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Research Article

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Posted Date: October 16th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11385034/v1>

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Additional Declarations: The authors declare no competing interests.

Generalized Sine-Type Estimators for Population Median: Simulation-Driven Performance Analysis

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Abstract

This study introduces a generalized family of sine-type estimators for the finite population median in the framework of simple random sampling without replacement, incorporating the known values of auxiliary variable. Auxiliary information is frequently employed to enhance the efficiency and precision of estimators in population parameter estimation, often leading to lower mean squared error (MSE) and higher percentage relative efficiency (PRE). This study modifies traditional estimators and develops novel sine-type and exponential-sine-type estimators to improve finite population median estimation using auxiliary information. We evaluate the bias and MSE of the proposed estimator through first-order approximation. The proposed estimator's effectiveness and practical relevance are examined using real-world data as well as simulated studies. The suggested estimator consistently outperforms the sample median as well as other sine-type estimators such as ratio, product, exponential ratio, and exponential product considered in this study. The proposed estimator demonstrates improved efficiency and practical utility in real data analysis.

Keywords: Sine-type estimator, Population median, Exponential-cum-sine-type estimator, MSE, Auxiliary information.

1. Introduction

In survey sampling, several attempts have been directed toward improving the efficiency of estimators by incorporating auxiliary information. Considerable work has been devoted to developing ratio, product, and regression estimators for parameters such as the population mean, variance, proportion, and regression coefficient with auxiliary information, but efficient estimation of the median has received comparatively less emphasis. In real-world applications, many variables such as production, household income, consumption, taxes, and the poverty index exhibit extreme fluctuations ranging from very high to very low values. Since these variables typically follow skewed and non-normal distributions, the median is generally considered a better indicator of location compared to the mean. The initial analysis of median estimation without the use of an auxiliary variable was carried out by Gross [1]. Subsequently auxiliary information was incorporated in median estimation by several researchers such as Kuk and Mak [2] and Meeden [3]. The earliest attempt to estimate the population median of a study variable with the aid of auxiliary information was made by Kuk and Mak [2]. Building on this, many others including [4-8], have suggested different estimators of the finite population median employing known auxiliary variables in several sampling frameworks. A new technique for median estimation under simple random sampling with auxiliary information was suggested by Ahmad et al. [9]. The study conducted by Baig et al. [10] proposed a refined class of difference-type estimators aimed at estimating

the population median under survey sampling. The estimation of finite population parameters in earlier studies has been carried out using various sampling strategies, as reported in [11-14]. In their work, Daraz et al. [18] proposed an approach for estimating the population median using auxiliary variables, validated through simulation experiments with real data across different sample sizes and parameters. In recent years, significant research efforts have focused on enhancing finite population median estimation through the use of auxiliary information under diverse sampling designs, including stratified, two-phase, and bivariate auxiliary frameworks (Zohaib and Irfan [23]; Alshanbari [24]; Hussain et al. [25]).

Although considerable work has been carried out on median estimation, the development of sine-type estimators remains limited and underdeveloped. The primary aim of this study is to develop some improved sine-type estimators for estimating the finite population median of a study variable by utilizing known auxiliary information under simple random sampling. Sine-type approaches have been explored in a few important studies, particularly by Bhattacharyya et al. [15] using an exponential-cum-sine-type hybrid imputation strategy. The study by Yunusa et al. [16] focused on a sine-type median-based estimator designed to enhance population mean estimation. The study by Singh et al. [17] recommends some sine-based estimators for the finite population mean by incorporating known auxiliary variable. To overcome gaps in finite population median estimation, this research proposes several sine-type estimators and an improved exponential-cum-sine-type estimator using known auxiliary information, derives their theoretical measures such as MSE and PRE, and evaluates their performance against existing estimators using empirical and simulation results.

Exponential and sine functions were utilized in the construction of the estimator because of their established role in practical implementations. Exponential functions are widely recognized for modeling natural growth and decay processes such as bacterial reproduction, human population growth, drug absorption and elimination (pharmacokinetics), the spread of viral content on social media, and financial phenomena like compound interest [15]. Sine functions are particularly useful for modeling periodic phenomena, including sound and light waves, quantum wave functions, tidal cycles, seasonal temperature variations, and electrical oscillations [15]. Sine and exponential-cum-sine type estimators are employed to account for nonlinearity, skewness, and periodic patterns commonly observed in real-world data, including seasonal trends, temperature variations, and viral spread. This hybrid approach enhances estimation in situations where conventional methods are inadequate. Existing estimators that use linear or exponential forms often fail to model the intricate nonlinear or oscillatory behaviours present in practical datasets. The use of sine-type estimators for estimating population median remains underexplored. To address this gap, we propose novel sine-type and exponential-cum-sine-type estimators aimed at improving estimation efficiency.

This paper is organized into the following sections: An introduction to the background is provided in Section 1, followed by Section 2, which defines the key terms and symbols. Section 3 presents a review of existing and modified conventional estimators. Section 4 outlines the proposed exponential-cum-sine-type estimator, including the first-order approximation of its bias and mean squared error. Section 5 discusses various members of the proposed estimator. A theoretical efficiency comparison is provided in Section 6. An empirical study is presented in Section 7. Additionally, a simulation study using Bivariate Normal and Gamma distributions was conducted to validate the theoretical properties of the proposed estimator. The following sections present the results and discussion, followed by the conclusion and suggestions for future research.

2. Symbols and Terminology

Suppose we are given a finite population $U = \{u_1, u_2, \dots, u_N\}$ consisting of N distinct units. A random sample of n units is drawn from it through the method of simple random sampling without replacement. The symbols and notations used throughout the study are defined in Table 1.

Table 1: The symbols and terminology.

Symbols	Terminology
N	Population size
n	Sample size
Y	Study variable
X	Auxiliary variable
$\hat{M}_y$	The sample median of study variable
$\hat{M}_x$	The sample median of auxiliary variable
M_x	The Population median of auxiliary variable
M_y	The Population median of study variable
$\rho_{(M_y, M_x)}$	The correlation coefficient between Y and X
$f_x(M_x)$	The probability density function for M_x
$f_y(M_y)$	The probability density function for M_y
C_{Mx}	The coefficient of variation of X
C_{My}	The coefficient of variation of Y

For analyzing the bias and MSE behavior of the sine-type estimator against traditional estimators, we define the following relative error terms:

$$\eta_y = \frac{\hat{M}_y - M_y}{M_y}, \quad \eta_x = \frac{\hat{M}_x - M_x}{M_x}, \quad \text{given } E(\eta_y) = E(\eta_x) = 0.$$

$$E(\eta_y^2) = \phi C_{My}^2, \quad E(\eta_x^2) = \phi C_{Mx}^2 \quad \text{and} \quad E(\eta_x \eta_y) = \phi C_{Myx}.$$

$$\text{Where, } C_{Myx} = \rho_{Myx} C_{My} C_{Mx}, \quad \phi = \left\{ \frac{1}{4} \left(\frac{1}{n} - \frac{1}{N} \right) \right\}, \quad C_{Mx} = \frac{1}{M_x f_x(M_x)}, \quad C_{My} = \frac{1}{M_y f_y(M_y)}.$$

The correlation coefficient between M_y and M_x , denoted by ρ_{yx} or ρ_c , is expressed as,

$$\rho_{(M_y, M_x)} = \rho_c = 4P_{11}(y, x) - 1, \quad \text{where } P_{11}(y, x) = P(y \leq M_y \cap x \leq M_x).$$

To evaluate how well our newly proposed estimator perform, we make use of both existing and certain recommended sine-type estimators.

3. Existing and Adapted Conventional Estimators

This section emphasizes various estimators that have shown considerable flexibility in addressing a wide range of statistical challenges. These established approaches are reconsidered and integrated into the sine-type estimator framework.

3.1) Usual unbiased median estimator: The sample median serves as the basis for estimating the finite population median (M_y), as described by Gross [1].

$$\hat{M}_0 = \hat{M}_y \quad (3.1)$$

The variance of the estimator ($\hat{M}_0$) is obtained through a first-order approximation.

$$Var(\hat{M}_0) = \phi M_y^2 C_{My}^2. \quad (3.2)$$

3.2) Generalized conventional ratio and product estimator

In survey sampling, ratio estimators are commonly used to improve the estimation of the population median when auxiliary information is known. This method, first suggested by Kuk and Mak [2], is based on calculating the ratio of sample medians of the study and auxiliary variables. When a positive linear relationship exists between the auxiliary and study variables, the method enhances estimation accuracy by making use of auxiliary data. However, the product estimator, introduced by Murthy [19], offers an alternative but effective methods of incorporating auxiliary information. The technique essentially relies on the product of the sample medians of the auxiliary and study variables, which forms the basis of estimation. Building upon this idea, we propose sine-type estimators, formulated as follows.

Following the approach of Kuk and Mak [2], we develop a sine-type ratio estimator as presented below:

$$\hat{M}_R = \hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} \quad (3.3)$$

$$\hat{M}_R = (1 + \eta_y) M_y [1 + \sin(-\eta_x)] \quad (3.4)$$

$$\hat{M}_R = (1 + \eta_y) M_y \left[1 - \eta_x + \frac{\eta_x^3}{3!} \right] \quad (3.5)$$

The estimator ($\hat{M}_R$) has its bias and MSE, under first-order approximation are provided by:

$$Bias(\hat{M}_R) = M_y (-\phi \rho_{Myx} C_{My} C_{Mx}) \quad (3.6)$$

$$MSE(\hat{M}_R) = M_y^2 \phi [C_{Mx}^2 + C_{My}^2 - 2\rho_{Myx} C_{Mx} C_{My}]. \quad (3.7)$$

This methodological framework is applied to each of the estimators presented.

Motivated by the work of Murthy [19], we introduce a sine-type product estimator for estimating the finite population median by employing auxiliary information.

$$\hat{M}_P = \hat{M}_y \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} \quad (3.8)$$

Using the general derivation method outlined earlier, the first-order approximations of the bias and MSE of estimator ($\hat{M}_P$) are obtained as:

$$Bias(\hat{M}_P) = M_y (\phi \rho_{Myx} C_{My} C_{Mx}) \quad (3.9)$$

$$MSE(\hat{M}_P) = M_y^2 \phi [C_{Mx}^2 + C_{My}^2 + 2\rho_{Myx} C_{Mx} C_{My}]. \quad (3.10)$$

3.3) Generalized Bahl and Tuteja (1991) Estimator

Bahl and Tuteja's 1991 work significantly advanced statistical techniques through the proposal of exponential estimators of the ratio and product type. The estimators are characterized using multiplication and division, which emphasize the structural relationships between study and auxiliary variables.

Drawing inspiration from Bahl and Tuteja [20], we construct an exponential-cum-sine type ratio estimator, which is defined as:

$$\hat{M}_{ER} = \hat{M}_y \exp\left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x}\right) \quad (3.11)$$

Applying the general approximation approach described earlier, the expressions for the bias and MSE of estimator ($\hat{M}_{ER}$) are derived as:

$$Bias(\hat{M}_{ER}) = M_y \phi \left[\frac{3}{8} C_{Mx}^2 - \frac{\rho_{Myx} C_{My} C_{Mx}}{2} \right] \quad (3.12)$$

$$MSE(\hat{M}_{ER}) = M_y^2 \phi \left[C_{My}^2 + \frac{C_{Mx}^2}{4} - \rho_{Myx} C_{Mx} C_{My} \right]. \quad (3.13)$$

Building on the approach of Bahl and Tuteja [20], an exponential-cum-sine type product estimator has been constructed and is expressed as below.

$$\hat{M}_{EP} = \hat{M}_y \exp\left(\frac{\hat{M}_x - M_x}{M_x + \hat{M}_x}\right) \quad (3.14)$$

The bias and MSE corresponding to the estimator ($\hat{M}_{EP}$) are presented as follows.

$$Bias(\hat{M}_{EP}) = M_y \phi \left[\frac{\rho_{Myx} C_{My} C_{Mx}}{2} - \frac{C_{Mx}^2}{8} \right] \quad (3.15)$$

$$MSE(\hat{M}_{EP}) = M_y^2 \phi \left[C_{My}^2 + \frac{C_{Mx}^2}{4} + \rho_{Myx} C_{Mx} C_{My} \right]. \quad (3.16)$$

4. Proposed Sine-Type Median Estimator

Following the approaches of Bhattacharyya et al. [15] and Singh et al. [17], we introduce an enhanced exponential-cum-sine type estimator for estimating the finite population median, which utilizes known auxiliary information under simple random sampling, defined as:

$$\hat{M}_y^* = \left[\hat{M}_y \left\{ 1 + \sin \left(k_1 \frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right)^{k_2} \right] \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(k_3 \frac{\hat{M}_x - M_x}{M_x} \right) \right\} \quad (4.1)$$

The scalar parameters (k_1, k_2, k_3) may assume values of $(0, -1, 1)$. whereas the scalars β are determined to minimize the mean squared error (MSE) of the suggested estimator ($\hat{M}_y^*$).

Equation (4.1) can be restated using error terms (η 's) up to the first-order approximation, as given below:

$$\hat{M}_y^* = \left[(1 + \eta_y) M_y \left\{ 1 + \sin \left(k_1 \frac{M_x - (1 + \eta_x) M_x}{M_x} \right) \right\} + \beta (M_x - (1 + \eta_x) M_x) \left(\frac{M_x}{(1 + \eta_x) M_x} \right)^{k_2} + \exp \left(\frac{M_x - (1 + \eta_x) M_x}{M_x + (1 + \eta_x) M_x} \right) \left\{ 1 + \sin \left(k_3 \frac{(1 + \eta_x) M_x - M_x}{M_x} \right) \right\} \right] \quad (4.2)$$

On simplifying Equation (4.2), the expression becomes:

$$\hat{M}_y^* = \left[M_y (1 + \eta_y) \left(1 - k_1 \eta_x + \frac{k_1^3 \eta_x^3}{3!} \right) - \beta M_x \eta_x (1 + \eta_x)^{-k_2} \right] \left(1 - \frac{\eta_x}{2} + \frac{3}{8} \eta_x^2 \right) \{ 1 + k_3 \eta_x \} \quad (4.3)$$

On expanding Equation (4.3), the expression below is obtained.

$$\hat{M}_y^* = \left[M_y(1 + \eta_y) \left(1 - k_1\eta_x + \frac{k_1^3\eta_x^3}{3!} \right) - \beta M_x\eta_x(1 + \eta_x)^{-k_2} \right] \left(1 + \frac{(2k_3-1)}{2}\eta_x + \frac{(3-4k_3)}{8}\eta_x^2 \right) \quad (4.4)$$

On subtracting the population median (M_y) from each side of equation (4.4) yields the expression below:

$$\hat{M}_y^* - M_y = M_y \left[\frac{(2k_3-1)}{2}\eta_x + \frac{(3-4k_3)}{8}\eta_x^2 - k_1\eta_x - \frac{k_1(2k_3-1)}{2}\eta_x^2 \right] - \left[\frac{\beta M_x\eta_x + \frac{(2k_3-1)}{2}\beta M_x\eta_x^2}{-k_2\beta M_x\eta_x^2} \right] \quad (4.5)$$

By computing the expectation of equation (4.5), we derive the first-order bias of the estimator ($\hat{M}_y^*$) as follows,

$$\text{Bias}(\hat{M}_y^*) = M_y \left[\frac{(3-4k_3)}{8}\phi C_{Mx}^2 - \frac{k_1(2k_3-1)}{2}\phi C_{Mx}^2 \right] - \left[\frac{(2k_3-1)}{2}\beta M_x\phi C_{Mx}^2 - k_2\beta M_x\phi C_{Mx}^2 \right] \quad (4.6)$$

Based on Equation (4.5), we obtain

$$\hat{M}_y^* - M_y = M_y \left[\eta_y + \frac{(2k_3-1-2k_1)}{2}\eta_x \right] - \beta M_x\eta_x. \quad (4.7)$$

By taking the square of each side of Equation (4.7), we obtain the result below:

$$(\hat{M}_y^* - M_y)^2 = \left[M_y^2 \left\{ \eta_y^2 + \frac{(2k_3-1-2k_1)^2}{4}\eta_x^2 + (2k_3-1-2k_1)\eta_x\eta_y \right\} + \beta^2 M_x^2\eta_x^2 \right] + 2\beta M_x M_y \left\{ \frac{(1-2k_3+2k_1)}{2}\eta_x^2 - \eta_x\eta_y \right\} \quad (4.8)$$

The expectation of equation (4.8) leads to the following expression for the MSE of the estimator ($\hat{M}_y^*$).

$$\text{MSE}(\hat{M}_y^*) = \left[M_y^2 \left\{ \phi C_{My}^2 + \frac{(2k_3-1-2k_1)^2}{4}\phi C_{Mx}^2 + (2k_3-1-2k_1)\phi C_{Myx}^2 \right\} + \beta^2 M_x^2\phi C_{Mx}^2 \right] + 2\beta M_x M_y \left\{ \frac{(1-2k_3+2k_1)}{2}\phi C_{Mx}^2 - \phi C_{Myx}^2 \right\} \quad (4.9)$$

$$\text{MSE}(\hat{M}_y^*) = P + Q\beta^2 + 2R\beta. \quad (4.10)$$

where,

$$P = M_y^2 \left\{ \phi C_{My}^2 + \frac{(2k_3-1-2k_1)^2}{4}\phi C_{Mx}^2 + (2k_3-1-2k_1)\phi C_{Myx}^2 \right\}.$$

$$Q = M_x^2\phi C_{Mx}^2.$$

$$R = M_x M_y \left\{ \frac{(1-2k_3+2k_1)}{2}\phi C_{Mx}^2 - \phi C_{Myx}^2 \right\}.$$

The estimator ($\hat{M}_y^*$) attains its minimum MSE when β takes the optimal values given below.

$$\beta_{(opt)} = -\frac{R}{Q}.$$

The minimum MSE of the estimator ($\hat{M}_y^*$) is derived by substituting the optimal values $\beta_{(opt)}$ into Equation (4.10).

$$MSE_{min}(\hat{M}_y^*) = P - \frac{R^2}{Q}. \quad (4.11)$$

5. Several members of the proposed estimator ($\hat{M}_y^*$).

By substituting suitable constants for k_1 , k_2 , and k_3 in equation (4.1), several optimal estimators can be obtained, as outlined in Table 2.

Table 2: Different estimators of the population median emerge by assigning suitable combinations to the constants k_i 's ($i = 1, 2, 3$).

Estimators	Values of		
	k_1	k_2	k_3
$\hat{M}_{y1}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\}} \right]$	1	1	1
$\hat{M}_{y2}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\}} \right]$	1	1	-1
$\hat{M}_{y3}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{\hat{M}_x}{M_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\}} \right]$	1	-1	1
$\hat{M}_{y4}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{\hat{M}_x}{M_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\}} \right]$	1	-1	-1
$\hat{M}_{y5}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\}} \right]$	-1	1	1
$\hat{M}_{y6}^* = \left[\frac{\hat{M}_y \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} + \beta (M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right)}{\exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\}} \right]$	-1	1	-1

$\hat{M}_{y7}^* = \left[\begin{array}{c} \hat{M}_y \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} + \beta(M_x - \hat{M}_x) \left(\frac{\hat{M}_x}{M_x} \right) \\ \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} \end{array} \right]$	-1	-1	1
$\hat{M}_{y8}^* = \left[\begin{array}{c} \hat{M}_y \left\{ 1 + \sin \left(\frac{\hat{M}_x - M_x}{M_x} \right) \right\} + \beta(M_x - \hat{M}_x) \left(\frac{\hat{M}_x}{M_x} \right) \\ \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} \end{array} \right]$	-1	-1	-1
$\hat{M}_{y9}^* = \left[\hat{M}_y + \beta(M_x - \hat{M}_x) \right] \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right)$	0	0	0
$\hat{M}_{y10}^* = \left[\begin{array}{c} \hat{M}_y \left\{ 1 + \sin \left(\frac{M_x - \hat{M}_x}{M_x} \right) \right\} + \beta(M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right) \\ \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \end{array} \right]$	1	1	0
$\hat{M}_{y11}^* = \left[\left\{ \hat{M}_y + \beta(M_x - \hat{M}_x) \left(\frac{M_x}{\hat{M}_x} \right) \right\} \exp \left(\frac{M_x - \hat{M}_x}{M_x + \hat{M}_x} \right) \right]$	0	1	0

6. Efficiency Comparison

In this section, the performance of the newly proposed estimator is evaluated against the sample median and some recommended sine-type estimators for the finite population median (M_y) that make use of auxiliary information. The mean squared error (MSE) is used to assess efficiency, and scenarios in which the proposed estimator shows superior performance are determined.

When the stated condition holds, estimator ($\hat{M}_y^*$) outperforms estimator ($\hat{M}_0$) in terms of efficiency.

$$Var(\hat{M}_0) - MSE(\hat{M}_y^*) > 0 \text{ or,}$$

$$\phi M_y^2 C_{M_y}^2 - \left[P - \frac{R^2}{Q} \right] > 0.$$

Estimator ($\hat{M}_y^*$) demonstrates higher efficiency than estimator ($\hat{M}_R$) when the specified condition is satisfied.

$$MSE(\hat{M}_R) - MSE(\hat{M}_y^*) > 0 \text{ or,}$$

$$M_y^2 \phi [C_{M_x}^2 + C_{M_y}^2 - 2\rho_{Myx} C_{M_x} C_{M_y}] - \left[P - \frac{R^2}{Q} \right] > 0.$$

Estimator ($\hat{M}_y^*$) is more efficient than estimator ($\hat{M}_P$) when the given condition holds.

$$MSE(\hat{M}_P) - MSE(\hat{M}_y^*) > 0 \text{ or,}$$

$$M_y^2 \phi [C_{M_x}^2 + C_{M_y}^2 + 2\rho_{Myx} C_{M_x} C_{M_y}] - \left[P - \frac{R^2}{Q} \right] > 0.$$

The efficiency of estimator $(\hat{M}_y^*)$ exceeds that of estimator $(\hat{M}_{ER})$ whenever the stated condition is fulfilled.

$$MSE(\hat{M}_{ER}) - MSE(\hat{M}_y^*) > 0 \text{ or,}$$

$$M_y^2 \phi \left[C_{My}^2 + \frac{C_{Mx}^2}{4} - \rho_{Myx} C_{Mx} C_{My} \right] - \left[P - \frac{R^2}{Q} \right] > 0.$$

Provided that the specified condition is met, estimator $(\hat{M}_y^*)$ exhibits superior efficiency compared to estimator $(\hat{M}_{EP})$.

$$MSE(\hat{M}_{EP}) - MSE(\hat{M}_y^*) > 0 \text{ or,}$$

$$M_y^2 \phi \left[C_{My}^2 + \frac{C_{Mx}^2}{4} + \rho_{Myx} C_{Mx} C_{My} \right] - \left[P - \frac{R^2}{Q} \right] > 0.$$

7. Empirical Study

This section presents a comparison between the proposed estimator $(\hat{M}_y^*)$ with other sine type estimators using real-life application data sets. The performance of the suggested class of estimators is examined using two real-world data sets and compared with other sine-type estimators in terms of MSE and PRE. A detailed description of the data sets and their main statistical characteristics is provided below:

Population 1: (Source: Murthy [21]).

Y: Represents the total number of households;

X: Represents the total area in square miles.

Population 2: (Source: Koyuncu and Kadilar [22]).

Y: Total teaching staff in educational institution;

X: The overall student enrolment in educational institutions in 2012.

A detailed statistical summary of each of the populations are given in Table 3.

Table 3: Descriptive statistics for two distinct populations.

Values	Population 1	Population 2
N	128	923
n	45	180
M_y	686	171
M_x	4.715	4123
$f_y(M_y)$	0.00092	0.002676
$f_x(M_x)$	0.1154	0.00009409
ρ_c	0.468	0.855
C_{Mx}	1.8379	2.5778
C_{My}	1.5849	2.1853

Table 4: The MSE values for various estimators based on real data sets.

Estimator	Population 1	Population 2
$\hat{M}_0$	1064.045	39.207
$\hat{M}_R$	1340.394	14.609
$\hat{M}_P$	3650.811	172.424
$\hat{M}_{ER}$	844.330	13.331
$\hat{M}_{EP}$	1999.539	92.238
$\hat{M}_y^*$	830.994	10.545

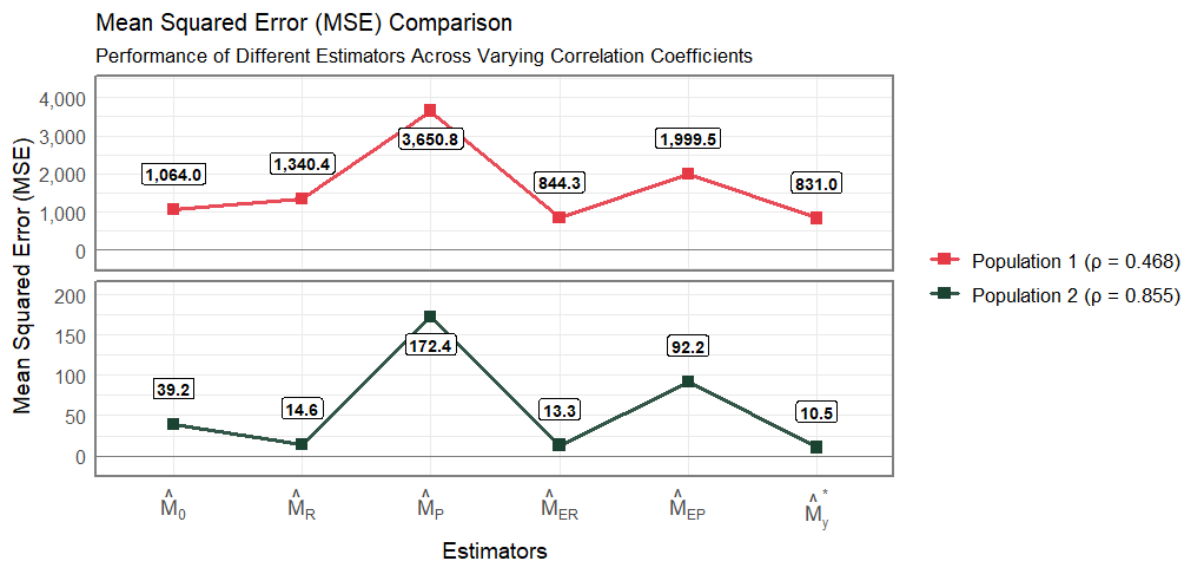


Figure 1. Line graph of MSE values for various estimators across two distinct populations.

The effectiveness of the proposed estimators is measured using the percent relative efficiency (PRE). The following expression is used to evaluate the PRE of several estimators in relation to the sample median.

$$PRE = \frac{MSE(\hat{M}_0)}{MSE(\text{Proposed sine type estimator})} \times 100$$

The PRE measures the mean squared error (MSE) of the conventional sample median in comparison with the proposed sine-type estimators. Expressed as a percentage, it indicates the extent of MSE reduction achieved by the proposed estimators over the traditional median.

Table 5: The PRE values for various estimators based on real data sets.

Estimator	Population 1	Population 2
$\hat{M}_0$	100.000	100.000
$\hat{M}_R$	79.383	268.374
$\hat{M}_P$	29.145	22.739

$\hat{M}_{ER}$	126.022	294.103
$\hat{M}_{EP}$	53.214	42.506
$\hat{M}_y^*$	128.045	371.781

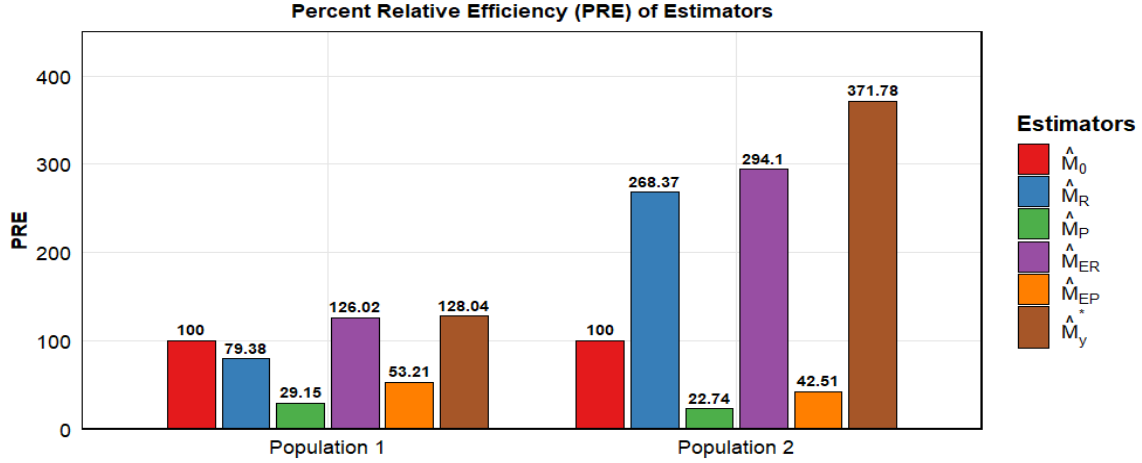


Figure 2. PRE Values Across Different Estimators.

From Tables 5, it can be observed that the proposed estimator ($\hat{M}_y^*$) yields PRE values of 128.04 for Population 1 and 371.78 for Population 2. This demonstrates that estimator ($\hat{M}_y^*$) performs with about 1.28 times greater efficiency than the conventional median estimator for Population 1 and 3.718 times for Population 2.

8. Simulation Study

A simulation study is carried out to analyse and assess the effectiveness of the proposed median estimation methods, which are compared with alternative estimators across multiple scenarios. The simulation analysis was conducted following the methodology outlined in Ahmad et al. [9] and Singh et al. [17]. The findings are summarized in the simulation tables, reporting MSE and PRE values for various estimators under different sample sizes and correlation coefficients. The simulation study is executed through a sequence of steps:

- The R software has been employed to simulate the Bivariate Normal distribution and the continuous Gamma distribution for X and Y, each with a size of $N = 1000$ under the specified parameter values.
- The Normal distribution: specified by the mean vector μ and the covariance matrix Σ , with, $\mu = [14, 8]$ and

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_y\sigma_x \\ \rho_{xy}\sigma_y\sigma_x & \sigma_y^2 \end{bmatrix}$$

where, $\sigma_x^2 = 4$, $\sigma_y^2 = 9$ and $\rho_c = 0.3, 0.5, 0.7, 0.9$;

The Gamma distribution: generated in R using the 'extraDistr' package, which provides functions for simulating correlated random variables from specialized probability distributions, with parameters:

$\lambda_x = 18, k_x = 8; \lambda_y = 14, k_y = 7$ and $\rho_c = 0.3, 0.5, 0.7, 0.9$; where λ is the rate parameter and k is the shape parameter.

- (c) The statistical characteristics corresponding to the simulated population of $N = 1000$ were obtained.

Table 6: MSE and PRE analysis of estimators using data from a simulated Bivariate Normal distribution.

$N = 1000$				
ρ_c	n	Estimator	MSE	PRE
0.3	80	$\hat{M}_0$	0.025888	100.00
		$\hat{M}_R$	0.023731	109.09
		$\hat{M}_P$	0.035559	72.80
		$\hat{M}_{ER}$	0.023871	108.45
		$\hat{M}_{EP}$	0.029784	86.92
		$\hat{M}_y^*$	0.023558	109.89
	150	$\hat{M}_0$	0.012756	100.00
		$\hat{M}_R$	0.011693	109.08
		$\hat{M}_P$	0.017521	72.80
		$\hat{M}_{ER}$	0.011762	108.45
		$\hat{M}_{EP}$	0.014676	86.91
		$\hat{M}_y^*$	0.011608	109.90
	250	$\hat{M}_0$	0.006753	100.00
		$\hat{M}_R$	0.006191	109.09
		$\hat{M}_P$	0.009276	72.81
		$\hat{M}_{ER}$	0.006227	108.45
		$\hat{M}_{EP}$	0.007769	86.91
		$\hat{M}_y^*$	0.006145	109.89
0.5	80	$\hat{M}_0$	0.025889	100.00
		$\hat{M}_R$	0.019788	130.83
		$\hat{M}_P$	0.039502	65.53
		$\hat{M}_{ER}$	0.021900	118.21
		$\hat{M}_{EP}$	0.031756	81.52
		$\hat{M}_y^*$	0.019417	133.33
	150	$\hat{M}_0$	0.012757	100.00
		$\hat{M}_R$	0.009750	130.83
		$\hat{M}_P$	0.019464	65.54
		$\hat{M}_{ER}$	0.010791	118.22
		$\hat{M}_{EP}$	0.015648	81.53
		$\hat{M}_y^*$	0.009567	133.33
	250	$\hat{M}_0$	0.006753	100.00
		$\hat{M}_R$	0.005162	130.84
		$\hat{M}_P$	0.010304	65.54
		$\hat{M}_{ER}$	0.005713	118.22

		$\hat{M}_{EP}$	0.008284	81.52
		$\hat{M}_y^*$	0.005065	133.34
0.7	80	$\hat{M}_0$	0.025889	100.00
		$\hat{M}_R$	0.015844	163.40
		$\hat{M}_P$	0.043447	59.58
		$\hat{M}_{ER}$	0.019927	129.91
		$\hat{M}_{EP}$	0.033729	76.75
		$\hat{M}_y^*$	0.013204	196.07
	150	$\hat{M}_0$	0.012757	100.00
		$\hat{M}_R$	0.007807	163.40
		$\hat{M}_P$	0.021408	59.58
		$\hat{M}_{ER}$	0.009819	129.92
		$\hat{M}_{EP}$	0.016620	76.76
		$\hat{M}_y^*$	0.006506	196.08
	250	$\hat{M}_0$	0.006753	100.00
		$\hat{M}_R$	0.004133	163.41
		$\hat{M}_P$	0.011333	59.59
		$\hat{M}_{ER}$	0.005198	129.92
		$\hat{M}_{EP}$	0.008798	76.75
		$\hat{M}_y^*$	0.003444	196.08
0.9	80	$\hat{M}_0$	0.025888	100.00
		$\hat{M}_R$	0.011896	217.61
		$\hat{M}_P$	0.047392	54.62
		$\hat{M}_{ER}$	0.017953	144.19
		$\hat{M}_{EP}$	0.035701	72.51
		$\hat{M}_y^*$	0.004918	526.31
	150	$\hat{M}_0$	0.012756	100.00
		$\hat{M}_R$	0.005861	217.61
		$\hat{M}_P$	0.023352	54.62
		$\hat{M}_{ER}$	0.008846	144.19
		$\hat{M}_{EP}$	0.017591	72.51
		$\hat{M}_y^*$	0.002423	526.32
	250	$\hat{M}_0$	0.006753	100.00
		$\hat{M}_R$	0.003103	217.62
		$\hat{M}_P$	0.012363	54.63
		$\hat{M}_{ER}$	0.004683	144.20
		$\hat{M}_{EP}$	0.009313	72.51
		$\hat{M}_y^*$	0.001283	526.32

(d) The simulated population was used to generate samples of sizes $n = 80, 150$ and 250 by employing simple random sampling without replacement.

(e) The values of MSE and PRE corresponding to the proposed and existing estimators were obtained for several correlation levels (ρ_c) using these samples.

Table 7: MSE and PRE analysis of estimators using data from a simulated Gamma distribution.

$N = 1000$				
ρ_c	n	Estimator	MSE	PRE
0.3	80	$\hat{M}_0$	0.000103	100.00
		$\hat{M}_R$	0.000136	75.50
		$\hat{M}_P$	0.000249	41.25
		$\hat{M}_{ER}$	0.000097	105.97
		$\hat{M}_{EP}$	0.000154	66.96
		$\hat{M}_y^*$	0.000094	109.58
	150	$\hat{M}_0$	0.000051	100.00
		$\hat{M}_R$	0.000012	75.49
		$\hat{M}_P$	0.000123	41.24
		$\hat{M}_{ER}$	0.000048	105.97
		$\hat{M}_{EP}$	0.000076	66.95
		$\hat{M}_y^*$	0.000046	109.57
	250	$\hat{M}_0$	0.000027	100.00
		$\hat{M}_R$	0.000035	75.50
		$\hat{M}_P$	0.000065	41.25
		$\hat{M}_{ER}$	0.000025	105.96
		$\hat{M}_{EP}$	0.000040	66.96
		$\hat{M}_y^*$	0.000024	109.58
0.5	80	$\hat{M}_0$	0.000102	100.00
		$\hat{M}_R$	0.000097	104.95
		$\hat{M}_P$	0.000287	35.76
		$\hat{M}_{ER}$	0.000078	131.95
		$\hat{M}_{EP}$	0.000173	59.54
		$\hat{M}_y^*$	0.000077	132.21
	150	$\hat{M}_0$	0.000051	100.00
		$\hat{M}_R$	0.000048	104.95
		$\hat{M}_P$	0.000142	35.77
		$\hat{M}_{ER}$	0.000038	131.96
		$\hat{M}_{EP}$	0.000085	59.54
		$\hat{M}_y^*$	0.000038	132.22
	250	$\hat{M}_0$	0.000027	100.00
		$\hat{M}_R$	0.000025	104.96
		$\hat{M}_P$	0.000075	35.75
		$\hat{M}_{ER}$	0.000020	131.96
		$\hat{M}_{EP}$	0.000045	59.55
		$\hat{M}_y^*$	0.000020	132.22

0.7	80	$\hat{M}_0$	0.000102	100.00
		$\hat{M}_R$	0.000059	173.40
		$\hat{M}_P$	0.000326	31.52
		$\hat{M}_{ER}$	0.000058	175.52
		$\hat{M}_{EP}$	0.000192	53.54
		$\hat{M}_y^*$	0.000053	193.08
	150	$\hat{M}_0$	0.000050	100.00
		$\hat{M}_R$	0.000029	173.40
		$\hat{M}_P$	0.000160	31.52
		$\hat{M}_{ER}$	0.000028	175.53
		$\hat{M}_{EP}$	0.000095	53.54
		$\hat{M}_y^*$	0.000026	193.09
	250	$\hat{M}_0$	0.000026	100.00
		$\hat{M}_R$	0.000015	173.40
		$\hat{M}_P$	0.000085	31.52
		$\hat{M}_{ER}$	0.000015	175.52
		$\hat{M}_{EP}$	0.000050	53.54
		$\hat{M}_y^*$	0.000013	193.08
0.9	80	$\hat{M}_0$	0.000102	100.00
		$\hat{M}_R$	0.000020	509.68
		$\hat{M}_P$	0.000365	28.14
		$\hat{M}_{ER}$	0.000038	263.55
		$\hat{M}_{EP}$	0.000211	48.59
		$\hat{M}_y^*$	0.000019	514.03
	150	$\hat{M}_0$	0.000050	100.00
		$\hat{M}_R$	0.000009	509.68
		$\hat{M}_P$	0.000179	28.14
		$\hat{M}_{ER}$	0.000019	263.55
		$\hat{M}_{EP}$	0.000104	48.59
		$\hat{M}_y^*$	0.000008	514.04
	250	$\hat{M}_0$	0.000026	100.00
		$\hat{M}_R$	0.000005	509.68
		$\hat{M}_P$	0.000095	28.143
		$\hat{M}_{ER}$	0.000010	263.55
		$\hat{M}_{EP}$	0.000055	48.59
		$\hat{M}_y^*$	0.000005	514.06

(f) The entire procedure of obtaining the MSEs of the estimators under simple random sampling is replicated 10,000 times.

Tables 6 and 7 provide the results demonstrating estimator performance as assessed by MSEs and PREs.

9. Results and Discussion

The results reported in Table 4 reveal that the traditional median estimator ($\hat{M}_0$) has a variance of 1064.04 for population 1 and 39.207 for population 2, while both the sine-type product estimator ($\hat{M}_P$) and exponential-sin-type product estimator ($\hat{M}_{EP}$) show relatively higher MSEs for both populations. These findings suggest that the estimator ($\hat{M}_0$) outperforms certain sine-type estimators. The reason behind the higher MSEs of estimators ($\hat{M}_P$) and ($\hat{M}_{EP}$) is the presence of a moderate positive correlation between the study variable and the auxiliary variable. The results in Tables 4 and 5 show that the proposed estimator ($\hat{M}_y^*$) attains lower MSE and higher PRE, thereby validating its superior performance over conventional and sine-type estimators. Furthermore, Figures 1 and 2 illustrate line plots of MSE values and multiple bar chart of PRE values for various estimators based on Population 1 and Population 2, respectively. These depictions provide a clear visual demonstration of the comparative performance of the estimators.

We performed a simulation study employing Bivariate Normal and Gamma distributions, where the effectiveness of the proposed estimator relative to existing ones was examined through MSE and PRE values. The simulation based on generated data provided the following insights:

- We performed a simulation study with generated data to examine the performance of the proposed estimators across varying conditions such as correlation levels and sample sizes. The findings in Tables 6 and 7 highlight that the proposed estimator consistently provides superior results relative to other existing and sine-type estimators. Its efficiency was assessed using MSE and PRE across correlation levels of ρ_c (0.3, 0.5, 0.7 and 0.9). As shown in Tables 6 and 7, our estimators provide superior outcomes for each considered correlation coefficient.
- In addition, as the sample size grows from $n = 80, 150$ and 250 , the proposed estimators ($\hat{M}_y^*$) maintain superior efficiency, thereby outperforming conventional approaches.
- The proposed estimator ($\hat{M}_y^*$) consistently demonstrates superior performance compared to the usual sample median and other sine-type estimators across all correlation coefficient values. Estimators ($\hat{M}_R, \hat{M}_{ER}$) are particularly effective when the correlation is positive, whereas estimators $\hat{M}_P$ and $\hat{M}_{EP}$ offer better performance when the correlation is negative.

10. Conclusion

This study introduces several sine-type estimators along with a family of sine-type estimators for estimating the finite population median under the framework of simple random sampling without replacement. The bias and MSE of the proposed estimator have been derived using first-order approximation methods. Supported by theoretical derivations and extensive numerical analysis employing real-life datasets and simulated data from Bivariate Normal and Gamma distributions, we establish that the proposed estimator consistently outperforms the conventional median estimator and various sine-type estimators. This study is subject to certain limitations. The population median of the auxiliary variable has been assumed to be known and fixed, along with the assumption of a non-zero correlation between the study and auxiliary variables. In addition, the study assumes that the relative difference between the sample median and population median of the auxiliary variable is small, as this condition underpins the validity of first-order approximations for bias and MSE. This study provides valuable contributions toward advancing statistical methods aimed at enhancing the estimation of population parameters in real-life contexts. Thus, it is advisable to adopt the proposed estimator to enhance the accuracy and efficiency of population median estimation within the framework of simple random sampling.

11. Future works

The following directions are suggested for advancing the present study:

Extending the enhanced finite population median estimators to additional parameters such as mean, CDF, proportions, and variance, by:

- (1) Leveraging multiple auxiliary variables for improved estimation.
- (2) Incorporating the impact of measurement errors and non-response.
- (3) Applying different sampling techniques such as double sampling, stratified random sampling, and ranked set sampling.
- (4) Adopting two-phase sampling and stratified two-phase sampling approaches.

Funding

The authors declare that no funds or other grants were received for the preparation of this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Data availability statement

The datasets used are already present in the manuscript.

Reference

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