

Closed-form autonomous optimal cloning on spin networks beyond the star: fractional revival, perfect transfer and balanced beam splitters in symmetric sectors

Hikaru Wakaura

hikaruwakaura@gmail.com

QuantScape Inc

Research Article

Keywords:

Posted Date: October 19th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11396207/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: The authors declare no competing interests.

Closed-form autonomous optimal cloning on spin networks beyond the star: fractional revival, perfect transfer and balanced beam splitters in symmetric sectors

Hikaru Wakaura^{1,*}

¹*QIRI (Quantum Integrated Research Institute Inc.),
1-16-3 Akasaka, Minato-ku, Tokyo 107-0061, Japan*

(Dated: October 16, 2026)

Copying quantum information is forbidden by the no-cloning theorem, yet approximate copies with the highest fidelity allowed by quantum mechanics are a basic resource for quantum networks and for assessing the security of quantum key distribution. Such cloners are usually gate sequences, but a spin network with fixed couplings can perform the transformation autonomously: the unknown qubit and blank qubits sit on the vertices of a graph and the free evolution does the rest. Until now, the only spin graph known to reach the optimal phase-covariant fidelity for every number of clones M was the star with a tuned anisotropy. Which other graphs can clone optimally, and why, has remained open. Here we show that the complete bipartite graph $K_{2,M-1}$ —the unknown qubit and one target on one side, the remaining targets on the other—reaches the optimal $1 \rightarrow M$ phase-covariant fidelity for every $M \geq 3$, with all couplings given in closed form. For odd M a staggered Ising anisotropy suffices, with the parameters fixed by two integers; for even M an additional Ising coupling among the middle targets is required, with its value an explicit algebraic number. Both families reduce, in the permutation-symmetric sectors, to two four-level problems: a perfect-state-transfer ladder on a four-cycle and a freeze block whose characteristic polynomial we obtain analytically. We verify the formulas with no free parameter up to $M = 10$ and classify exactness for all four-vertex graphs with a least-squares test of the exact revival and transfer equations. The results connect the fractional-revival engineering of spin chains to optimal cloning and identify the hidden involution behind both. Beyond cloning, optimal quantum machines can thus be read off from the spectrum of small symmetric blocks, suggesting autonomous implementations of other covariant transformations on spin and Rydberg arrays.

I. INTRODUCTION

Information that can be copied can be distributed, stored redundantly and verified. Quantum information cannot be copied exactly [1], but it can be copied approximately with a state-independent fidelity whose maximum is fixed by quantum mechanics [2–4]. These optimal cloning machines are the standard model of an eavesdropper on quantum key distribution [5] and the building block of multi-user protocols such as telecloning [6], which has recently been demonstrated with eight-photon Dicke states [7] and across silicon photonic chips [8]. For qubits prepared on the equator of the Bloch sphere, the relevant limit is the phase-covariant bound of Fan *et al.* [9], $F_{\text{opt}}(M) = \frac{1}{2} + \frac{M+1}{4M}$ for odd M and $\frac{1}{2} + \frac{\sqrt{M(M+2)}}{4M}$ for even M .

An optimal cloner is a unitary and can be realised by a sequence of gates. It can also be realised *autonomously*: De Chiara *et al.* showed that the free evolution of an XY spin star, with the unknown qubit at the centre and blank qubits on the leaves, performs phase-covariant cloning without any external control and with better resilience to decoherence than the equivalent gate circuit [10, 11]. Chen *et al.* then showed that, if the leaves are prepared in a Dicke state and the star carries an XXZ anisotropy, the optimal bound is reached for every M [12]. In parallel,

a large literature has engineered spin chains and graphs whose free dynamics transfers or splits a single excitation perfectly—perfect state transfer [13, 18] and fractional revival [19–23]—and it was noted that fractional revival between two vertices generates entanglement but does not perfectly clone, and cannot by the no-cloning theorem [21]; cloning on chains was instead pursued through engineered boundary entanglement [25].

However, the star has remained the only spin graph for which autonomous optimal cloning is known for every M (generalised stars with ring couplings [26], open chains [25] and Heisenberg-star parametrisations of state-dependent cloning [27] have been studied, but not shown to reach $F_{\text{opt}}(M)$ for all M), and no principle has been available to decide which graphs can clone optimally or how the fractional-revival design rules relate to cloning. In this study, we answer both questions. We first show that in the single-excitation sector the fractional-revival amplitude vector of any graph is literally the clone-fidelity vector, $F_s = \frac{1}{2} + |\beta_s|/2$, which bounds symmetric cloning by $\frac{1}{2} + 1/(2\sqrt{M})$ and explains why fractional-revival design reaches the optimum only for $M = 2$. We then find that the complete bipartite graph $K_{2,M-1}$ reaches $F_{\text{opt}}(M)$ for every $M \geq 3$ and we give all couplings in closed form (Theorems 1 and 2). The result shows that optimal cloning on a graph is a property of two four-dimensional symmetric blocks—a perfect-state-transfer ladder and a freeze block with a two-fold zero mode—and that a hidden involution, not a graph automorphism, is what makes the star special and what can

* h.wakaura@qiri.co.jp

be re-created elsewhere.

II. SETTING AND THE SINGLE-EXCITATION SECTOR

We consider $n = M + 1$ qubits on a graph $G = (V, E)$ with

$$H = \sum_{(i,j) \in E} \left[\frac{J_{ij}}{2} (X_i X_j + Y_i Y_j) + \frac{g_{ij}}{2} Z_i Z_j \right], \quad (1)$$

so that in the single-excitation sector the XX part is a continuous-time quantum walk with hopping matrix J . Vertex a holds the unknown equatorial qubit $|\psi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$; the M targets $T = V \setminus \{a\}$ are prepared in the Dicke state $|D_k\rangle$ with k excitations; the system evolves freely for a time t ; each target is read out after a fixed, input-independent local Z rotation. The figure of merit is the equatorial average of the single-site fidelity F_s .

Proposition 1. With all targets in $|0\rangle$ ($k = 0$) and $U(t)|a\rangle = \sum_s \beta_s(t)|s\rangle$ the quantum-walk amplitudes,

$$F_s(t) = \frac{1}{2} + \frac{1}{2}|\beta_s(t)|. \quad (2)$$

Proof. Total magnetisation is conserved, so the input $(\alpha|0\rangle_a + \beta|1\rangle_a)|0 \cdots 0\rangle$ evolves as $\alpha e^{-iE_0 t}|0 \cdots 0\rangle + \beta \sum_s \beta_s(t)|s\rangle$, where $E_0 = \sum_{(ij)} g_{ij}/2$ is the energy of the vacuum, $|s\rangle$ denotes the single excitation at vertex s , and $\beta_s(t) = \langle s|e^{-iHt}|a\rangle$ is the amplitude of the continuous-time quantum walk generated by H restricted to the single-excitation sector (the hopping matrix J plus the diagonal ZZ potential) [13, 15]. Tracing out all vertices but s gives $\rho_{00} = |\alpha|^2 + |\beta|^2(1 - |\beta_s|^2)$, $\rho_{11} = |\beta|^2|\beta_s|^2$ and $\rho_{01} = \alpha\beta^*\beta_s^*e^{-iE_0 t}$, because only the component $|s\rangle$ leaves the other vertices in the vacuum. With $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$ the fidelity after one input-independent Z rotation (which absorbs the phases of β_s and $e^{-iE_0 t}$, Appendix D) is $F_s = \frac{1}{4} + \frac{1}{4}(1 - |\beta_s|^2) + \frac{1}{4}|\beta_s|^2 + \frac{1}{2}|\beta_s| = \frac{1}{2} + \frac{1}{2}|\beta_s|$. $\square$ We verified the identity to 10^{-9} on paths, cycles, the star and the Krawtchouk chain. For a symmetric $1 \rightarrow q$ cloner all $|\beta_s|$, $s \in T$, are equal, so $q|\beta_s|^2 \leq \sum_s |\beta_s|^2 \leq 1$ gives the bound below. Hence symmetric $1 \rightarrow q$ cloning in this sector obeys $F \leq \frac{1}{2} + 1/(2\sqrt{q})$, which equals F_{opt} only for $q = 2$ (0.8536) and falls below it for $q \geq 3$ (0.789 vs 0.833 at $q = 3$). Fractional revival over a vertex subset [21, 22, 24] is therefore an optimal cloner design only for two clones; for more clones the multi-excitation sectors are unavoidable. On the star, Chen's bound $F \leq \frac{1}{2} + \max_{\pm} |\langle D_{k\pm 1}|S_{\pm}|D_k\rangle|/(2M)$ [12] (Ref. [12] counts the $|0\rangle$'s of the leaves; our k counts excitations, the two being related by a global spin flip) is the multi-excitation version of Proposition 1, and we find (Appendix A) that it is attained at finite time only for $k \in \{0, M, \lfloor M/2 \rfloor, \lceil M/2 \rceil\}$; for other k it is a supremum approached by Diophantine alignment of two incommensurate Rabi frequencies.

III. ODD M : STAGGERED ANISOTROPY ON $K_{2,m}$

Let $M = 2k + 1$, $m = 2k$, and let $G = K_{2,m}$ with the input a and one target f on one side and m ‘‘middle’’ targets on the other [Fig. 1(c)]. Put $J_{ai} = J_{if} = J$ and the staggered anisotropy $g_{ai} = -g_{if} = g$.

Theorem 1. With the targets in $|D_k\rangle$,

$$t = \frac{n\pi}{2J\sqrt{k(k+1)}}, \quad n \text{ odd}, \quad (3)$$

$$\frac{g^2}{J^2} = 4\left(\frac{n'}{n}\right)^2 k(k+1) - k(k+1) + 1, \quad n' \in \mathbb{N}, \quad (4)$$

gives $F_s = F_{\text{opt}}(M)$ on all M targets.

Derivation. H is invariant under the permutation group S_m of the middle vertices and the input lies in the S_m -symmetric subspace, spanned by the normalised states $|n_a, n_{\text{mid}}, n_f\rangle = |n_a\rangle_a |D_{n_{\text{mid}}}^{(m)}\rangle |n_f\rangle_f$ labelled by the occupations of the input, of the middle set and of f . In this basis the XY part acts through the Dicke ladder elements [16]

$$\langle D_{j+1}^{(m)} | S_+^{\text{mid}} | D_j^{(m)} \rangle = \sqrt{(j+1)(m-j)}, \quad (5)$$

where $S_{\pm}^{\text{mid}} = \sum_{i \in \text{mid}} \sigma_i^{\pm}$; e.g. $\langle 0, j+1, n_f | H | 1, j, n_f \rangle = J\sqrt{(j+1)(m-j)}$ (the input hands its excitation to the middle set) and $\langle n_a, j, 1 | H | n_a, j+1, 0 \rangle = J\sqrt{(j+1)(m-j)}$, while the staggered ZZ part is diagonal, $\frac{g}{2}(Z_a - Z_f) \sum_{i \in \text{mid}} Z_i \rightarrow \frac{g}{2}(z_a - z_f)(m - 2n_{\text{mid}})$ with $z = +1, -1$ for $n = 0, 1$. The Dicke state of all $M = m + 1$ targets decomposes with binomial weights, $|D_j\rangle = \sqrt{\frac{m-j+1}{m+1}} |D_j^{(m)}\rangle |0\rangle_f + \sqrt{\frac{j}{m+1}} |D_{j-1}^{(m)}\rangle |1\rangle_f$. Since the total magnetisation is conserved, the two branches of the input, $|0\rangle_a |D_k\rangle$ with k excitations and $|1\rangle_a |D_k\rangle$ with $k+1$, evolve in two separate blocks.

Transfer block ($k+1$ excitations). With $b_1 = |1, k, 0\rangle$, $b_2 = |0, k+1, 0\rangle$, $b_3 = |0, k, 1\rangle$, $b_4 = |1, k-1, 1\rangle$ and $m = 2k$, the diagonal $\frac{g}{2}(z_a - z_f)(2k - 2n_{\text{mid}})$ vanishes on all four states ($z_a = z_f$ on b_2, b_4 and $n_{\text{mid}} = k$ on b_1, b_3). By Eq. (5) the hoppings $b_1 b_2, b_2 b_3, b_3 b_4, b_4 b_1$ all equal $J\sqrt{k(k+1)} = Js$, and $b_1 b_3, b_2 b_4$ vanish (a and f are not adjacent). Hence $H_T = Js A(C_4)$ with the four-cycle $b_1 b_2 b_3 b_4$, i.e. the hypercube $K_2 \square K_2$, whose walk $e^{-i\tau A} = e^{-i\tau X} \otimes e^{-i\tau X}$ sends every vertex to its antipode, with the sign -1 , at $\tau = n\pi/2$, n odd [13]. The initial and target vectors, $|1\rangle_a |D_k\rangle = \sqrt{\frac{k+1}{2k+1}} b_1 + \sqrt{\frac{k}{2k+1}} b_4$ and $|0\rangle_a |D_{k+1}\rangle = \sqrt{\frac{k}{2k+1}} b_2 + \sqrt{\frac{k+1}{2k+1}} b_3$, are supported on the two antipodal pairs with matching weights, so

$$e^{-iH_T t} |1\rangle_a |D_k\rangle = -|0\rangle_a |D_{k+1}\rangle \quad \text{iff} \quad 2Jst \in \pi(2\mathbb{Z} + 1), \quad (6)$$

the first condition of Theorem 1.

Freeze block (k excitations). With $c_1 = |0, k, 0\rangle$, $c_2 = |0, k-1, 1\rangle$, $c_3 = |1, k-1, 0\rangle$, $c_4 = |1, k-2, 1\rangle$ the diagonal

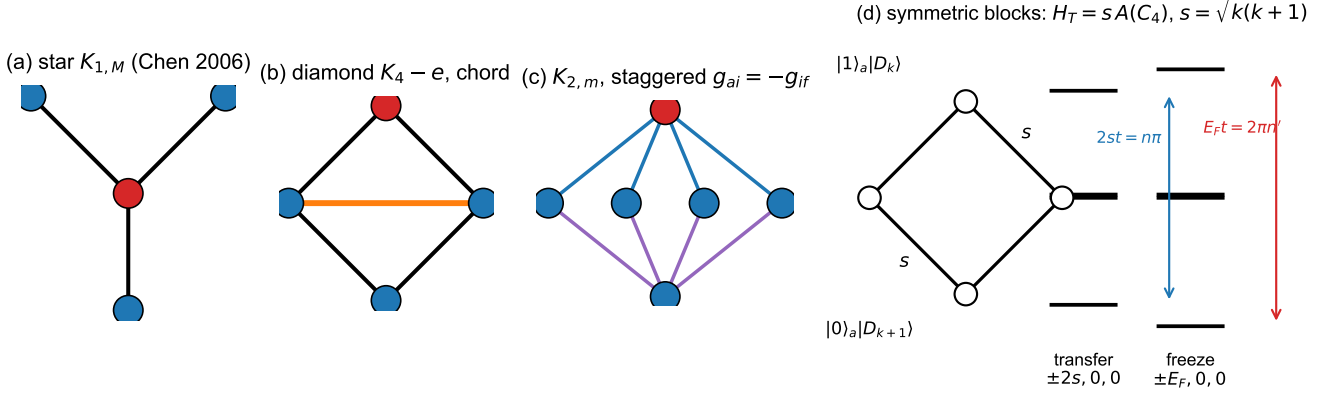


FIG. 1. Autonomous optimal cloning needs two commensurate four-level symmetric blocks, not a special graph. (a) The star of Chen *et al.*; (b) the diamond $K_4 - e$ with a chord (orange), exact with uniform anisotropy; (c) $K_{2,m}$ with staggered anisotropy, exact for all odd M (Theorem 1), and with a middle Ising coupling for all even M (Theorem 2). Red: input; blue: targets. (d) The mechanism in the permutation-symmetric sectors.

is $(0, 2g, -2g, 0)$ and Eq. (5) gives the hoppings $c_1c_2 = c_1c_3 = Js$ and $c_2c_4 = c_3c_4 = Jr$ with $r^2 = (k-1)(k+2)$:

$$H_F = J \begin{pmatrix} 0 & s & s & 0 \\ s & 2g/J & 0 & r \\ s & 0 & -2g/J & r \\ 0 & r & r & 0 \end{pmatrix}, \quad (7)$$

whose characteristic polynomial is $\lambda^2(\lambda^2 - E_F^2)$ with $E_F^2 = J^2(2s^2 + 2r^2) + 4g^2 = 4(J^2(k^2 + k - 1) + g^2)$ (expand along the first row; Appendix D). The spectrum $\{0, 0, \pm E_F\}$ gives $e^{-iH_F t} = P_0 + \cos(E_F t)(P_+ + P_-) - i \sin(E_F t)(P_+ - P_-) = \mathbb{I}$ whenever $E_F t \in 2\pi\mathbb{Z}$, so the whole block, and in particular $|0\rangle_a|D_k\rangle = \sqrt{\frac{k+1}{2k+1}}c_1 + \sqrt{\frac{k}{2k+1}}c_2$, revives with phase $+1$. Inserting $t = n\pi/(2Js)$ into $E_F t = 2\pi n'$ gives the second formula of Theorem 1. The output $\alpha|D_k\rangle - \beta|D_{k+1}\rangle$ on the targets is the economical optimal transformation of Fan *et al.* [9] [their Eq. (33) with $k = (M-1)/2$] up to the relative sign, which the readout removes by Z on every target $[\prod_s Z_s$ multiplies $|D_j\rangle$ by $(-1)^j$]; its fidelity $\frac{1}{2} + \langle D_{k+1}|S_+|D_k\rangle/(2M) = \frac{1}{2} + (k+1)/(2M) = F_{\text{opt}}(M)$ follows from Eq. (5) on the M targets, as in Eq. (7) of Ref. [12]. $\square$

We verified the formulas, with no free parameter, for $M = 3, 5, 7, 9$ to $|F - F_{\text{opt}}| \leq 2 \times 10^{-15}$ [Fig. 2].

Two remarks. (i) $m = 2$ is the four-cycle: with a uniform anisotropy $g_{ij} = \lambda J_{ij}$ the cycle is *not* an exact cloner (the least-squares residual of the exact equations stays above 10^{-3}), with staggered anisotropy it is. (ii) An isotropic Heisenberg chord inside the middle set is a constant on the symmetric sector; the “diamond” $K_4 - e$ with uniform anisotropy and a chord, which we found first by numerical search, and the $K_{2,m}$ family are two ways to create the same two-level freeze block, by a chord or by staggering (Appendix B gives the diamond’s own countable family).

IV. EVEN M : A MIDDLE ISING COUPLING AND A BALANCED BEAM SPLITTER

Let $M = 2k$, $m = 2k - 1$, and add to $K_{2,m}$ Ising couplings g_c between all pairs of middle targets, with $J_{ai} = 1$, $J_{if} = J_q$, $g_{ai} = g_p$, $g_{if} = g_q$.

Theorem 2. Define $B = k(k+1)m^2$, $A = B + M$ and

$$g_p = \sqrt{A} - \sqrt{B}, \quad g_q = J_q + g_c, \quad g_c = \frac{(2k-1)\sqrt{A} - (2k+1)\sqrt{B}}{2k}, \quad (8)$$

so that $g_p(g_p - g_c) = 1$. Let $E_1 < \dots < E_4$ be the eigenvalues of the k -excitation symmetric block (identically $E_2 - E_1 = E_4 - E_3$). For every (J_q, t) with $(E_2 - E_1)t = n_1\pi$ (n_1 odd) and $(E_3 - E_2)t = n_2\pi$ (n_2 even), the targets in $|D_k\rangle$ give $F_s = F_{\text{opt}}(M)$ on all targets.

Derivation. For even M the optimal transformation of Fan *et al.* [9] (their Sec. V, $\alpha_j = 1/\sqrt{2}$ for $j = M/2 - 1, M/2$) is the balanced mixture of two Dicke states, which Chen *et al.* realise on the star with the input qubit as ancilla [12]. On the targets we therefore require

$$\begin{aligned} \sqrt{2}U|0\rangle_a|D_k\rangle &= e^{i\varphi_1}|0\rangle_a|D_k\rangle + e^{i\varphi_2}|1\rangle_a|D_{k-1}\rangle, \\ \sqrt{2}U|1\rangle_a|D_k\rangle &= e^{i(\varphi_1+\delta)}|0\rangle_a|D_{k+1}\rangle + e^{i(\varphi_2+\delta)}|1\rangle_a|D_k\rangle, \end{aligned} \quad (9)$$

up to one local Z phase per target. The single-site coherence of this output is, by Eq. (5) on the M targets, $\alpha^*\beta[\sqrt{(k+1)(M-k)} + \sqrt{k(M-k+1)}]/(2M) = \alpha^*\beta\sqrt{M(M+2)}/(4M)$, giving $F_{\text{opt}}(M)$ for even M . The global spin flip $X^{\otimes n}$ leaves H invariant and maps the $(k+1)$ -excitation problem in Eq. (9) onto the k -excitation one, so one block suffices.

The k -excitation block. With $m = 2k - 1$ and the basis $b_1 = |0, k, 0\rangle$, $b_2 = |0, k - 1, 1\rangle$, $b_3 = |1, k - 1, 0\rangle$, $b_4 = |1, k - 2, 1\rangle$, Eq. (5) gives the hoppings $b_1 b_2 = k J_q$, $b_1 b_3 = k$, $b_2 b_4 = \sqrt{k^2 - 1}$, $b_3 b_4 = \sqrt{k^2 - 1} J_q$, and the ZZ terms the diagonal $\frac{g_p}{2} z_a S + \frac{g_c}{2} z_f S + \frac{g_c}{4} (S^2 - m)$ with $S = m - 2n_{\text{mid}}$ [the middle Ising sum is $\frac{g_c}{2} \sum_{i < j} Z_i Z_j = \frac{g_c}{4} ((\sum_i Z_i)^2 - m)$]. Because $\binom{2k-1}{k} = \binom{2k-1}{k-1}$, the input is $v = |0\rangle_a |D_k\rangle = (b_1 + b_2)/\sqrt{2}$ for every k , and $w = |1\rangle_a |D_{k-1}\rangle = \sqrt{\frac{k+1}{2k}} b_3 + \sqrt{\frac{k-1}{2k}} b_4$. Complete $\{v, w\}$ to an orthonormal basis by $u_1 = (b_1 - b_2)/\sqrt{2}$ and $u_2 = \sqrt{\frac{k-1}{2k}} b_3 - \sqrt{\frac{k+1}{2k}} b_4$. In this basis the entries of H that matter are

$$\begin{aligned} H_{vv} &= (k - \frac{1}{2}) \sqrt{\frac{k+1}{k}}, & H_{u_1 u_2} &= (k + \frac{1}{2}) \sqrt{\frac{k-1}{k}}, \\ H_{vu_1} &= -\frac{g_p}{2}, & H_{vu_2} &= -\frac{1}{2} \sqrt{\frac{k-1}{k}}, & H_{wu_1} &= \frac{1}{2} \sqrt{\frac{k+1}{k}}, \\ H_{wu_2} &= \frac{\sqrt{k^2-1}}{2k} (g_p + 2\Delta), \\ H_{vv} - H_{ww} &= \frac{2k-1}{2k} g_p - g_c - \frac{\Delta}{k}, \\ H_{u_1 u_1} - H_{u_2 u_2} &= \frac{2k+1}{2k} g_p - g_c + \frac{\Delta}{k}, \end{aligned} \quad (10)$$

with $\Delta \equiv g_q - J_q - g_c$; the hopping J_q enters only as $J_q(k - \frac{1}{2})$ on v, w and $-J_q(k + \frac{1}{2})$ on u_1, u_2 , i.e. as a scalar on each of the two planes.

The involution. Equation (9) asks the evolution to map $v \mapsto (v + w)/\sqrt{2}$ up to phases. Let R be the reflection of the plane $\text{span}\{v, w\}$ across the bisector at 22.5° , $Rv = (v + w)/\sqrt{2}$, $Rw = (v - w)/\sqrt{2}$, combined with a reflection of the complement across the axis at angle ψ , $R = \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}$ on $\{u_1, u_2\}$. $R^2 = \mathbb{I}$, and if $[H, R] = 0$ then e^{-iHt} equals $e^{i\varphi} R$ on the block exactly when $e^{-iE_j t} = e^{i\varphi} r_j$ for every eigenvector, $r_j = \pm 1$ being its R -parity—the parity condition of perfect state transfer [14, 18]. $[H, R] = 0$ means that H has no matrix element between the $+1$ eigenvectors of R , $e_1 = \cos \frac{\pi}{8} v + \sin \frac{\pi}{8} w$, $e_2 = \cos \psi u_1 + \sin \psi u_2$, and the -1 eigenvectors $f_1 = -\sin \frac{\pi}{8} v + \cos \frac{\pi}{8} w$, $f_2 = -\sin \psi u_1 + \cos \psi u_2$. The scalar J_q terms drop out and the four conditions read, with Eq. (10),

$$H_{ww} - H_{vv} + 2H_{vw} = 0, \quad (11)$$

$$\frac{1}{2} \sin 2\psi (H_{u_2 u_2} - H_{u_1 u_1}) + \cos 2\psi H_{u_1 u_2} = 0, \quad (12)$$

$$\langle e_1 | H | f_2 \rangle = \langle e_2 | H | f_1 \rangle = 0, \quad (13)$$

[Eqs. (11) and (12) are $\langle e_1 | H | f_1 \rangle = 0$ and $\langle e_2 | H | f_2 \rangle = 0$ evaluated with $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$], the last two being linear and homogeneous in $(\sin \psi, \cos \psi)$. Four equations for the four unknowns (g_p, g_c, Δ, ψ) have four solutions (found numerically to 10^{-20} , Appendix C): two with $\Delta = 0$ and two with $\Delta \neq 0$; only the former yield optimal cloners (the latter stall at 0.745 and 0.794 for $M = 4$), so we set $g_q = J_q + g_c$. With $\Delta = 0$, eliminating ψ from Eq. (13) (the determinant of the 2×2 homogeneous system must vanish) gives, after using $\sin \frac{\pi}{8} \cos \frac{\pi}{8} = \frac{\sqrt{2}}{4}$ and $\cos \frac{\pi}{4} =$

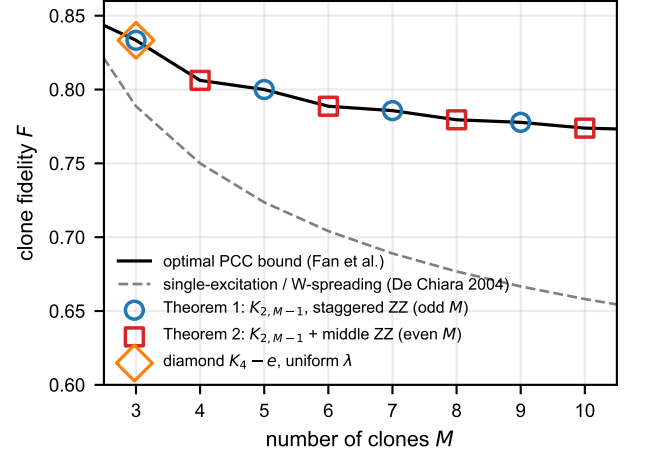


FIG. 2. The closed-form families reach the optimal phase-covariant bound for every M , while single-excitation (fractional-revival) design does so only for $M = 2$. Symbols: fidelities computed from Theorems 1 and 2 with no free parameter ($|F - F_{\text{opt}}| \leq 10^{-12}$); the $M = 10$ point is a prediction of the even- M formula. Lines: Fan bound and the single-excitation bound $\frac{1}{2} + 1/(2\sqrt{M})$ of Proposition 1 (the W-spreading cloner of De Chiara *et al.*).

$$\frac{\sqrt{2}}{2},$$

$$g_p^2 + 2(2k - 1)\sqrt{k(k+1)}g_p - 2k = 0, \quad (14)$$

whose two roots are $-cm \pm \sqrt{B + M}$; we take the positive one, $g_p = \sqrt{A} - \sqrt{B}$. Equation (11) then gives $g_c = (2k - 1)[g_p/(2k) - \sqrt{(k+1)/k}]$, which by Eq. (14) equals $g_p - 1/g_p$ and the expression in Theorem 2; ψ follows from either line of Eq. (13), $\tan \psi = -[\sin \frac{\pi}{8} g_p + \cos \frac{\pi}{8} \sqrt{\frac{k+1}{k}}]/[\sin \frac{\pi}{8} \sqrt{\frac{k-1}{k}} + \cos \frac{\pi}{8} g_p \sqrt{\frac{k^2-1}{k}}]$, which simplifies to $\cos^2 2\psi = A/2B$, and Eq. (12) is then satisfied identically (Appendix C). Finally, ordering the eigenvalues $E_1 < \dots < E_4$, the parity pattern $(r_j) = (+, -, -, +)$ translates into $(E_2 - E_1)t = n_1\pi$ with n_1 odd and $(E_3 - E_2)t = n_2\pi$ with n_2 even, which fixes (J_q, t) and completes the proof; $E_4 - E_3 = E_2 - E_1$ holds because the two R -sectors have the same centre of spectrum for these couplings. $\square$

With $c = \sqrt{k(k+1)}$ the collective ladder element $\langle D_{k+1} | S_+ | D_k \rangle$ on the M targets and m the middle multiplicity, $B = (cm)^2$ and Eq. (14) reads $g^2 + 2cmg = M$, while $g_c = g_p - 1/g_p$ and $\cos^2 2\psi = \frac{1}{2} + M/2(cm)^2$. We verified $M = 4, 6, 8$ and, as a prediction of the formula, $M = 10$ (eleven qubits), to $|F - F_{\text{opt}}| \leq 10^{-12}$ [Fig. 2]. Without the middle coupling the best $M = 4$ fidelity stalls 1.5×10^{-4} below F_{opt} ; the Ising coupling among the targets is the missing diagonal freedom.

V. WHICH GRAPHS CLONE OPTIMALLY

To decide exactness we do not rely on fidelity maximisation, whose landscape is flat near a solution, but solve the exact revival and transfer equations (freeze branch returns to itself, transfer branch reaches the Dicke target, up to one local Z phase per target orbit and, for even M , a shared relative phase between the branches) by Levenberg–Marquardt least squares; a residual norm of 10^{-29} certifies exactness. For $1 \rightarrow 3$ on all connected four-vertex graphs and input classes, the exact cases are the star, K_4 and the diamond with the input at a degree-two vertex; the four-cycle, the paw and the star with a leaf input are near misses (10^{-5} – 10^{-3} below F_{opt} for the best input class and k , up to 7×10^{-3} otherwise) and paths are far (10^{-2}). Neither a transitive automorphism action on the targets nor adjacency of the input to every target orbit characterises exactness; what matters is whether the couplings supply the algebraic degeneracy—a chord or an edge-resolved anisotropy—that lets the two symmetric blocks satisfy the ladder and freeze conditions simultaneously.

VI. ROBUSTNESS

Figure 3 compares the star, the diamond and the $K_{2,m}$ families under static coupling disorder, readout-time error and dephasing (Lindblad Z noise on all sites). The star remains the most robust, being almost insensitive to timing because its ladder is the minimal one and its time $\pi/(M+1)$ is the shortest. The diamond matches or slightly exceeds the star under static disorder and stays within the same order for the other errors, whereas the staggered $K_{2,m}$ families, with large anisotropy and longer times, are five to ten times more sensitive to disorder and 50–150 times more sensitive to timing. Per unit γt all systems dephase at the same rate; the differences are set by the evolution time alone. Random local fields $h_i Z_i/2$ (Zeeman disorder) are far less harmful than coupling disorder, because the targets stay in a fixed S_z sector and only field differences act: with $\sigma_h = 0.03$ the loss is below 10^{-4} for the star and the diamond alike, and 3 – 4×10^{-4} at $\sigma_h = 0.1$, thirty times smaller than for coupling disorder of the same relative size (Appendix D).

VII. CONCLUSION

We have shown that autonomous optimal phase-covariant cloning is not a privilege of the star: the complete bipartite graph $K_{2,M-1}$ clones optimally for every $M \geq 3$, with closed-form couplings (Theorems 1 and 2), and we have identified the two four-level symmetric blocks—a perfect-state-transfer ladder and a freeze block with a double zero mode—whose simultaneous commensurability is the actual condition. Compared

with the star [12], the new families need one extra coupling type (a staggered or a middle Ising term) and are less robust to timing, while the diamond with uniform anisotropy is as robust as the star under static disorder. Compared with fractional-revival engineering [19, 21, 22], the single-excitation design is shown to be an optimal cloner only for two clones, and the multi-excitation extension is given here. In the broader picture these results place optimal cloning among the quantum-transport phenomena of spin networks that can be read off from small symmetric blocks, next to perfect transfer and fractional revival. Without this identification, searches for autonomous quantum machines on graphs are likely to converge on near-miss solutions a fraction of a percent below the bound, as our four-vertex classification shows they do for most graphs. The next step is to carry the same block analysis to other covariant tasks—universal cloning, state-dependent cloning and telecloning resources—and to implement the diamond on Rydberg arrays. There the dipolar exchange ($\propto r^{-3}$) and van der Waals Ising ($\propto r^{-6}$) interactions impose $g_{ij} = \kappa J_{ij}^2$ with one positive κ ; the diamond admits an exact solution under this constraint, and with all pair interactions included a four-atom kite reaches $F_{\text{opt}} - 7 \times 10^{-7}$ and a six-atom axial geometry ($M = 5$) $F_{\text{opt}} - 8 \times 10^{-5}$ (Appendix B).

DATA AVAILABILITY

All numerical results of this paper are reproducible from the open-source Python package **acloner** (version 0.1.0), archived at Zenodo (DOI: 10.5281/zenodo.23392981) and maintained at https://github.com/deeptell-inc/autonomous_cloning. The package contains (i) the closed-form constructions of Theorems 1 and 2, of Chen’s star and of the diamond family, including the Rydberg-constrained member of Appendix B; (ii) the Levenberg–Marquardt exactness certification and the **sympy** checks of the characteristic polynomial and of the involution identity of Appendix C; (iii) the robustness protocols of Appendix D; and (iv) the scripts that regenerate the data files **results/families.json** (Fig. 2), **results/robustness.json** (Fig. 3) and **results/rydberg.json**, together with Figs. 2 and 3. A full reproduction (**make all**: $M \leq 8$, 200 disorder samples) runs in under one minute on a laptop; **make test** executes the 17 unit tests that verify every theorem and bound quoted in the text to the stated precision. The RIS bibliographic corpus used for the literature survey and the raw optimisation logs are available from the corresponding author upon reasonable request.

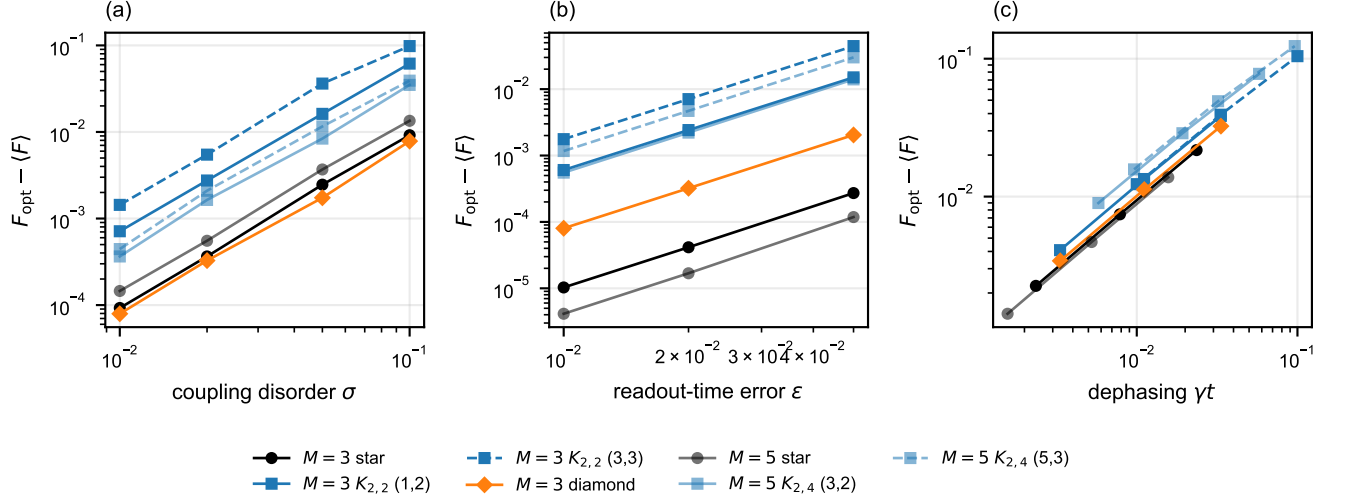


FIG. 3. The star is the most robust autonomous cloner, but the diamond matches it under static disorder. Loss of fidelity below the optimum under (a) relative Gaussian disorder σ of all couplings (100 samples), (b) relative readout-time error ϵ , (c) dephasing γt ; $M = 3$ (full) and $M = 5$ (faint). Static disorder and timing error enter quadratically, dephasing linearly in γt ; the staggered $K_{2,m}$ members are 5–10 times (disorder) and 50–150 times (timing) more sensitive because of their larger anisotropy and longer time.

Appendix A: Attainability of Chen's bound on the star

For the star with Dicke- k leaves, Eq. (12) of Ref. [12] gives, in units of J ,

$$F = \frac{1}{2} + \frac{k(M-k+1)\chi_1 - (M-k)(k+1)\chi_2}{M\eta_1\eta_2}, \quad (\text{A1})$$

$\chi_1 = \eta_1 \cos \frac{\eta_1 t}{2} \sin Bt \sin \frac{\eta_2 t}{2} - \lambda(M-2k-1) \sin \frac{\eta_1 t}{2} \cos Bt \sin \frac{\eta_2 t}{2}$, $\chi_2 = \eta_2 \cos \frac{\eta_2 t}{2} \sin Bt \sin \frac{\eta_1 t}{2} - \lambda(M-2k+1) \sin \frac{\eta_2 t}{2} \cos Bt \sin \frac{\eta_1 t}{2}$, with $\eta_1 = \sqrt{4k(M-k)(k+1) + (M-2k-1)^2 \lambda^2}$ and $\eta_2 = \sqrt{4k(M-k+1) + (M-2k+1)^2 \lambda^2}$ (Ref. [12] counts the leaves in $|0\rangle$; the expression is invariant under $k \rightarrow M-k$, $\eta_1 \leftrightarrow \eta_2$, so it holds for our k as well). The field B enters only through $\sin Bt$ and $\cos Bt$, because the total magnetisation is conserved and B merely rotates the two branches of the input relative to each other. Collecting the two terms, $F = \frac{1}{2} + (P \sin Bt + Q \cos Bt)/(M\eta_1\eta_2)$, and this rotation is exactly the input-independent local Z correction, so maximising over Bt gives the closed form

$$F(\lambda, t) = \frac{1}{2} + \frac{\sqrt{P^2 + Q^2}}{M\eta_1\eta_2}, \quad (\text{A2})$$

with $P = k(M-k+1)\eta_1 \cos \frac{\eta_1 t}{2} \sin \frac{\eta_2 t}{2} - (M-k)(k+1)\eta_2 \cos \frac{\eta_2 t}{2} \sin \frac{\eta_1 t}{2}$ and $Q = \lambda \sin \frac{\eta_1 t}{2} \sin \frac{\eta_2 t}{2} [(M-k)(k+1)(M-2k+1) - k(M-k+1)(M-2k-1)]$. We

checked this against the full 2^{M+1} -dimensional dynamics at random points to 10^{-7} . Chen's bound $\frac{1}{2} + \max\{\sqrt{k(M-k+1)}, \sqrt{(M-k)(k+1)}\}/2M$ follows from the inequality chain of Ref. [12] [its Eqs. (6)–(7)]; one sees from it that equality requires all weight on the dominant branch unless the two ladder elements coincide ($k = M/2$). Full transfer of the dominant branch needs its detuning $\lambda(M-2k \mp 1)$ to vanish, so $\lambda \neq 0$ helps only at $k = (M \mp 1)/2$; otherwise $\lambda = 0$ and the frozen and transferred branches must satisfy $\varepsilon_{\text{frozen}}/\varepsilon_{\text{transfer}} = 2n/(2m+1)$ with their bare Rabi frequencies ε , which has no integer solutions for $0 < k < \lfloor M/2 \rfloor$. Hence the bound is attained at finite time iff $k \in \{0, M, \lfloor M/2 \rfloor, \lceil M/2 \rceil\}$. A global scan over $\lambda \in [-10, 10]$ and $t \leq 10^3$ for $M = 2-8$ confirms this: the exact set coincides with the prediction, and for all other k the optimum has $\lambda = 0$ and approaches the bound only slowly (10^{-3} below it for $t \leq 10$, 10^{-7} for $t \leq 10^3$).

Appendix B: The diamond family

Let the diamond $K_4 - e$ have middle vertices 0, 1 (chord $c = J_{01}$), input $a = 2$ and far vertex 3, with $J_{a0} = J_{a1} = p$, $J_{03} = J_{13} = q$ and a uniform anisotropy $g_{ij} = \lambda J_{ij}$. With $k = 2$ (targets in $|D_2\rangle$) the exact structure freezes the $|1\rangle_a$ branch and transfers the $|0\rangle_a$ branch to $|1\rangle_a |D_1\rangle$. In the swap-symmetric basis, with $|m_s\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$ on the middle pair, the transfer block (two excitations, basis $\{|a m_s\rangle, |a 3\rangle, |01\rangle, |m_s 3\rangle\}$) and the freeze block (three excitations, basis $\{|a 01\rangle, |a m_s 3\rangle, |01 3\rangle\}$) are obtained from Eq. (1) as

$$\begin{aligned}
H_\alpha &= \begin{pmatrix} \frac{c(2-\lambda)}{2} & \sqrt{2}q & \sqrt{2}p & 0 \\ \sqrt{2}q & \frac{\lambda(c-2p-2q)}{2} & 0 & \sqrt{2}p \\ \sqrt{2}p & 0 & \frac{\lambda(c-2p-2q)}{2} & \sqrt{2}q \\ 0 & \sqrt{2}p & \sqrt{2}q & \frac{c(2-\lambda)}{2} \end{pmatrix}, \\
H_\beta &= \begin{pmatrix} \frac{\lambda(c+2p-2q)}{2} & \sqrt{2}q & 0 & \\ \sqrt{2}q & \frac{c(2-\lambda)}{2} & \sqrt{2}p & \\ 0 & \sqrt{2}p & \frac{\lambda(c-2p+2q)}{2} & \end{pmatrix} \quad (B1)
\end{aligned}$$

(the $\sqrt{2}$ is the ladder element of the symmetric pair, the chord contributes $+c$ on $|m_s\rangle$ and $-\lambda c/2$ from its ZZ term). The initial vector $v_\alpha = |0\rangle_a |D_2\rangle = (0, 0, 1, \sqrt{2})/\sqrt{3}$ and the target $|1\rangle_a |D_1\rangle = (\sqrt{2}, 1, 0, 0)/\sqrt{3}$ are exchanged by the anti-diagonal reflection $R : |a m_s\rangle \leftrightarrow |m_s 3\rangle$, $|a 3\rangle \leftrightarrow |01\rangle$, and $[H_\alpha, R] = 0$ holds identically because H_α is persymmetric. Its characteristic polynomial factorises into the two R -sectors, $\prod_{\pm} [4x^2 - 4(c - \lambda p - \lambda q)x + \kappa_{\pm}]$ with $\kappa_{\pm} = 2c^2\lambda - c^2\lambda^2 + 2c\lambda^2(p+q) - 4c\lambda(p+q) - 8(p \mp q)^2$, so both sectors are centred at $(c - \lambda p - \lambda q)/2$ and $E_2 - E_1 = E_4 - E_3$ identically; the parity condition of perfect state transfer [14, 18] then reduces to the ladder below. In the freeze block the initial vector $v_\beta = |1\rangle_a |D_2\rangle = (1, \sqrt{2}, 0)/\sqrt{3}$ must lie in a two-dimensional Krylov space, $\det[v_\beta, H_\beta v_\beta, H_\beta^2 v_\beta] = 0$, which with the matrices above is the cubic

$$\begin{aligned}
&2\lambda^2(p-q)^2 - 2p^2 - q^2 + \lambda q(p-q) \\
&+ c[2\lambda^2(p-q) - 2\lambda p + \lambda q + q] = 0; \quad (B2)
\end{aligned}$$

then v_β is a combination of two eigenvectors and revives when their gap ΔE_β satisfies $\Delta E_\beta t \in 2\pi\mathbb{Z}$. The exact conditions are therefore

$$\begin{aligned}
&\det[v_\beta, H_\beta v_\beta, H_\beta^2 v_\beta] = 0, \quad (E_2 - E_1)t = n_1\pi \quad (n_1 \text{ odd}), \\
&(E_3 - E_2)t = n_2\pi \quad (n_2 \text{ even}), \quad \Delta E_\beta t = 2\pi n_b, \quad (B3)
\end{aligned}$$

four equations for the four unknowns $(q/p, c/p, \lambda, tp)$. Solving them directly reproduces the least-squares solution $[(n_1, n_2, n_b) = (1, 2, 2): q/p = 2.985643, c/p = 2.048572, \lambda = -1.342497, tp = 1.113388]$ and yields a distinct exact cloner for every admissible integer triple, e.g. $(1, 2, 1)$, $(1, 2, 3)$, $(3, 2, 2)$, $(1, 4, 2)$, $(1, 4, 3)$, $(3, 4, 2)$, all with transfer and revival probabilities equal to one to 10^{-15} . With $c = 0$ (the four-cycle) the dark-state condition becomes a constraint on (p, q, λ) and the system is overdetermined, which is why the four-cycle with uniform anisotropy is not exact. For $m \geq 4$ middle vertices $(K_{m+2} - e)$ the freeze block is four dimensional and needs two dark-state conditions; with uniform λ the system is again overdetermined, while edge-resolved anisotropies (g_p, g_q, g_c) restore a continuous family of exact solutions (verified by least squares for $M = 5$ and 7), of which the staggered $K_{2,m}$ family of Theorem 1 is the member with an isotropic (hence irrelevant) chord.

Under the Rydberg constraint $g_{ij} = \kappa J_{ij}^2$ with a common κ , the diamond graph still admits an exact solution ($\kappa > 0$, $\lambda_e = g_e/J_e \simeq 0.37, 0.28, 0.21$ on the chord, input and far edges). Including the interactions of all atom pairs with $J_{ij} = C_3/r_{ij}^3$ and $g_{ij} = C_6/r_{ij}^6$, and optimising the atom positions, a planar kite (middle pair at $(\pm 1, 0)$, input at $(0, 3.69)$, far atom at $(0, -1.72)$) reaches $F = F_{\text{opt}} - 7 \times 10^{-7}$ at $t_{J_{\text{nn}}} \simeq 7$, and a six-atom axial geometry for $M = 5$ (four middle atoms on a ring, input and far atom on the axis) reaches $F_{\text{opt}} - 8 \times 10^{-5}$.

Appendix C: Even M : the involution identity for symbolic k

In the S_m -symmetric basis $\{(0, k, 0), (0, k-1, 1), (1, k-1, 0), (1, k-2, 1)\}$ (occupations of input, middle set and far vertex) the k -excitation block of $K_{2,2k-1}$ with middle Ising coupling has hoppings kJ_q , k , $\sqrt{k^2-1}$, $\sqrt{k^2-1}J_q$ and diagonal entries fixed by the occupations, e.g. $-\frac{g_p+g_q}{2}S - \frac{g_c}{4}(m-S^2)$ with $S = m - 2n_{\text{mid}}$ up to the signs of the input and far spins. The initial vector is $v = (1, 1, 0, 0)/\sqrt{2}$ for every k (since $\binom{2k-1}{k} = \binom{2k-1}{k-1}$) and the target $w = (0, 0, \sqrt{k+1}, \sqrt{k-1})/\sqrt{2k}$. Solving the four conditions (11)–(13) numerically for (g_p, g_c, Δ, ψ) from 200 random starts returns exactly four solutions at each $k = 2, 3, 4$: the two roots of Eq. (14) with $\Delta = 0$ [for $k = 2$: $g_p = 0.267304$ and -14.964242] and two with $\Delta \neq 0$ $[(g_p, g_c, \Delta) = (-6.4209, -11.7426, 6.5054)$ and $(3.9714, 1.5364, -4.4642)$ for $k = 2$]; the ladder-compatible members of the latter reach only $F = 0.7449$ and 0.7942 at $M = 4$, while both $\Delta = 0$ roots reach F_{opt} . Substituting the closed form of Theorem 2 together with $\cos 2\psi = -\sqrt{A/2B}$, $\sin 2\psi = -\sqrt{(B-M)/2B}$ into $[H, R]$ gives identically zero: exactly at $k = 2, \dots, 10$ and to forty digits at non-integer k , with vanishing J_q coefficient. The ladder condition then reads $(E_2 - E_1)t = n_1\pi$ (odd), $(E_3 - E_2)t = n_2\pi$ (even), and we verified $|F - F_{\text{opt}}| \leq 10^{-12}$ for forty members at $M = 4$, four each at $M = 6, 8$ and two at $M = 10$.

Appendix D: Numerical methods

Fidelity. For each target s we compute the reduced state for twelve equatorial inputs ϕ_j and define $F_s = \langle \rho_{00} |\alpha|^2 + \rho_{11} |\beta|^2 \rangle_\phi + 2 |\langle \alpha^* \beta \rho_{01} \rangle_\phi|$, i.e. one input-independent Z rotation per site; a per-input correction would make any pure equatorial state look like a perfect clone.

Exactness. Odd M : we solve $U(t)v_\alpha = e^{i\varphi} Z_\theta v_\alpha$, $U(t)v_\beta = e^{i\varphi'} Z_\theta v_\beta$ (or the dual assignment) for the couplings, t , the global phases and one local Z phase per target orbit by Levenberg–Marquardt least squares [17] from 6–16 random starts; a residual norm $\leq 10^{-29}$ certifies exactness, while failures saturate at $\geq 10^{-7}$ with the solver drifting to large couplings and times (long-time Diophantine approximation). Even M : the output is the balanced mixture $Uv_\alpha = (e^{i\varphi_1} Z v_\alpha + e^{i\varphi_2} Z |1\rangle_a |D_{k-1}\rangle) / \sqrt{2}$, $Uv_\beta = (e^{i(\varphi_1+\delta)} Z |0\rangle_a |D_{k+1}\rangle + e^{i(\varphi_2+\delta)} Z v_\beta) / \sqrt{2}$ with a shared relative phase δ ; omitting δ misses the $M = 6$ so-

lution. The mixing angle must be fixed at $\pi/4$, otherwise the solver finds exact but non-optimal dynamics. Hamiltonians are assembled as linear combinations of precomputed operator sums, and the phases are pre-fitted to the output of a Nelder–Mead near-optimal point before the full least-squares step; with these two measures a seven-qubit certification takes seconds.

Characteristic polynomials. Expanding $\det(H_F - \lambda)$ along the first row gives $\lambda^2(\lambda^2 - 2s^2 - 2r^2 - 4g^2)$ for the freeze block of Theorem 1; this and the factorisation of the diamond block were checked with `sympy`, as was the identity $[H, R] = 0$ of Theorem 2 for symbolic k (Appendix C).

Robustness. Static disorder multiplies every J_e and g_e by $1 + \sigma \xi_e$ with independent standard normal ξ_e (100 samples); timing error evaluates $t(1 \pm \epsilon)$; dephasing integrates $\dot{\rho} = -i[H, \rho] + \gamma \sum_i (Z_i \rho Z_i - \rho)$ with a vectorised Liouvillian and `expm_multiply`; Zeeman disorder adds $h_i Z_i / 2$ with $h_i \sim \mathcal{N}(0, \sigma_h^2)$. All code is available in the repository described in the Data Availability statement.

-
- [1] W. K. Wootters and W. H. Zurek, *Nature* **299**, 802 (1982).
 - [2] V. Bužek and M. Hillery, *Phys. Rev. A* **54**, 1844 (1996).
 - [3] N. Gisin and S. Massar, *Phys. Rev. Lett.* **79**, 2153 (1997).
 - [4] V. Scarani, S. Iblisdir, N. Gisin, and A. Acín, *Rev. Mod. Phys.* **77**, 1225 (2005).
 - [5] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, *Phys. Rev. Lett.* **88**, 127902 (2002); H. Fan, Y.-N. Wang, L. Jing, J.-D. Yue, H.-D. Shi, Y.-L. Zhang, and L.-Z. Mu, *Phys. Rep.* **544**, 241 (2014).
 - [6] M. Murao, D. Jonathan, M. B. Plenio, and V. Vedral, *Phys. Rev. A* **59**, 156 (1999).
 - [7] B. Wang, H. Yang, J. Sun, W. Zhou, J. Wu, Y. Xiao, D. Wu, and J. Wu, Loss-resilient quantum networking with Dicke entanglement, arXiv:2609.33094 (2026).
 - [8] Z. Wen, K. Wang, B. Wu, L. Chen, Y.-Q. Lu, S. Zhu, and X.-S. Ma, Experimental quantum telecloning across silicon photonic chips, arXiv:2608.11718 (2026).
 - [9] H. Fan, K. Matsumoto, X.-B. Wang, and M. Wadati, *Phys. Rev. A* **65**, 012304 (2001).
 - [10] G. De Chiara, R. Fazio, C. Macchiavello, S. Montangero, and G. M. Palma, *Phys. Rev. A* **70**, 062308 (2004).
 - [11] G. De Chiara, R. Fazio, C. Macchiavello, S. Montangero, and G. M. Palma, *Phys. Rev. A* **72**, 012328 (2005).
 - [12] Q. Chen, J. Cheng, K.-L. Wang, and J. Du, *Phys. Rev. A* **74**, 034303 (2006).
 - [13] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, *Phys. Rev. Lett.* **92**, 187902 (2004).
 - [14] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, *Phys. Rev. A* **71**, 032312 (2005).
 - [15] S. Bose, *Phys. Rev. Lett.* **91**, 207901 (2003).
 - [16] R. H. Dicke, *Phys. Rev.* **93**, 99 (1954).
 - [17] J. J. Moré, in *Numerical Analysis*, Lecture Notes in Mathematics Vol. 630, edited by G. A. Watson (Springer, Berlin, 1978), p. 105.
 - [18] A. Kay, *Int. J. Quantum Inf.* **8**, 641 (2010).
 - [19] V. X. Genest, L. Vinet, and A. Zhedanov, *Ann. Phys.* **371**, 348 (2016).
 - [20] L. Bianchi, E. Compagno, and S. Bose, *Phys. Rev. A* **91**, 052323 (2015).
 - [21] A. Chan, G. Coutinho, C. Tamon, L. Vinet, and H. Zhan, *Discrete Appl. Math.* **269**, 86 (2019).
 - [22] A. Kay, *Quantum* **1**, 24 (2017).
 - [23] M. Christandl, L. Vinet, and A. Zhedanov, *Phys. Rev. A* **96**, 032335 (2017).
 - [24] A. Chan, G. Coutinho, W. Drazen, O. Eisenberg, C. Godsil, G. Lippner, M. Kempton, C. Tamon, and H. Zhan, Fundamentals of fractional revival in graphs, arXiv:2004.01129 (2020).
 - [25] L. Dai, Y. P. Feng, and L. C. Kwek, *J. Phys. A: Math. Theor.* **43**, 035302 (2010).
 - [26] H.-L. Deng and X.-M. Fang, *Chin. Phys. B* **17**, 702 (2008).
 - [27] A. Kay, R. Ramanathan, and D. Kaszlikowski, Optimal asymmetric quantum cloning, arXiv:1208.5574 (2012).