\int_0^\infty \left[\frac{s}{\omega} - \frac{\omega}{s}\right]e^{-|y-\eta|s}\sin[\omega(x-\xi)]\,d\omega,$$

$$K^{y}_{xy}(x,y;\xi,\eta) = -\frac{\beta^2-\gamma^2}{2\pi\gamma}\left(\frac{y-\eta}{r^2}\right)$$
$$+\mathrm{sgn}(y-\eta)\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega^2\left[\frac{2}{\gamma\kappa_1^2}e^{-|y-\eta|s} - \frac{1}{\kappa}e^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega$$
$$-\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{2\pi}\int_0^\infty \left[\frac{\beta}{\gamma}e^{-|y-\eta|s} - e^{-|y-\eta|t} - \frac{\beta-\gamma}{\gamma}e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$K^{z}_{yx}(x,y;\xi,\eta) = -\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{4\pi}\int_0^\infty \omega\left[e^{-|y-\eta|t} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$K^{x}_{yx}(x,y;\xi,\eta) = \frac{\beta^2-\gamma^2}{2\pi\gamma}\left(\frac{x-\xi}{r^2}\right) - \frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega\left[\frac{2}{\gamma\kappa_1^2}se^{-|y-\eta|s} - \frac{1}{\kappa}te^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega$$
$$-\frac{1}{2\pi\gamma}\int_0^\infty \left\{\left[\gamma^2\frac{s}{\omega} - \beta^2\frac{\omega}{s}\right]e^{-|y-\eta|s} + (\beta^2-\gamma^2)e^{-|y-\eta|\omega}\right\}\sin[\omega(x-\xi)]\,d\omega,$$

$$K^{y}_{yx}(x,y;\xi,\eta) = -\mathrm{sgn}(y-\eta)\frac{(\beta+\gamma)^2}{4\pi}\int_0^\infty \omega^2\left[\frac{2}{\gamma\kappa_1^2}e^{-|y-\eta|s} - \frac{1}{k}e^{-|y-\eta|t} - \frac{1}{2\mu+\kappa}e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega$$
$$-\mathrm{sgn}(y-\eta)\frac{\beta+\gamma}{2\pi}\int_0^\infty \left[e^{-|y-\eta|s} - e^{-|y-\eta|t}\right]\cos[\omega(x-\xi)]\,d\omega,$$

In Eqs (20) $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$, is the distance from the dislocation. It is noteworthy that all the definite integrals are convergent and prone to numerical evaluation. Furthermore, $\sigma_{ij}(x,y) \sim 1/r$ and $m_{ij}(x,y) \sim 1/r$, as $r \to 0$. In the anti-plane analysis of cracks parallel to x-axis, only stress component σ_{yz} and couple-stress components m_{yy} and m_{yx} do contribute.

Edge dislocation in micro-polar planes

We consider an edge dislocation situated at the origin of the Cartesian coordinate system where the line of discontinuity is the half-line $x > 0$. The stress analysis is analogous to that of screw dislocation; thus, the mathematical manipulation is briefed. The edge dislocation is defined as

$$\frac{\partial u(x,0^+)}{\partial x} - \frac{\partial u(x,0^-)}{\partial x} = B_x\delta(x),$$
$$\frac{\partial v(x,0^+)}{\partial x} - \frac{\partial v(x,0^-)}{\partial x} = B_y\delta(x),$$
$$\frac{\partial \varphi_z(x,0^+)}{\partial x} - \frac{\partial \varphi_z(x,0^-)}{\partial x} = \mathrm{D}_z\delta(x). \qquad (21)$$

In Eqs (21), B_x and B_y are the components of Burgers vector, identifying glid and climb of the edge dislocation, and D_z is the disclination in z-direction. Moreover, continuity of traction vectors (7) on the half-line $x > 0$ implies that

$$\sigma_{yi}(x,0^+) = \sigma_{yi}(x,0^-), \qquad i \in \{x,y\}$$
$$m_{yz}(x,0^+) = m_{yz}(x,0^-). \qquad (22)$$

In view of Eqs (3), the above continuity conditions are expressed as

$$\mu\frac{\partial v}{\partial x}(x,0^+)+(\mu+\kappa)\frac{\partial u}{\partial y}(x,0^+)+\kappa\varphi_z(x,0^+)$$
$$=\mu\frac{\partial v}{\partial x}(x,0^-)+(\mu+\kappa)\frac{\partial u}{\partial y}(x,0^-)+\kappa\varphi_z(x,0^-),$$
$$\lambda\frac{\partial u}{\partial x}(x,0^+)+(\lambda+2\mu+\kappa)\frac{\partial v}{\partial y}(x,0^+)=\lambda\frac{\partial u}{\partial x}(x,0^-)+(\lambda+2\mu+\kappa)\frac{\partial v}{\partial y}(x,0^-),$$
$$\frac{\partial\varphi_z}{\partial y}(x,0^+)=\frac{\partial\varphi_z}{\partial y}(x,0^-). \tag{23}$$

The complex Fourier transform, $\mathcal{F}$, of Eqs (9) yields

$$(\mu+\kappa)\frac{d^2U}{dy^2}-i\omega(\lambda+\mu)\frac{dV}{dy}+\kappa\frac{d\Phi_z}{dy}-\omega^2(\lambda+2\mu+\kappa)U=0,$$
$$(\lambda+2\mu+\kappa)\frac{d^2V}{dy^2}-i\omega(\lambda+\mu)\frac{dU}{dy}-\omega^2(\mu+\kappa)V+i\omega\kappa\Phi_z=0,$$
$$\gamma\frac{d^2\Phi_z}{dy^2}-\kappa\frac{dU}{dy}-i\omega\kappa V-(\omega^2\gamma+2\kappa)\Phi_z=0. \tag{24}$$

In the above equations, $[U,V,\Phi_z]=\mathcal{F}\{[u,v,\varphi_z];x\to\omega\}$. The Fourier transforms of Eqs (21) and (23) are

$$U(\omega,0^+)-U(\omega,0^-)=\frac{iB_x}{\omega},$$
$$V(\omega,0^+)-V(\omega,0^-)=\frac{iB_y}{\omega},$$
$$\Phi_z(\omega,0^+)-\Phi_z(\omega,0^-)=\frac{iD_z}{\omega},$$
$$(\mu+\kappa)\frac{dU}{dy}(\omega,0^+)-i\omega\mu V(\omega,0^+)+\kappa\Phi_z(\omega,0^+)$$
$$=(\mu+\kappa)\frac{dU}{dy}(\omega,0^-)-i\omega\mu V(\omega,0^-)+\kappa\Phi_z(\omega,0^-),$$
$$(\lambda+2\mu+\kappa)\frac{dV}{dy}(\omega,0^+)-i\omega\lambda U(\omega,0^+)=(\lambda+2\mu+\kappa)\frac{dV}{dy}(\omega,0^-)-i\omega\lambda U(\omega,0^-),$$
$$\frac{d\Phi_z}{dy}(\omega,0^+)=\frac{d\Phi_z}{dy}(\omega,0^-). \tag{25}$$

The inverse complex Fourier transformations of the solution to Eqs (24), subject to conditions (25) are used to determine the derivatives $(\partial/\partial\zeta)[u(x,y),v(x,y),\varphi_z(x,y)]$, $\zeta\in\{x,y\}$. We substitute these quantities into Eqs (3) and carry out the asymptotic analysis of the integrals in the Fourier inversions as $\omega\to\infty$ to obtain stress and couple-stress components of an edge dislocation located at (ξ,η) as

$$\sigma_{ij}(x,y;\xi,\eta)=L_{ij}^x(x,y;\xi,\eta)B_x+L_{ij}^y(x,y;\xi,\eta)B_y+L_{ij}^z(x,y;\xi,\eta)D_z,\qquad ij\in\{xx,yy,zz,xy,yx\}$$
$$m_{ij}(x,y;\xi,\eta)=L_{ij}^x(x,y;\xi,\eta)B_x+L_{ij}^y(x,y;\xi,\eta)B_y+L_{ij}^z(x,y;\xi,\eta)D_z,\qquad ij\in\{xz,yz,zx,zy\} \tag{26}$$

The functions in Eqs (26) are

$$L^x_{xx}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(y-\eta)[3(x-\xi)^2+(y-\eta)^2]}{r^4}-\mathrm{sgn}(y-\eta)\int_0^\infty\omega^2\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$L^y_{xx}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(x-\xi)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\int_0^\infty\omega\left[se^{-|y-\eta|s}-\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$L^z_{xx}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\int_0^\infty\left[se^{-|y-\eta|s}-\omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^x_{yy}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(y-\eta)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\mathrm{sgn}(y-\eta)\int_0^\infty\omega^2\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$L^y_{yy}(x,y;\xi,\eta)=-\frac{\gamma}{2\pi}\left\{\psi\frac{(x-\xi)[(x-\xi)^2+3(y-\eta)^2]}{r^4}+\int_0^\infty\omega\left[se^{-|y-\eta|s}-\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$L^z_{yy}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\int_0^\infty\left[\omega e^{-|y-\eta|\omega}-se^{-|y-\eta|s}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^x_{zz}(x,y;\xi,\eta)=\frac{\lambda(2\mu+\kappa)}{2\pi(\lambda+2\mu+\kappa)}\frac{(y-\eta)}{r^2},$$

$$L^y_{zz}(x,y;\xi,\eta)=-\frac{\lambda(2\mu+\kappa)}{2\pi(\lambda+2\mu+\kappa)}\frac{(x-\xi)}{r^2},$$

$$L^z_{zz}(x,y;\xi,\eta)=0,$$

$$L^x_{xy}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(x-\xi)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\int_0^\infty\omega\left(se^{-|y-\eta|s}-\omega e^{-|y-\eta|\omega}\right)\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$L^y_{xy}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(y-\eta)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\mathrm{sgn}(y-\eta)\int_0^\infty\left[s^2e^{-|y-\eta|s}-\omega^2e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$L^z_{xy}(x,y;\xi,\eta)=-\mathrm{sgn}(y-\eta)\frac{\gamma}{2\pi}\int_0^\infty\left[\frac{s^2}{\omega}e^{-|y-\eta|s}-\omega e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$L^x_{yx}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(x-\xi)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\int_0^\infty\omega^2\left(\frac{\omega}{s}e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right)\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$L^y_{yx}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\psi\frac{(y-\eta)[(y-\eta)^2-(x-\xi)^2]}{r^4}+\mathrm{sgn}(y-\eta)\int_0^\infty\omega^2\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$L^z_{yx}(x,y;\xi,\eta)=-\mathrm{sgn}(y-\eta)\frac{\gamma}{2\pi}\int_0^\infty\omega\left(e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right)\sin[\omega(x-\xi)]\,d\omega,$$

$$L^x_{xz}(x,y;\xi,\eta)=-\frac{\gamma}{2\pi}\int_0^\infty\omega\left[\frac{\omega}{s}e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^y_{xz}(x,y;\xi,\eta)=\mathrm{sgn}(y-\eta)\frac{\gamma}{2\pi}\int_0^\infty\omega\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$L^z_{xz}(x,y;\xi,\eta)=\frac{\gamma}{2\pi}\left\{\frac{y-\eta}{r^2}+\mathrm{sgn}(y-\eta)\int_0^\infty\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},$$

$$L^x_{zx}(x,y;\xi,\eta)=-\frac{\beta}{2\pi}\int_0^\infty\omega\left[\frac{\omega}{s}e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^y_{zx}(x,y;\xi,\eta)=\mathrm{sgn}(y-\eta)\frac{\beta}{2\pi}\int_0^\infty\omega\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$L^z_{zx}(x,y;\xi,\eta)=\frac{\beta}{2\pi}\left\{\frac{y-\eta}{r^2}+\mathrm{sgn}(y-\eta)\int_0^\infty\left[e^{-|y-\eta|s}-e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega\right\},\qquad(27)$$

$$L^x_{yz}(x,y;\xi,\eta) = \operatorname{sgn}(y-\eta)\frac{\gamma}{2\pi}\int_0^\infty \omega\left[e^{-|y-\eta|s} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$L^y_{yz}(x,y;\xi,\eta) = \frac{\gamma}{2\pi}\int_0^\infty \left[se^{-|y-\eta|s} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^z_{yz}(x,y;\xi,\eta) = -\frac{\gamma}{2\pi}\left\{\frac{x-\xi}{r^2} + \int_0^\infty \left[\frac{s}{\omega}e^{-|y-\eta|s} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega\right\},$$

$$L^x_{zy}(x,y;\xi,\eta) = \operatorname{sgn}(y-\eta)\frac{\beta}{2\pi}\int_0^\infty \omega\left[e^{-|y-\eta|s} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega,$$

$$L^y_{zy}(x,y;\xi,\eta) = \frac{\beta}{2\pi}\int_0^\infty \left[se^{-|y-\eta|s} - \omega e^{-|y-\eta|\omega}\right]\cos[\omega(x-\xi)]\,d\omega,$$

$$L^z_{zy}(x,y;\xi,\eta) = -\frac{\beta}{2\pi}\left\{\frac{x-\xi}{r^2} + \int_0^\infty \left[\frac{s}{\omega}e^{-|y-\eta|s} - e^{-|y-\eta|\omega}\right]\sin[\omega(x-\xi)]\,d\omega\right\},$$

where the constant $\psi = (\kappa+2\mu)(2\lambda+2\mu+\kappa)/2\gamma(\lambda+2\mu+\kappa)$.

Stress singularity at a crack tip

The displacement and micro-rotation vectors in polar coordinates may be written as

$$U = u(r,\theta)e_r + v(r,\theta)e_\theta + w(r,\theta)e_z,$$
$$\Phi = \varphi_r(r,\theta)e_r + \varphi_\theta(r,\theta)e_\theta + \varphi_z(r,\theta)e_z. \tag{28}$$

Substituting Eqs (28) into (4), results in

$$(\lambda+2\mu+\kappa)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - \frac{\lambda+3\mu+2\kappa}{r^2}\frac{\partial v}{\partial\theta} + \frac{\lambda+\mu}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{\mu+\kappa}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\kappa}{r}\frac{\partial\varphi_z}{\partial\theta} = 0,$$

$$\frac{\lambda+\mu}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right) + (\mu+\kappa)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2}\right) - \kappa\frac{\partial\varphi_z}{\partial r} = 0,$$

$$(\mu+\kappa)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) + \kappa\left(\frac{\partial\varphi_\theta}{\partial r} + \frac{\varphi_\theta}{r} - \frac{1}{r}\frac{\partial\varphi_r}{\partial\theta}\right) = 0,$$

$$(\alpha+\beta+\gamma)\frac{\partial}{\partial r}\left(\frac{\partial\varphi_r}{\partial r} + \frac{\varphi_r}{r} + \frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}\right) - \frac{\gamma}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi_\theta}{\partial r} + \frac{\varphi_\theta}{r} - \frac{1}{r}\frac{\partial\varphi_r}{\partial\theta}\right) + \frac{\kappa}{r}\frac{\partial w}{\partial\theta} - 2\kappa\varphi_r = 0,$$

$$\frac{\alpha+\beta+\gamma}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi_r}{\partial r} + \frac{\varphi_r}{r} + \frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}\right) + \gamma\frac{\partial}{\partial r}\left(\frac{\partial\varphi_\theta}{\partial r} + \frac{\varphi_\theta}{r} - \frac{1}{r}\frac{\partial\varphi_r}{\partial\theta}\right) - \kappa\frac{\partial w}{\partial r} - 2\kappa\varphi_\theta = 0,$$

$$\gamma\left(\frac{\partial^2\varphi_z}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi_z}{\partial\theta^2}\right) + \kappa\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial\theta}\right) - 2\kappa\varphi_z = 0. \tag{29}$$

We consider a plane containing a semi-infinite crack, $r > 0, -\pi < \theta < \pi$. Utilizing Eqs (6) boundary conditions on the crack surface become

$$n.\boldsymbol{\sigma} = 0.$$
$$n.\boldsymbol{m} = 0, \qquad \theta = \pm\pi \tag{30}$$

where, n is the unit vector perpendicular to the crack surface. In view of Eqs (7), Eqs (30) become

$$
\begin{aligned}
&(3\mu+\kappa)\frac{\partial v}{\partial r}+\frac{\mu}{r}\left(v-\frac{\partial u}{\partial \theta}\right)+\kappa\varphi_z=0,\\
&\lambda\frac{\partial u}{\partial r}+\frac{\lambda+2\mu+\kappa}{r}\left(u+\frac{\partial v}{\partial \theta}\right)=0,\\
&\frac{\mu}{r}\frac{\partial w}{\partial \theta}+\kappa\varphi_r=0,\\
&(\beta+2\gamma)\frac{\partial \varphi_\theta}{\partial r}-\frac{\gamma}{r}\left(\frac{\partial \varphi_r}{\partial \theta}-\varphi_\theta\right)=0,\\
&\alpha\frac{\partial \varphi_r}{\partial r}+\frac{\alpha+\beta+\gamma}{r}\left(\varphi_r+\frac{\partial \varphi_\theta}{\partial \theta}\right)=0,\\
&\frac{\partial \varphi_z}{\partial \theta}=0, \qquad \theta=\pm\pi
\end{aligned}
\tag{31}
$$

From Eqs (29), we observe that near the crack tip, $r\to 0$, components of the displacement and micro-rotation vectors should behave as

$$
\begin{aligned}
&\{u(r,\theta),v(r,\theta),w(r,\theta)\}=r^{\xi+1}\{U(\theta),V(\theta),W(\theta)\},\\
&\{\varphi_r(r,\theta),\varphi_\theta(r,\theta),\varphi_z(r,\theta)\}=r^{\xi}\{\Phi_r(\theta),\Phi_\theta(\theta),\Phi_z(\theta)\}, \qquad \xi>0
\end{aligned}
\tag{32}
$$

We substitute the above equations into Eqs (29) and (31) then let $r\to 0$. For anti-plane deformation we arrive at

$$
\begin{aligned}
&(\mu+\kappa)\left[\frac{d^2W}{d\theta^2}+(\xi+1)^2W\right]-\kappa\left[\frac{d\Phi_r}{d\theta}-(\xi+1)\Phi_\theta\right]=0,\\
&\gamma\frac{d^2\Phi_r}{d\theta^2}+[(\alpha+\beta)(\xi-1)-2\gamma]\frac{d\Phi_\theta}{d\theta}+(\alpha+\beta+\gamma)(\xi^2-1)\Phi_r=0,\\
&(\alpha+\beta+\gamma)\frac{d^2\Phi_\theta}{d\theta^2}+[(\alpha+\beta)(\xi+1)+2\gamma]\frac{d\Phi_r}{d\theta}+\gamma(\xi^2-1)\Phi_\theta=0.
\end{aligned}
\tag{33}
$$

Subject to the homogeneous boundary conditions

$$
\begin{aligned}
&\mu\frac{dW}{d\theta}+\kappa\Phi_r=0,\\
&\gamma\frac{d\Phi_r}{d\theta}-[(\beta+2\gamma)\xi+\gamma]\Phi_\theta=0,\\
&(\alpha+\beta+\gamma)\frac{d\Phi_\theta}{d\theta}+(\alpha\xi+\alpha+\beta+\gamma)\Phi_r=0, \qquad \theta=\pm\pi
\end{aligned}
\tag{34}
$$

For in-plane deformation, we have

$$
\begin{aligned}
&(\mu+\kappa)\frac{d^2U}{d\theta^2}+[(\lambda+\mu)\xi-2(\mu+\kappa)]\frac{dV}{d\theta}+(\lambda+2\mu+\kappa)\xi(\xi+2)U+\kappa\frac{d\Phi_z}{d\theta}=0,\\
&(\lambda+2\mu+\kappa)\frac{d^2V}{d\theta^2}+[(\lambda+\mu)\xi+2(\lambda+2\mu+\kappa)]\frac{dU}{d\theta}+(\mu+\kappa)\xi(\xi+2)V-\kappa\xi\Phi_z=0,\\
&\frac{d^2\Phi_z}{d\theta^2}+\xi^2\Phi_z=0.
\end{aligned}
\tag{35}
$$

Following homogeneous boundary conditions should be imposed to the solution of Eqs (35)

$$\mu\frac{dU}{d\theta}-[(3\mu+\kappa)(\xi+1)+\mu]V-k\Phi_z=0,$$
$$(\lambda+2\mu+\kappa)\frac{dV}{d\theta}+[\lambda(\xi+1)+(\lambda+2\mu+\kappa)]U=0,$$
$$\frac{d\Phi_z}{d\theta}=0,\qquad \theta=\pm\pi \tag{36}$$

The analytical solutions to Eqs (33) and (35) under the relevant homogeneous boundary conditions lead to the eigenvalues ξ, which are for anti-plane and in-plane deformations, respectively, the solutions of

$$\frac{\mu^2(\beta+\gamma)^4(\alpha+\beta+2\gamma)^2}{4\gamma(\alpha+\beta+\gamma)}\xi^2(1+\xi)\sin^3(2\pi\xi)=0.$$

and

$$\frac{(2\mu+\kappa)^2(\kappa(\lambda-3\mu)-2\mu(\lambda+3\mu))^2}{4(\mu+\kappa)(\lambda+2\mu+\kappa)}\xi^2(1+\xi)^2\sin^3(2\pi\xi)=0. \tag{37}$$

Therefore, the eigenvalues for both cases of deformation are

$$\xi=\frac{\mathrm{n}+1}{2},\qquad n\in\mathbb{Z} \tag{38}$$

The micro-rotation vector and strain energy, defined in Eq. (5), are finite in the regions engulfing a crack tip; therefore $n\geq 0$ and stress and couple-stress tensors behave as $\boldsymbol{\sigma}\sim r^{1/2}, \boldsymbol{m}\sim r^{-1/2}$ as $r\to 0$. Analogous behavior has been reported in the analysis of cracks under anti-plane deformation in flexoelectric materials, Xia and Linder (2024). It is noteworthy that by setting $\kappa=0$ in Eqs (29) i.e., ignoring the interaction between dislocation and disclination in the vicinity of a crack tip in micro-polar materials, governing equations for displacement and micro-rotation fields decouple. As a result, we may take

$$w(r,\theta)=r^{\xi_1}W(\theta),\qquad \xi_1>0$$
$$\{\varphi_r(r,\theta),\varphi_\theta(r,\theta)\}=r^{\xi_2}\{\Phi_r(\theta),\Phi_\theta(\theta)\},\qquad \xi_2>0 \tag{39}$$

It is rather straightforward to show that, Garajeu and Soós (2003), the smallest admissible eigenvalues $\xi_1=\xi_2=1/2$, making both stress and couple-stress tensors square-root singular at a crack tip.

Crack formulation

We consider a micro-polar plane containing a crack with length $2l$ situated on the x-axis. Crack in parametric form is represented as

$$x=ql,\qquad -1<q<1 \tag{40}$$

$y = 0,$

In what follows, crack under anti-plane and in-plane deformations are treated separately.

Anti-plane deformation

The infinite plane is under a constant anti-plane traction $\sigma_{yz}(x,y) = \tau_{z0}$, as $|y| \to \infty$. Employing the distributed dislocation technique crack may be modeled by the distribution of the screw dislocations. From Eqs (19), the components of stress and couple-stress in the plane become

$$\sigma_{yz}(x,y) = \tau_{z0} + l\int_{-1}^{1}\left[K_{yz}^{z}(x,y;ql,0)b_z(q) + K_{yz}^{x}(x,y;ql,0)d_x(q) + K_{yz}^{y}(x,y;ql,0)d_y(q)\right]dq,$$

$$\sigma_{ij}(x,y) = l\int_{-1}^{1}\left[K_{ij}^{z}(x,y;ql,0)b_z(q) + K_{ij}^{x}(x,y;ql,0)d_x(q) + K_{ij}^{y}(x,y;ql,0)d_y(q)\right]dq,$$

$$ij \in \{xz, zx, zy\}$$

$$m_{ij}(x,y) = l\int_{-1}^{1}\left[K_{ij}^{z}(x,y;ql,0)b_z(q) + K_{ij}^{x}(x,y;ql,0)d_x(q) + K_{ij}^{y}(x,y;ql,0)d_y(q)\right]dq,$$

$$ij \in \{xx, yy, zz, xy, yx\} \quad (41)$$

In Eqs (41), the unknown density functions on the crack surface are defined as

$$\{b_z(q), d_x(q), d_y(q)\} = \frac{d}{dq}\{B_z(q), D_x(q), D_y(q)\}, \qquad -1 < q < 1 \quad (42)$$

On the crack surface, the free traction conditions are

$$\sigma_{yz}(x(p),0) = 0,$$
$$m_{yx}(x(p),0) = 0,$$
$$m_{yy}(x(p),0) = 0, \qquad -1 < p < 1 \quad (43)$$

From Eqs (41) and (43), we arrive at the system of Cauchy singular integral equations for the density functions as

$$\tau_{z0} + l\int_{-1}^{1}\left[K_{yz}^{z}(x(p),y;ql,0)b_z(q) + K_{yz}^{x}(x(p),y;ql,0)d_x(q) + K_{yz}^{y}(x(p),y;ql,0)d_y(q)\right]dq = 0,$$

$$\int_{-1}^{1}\left[K_{yx}^{z}(x(p),y;ql,0)b_z(q) + K_{yx}^{x}(x(p),y;ql,0)d_x(q) + K_{yx}^{y}(x(p),y;ql,0)d_y(q)\right]dq = 0,$$

$$\int_{-1}^{1}\left[K_{yy}^{z}(x(p),y;ql,0)b_z(q) + K_{yy}^{x}(x(p),y;ql,0)d_x(q) + K_{yy}^{y}(x(p),y;ql,0)d_y(q)\right]dq = 0,$$

$$y \to 0, \qquad -1 < p < 1 \quad (44)$$

By virtue of the asymptotic behavior of stress and couple-stress tensors at the embedded tips of a crack in a micro-polar plane, Eqs (32) and (38), we may express

$$b_z(q) = \sqrt{1-q^2}\, g_z(q), \qquad d_x(q) = \frac{g_x(q)}{\sqrt{1-q^2}}, \qquad d_y(q) = \frac{g_y(q)}{\sqrt{1-q^2}}. \tag{45}$$

where, $g_z(q)$, $g_x(q)$, and $g_y(q)$ are unknown continuous functions. For embedded cracks system of the Cauchy singular integral Eqs (44) has unique solution provided that the following closure conditions are satisfied, Muskhelishvili (1963)

$$\int_{-1}^{1} b_z(q)dq = 0\,, \qquad \int_{-1}^{1} d_x(q)dq = 0\,, \qquad \int_{-1}^{1} d_y(q)dq = 0. \tag{46}$$

Substitution of Eqs (45) into Eqs (44) and (46) results in a system of integral equations consisting of equations with multiple Cauchy singular terms for $g_z(q)$, $g_x(q)$, and $g_y(q)$. This necessitates the generalization of the well-established numerical method for integral equations containing only one singular term in each equation, Erdogan et al. (1973). The detail of the general procedure is rendered in the Appendix. The resultant system of algebraic equations which is the discretized form of Eqs (44) and (46) at the collocation points yields

$$\frac{\tau_{z0}}{l} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{yz}^{z}(lp_{br}, y; lq_{bk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) K_{yz}^{x}(lp_{br}, y; lq_{uk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{yz}^{y}(lp_{br}, y; lq_{uk}, 0) = 0,$$

$$\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{yx}^{z}(lp_{us}, y; lq_{bk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) K_{yx}^{x}(lp_{us}, y; lq_{uk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{yx}^{y}(lp_{us}, y; lq_{uk}, 0) = 0,$$

$$\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{yy}^{z}(lp_{us}, y; lq_{bk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) K_{yy}^{x}(lp_{us}, y; lq_{uk}, 0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{yy}^{y}(lp_{us}, y; lq_{uk}, 0) = 0,$$

$$\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) = 0,$$

$$\frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) = 0,$$

$$\frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) = 0,$$

wherein, $y \to 0$, $r \in \{1, \dots, N-1\}$, and

$$s \in \begin{cases} \left\{1, \dots, \dfrac{N-1}{2}\right\} \cup \left\{\dfrac{N+3}{2}, \dots, N+1\right\}, & N \text{ is odd} \\ \left\{1, \dots, \dfrac{N}{2}-1\right\} \cup \left\{\dfrac{N}{2}+1\right\} \cup \left\{\dfrac{N}{2}+3, \dots, N+1\right\}, & N \text{ is even} \end{cases} \tag{47}$$

The system of linear algebraic equations (47) is solved for the unknowns $g_z(q_{bk})$, $g_x(q_{uk})$, and $g_y(q_{uk}), k \in \{1,2,\dots N\}$. The components of stress and couple-stress tensors are derived by applying the quadrature rule of integration mentioned in the Appendix on Eqs (41), leading to

$$\sigma_{yz}(x,y) = \tau_{z0} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{yz}^{z}(x,y;lq_{bk},0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) K_{yz}^{x}(x,y;lq_{uk},0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{yz}^{y}(x,y;lq_{uk},0),$$

$$\sigma_{ij}(x,y) = \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{ij}^{z}(x,y;lq_{bk},0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk}) K_{ij}^{x}(x,y;lq_{uk},0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{ij}^{y}(x,y;lq_{uk},0), \qquad ij \in \{xz,zx,zy\}$$

$$m_{ij}(x,y) = \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, g_z(q_{bk}) K_{ij}^{z}(x,y;lq_{bk},0) + \frac{1}{N}\sum_{k=1}^{N} g_x(q_{uk})\, K_{ij}^{x}(x,y;lq_{uk},0) + \frac{1}{N}\sum_{k=1}^{N} g_y(q_{uk}) K_{ij}^{y}(x,y;lq_{uk},0), \qquad ij \in \{xx,yy,zz,xy,yx\}. \tag{48}$$

It is noteworthy that by eliminating micro-polar coefficients in Eqs (48), the couple-stress tensor vanishes.

In-plane deformation

Plane is subjected to the normal and shear tractions $\sigma_{yy}(x,y) = \sigma_{y0}$ and $\sigma_{yx}(x,y) = \tau_{y0}$ as $|y| \to \infty$. Utilizing Eqs (26), components of the stress and couple-stress tensors yield

$$\sigma_{yy}(x,y) = \sigma_{y0} + l\int_{-1}^{1}\left[L_{yy}^{x}(x,y;ql,0)b_x(q) + L_{yy}^{y}(x,y;ql,0)b_y(q) + L_{yy}^{z}(x,y;ql,0)d_z(q)\right]dq,$$

$$\sigma_{yx}(x,y) = \tau_{y0} + l\int_{-1}^{1}\left[L_{yx}^{x}(x,y;ql,0)b_x(q) + L_{yx}^{y}(x,y;ql,0)b_y(q) + L_{yx}^{z}(x,y;ql,0)d_z(q)\right]dq,$$

$$\sigma_{ij}(x,y) = l\int_{-1}^{1}\left[L_{ij}^{x}(x,y;ql,0)b_x(q) + L_{ij}^{y}(x,y;ql,0)b_y(q) + L_{ij}^{z}(x,y;ql,0)d_z(q)\right]dq, \qquad ij \in \{xx,zz,xy\}$$

$$m_{ij}(x,y) = l\int_{-1}^{1}\left[L_{ij}^{x}(x,y;ql,0)b_x(q) + L_{ij}^{y}(x,y;ql,0)b_y(q) + L_{ij}^{z}(x,y;ql,0)d_z(q)\right]dq, \qquad ij \in \{xz,yz,zx,zy\} \tag{49}$$

In Eqs (49), the density functions are

$$\{b_x(q), b_y(q), d_z(q)\} = \frac{d}{dq}\{B_x(q), B_y(q), D_z(q)\}, \qquad -1 < q < 1 \tag{50}$$

The closure conditions for an embedded crack imply that

$$\int_{-1}^{1} b_x(q)dq = 0\,, \qquad \int_{-1}^{1} b_y(q)dq = 0\,, \qquad \int_{-1}^{1} d_z(q)dq = 0. \tag{51}$$

The free boundary conditions on the crack surface requires that

$$\sigma_{yy}(x(p),0) = 0, \qquad \sigma_{yx}(x(p),0) = 0, \quad m_{yz}(x(p),0) = 0, \qquad -1 < p < 1 \tag{52}$$

In view of the behavior of stress and couple-stress tensors at a crack tip, we take

$$b_x(q) = \sqrt{1-q^2}\, f_x(q), \qquad b_y(q) = \sqrt{1-q^2}\, f_y(q), \qquad d_z(q) = \frac{f_z(q)}{\sqrt{1-q^2}}. \tag{53}$$

where, $f_x(q), f_y(q)$, and $f_z(q), -1 < q < 1$ are bounded functions. From Eqs (49) and (52) the system of Cauchy singular integral equations becomes

$$\begin{aligned}
&\sigma_{y0} + l \int_{-1}^{1} \left[L^x_{yy}(x(p),y;ql,0)b_x(q) + L^y_{yy}(x(p),y;ql,0)b_y(q) + L^z_{yy}(x(p),y;ql,0)d_z(q)\right] dq = 0,\\
&\tau_{y0} + l \int_{-1}^{1} \left[L^x_{yx}(x(p),y;ql,0)b_x(q) + L^y_{yx}(x(p),y;ql,0)b_y(q) + L^z_{yx}(x(p),y;ql,0)d_z(q)\right] dq = 0,\\
&\int_{-1}^{1} \left[L^x_{yz}(x(p),y;ql,0)b_x(q) + L^y_{yz}(x(p),y;ql,0)b_y(q) + L^z_{yz}(x(p),y;ql,0)d_z(q)\right] dq = 0,\\
&\qquad\qquad y \to 0, \qquad -1 < p < 1
\end{aligned} \tag{54}$$

We substitute Eqs (53) into (54) and (51) and solve the ensuing system of integral equations by utilizing the numerical procedure detailed in the Appendix to obtain the system of linear algebraic equations for $f_x(q_{bk}), f_y(q_{bk})$, and $f_z(q_{uk}), k \in \{1,2,\dots,N\}$ as

$$\begin{aligned}
&\frac{\sigma_{y0}}{l} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_x(q_{bk}) L^x_{yy}(lp_{bs},y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_y(q_{bk}) L^y_{yy}(lp_{bs},y;lq_{bk},0)\\
&\qquad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) L^z_{yy}(lp_{bs},y;lq_{uk},0) = 0,\\
&\frac{\tau_{y0}}{l} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_x(q_{bk}) L^x_{yx}(lp_{bs},y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_y(q_{bk}) L^y_{yx}(lp_{bs},y;lq_{bk},0)\\
&\qquad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) L^z_{yx}(lp_{bs},y;lq_{uk},0) = 0,\\
&\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_x(q_{bk}) L^x_{yz}(lp_{ur},y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1} f_y(q_{bk}) L^y_{yz}(lp_{ur},y;lq_{bk},0)\\
&\qquad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) L^z_{yz}(lp_{ur},y;lq_{uk},0) = 0,
\end{aligned} \tag{55}$$

$$\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_x(q_{bk}) = 0,$$

$$\sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_y(q_{bk}) = 0,$$

$$\frac{1}{n}\sum_{k=1}^{N} f_z(q_{uk}) = 0,$$

wherein, $y \to 0,\ r \in \{1, \dots, N-1\}$, and

$$s \in \begin{cases} \left\{1, \dots, \dfrac{N-1}{2}\right\} \cup \left\{\dfrac{N+3}{2}, \dots, N+1\right\}, & N \text{ is odd} \\ \left\{1, \dots, \dfrac{N}{2}-1\right\} \cup \left\{\dfrac{N}{2}+1\right\} \cup \left\{\dfrac{N}{2}+3, \dots, N+1\right\}, & N \text{ is even} \end{cases}$$

Application of the quadrature rule of integration to Eqs (49) leads to

$$\begin{aligned}
\sigma_{yy}(x,y) &= \sigma_{y0} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_x(q_{bk}) L_{yy}^{x}(x,y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_y(q_{bk}) L_{yy}^{y}(x,y;lq_{bk},0) \\
&\quad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) K_{yy}^{z}(x,y;lq_{uk},0), \\
\sigma_{yx}(x,y) &= \tau_{y0} + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_x(q_{bk}) L_{yx}^{x}(x,y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_y(q_{bk}) L_{yx}^{y}(x,y;lq_{bk},0) \\
&\quad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) K_{yx}^{z}(x,y;lq_{uk},0), \\
\sigma_{ij}(x,y) &= \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_x(q_{bk}) L_{ij}^{x}(x,y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_y(q_{bk}) L_{ij}^{y}(x,y;lq_{bk},0) \\
&\quad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) L_{ij}^{z}(x,y;lq_{uk},0), \qquad ij \in \{xx, zz, xy\} \\
m_{ij}(x,y) &= \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_x(q_{bk}) L_{ij}^{x}(x,y;lq_{bk},0) + \sum_{k=1}^{N} \frac{1-q_{bk}^2}{N+1}\, f_y(q_{bk}) L_{ij}^{y}(x,y;lq_{bk},0) \\
&\quad + \frac{1}{N}\sum_{k=1}^{N} f_z(q_{uk}) L_{ij}^{z}(x,y;lq_{uk},0), \qquad ij \in \{xz, zx, yz, zy\}.
\end{aligned} \tag{56}$$

The solution of Eqs (55) is substituted into Eqs (56) to evaluate stress and couple-stress fields. Under in-plane loading, crack opening and sliding displacements are defined as

$$\begin{aligned}
\Delta u(x) &= l \int_{-1}^{x \le 1} b_x(q)\, dq, \\
\Delta v(x) &= l \int_{-1}^{x \le 1} b_y(q)\, dq.
\end{aligned} \tag{57}$$

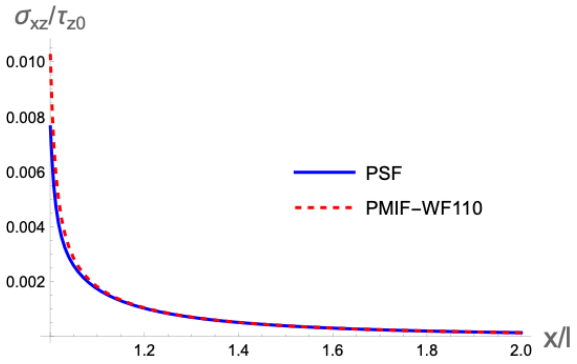
Numerical results

The numerical analysis is carried out for two infinite cracked micro-polar planes, one is composed of Polystyrene foam (PSF), and the other is made of Polymethacrylimide foam grade WFI 10 (PMIF-WFI 10). The materials properties are, Hassanpour and Heppler (2015)

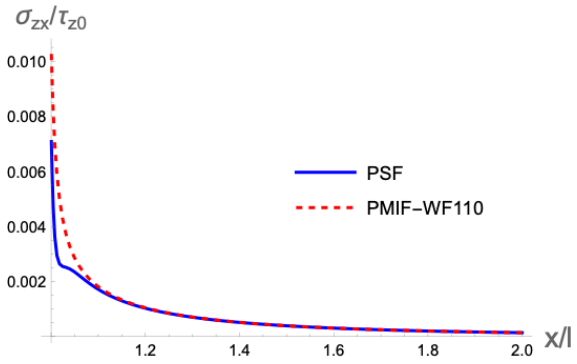
$$\begin{aligned} &\text{for (PSF):} && \mu = 0.607\ (\text{MPa}), \quad \lambda = 0.099\ (\text{MPa}), \quad \alpha = -5.84\ (\text{N}), \\ &&& \beta = 52\ (\text{N}), \quad \gamma = 8.77\ (\text{N}), \quad \kappa = 0.06\ (\text{MPa}) \\ &\text{for (PMIF-WFI 10):} && \mu = 75\ (\text{MPa}), \quad \lambda = 487.4\ (\text{MPa}), \quad \alpha = -13.52\ (\text{N}), \\ &&& \beta = 16.47\ (\text{N}), \quad \gamma = 2.28\ (\text{N}), \quad \kappa = 0.7576\ (\text{MPa}). \end{aligned} \tag{58}$$

In all examples, planes are weakened by a crack described in Eqs (40) and as $|y| \to \infty$, planes are subjected to a constant component of traction t^n whereas micro-traction $M^n = 0$, in Eqs (7). Therefore, in comparison to the normalized stress components normalized couple-stresses, except in the near crack tip region which is affected by the square-root singularity of the couple-stress tensor, are small. The effect of friction on the crack surface is neglected; thus, crack sliding takes place freely. The convergence of the numerical solution of Eqs (47) and (55) is attained by taking, at a maximum, $N = 21$. Lastly, in the first two examples, because of the symmetry, results at the right crack tip are reported.

In the first example, plane is subjected to the anti-plane traction $\lim_{|y|\to\infty}\sigma_{yz}(x,y) = \tau_{z0}$. The solution of Eqs (47) is substituted into Eqs (48) to determine the normalized stress $\sigma_{ij}(x,0)/\tau_{z0}$ and couple-stress $m_{ij}(x,0)/l\tau_{z0}$ for, $x > l$, Fig (1). In both micro-polar planes $\sigma_{yz}(\pm l,0) = 0$, contradicting the square-root singularity behavior of the stress component in the local elasticity solution. For a plane made up of PSF, the normalized stress and couple-stress components in the region $r/l < 0.02$, $-11\pi/12 < \theta < 11\pi/12$, r is the distance from the crack tip, are shown in Fig. (2), illustrating the asymmetry of both stress tensors.



(a)



(b)

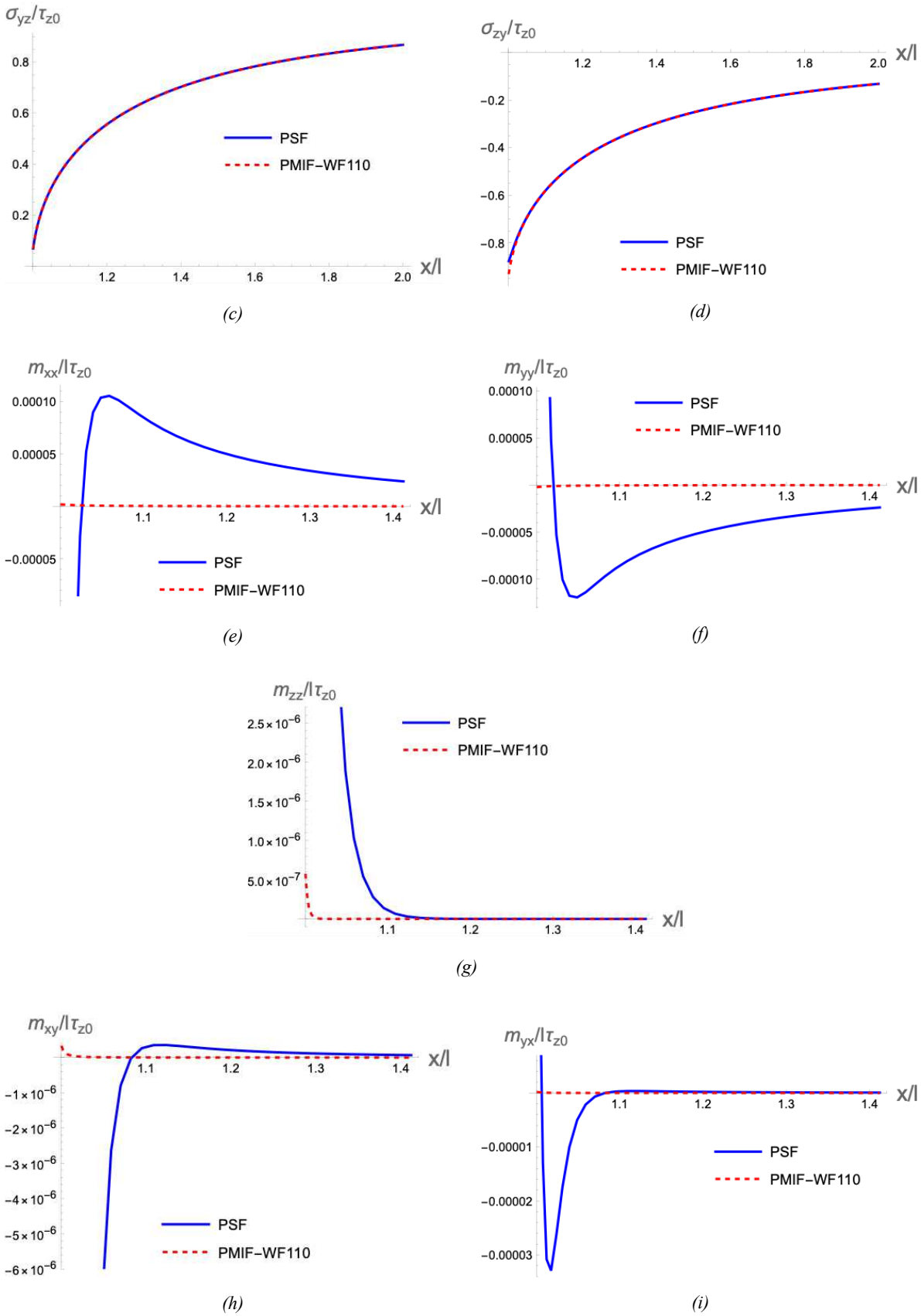
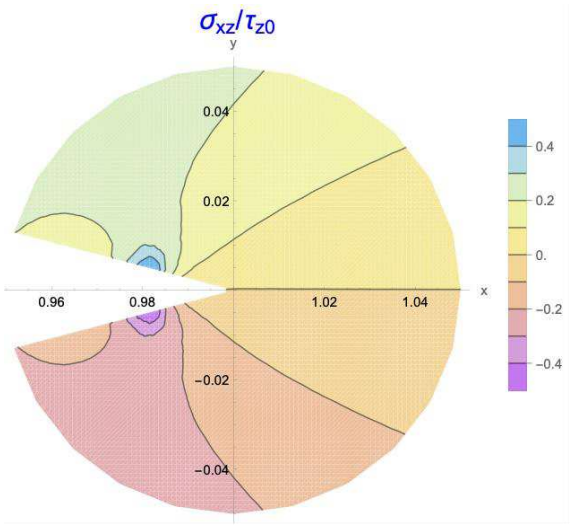
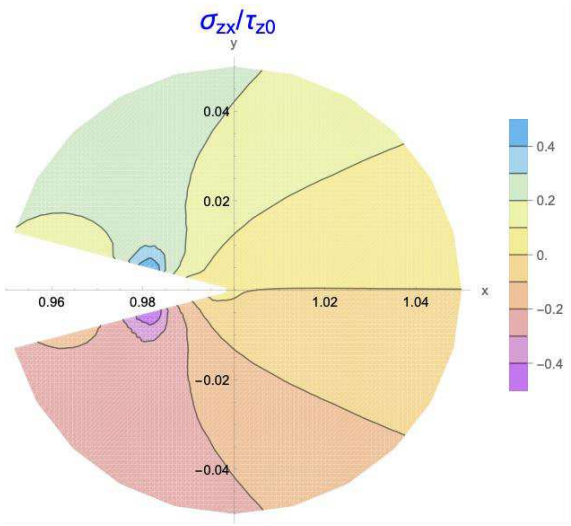


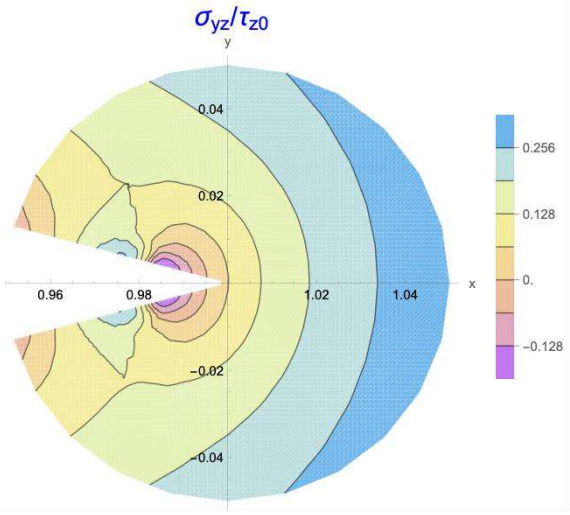
Fig. (1) Normalized stress and couple-stress components on the half-line, $x > l, y = 0$, under far-field anti-plane traction $\lim_{|y|\to\infty} \sigma_{yz}(x,y) = \tau_{z0}$.


σxz/τz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
0.4
0.2
0.
−0.2
−0.4

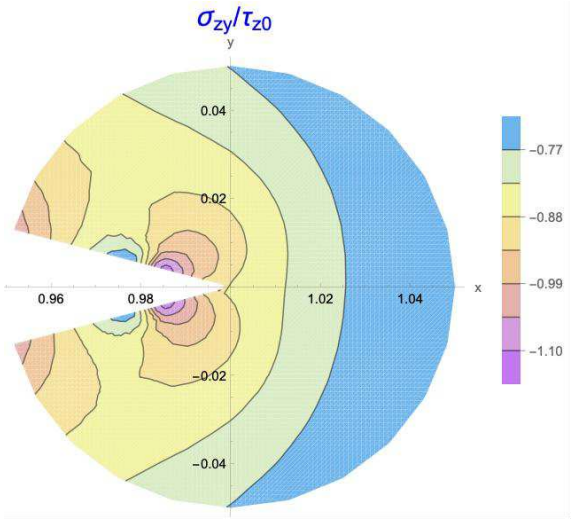
(a)


σzx/τz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
0.4
0.2
0.
−0.2
−0.4

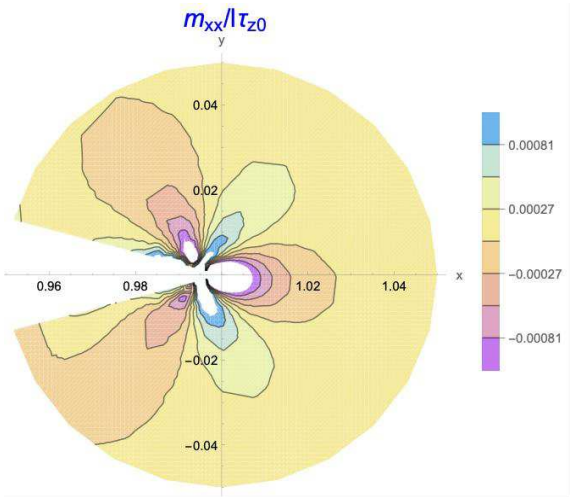
(b)


σyz/τz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
0.256
0.128
0.
−0.128

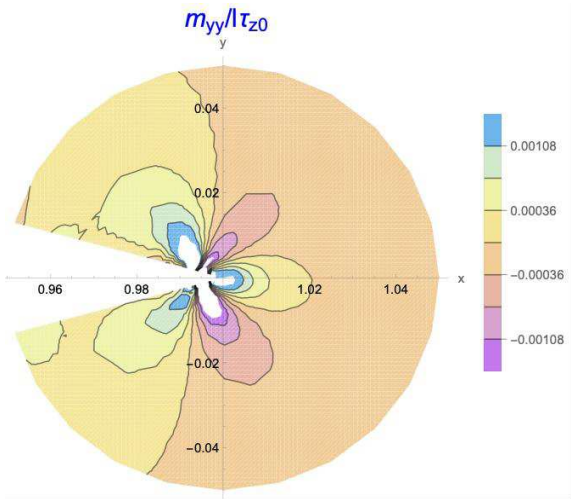
(c)


σzy/τz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
−0.77
−0.88
−0.99
−1.10

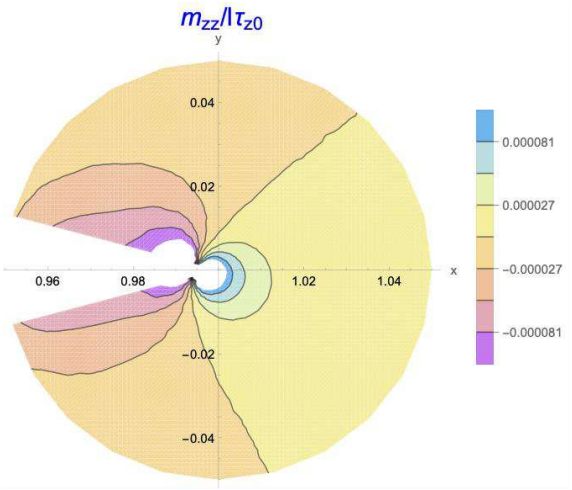
(d)


mxx/lτz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
0.00081
0.00027
−0.00027
−0.00081

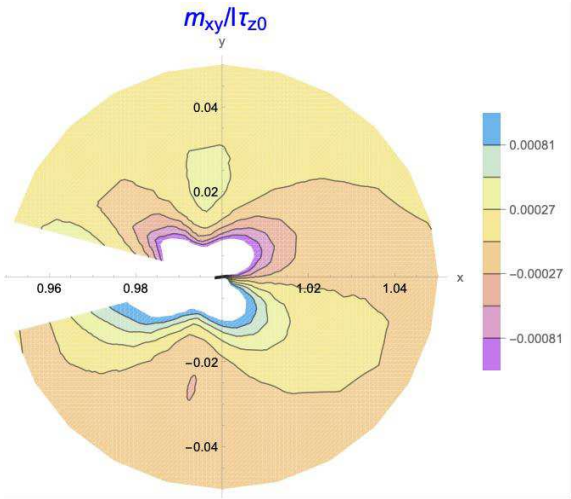
(e)


myy/lτz0
y
0.04
0.02
0.96
0.98
1.02
1.04
x
−0.02
−0.04
0.00108
0.00036
−0.00036
−0.00108

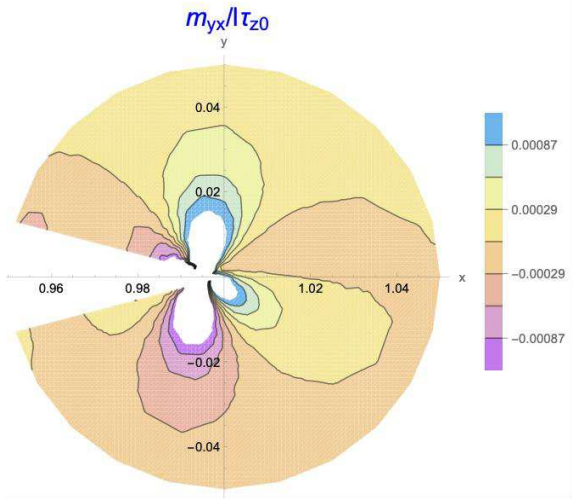
(f)



(g)



(h)



(i)

Fig. (2) Variations of stress and couple-stress components around a crack tip in a PSF plane under far-field anti-plane traction $\lim_{|y|\to\infty} \sigma_{yz}(x,y) = \tau_{z0}$.

In the second example, the cracked plane is under the far-field compressive traction $\lim_{|y|\to\infty}\sigma_{yy}(x,y) = \sigma_{yo} < 0$. The solution of Eqs (55) is substituted into Eqs (57) and the normalized crack displacements $\Delta u(x)\mu/l|\sigma_{yo}|$ and $\Delta v(x)\mu/l\,|\sigma_{yo}|$, $-l \le x \le l$, are plotted, Fig. (3). As a result of symmetry $\Delta u(-l < x < 0) = -\Delta u(0 < x < l)$ and $\Delta v(-l < x < 0) = \Delta v(0 < x < l)$. Since crack surface sliding takes place under the application of the pure normal traction, i.e., mixed mode I-II fracture prevails, negating the existence of the pure mode I or mode II in micro-polar planes. Furthermore, crack is opening (closing) under a compressive (tensile) far-field traction. The above two phenomena, crack opening under far-field compressive traction and the impossibility of a single in-plane mode of deformation, may be considered as the distinct features of the cracked micro-polar planes. The normalized

components of stress $\sigma_{ij}(x,0)/\sigma_{y0}$ and couple-stress $m_{ij}(x,0)/l\sigma_{y0}$, $x > l$ are depicted in Fig. (4).

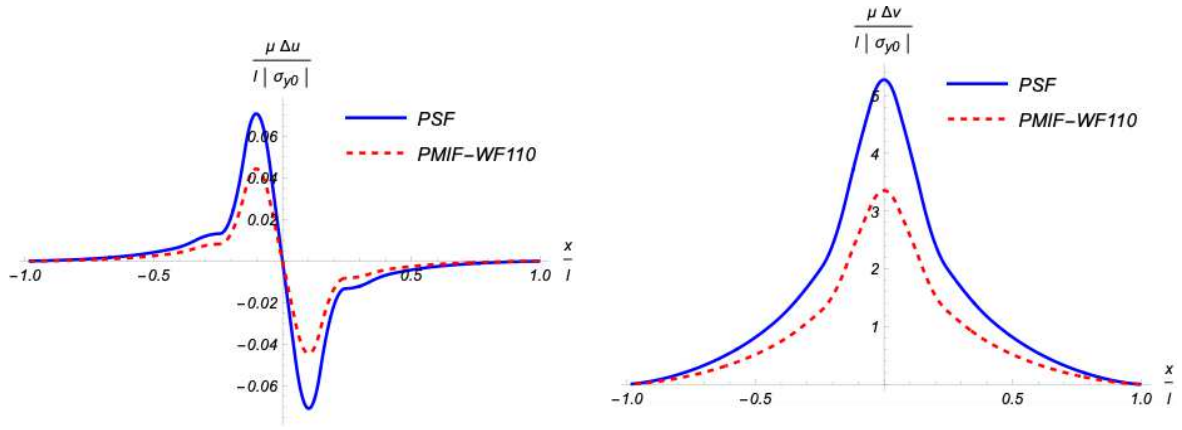
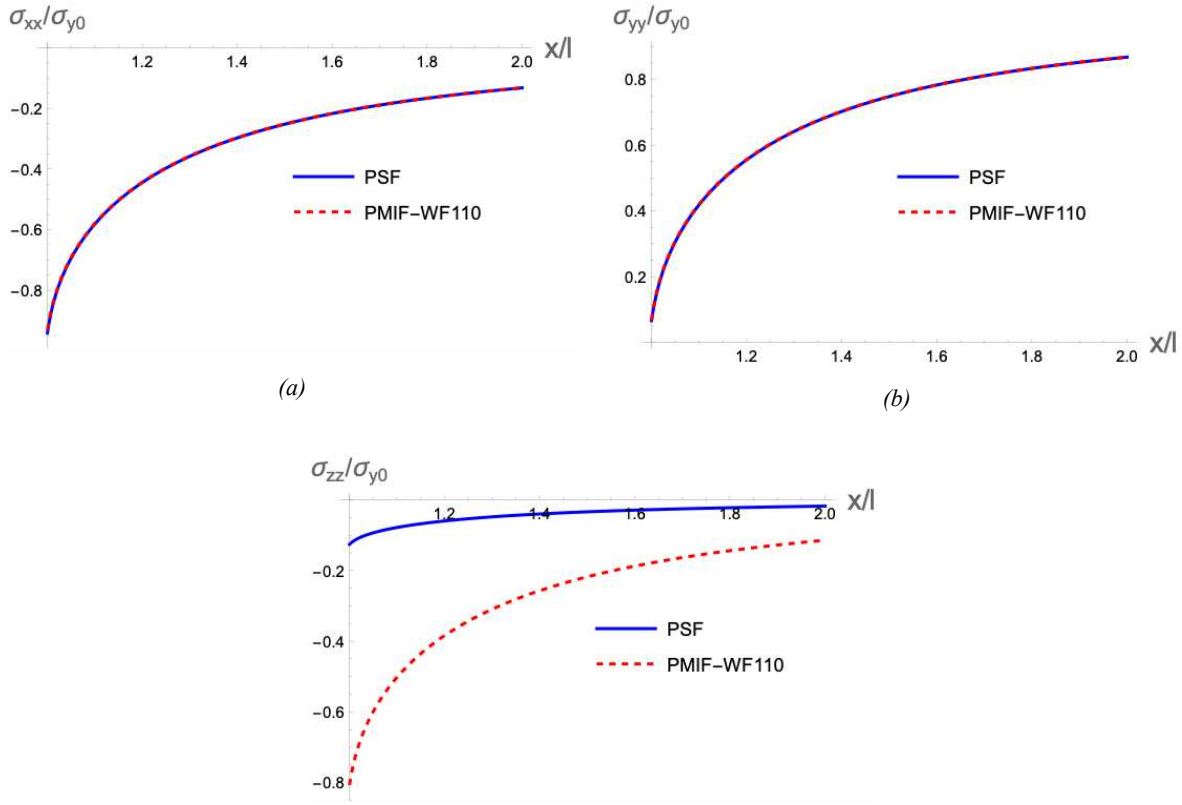


Fig. (3) Normalized crack opening and sliding for two micro-polar materials under far-field compressive traction $\lim_{|y|\to\infty} \sigma_{yy}(x,y) = \sigma_{yo} < 0$



(a)

(b)

(c)

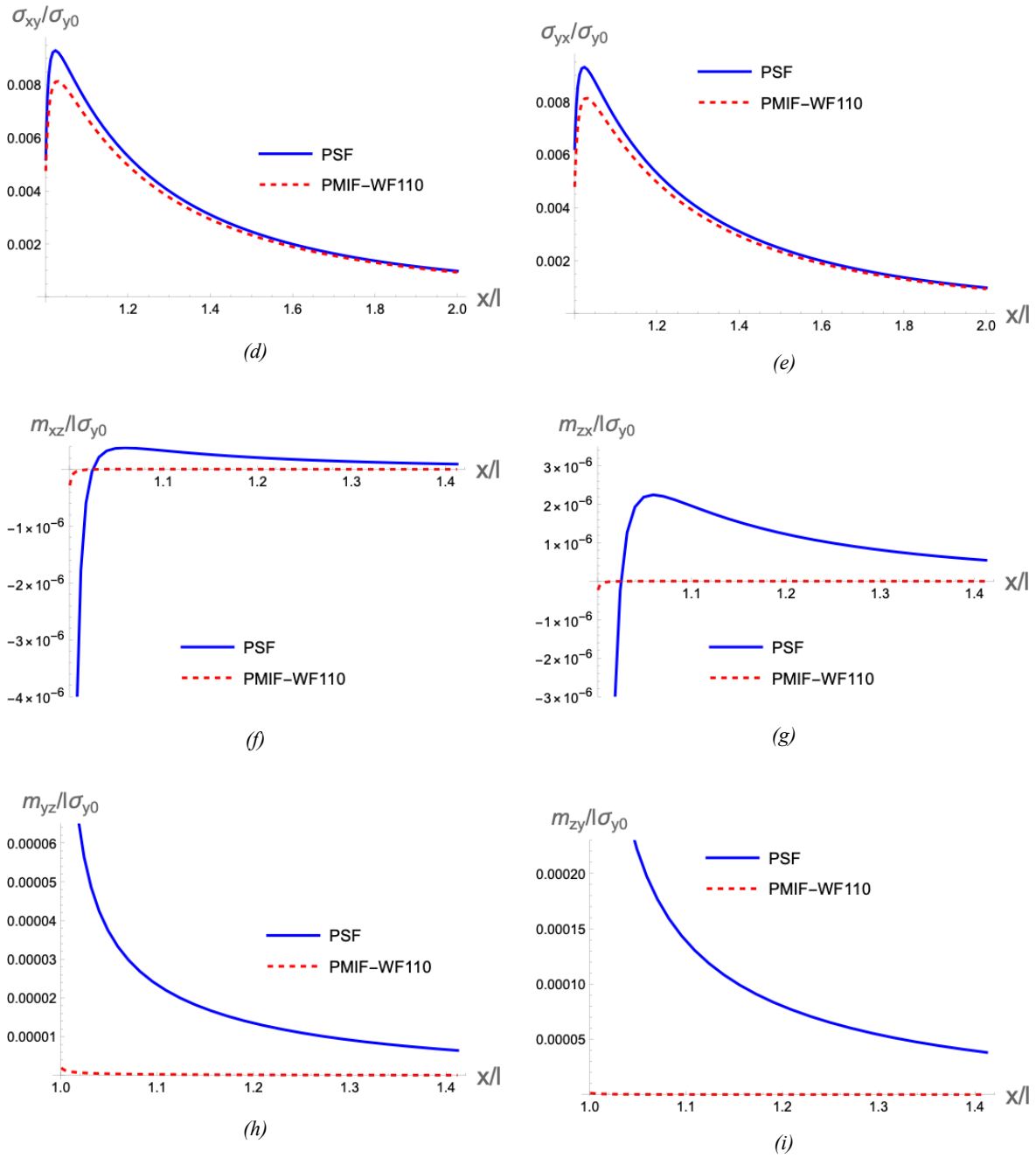
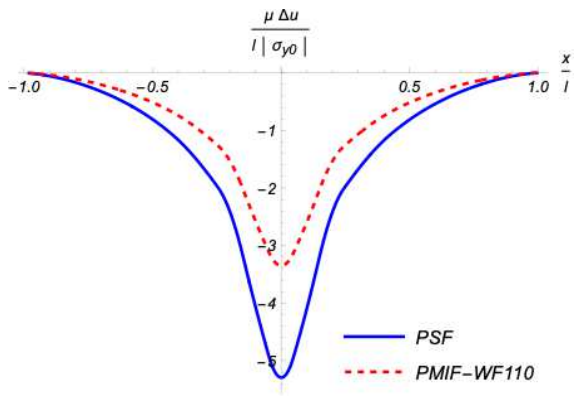
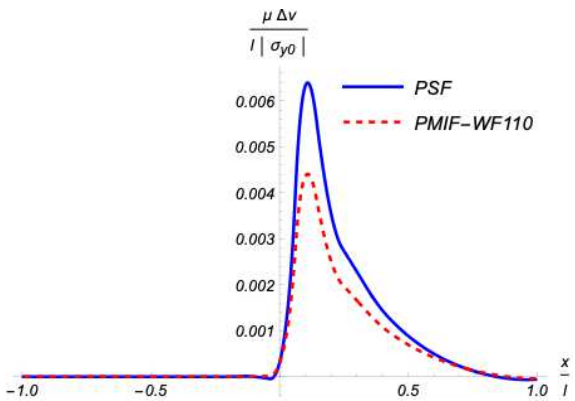


Fig. (4) Normalized stress and couple-stress components on the half-line, $x > l, y = 0$, under far-field traction $\sigma_{yy} = \sigma_{y0} < 0$

In the last example, plane is subjected to the far-field shear traction $\lim_{|y|\to\infty}\sigma_{yx}(x,y) = \tau_{yo}$. From the solution of Eqs (55) and (57), we observe that half of the crack surface $-l < x < 0$ is closed and the other half $0 < x < l$ is open, therefore, free of the normal traction. However, Eqs (55) are not valid for closed cracks and we should set $b_y(q) = 0$, $-1 < q \leq 0$ in Eqs (55) and (57), leading to a smaller system of the algebraic equations. The resultant normalized displacement components are shown in Fig. (5), wherein $\Delta u(-l < x < 0) = \Delta u(0 < x < l)$. Due to the lack of symmetry of the crack tips, variations of $\sigma_{ij}(x,0)/\tau_{y0}$ and $m_{ij}(x,0)/l\tau_{y0}$ on the half-lines $|x| > l$ are shown, Figs (6).

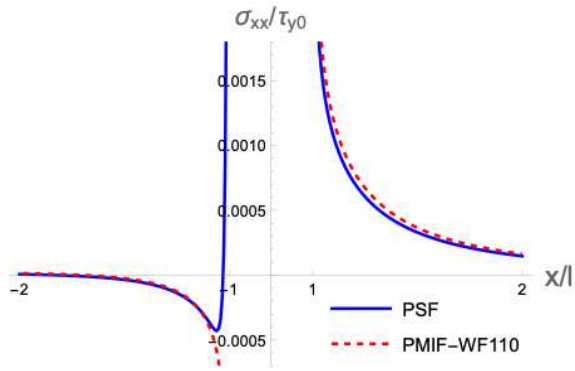


(a)

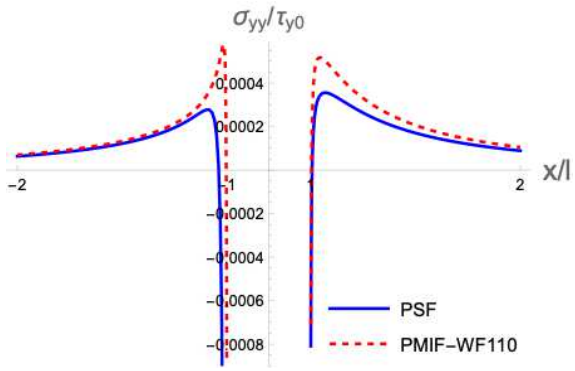


(b)

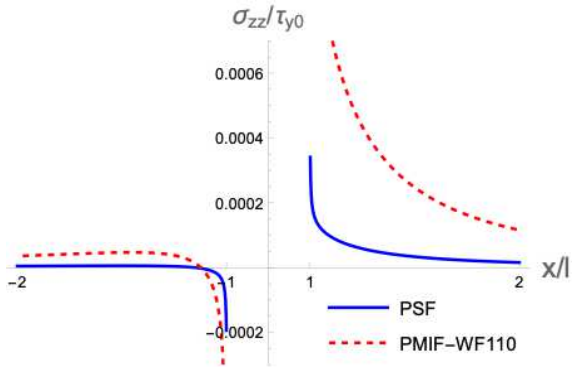
Fig. (5) Normalized crack opening and sliding for two materials under far-field shear $\lim_{|y|\to\infty} \sigma_{yx}(x,y) = \tau_{y0}$



(a)



(b)



(c)

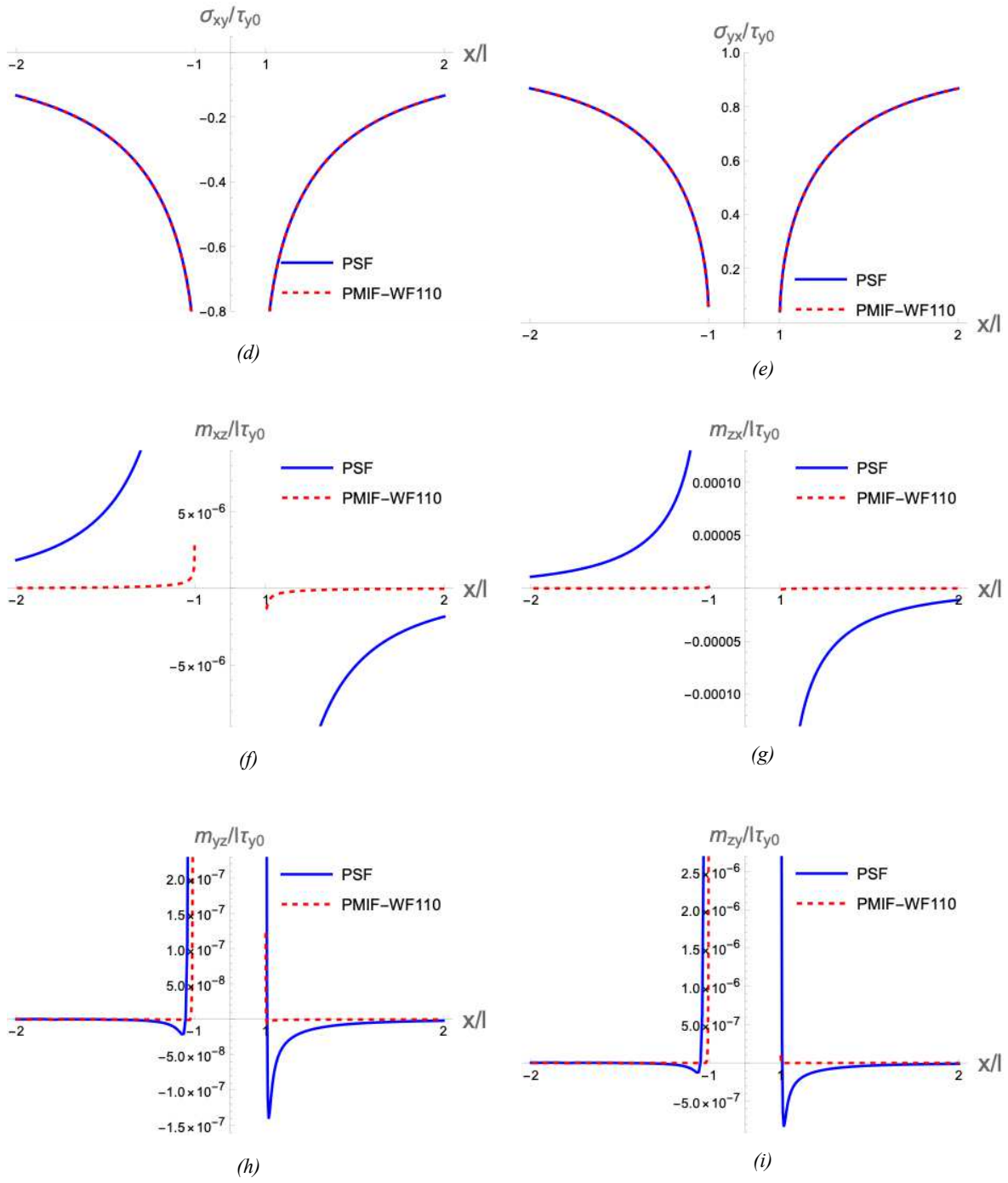


Fig. (6) Normalized stress and couple-stress components on the half-lines, $|x| > l, y = 0$, *under far-field shear* $\lim_{|y|\to\infty} \sigma_{yx}(x,y) = \tau_{y0}$

From Figs (1), (4), and (6), we observe that $\lim_{x\to\infty}\sigma_{ij}(x,0) = \lim_{|y|\to\infty}\sigma_{ij}(x,y)$, and $\lim_{x\to\infty} m_{ij}(x,0) = 0$. Thus, present solutions satisfy the applied far-field boundary conditions. Comparing the ratios of the elasticity constants with those of micro-polar constants of the two materials shows that the effects of micro-polar constants of PSF plane should be more pronounced than those of PMIF-WFI 10. These are in accord with the above mentioned figures, wherein the couple-stress components near the crack tip in the former material are much larger than those of the latter.

Conclusions

The solution to screw and edge dislocations in a micro-polar plane are derived, revealing that stress and couple-stress tensors are Cauchy singular at the dislocation location. An asymptotic analysis in the vicinity of a crack tip under anti-plane and in-plane deformations show that the stress tensor is bounded while the couple-stress tensor is square-root singular. These are in contrast with the square-root singularity of both tensors reported in the literature. However, by eliminating a micro-polar constant, $\kappa = 0$, both tensors become square-root singular, aligning with the earlier cited results. Systems of integral equations are developed for a cracked micro-polar plane under far-field anti-plane and in-plane tractions, utilizing the distributed dislocation technique. The systems include equations with multiple Cauchy singular terms. The solution to such a system of integral equations is not readily available. Therefore, generalization of an existing numerical procedure is required. Two infinite cracked micro-polar planes with different materials under far-field loading are analyzed, and stress and couple-stress tensors are determined in the vicinity of a crack tip. In the case of anti-plane deformation variations of the stress and couple-stress fields round a crack tip are illustrated and the effects of material micro-polar constants are investigated. The behavior of a crack in micro-polar planes under in-plane far-field traction is quite different from that using local elasticity formulation. First of all, mixed mode I-II prevails under either of the far-field normal and shear traction. Moreover, in micro-polar planes crack opening occurs while far-field traction is compressive and closes under tensile traction. Under a far-field shear traction half of the crack surface is opening.

Appendix

A numerical procedure is devised for the solution of the following system of three integral equations, having multiple Cauchy singular terms

$$\frac{1}{\pi}\int_{-1}^{1}\left\{b_z(q)\left[\frac{\mathrm{a}_{zi}}{p-q}+K_{zi}(p,q)\right]+d_x(q)\left[\frac{\mathrm{a}_{xi}}{p-q}+K_{xi}(p,q)\right]+d_y(q)\left[\frac{\mathrm{a}_{yi}}{p-q}+K_{yi}(p,q)\right]\right\}dq=f_i(p),\quad i\in\{1,2,3\} \tag{A.1}$$

where, $b_z(q)$, $d_x(q)$, $d_y(q)$, are unknown functions, a_{zi}, a_{xi}, a_{yi}, $i \in \{1,2,3\}$ are constants, the kernels $K_i(p,q)$ are bounded in the closed domain $-1 \le (p,q) \le 1$, and $f_i(p)$ are given sufficiently smooth functions. Consider the unknown bounded and continuous functions g_z, g_x, g_y, $-1 \le q \le 1$. The solution to Eqs (A.1) is obtained under the assumption that the unknown functions $b_z(q)$, $d_x(q)$, $d_y(q)$ have the following representations

$$b_z(q)=\sqrt{1-q^2}\,g_z(q),\qquad d_x(q)=\frac{g_x(q)}{\sqrt{1-q^2}},\qquad d_y(q)=\frac{g_y(q)}{\sqrt{1-q^2}} \tag{A.2}$$

Substituting (A.2) in (A.1), yields

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\sqrt{1-q^2}g_z(q)\left[\frac{\mathrm{a}_{zi}}{p-q}+K_{zi}(p,q)\right]+\frac{g_x(q)}{\sqrt{1-q^2}}\left[\frac{\mathrm{a}_{xi}}{p-q}+K_{xi}(p,q)\right]\right.$$
$$\left.+\frac{g_y(q)}{\sqrt{1-q^2}}\left[\frac{\mathrm{a}_{yi}}{p-q}+K_{yi}(p,q)\right]\right\}dq=f_i(p),\quad i\in\{1,2,3\} \tag{A.3}$$

In Eqs (A.3), the unknown functions are approximated with a sufficient degree of accuracy by the truncated series of Chebyshev polynomials of the first, $T_s(q)$, and second, $U_s(q)$, kinds as

$$g_z(q)\cong\sum_{s=1}^{N}G_{zs}U_s(q),\qquad g_x(q)\cong\sum_{s=1}^{N}G_{xs}T_s(q),\qquad g_y(q)\cong\sum_{s=1}^{N}G_{ys}T_s(q). \tag{A.4}$$

Utilizing the above expansions, Eqs (A.3) become

$$\frac{1}{\pi}\left\{\mathrm{a}_{zi}\sum_{s=1}^{N}G_{zs}\int_{-1}^{1}\frac{U_s(q)\sqrt{1-q^2}}{p-q}dq+\int_{-1}^{1}K_{zi}(p,q)g_z(q)\sqrt{1-q^2}dq\right.$$
$$+\mathrm{a}_{xi}\sum_{s=1}^{N}G_{xs}\int_{-1}^{1}\frac{T_s(q)}{(p-q)\sqrt{1-q^2}}dq+\int_{-1}^{1}\frac{g_x(q)}{\sqrt{1-q^2}}K_{xi}(p,q)dq$$
$$\left.+a_{yi}\sum_{s=1}^{N}G_{ys}\int_{-1}^{1}\frac{T_s(q)}{(p-q)\sqrt{1-q^2}}dq+\int_{-1}^{1}\frac{g_y(q)}{\sqrt{1-q^2}}K_{yi}(p,q)dq\right\}=f_i(p),$$
$$i\in\{1,2,3\} \tag{A.5}$$

In view of the identities

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_j(q)dq}{(p-q)\sqrt{1-q^2}}=\begin{cases}0, & j=0\\ -U_{j-1}(p), & j>0\end{cases}$$
$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_j(q)\sqrt{1-q^2}dq}{p-q}=T_{j+1}(p),\qquad -1<p<1 \tag{A.6}$$

Eqs (A.5) are simplified as

$$\mathrm{a}_{zi}\sum_{s=1}^{N}G_{zs}T_{s+1}(p)+\frac{1}{\pi}\int_{-1}^{1}\sum_{s=1}^{N}K_{zi}(p,q)G_{zs}U_s(q)\sqrt{1-q^2}\,dq-\mathrm{a}_{xi}\sum_{s=1}^{N}G_{xs}U_{s-1}(p)$$
$$+\frac{1}{\pi}\int_{-1}^{1}K_{xi}(p,q)\sum_{s=1}^{N}\frac{G_{xs}T_s(q)}{\sqrt{1-q^2}}dq-\mathrm{a}_{yi}\sum_{s=1}^{N}G_{ys}U_{s-1}(p)$$
$$+\frac{1}{\pi}\int_{-1}^{1}K_{yi}(p,q)\sum_{s=1}^{N}\frac{G_{ys}T_s(q)}{\sqrt{1-q^2}}dq=f_i(p),\qquad i\in\{1,2,3\} \tag{A.7}$$

The quadrature q_{uk}, q_{bk} and collocation p_{ur}, p_{br} points are chosen as

$$q_{uk}=\cos\left[\frac{\pi(2k-1)}{2N}\right],\quad k\in\{1,\dots,N\}\qquad p_{uk}=\cos\left(\frac{\pi k}{N}\right),\quad k\in\{1,\dots,N-1\}$$

$$q_{bk}=\cos\left[\frac{\pi k}{N+1}\right],\quad k\in\{1,\dots,N\}\qquad p_{bk}=\cos\left[\frac{\pi(2k-1)}{2(N+1)}\right],\quad k\in\{1,\dots,N+1\} \tag{A.8}$$

These points are the solutions of the following equations

$$T_n(q_{uk}) = 0, \qquad k \in \{1, \dots, N\} \qquad\qquad U_{n-1}(p_{uk}) = 0, \qquad k \in \{1, \dots, N-1\}$$

$$U_n(q_{bk}) = 0, \qquad k \in \{1, \dots, N\} \qquad\qquad T_{n+1}(p_{bk}) = 0, \qquad k \in \{1, \dots, N+1\} \tag{A.9}$$

The following identities for $-1 < x < 1$, hold, Erdogan and Gupta (1972)

$$\sum_{k=1}^{N} \frac{T_j(q_{uk})}{N(x - q_{uk})} = -U_{j-1}(x) + \frac{T_j(x)U_{N-1}(x)}{T_N(x)}, \qquad 0 < j < N$$

$$\sum_{k=1}^{N} \frac{(1 - q_{bk}^2)U_j(q_{bk})}{(N+1)(x - q_{bk})} = T_{j+1}(x) - \frac{U_j(x)T_{n+1}(x)}{U_N(x)}, \qquad 0 < j < N \tag{A.10}$$

At the collocation points Eqs (A.10) simplify as

$$\sum_{k=1}^{N} \frac{T_j(q_{uk})}{N(p_{ur} - q_{uk})} = -U_{j-1}(p_{ur}), \qquad 0 < j < N$$

$$\sum_{k=1}^{N} \frac{(1 - q_{bk}^2)U_j(q_{bk})}{(N+1)(p_{br} - q_{bk})} = T_{j+1}(p_{br}), \qquad 0 < j < N \tag{A.11}$$

Making use of the above identities, Eqs (A.7) at the colocation points may be written as

$$\begin{aligned}
&\mathrm{a}_{zi} \sum_{k=1}^{N} \frac{(1 - q_{bk}^2)g_z(q_{bk})}{(N+1)(p_{ur} - q_{bk})} + \mathrm{a}_{zi} \sum_{s=1}^{N} \frac{G_{zs}U_s(p_{ur})T_{N+1}(p_{ur})}{U_N(p_{ur})} + \frac{1}{\pi}\int_{-1}^{1} K_{zi}(p_{ur}, q)g_z(q)\sqrt{1 - q^2}\,dq \\
&\quad + \mathrm{a}_{xi} \sum_{k=1}^{N} \frac{g_x(q_{uk})}{N(p_{ur} - q_{uk})} + \frac{1}{\pi}\int_{-1}^{1} K_{xi}(p_{ur}, q)\frac{g_x(q)}{\sqrt{1 - q^2}}\,dq + \mathrm{a}_{yi} \sum_{k=1}^{N} \frac{g_y(q_{uk})}{N(p_{ur} - q_{uk})} \\
&\quad + \frac{1}{\pi}\int_{-1}^{1} K_{yi}(p_{ur}, q)\frac{g_y(q)}{\sqrt{1 - q^2}}\,dq = f_i(p_{ur}),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{a}_{zi} \sum_{k=1}^{N} \frac{(1 - q_{bk}^2)g_z(q_{bk})}{(N+1)(p_{bs} - q_{bk})} + \frac{1}{\pi}\int_{-1}^{1} K_{zi}(p_{bs}, q)g_z(q)\sqrt{1 - q^2}\,dq + \mathrm{a}_{xi} \sum_{k=1}^{N} \frac{g_x(q_{uk})}{N(p_{bs} - q_{uk})} \\
&\quad - \mathrm{a}_{xi} \sum_{s=1}^{N} \frac{G_{xs}T_s(p_{bs})U_{N-1}(p_{bs})}{T_N(p_{bs})} + \frac{1}{\pi}\int_{-1}^{1} K_{xi}(p_{bs}, q)\frac{g_x(q)}{\sqrt{1 - q^2}}\,dq + \mathrm{a}_{yi} \sum_{k=1}^{N} \frac{g_y(q_{uk})}{N(p_{bs} - q_{uk})} \\
&\quad - \mathrm{a}_{yi} \sum_{s=1}^{N} \frac{G_{ys}T_t(p_{bs})U_{N-1}(p_{bs})}{T_N(p_{bs})} + \frac{1}{\pi}\int_{-1}^{1} K_{yi}(p_{bs}, q)\frac{g_y(q)}{\sqrt{1 - q^2}}\,dq = f_i(p_{bs}),
\end{aligned}$$

$$r \in \{1, \dots, N-1\}, \qquad s \in \{1, \dots, N+1\}, \qquad i \in \{1,2,3\} \tag{A.12}$$

To carry out the integrations in Eqs (A.12), the quadrature rule of integration is employed leading to

$$\sum_{k=1}^{N}\frac{1-q_{bk}^2}{N+1}g_z(q_{bk})\left[\frac{\mathrm{a}_{zi}}{p_{ur}-q_{bk}}+K_{zi}(p_{ur},q_{bk})\right]+\mathrm{a}_{zi}\frac{T_{N+1}(p_{ur})}{U_N(p_{ur})}g_z(p_{ur})$$
$$+\frac{1}{N}\sum_{k=1}^{N}g_x(q_{uk})\left[\frac{\mathrm{a}_{xi}}{p_{ur}-q_{uk}}+K_{xi}(p_{ur},q_{uk})\right]$$
$$+\frac{1}{N}\sum_{k=1}^{N}g_y(q_{uk})\left[\frac{\mathrm{a}_{yi}}{p_{ur}-q_{uk}}+K_{yi}(p_{ur},q_{uk})\right]=f_i(p_{ur}),$$

$$\sum_{k=1}^{N}\frac{1-q_{bk}^2}{N+1}g_z(q_{bk})\left[\frac{\mathrm{a}_{zi}}{p_{bs}-q_{bk}}+K_{zi}(p_{bs},q_{bk})\right]+\frac{1}{N}\sum_{k=1}^{N}g_x(q_{uk})\left[\frac{\mathrm{a}_{xi}}{p_{bs}-q_{uk}}+K_{xi}(p_{bs},q_{uk})\right]$$
$$-\mathrm{a}_{xi}\frac{U_{N-1}(p_{bs})}{T_N(p_{bs})}g_x(p_{bs})+\frac{1}{N}\sum_{k=1}^{N}g_y(q_{uk})\left[\frac{\mathrm{a}_{yi}}{p_{bs}-q_{uk}}+K_{yi}(p_{bs},q_{uk})\right]$$
$$-\mathrm{a}_{yi}\frac{U_{N-1}(p_{bs})}{T_N(p_{bs})}g_y(p_{bs})=f_i(p_{bs}),$$
$$r\in\{1,\dots,N-1\},\qquad s\in\{1,\dots,N+1\},\qquad i\in\{1,2,3\}\quad\text{(A.13)}$$

References

Atkinson, C., and F. G. Leppington. "*The effect of couple stresses on the tip of a crack*." International Journal of Solids and Structures 13, no. 11 (1977): 1103-1122. https://doi.org/10.1016/0020-7683(77)90080-4

Diegele, Eberhard, Rainer ElsÄßer, and Charalampos Tsakmakis. "*Linear micropolar elastic crack-tip fields under mixed mode loading conditions*." International journal of fracture 129, no. 4 (2004): 309-339. https://doi.org/10.1023/B:FRAC.0000049492.13523.5a

Dyszlewicz, Janusz. *Micropolar theory of elasticity*. Vol. 15. Springer Science & Business Media, 2004.

Erdogan, Fazil, and G.D. Gupta. "*On the numerical solution of singular integral equations*." Quarterly of Applied mathematics 29, no. 4 (1972): 525-534. https://doi.org/10.1090/qam/408277

Eremeyev, Victor A., Leonid P. Lebedev, and Holm Altenbach. *Foundations of micropolar mechanics*. Springer Science & Business Media, 2012.

Eringen, A. Cemal. "*Linear theory of micropolar elasticity.*" Journal of Mathematics and Mechanics (1966): 909-923. https://www.jstor.org/stable/24901442

Garajeu, Mihail, and Eugen Soós. "*Cosserat models versus crack propagation*." Mathematics and Mechanics of Solids 8, no. 2 (2003): 189-218. https://doi.org/10.1177/108128603029769

Hassanpour, Soroosh, and Glenn R. Heppler. "*Micropolar elasticity theory: a survey of linear isotropic equations, representative notations, and experimental investigations*." Mathematics and Mechanics of Solids (2015): 1-9, https://doi.org/10.1177/1081286515581183

Ieşan, Dorin. "*On the linear theory of micropolar elasticity*." International Journal of Engineering Science 7, no. 12 (1969): 1213-1220. https://doi.org/10.1016/0020-7225(69)90030-5

Li, Yong-Dong, and Kang Yong Lee. "*Fracture analysis in micropolar elasticity: anti-plane crack*." International journal of fracture 152, no. 2 (2008): 163-168. https://doi.org/10.1007/s10704-008-9277-4

Li, Yong-Dong, and Kang Yong Lee. "*Fracture analysis in micropolar elasticity: mode-I crack*." International journal of fracture 156, no. 2 (2009): 179-184. https://doi.org/10.1007/s10704-009-9358-z

Muskhelishvili, Nikolaĭ Ivanovich, Singular integral equations. Noordhoff International Pub. The Netherlands. (1963).

Nowacki, Witold, and International centre for mechanical sciences. courses and lectures; no. 25. *Theory of micropolar elasticity*. Vol. 494. Berlin: Springer, 1972.

Paul, H. S., and K. Sridharan. "*The problem of a griffith crack in micropolar elasticity*." International Journal of Engineering Science 19, no. 4 (1981): 563-579. https://doi.org/10.1016/0020-7225(81)90090-2

Shmoylova, E., S. Potapenko, and L. Rothenburg. "*Boundary element analysis of stress distribution around a crack in plane micropolar elasticity*." International journal of engineering science 45, no. 2-8 (2007): 199-209. https://doi.org/10.1016/j.ijengsci.2007.04.006

Tuna, Meral, Patrizia Trovalusci, and Nicholas Fantuzzi. "*On quasi-brittle static fracture analysis of micropolar plates via XFEM model*." International Journal of Engineering Science 206 (2025): 104168. https://doi.org/10.1016/j.ijengsci.2024.104168

Yu, Hongjun, Yingbin Zhang, and Jianshan Wang. "*Crack-tip fields of an anti-plane crack in micropolar elastic solids*." European Journal of Mechanics-A/Solids 105 (2024): 105267. https://doi.org/10.1016/j.euromechsol.2024.105267

Xie, Jinchen, and Christian Linder. "*Full field crack solutions in anti-plane flexoelectricity*." Theoretical and Applied Fracture Mechanics 134 (2024): 104674. https://doi.org/10.1016/j.tafmec.2024.104674