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Salem Saleh Al-Qasemi Barahmah

salemalqasemi@yahoo.com

University of Aden

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Symmetric Jacobi–Dirichlet Deformations: Finite Hypergeometric Closure, Positive Tilts, and Meromorphic Continuation

Dr. Salem Saleh Al-Qasemi Barahmah

Department of Mathematics, Aden University, Aden, Yemen

E-mail: salemalqasemi@yahoo.com

ORCID: 0000-0002-9766-3972

Abstract

We introduce a permutation-invariant Jacobi deformation of the three-parameter Dirichlet integral and separate its signed and positive roles. The signed family retains the oscillatory and differential structure of the Jacobi system and admits a finite beta expansion, an exact terminating generalized-hypergeometric representation, deformation and parameter derivatives, a simplex recurrence, a finite Mellin transform, a differential equation, a degree recurrence, and a weighted generating identity. The central positivity contribution is obtained by squaring the normalized shifted Jacobi kernel. The resulting quadratic family defines a strictly positive partition function and a polynomial tilt of the Dirichlet law. We prove joint log-convexity and strict convexity in the Dirichlet parameters, derive Turán-type inequalities, identify the Hessian of the log-normalizer with a covariance matrix, and obtain explicit mixed moments for the associated Jacobi–Dirichlet distribution. Appell- and Humbert-type companions follow from the same simplex mechanism. Meromorphic continuation is analyzed at the gamma-factor level, showing that the candidate poles lie only on coordinate hyperplanes at nonpositive integers. Independent finite-sum and adaptive-quadrature calculations confirm the formulas numerically. The construction therefore links positivity, convexity, special-function closure, and probabilistic normalization within a single symmetric simplex framework.

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1. Introduction

The Dirichlet integral is simultaneously a multivariate beta function, a normalizing constant for probability laws on a simplex, and a convenient mechanism for producing multiple hypergeometric identities. Its value is explicit because monomials in the barycentric coordinates integrate to gamma quotients. This algebraic transparency makes the Dirichlet integral an unusually effective testing ground for deformations: a successful deformation should be rich enough to create new structure, yet sufficiently controlled that evaluation, differentiation, continuation, and numerical verification remain possible.

A large part of the extension literature modifies endpoint behavior by inserting exponential or related non-polynomial factors. The classical extension of Chaudhry et al. [6], and later multivariate developments [7,8], belong to this broad direction. Such kernels can enlarge the useful analytic domain or regularize boundary singularities, but they usually lead to infinite expansions. Our objective is different. We insert a bounded symmetric polynomial statistic into a classical orthogonal polynomial. The convergence region of the original Dirichlet integral is therefore not enlarged; the compensation is exact finite closure.

Jacobi polynomials are particularly suitable for this purpose. They possess a terminating Gauss representation, a stable differentiation rule, explicit endpoint values, classical three-term recurrences, and direct specializations to Legendre, Gegenbauer, and Chebyshev systems [1–3,12,13]. Their role in multivariate probability and Dirichlet structures is well established [4,9,10], while recent work has emphasized efficient recursions for integrated products of Jacobi polynomials [11] and new integral representations [22]. The construction below is not a multivariate Jacobi polynomial in the conventional sense. Instead, a one-variable Jacobi kernel is composed with the fully symmetric barycentric product uvw .

The choice of $27uvw$ is deliberate. It is invariant under every permutation of u, v, w , it ranges exactly over $[0,1]$, and its powers shift the three Dirichlet exponents by the same amount. These three facts explain most of

the closed structure obtained later. A coordinate-dependent argument such as u or $u - v$ would break the parameter symmetry and replace the single diagonal shift $(a, b, c) \mapsto (a + k, b + k, c + k)$ by asymmetric shifts.

The Jacobi kernel changes sign for positive degree, so its analytic and probabilistic roles must be separated. We therefore introduce two complementary families. The signed deformation preserves oscillation, differential structure, recurrences, and zero information. The quadratic deformation squares the same kernel and is the positivity-driven component of the theory: it is a strictly positive partition function, generates a polynomially tilted Dirichlet law, and supports convexity and Turán-type inequalities. Thus positivity is not an auxiliary consequence of the construction but one of its organizing principles.

The main contributions, with emphasis on the positivity structure, are as follows.

1. The quadratic family is a strictly positive normalizer. Hölder's inequality and a covariance representation yield joint log-convexity, strict convexity, and Turán-type inequalities in the Dirichlet parameters.
2. The normalized positive kernel defines a Jacobi–Dirichlet law with explicit mixed moments, covariance identities for logarithmic derivatives, and an entire joint moment-generating function.
3. The signed deformation is evaluated as a finite sum of shifted three-variable beta functions and, after the gamma triplication formula, as a terminating generalized-hypergeometric polynomial.
4. For the signed family, full differential closure is established with respect to the deformation variable and beta parameters; a finite Mellin transform, a generalized-hypergeometric differential equation, a Jacobi degree recurrence, and a weighted generating identity are derived.
5. The same construction generates Appell- F_3 and Humbert- Φ_2 companions.
6. The meromorphic continuation is audited at the gamma-factor level. This removes a common but important mistake: the planes $a + b + c = -m$ are not candidate poles of the beta quotient, because $1/\Gamma(a + b + c)$ is entire.
7. Numerical values are computed independently from finite beta sums and adaptive simplex quadrature. The two routes agree at the expected floating-point scale.

The manuscript is written in a theorem–proof form, but the proofs are not reduced to one-line references. The key transformations are displayed explicitly so that each parameter shift and gamma quotient can be checked independently.

1.1 Position relative to related literature

Chaudhry et al. [6]. Their extension uses an exponential beta kernel and is primarily concerned with endpoint modification; it is not a finite Jacobi closure and does not produce the present tilted law.

Raïssouli and El-Soubhy [7]. Their framework develops multivariate beta generalizations and inequalities. The kernel and continuation mechanism differ from the scalar barycentric Jacobi construction used here.

Al-harbi and Raïssouli [8]. Their several-variable extension includes an application to multivariate Laguerre polynomials. The present work instead uses a symmetric Jacobi kernel and separates signed and positive families.

Griffiths and Spanò [9,10]. Their results connect multivariate Jacobi and Laguerre systems with Dirichlet measures and orthogonal kernels. Our deformation is not a multivariate orthogonal basis; it is a one-variable kernel composed with the symmetric statistic $27uvw$.

Beuchler et al. [11]. Their recursion theory concerns integrated products of Jacobi polynomials and efficient computation. The present finite closure arises from diagonal beta shifts and leads additionally to a positive probability tilt and meromorphic continuation.

Present work. The defining kernel is $R_n^{(\alpha, \beta)}(27\rho uvw)$. It is fully permutation-invariant, closes as a terminating ${}_5F_4$, and generates both a signed analytic family and a positive Jacobi–Dirichlet law.

The table is not meant to claim that the cited theories are special cases of one another. It identifies the distinct mechanism used here and clarifies which parts of the present contribution rely specifically on the barycentric product.

2. Preliminaries

Let

$$\Delta_2 = \{(u, v) \in \mathbb{R}^2 : u > 0, v > 0, u + v < 1\}, \quad w = 1 - u - v. \quad (2.1)$$

For $\Re a, \Re b, \Re c > 0$, the three-variable beta function is

$$B_3(a, b, c) = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} du dv = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}. \quad (2.2)$$

We write $\sigma = a + b + c$; the letter s is reserved for summation and moment indices. We use the Pochhammer symbol

$$(q)_k = \frac{\Gamma(q+k)}{\Gamma(q)}, \quad (q)_0 = 1. \quad (2.3)$$

The generalized hypergeometric function is

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}. \quad (2.4)$$

For $n \in \mathbb{N}_0$ and Jacobi indices $\alpha, \beta > -1$, define the normalized shifted Jacobi polynomial

$$R_n^{(\alpha, \beta)}(t) = \frac{P_n^{(\alpha, \beta)}(1-2t)}{P_n^{(\alpha, \beta)}(1)} = {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; t \right). \quad (2.5)$$

Because $-n$ is a nonpositive integer, the series terminates:

$$R_n^{(\alpha, \beta)}(t) = \sum_{k=0}^n c_{n,k}^{(\alpha, \beta)} t^k, \quad c_{n,k}^{(\alpha, \beta)} = \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(\alpha + 1)_k k!}. \quad (2.6)$$

In particular, $R_n^{(\alpha, \beta)}(0) = 1$.

Lemma 2.1. Range and equality case of the barycentric product

For $(u, v) \in \bar{\Delta}_2$,

$$0 \leq uvw \leq \frac{1}{27}, \quad 0 \leq \xi(u, v, w) := 27uvw \leq 1, \quad (2.7)$$

and equality in the upper bound occurs only at $u = v = w = 1/3$.

Proof. The three numbers u, v, w are nonnegative and satisfy $u + v + w = 1$. The arithmetic–geometric mean inequality gives

$$(uvw)^{1/3} \leq \frac{u + v + w}{3} = \frac{1}{3}.$$

Cubing yields $uvw \leq 1/27$. Equality in AM–GM holds precisely when $u = v = w$. Together with $u + v + w = 1$, this forces $u = v = w = 1/3$. The lower bound follows from nonnegativity.

Lemma 2.2. Differentiation of the normalized kernel

For $0 \leq m \leq n$,

$$\frac{d^m}{dt^m} R_n^{(\alpha, \beta)}(t) = \frac{(-n)_m (n + \alpha + \beta + 1)_m}{(\alpha + 1)_m} R_{n-m}^{(\alpha+m, \beta+m)}(t). \quad (2.8)$$

For $m > n$, the derivative vanishes.

Proof. Start from the terminating expansion (2.6):

$$\frac{d^m}{dt^m} R_n(t) = \sum_{k=m}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k}{(\alpha + 1)_k k!} \frac{k!}{(k-m)!} t^{k-m}.$$

Set $j = k - m$. The Pochhammer factorization $(q)_{j+m} = (q)_m (q + m)_j$ gives

$$\frac{d^m}{dt^m} R_n(t) = \frac{(-n)_m (n + \alpha + \beta + 1)_m}{(\alpha + 1)_m} \sum_{j=0}^{n-m} \frac{(-n+m)_j (n + \alpha + \beta + 1 + m)_j}{(\alpha + 1 + m)_j j!} t^j.$$

The last finite series is exactly $R_{n-m}^{(\alpha+m, \beta+m)}(t)$, because

$$(n - m) + (\alpha + m) + (\beta + m) + 1 = n + \alpha + \beta + m + 1.$$

This proves (2.8).

Lemma 2.3. Diagonal beta shift

For every $k \in \mathbb{N}_0$,

$$\frac{B_3(a+k, b+k, c+k)}{B_3(a, b, c)} = \frac{(a)_k (b)_k (c)_k}{(\sigma)_{3k}}. \quad (2.9)$$

Moreover,

$$(\sigma)_{3k} = 3^{3k} \left(\frac{\sigma}{3}\right)_k \left(\frac{\sigma+1}{3}\right)_k \left(\frac{\sigma+2}{3}\right)_k. \quad (2.10)$$

Proof. Divide the gamma quotient for $B_3(a+k, b+k, c+k)$ by (2.2), then use $\Gamma(q+k)/\Gamma(q) = (q)_k$. Formula (2.10) is the Pochhammer form of the gamma triplication formula.

3. The signed Jacobi–Dirichlet deformation

Definition 3.1

For $n \in \mathbb{N}_0$, $\rho \in \mathbb{C}$, $\alpha, \beta > -1$, and $\Re a, \Re b, \Re c > 0$, define

$$\mathcal{J}_n^{(\alpha, \beta)}(\rho; a, b, c) = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} R_n^{(\alpha, \beta)}(27\rho uvw) du dv. \quad (3.1)$$

The adjective *signed* refers to the fact that the Jacobi factor can change sign when the parameters are real. In intermediate proof lines, fixed Jacobi indices and unchanged arguments are suppressed only when no ambiguity can arise.

Theorem 3.1. Absolute convergence, polynomial dependence, and holomorphy

Assume $\Re a, \Re b, \Re c > 0$. Then (3.1) converges absolutely for every $\rho \in \mathbb{C}$. For fixed n, α, β , it is a polynomial in ρ of degree at most n and is jointly holomorphic in (a, b, c) on the product of right half-planes.

Proof. Fix ρ . By Lemma 2.1, $|27\rho uvw| \leq |\rho|$. Since R_n is a polynomial, it is bounded on the closed disk $|t| \leq |\rho|$; write $|R_n(t)| \leq M_{n, \rho}$. Therefore

$$|u^{a-1} v^{b-1} w^{c-1} R_n(27\rho uvw)| \leq M_{n, \rho} u^{\Re a-1} v^{\Re b-1} w^{\Re c-1},$$

and the majorant is integrable by (2.2). Absolute convergence follows.

Next insert the finite expansion (2.6). Since only $n+1$ terms are present, integration may be performed term by term, producing a polynomial in ρ . For holomorphy, let K be a compact subset of the parameter domain. There is an $\varepsilon > 0$ such that $\Re a, \Re b, \Re c \geq \varepsilon$ on K . Parameter derivatives introduce powers of $\log u, \log v, \log w$, and each such logarithmic factor is integrable against $u^{\varepsilon-1} v^{\varepsilon-1} w^{\varepsilon-1}$. Dominated differentiation on K proves joint holomorphy.

Corollary 3.1. Classical reductions

$$\mathcal{J}_n^{(\alpha, \beta)}(0; a, b, c) = B_3(a, b, c), \quad \mathcal{J}_0^{(\alpha, \beta)}(\rho; a, b, c) = B_3(a, b, c). \quad (3.2)$$

Proof. In both cases the normalized kernel equals one.

Theorem 3.2. Finite beta sum and terminating ${}_5F_4$ representation

The signed deformation has the exact finite representation

$$\mathcal{J}_n^{(\alpha, \beta)}(\rho; a, b, c) = \sum_{k=0}^n c_{n, k}^{(\alpha, \beta)} (27\rho)^k B_3(a+k, b+k, c+k). \quad (3.3)$$

Equivalently,

$$\frac{j_n^{(\alpha, \beta)}(\rho; a, b, c)}{B_3(a, b, c)} = {}_5F_4 \left(\begin{matrix} -n, n + \alpha + \beta + 1, a, b, c \\ \alpha + 1, \sigma/3, (\sigma + 1)/3, (\sigma + 2)/3 \end{matrix}; \rho \right). \quad (3.4)$$

Proof. Substitute (2.6) into (3.1):

$$\mathcal{J}_n = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} \sum_{k=0}^n c_{n,k} (27\rho uvw)^k du dv.$$

The sum is finite, hence

$$\mathcal{J}_n = \sum_{k=0}^n c_{n,k} (27\rho)^k \int_{\Delta_2} u^{a+k-1} v^{b+k-1} w^{c+k-1} du dv,$$

which is (3.3). Divide by $B_3(a, b, c)$ and apply (2.9):

$$\frac{\mathcal{J}_n}{B_3(a, b, c)} = \sum_{k=0}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k (a)_k (b)_k (c)_k (27\rho)^k}{(\alpha + 1)_k (\sigma)_{3k} k!}.$$

Using (2.10), the factor $27^k = 3^{3k}$ cancels exactly. The remaining coefficients are those of the ${}_5F_4$ series in (3.4). Termination follows from $(-n)_k = 0$ for $k > n$.

Corollary 3.2. Explicit first- and second-degree formulas

Let $\Lambda = \alpha + \beta + 3$. Then

$$\mathcal{J}_1 = B_3(a, b, c) - 27\rho \frac{\alpha + \beta + 2}{\alpha + 1} B_3(a + 1, b + 1, c + 1), \quad (3.5)$$

and

$$\begin{aligned} \mathcal{J}_2 &= B_3(a, b, c) - 54\rho \frac{\Lambda}{\alpha + 1} B_3(a + 1, b + 1, c + 1) \\ &\quad + 729\rho^2 \frac{\Lambda(\Lambda + 1)}{(\alpha + 1)(\alpha + 2)} B_3(a + 2, b + 2, c + 2). \end{aligned} \quad (3.6)$$

Proof. Evaluate $c_{n,k}$ from (2.6) for $n = 1, 2$ and substitute into (3.3).

Theorem 3.3. Permutation symmetry

For every permutation $\pi \in S_3$,

$$\mathcal{J}_n(\rho; a, b, c) = \mathcal{J}_n(\rho; \pi(a, b, c)). \quad (3.7)$$

Proof. A permutation of the barycentric coordinates maps Δ_2 onto itself, has unit Jacobian in absolute value, and leaves uvw invariant. It only permutes the exponents in the weight.

Theorem 3.4. Simplex recurrence

$$\mathcal{J}_n(\rho; a, b, c) = \mathcal{J}_n(\rho; a + 1, b, c) + \mathcal{J}_n(\rho; a, b + 1, c) + \mathcal{J}_n(\rho; a, b, c + 1). \quad (3.8)$$

Proof. Insert $u + v + w = 1$ into the integrand of (3.1), split the resulting integral into three terms, and absorb each coordinate into its corresponding exponent.

Theorem 3.5. Derivative ladder in the deformation variable

For $0 \leq m \leq n$,

$$\frac{\partial^m}{\partial \rho^m} \mathcal{J}_n^{(\alpha, \beta)}(\rho; a, b, c) = 27^m \frac{(-n)_m (n + \alpha + \beta + 1)_m}{(\alpha + 1)_m} \mathcal{J}_{n-m}^{(\alpha + m, \beta + m)}(\rho; a + m, b + m, c + m). \quad (3.9)$$

The derivative is zero for $m > n$.

Proof. Differentiate under the integral sign, justified by Theorem 3.1:

$$\frac{\partial^m}{\partial \rho^m} R_n(27\rho uvw) = (27uvw)^m R_n^{(m)}(27\rho uvw).$$

Apply Lemma 2.2:

$$\frac{\partial^m}{\partial \rho^m} R_n(27\rho uvw) = 27^m (uvw)^m \frac{(-n)_m (n + \alpha + \beta + 1)_m}{(\alpha + 1)_m} R_{n-m}^{(\alpha+m, \beta+m)}(27\rho uvw).$$

Multiplication by $(uvw)^m$ raises all three beta exponents by m , and the remaining integral is the shifted deformation on the right-hand side of (3.9).

At $\rho = 0$, Corollary 3.1 gives the explicit Taylor data

$$\mathcal{J}_n^{(m)}(0) = 27^m \frac{(-n)_m (n + \alpha + \beta + 1)_m}{(\alpha + 1)_m} B_3(a + m, b + m, c + m). \quad (3.10)$$

Theorem 3.6. Derivatives with respect to the beta parameters

Let $\psi = \Gamma'/\Gamma$ denote the digamma function. Then

$$\frac{\partial \mathcal{J}_n}{\partial a} = \sum_{k=0}^n c_{n,k} (27\rho)^k B_3(a + k, b + k, c + k) [\psi(a + k) - \psi(\sigma + 3k)]. \quad (3.11)$$

The formulas for $\partial_b \mathcal{J}_n$ and $\partial_c \mathcal{J}_n$ follow by symmetry.

Proof. Differentiate the finite sum (3.3). Since

$$\log B_3(a + k, b + k, c + k) = \log \Gamma(a + k) + \log \Gamma(b + k) + \log \Gamma(c + k) - \log \Gamma(\sigma + 3k),$$

its derivative with respect to a is $\psi(a + k) - \psi(\sigma + 3k)$. Multiplication by the beta factor proves (3.11).

A second differentiation produces trigamma terms and products of digamma differences. This gives explicit Hessian formulas without differentiating a singular integral directly.

Theorem 3.7. Finite Mellin transform on the unit deformation interval

For $\Re z > 0$, define

$$\mathfrak{M}_n(z) = \int_0^1 \rho^{z-1} \mathcal{J}_n(\rho; a, b, c) d\rho. \quad (3.12)$$

Then

$$\mathfrak{M}_n(z) = \frac{B_3(a, b, c)}{z} {}_6F_5 \left(\begin{matrix} -n, n + \alpha + \beta + 1, a, b, c, z \\ \alpha + 1, \sigma/3, (\sigma + 1)/3, (\sigma + 2)/3, z + 1 \end{matrix}; 1 \right). \quad (3.13)$$

Proof. By (3.3),

$$\mathfrak{M}_n(z) = \sum_{k=0}^n c_{n,k} 27^k B_3(a + k, b + k, c + k) \int_0^1 \rho^{z+k-1} d\rho.$$

The elementary integral is $(z + k)^{-1}$. Use

$$\frac{1}{z + k} = \frac{1}{z} \frac{(z)_k}{(z + 1)_k},$$

then repeat the beta-ratio and triplication calculation of Theorem 3.2.

Remark 3.1. The interval $[0, 1]$ is essential in (3.12). A Mellin transform over $(0, \infty)$ generally diverges because $\mathcal{J}_n$ is a nonzero polynomial in ρ .

Theorem 3.8. Generalized-hypergeometric differential equation

Let

$$y(\rho) = \frac{\mathcal{J}_n^{(\alpha, \beta)}(\rho; a, b, c)}{B_3(a, b, c)}, \quad \theta = \rho \frac{d}{d\rho}. \quad (3.14)$$

Then

$$\left[\theta(\theta + \alpha) \left(\theta + \frac{\sigma}{3} - 1 \right) \left(\theta + \frac{\sigma-2}{3} \right) \left(\theta + \frac{\sigma-1}{3} \right) - \rho(\theta - n)(\theta + n + \alpha + \beta + 1)(\theta + a)(\theta + b)(\theta + c) \right] y = 0. \quad (3.15)$$

Proof. A function ${}_pF_{p-1}(\mathbf{a}; \mathbf{b}; \rho)$ satisfies

$$\left[\theta \prod_{j=1}^{p-1} (\theta + b_j - 1) - \rho \prod_{i=1}^p (\theta + a_i) \right] y = 0.$$

Insert the five numerator and four denominator parameters from (3.4).

Theorem 3.9. Three-term recurrence in the Jacobi degree

Set

$$h_n = P_n^{(\alpha, \beta)}(1) = \frac{(\alpha+1)_n}{n!}, \quad (3.16)$$

and define

$$\begin{aligned} A_n &= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta), \\ B_n &= (2n+\alpha+\beta+1)(\alpha^2 - \beta^2), \\ C_n &= (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2), \\ D_n &= 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2). \end{aligned} \quad (3.17)$$

For $n \geq 1$,

$$\begin{aligned} A_n h_{n+1} \mathcal{J}_{n+1}(a, b, c) &= (B_n + C_n) h_n \mathcal{J}_n(a, b, c) - 54\rho C_n h_n \mathcal{J}_n(a+1, b+1, c+1) \\ &\quad - D_n h_{n-1} \mathcal{J}_{n-1}(a, b, c). \end{aligned} \quad (3.18)$$

Proof. The classical Jacobi recurrence is

$$A_n P_{n+1}(x) = (B_n + C_n x) P_n(x) - D_n P_{n-1}(x).$$

Put $x = 1 - 54\rho uvw$, use $P_j(x) = h_j R_j(27\rho uvw)$, multiply by the Dirichlet weight, and integrate. The term containing $uvw R_n$ is exactly $\mathcal{J}_n(a+1, b+1, c+1)$.

Theorem 3.10. Weighted generating identity

For $|r| < 1$, let

$$X = 1 - 54\rho uvw, \quad \mathcal{D} = (1 - 2Xr + r^2)^{1/2},$$

with the principal square root. Then

$$\sum_{n=0}^{\infty} h_n \mathcal{J}_n^{(\alpha, \beta)}(\rho; a, b, c) r^n = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} \frac{2^{\alpha+\beta}}{\mathcal{D}(1-r+\mathcal{D})^\alpha (1+r+\mathcal{D})^\beta} du dv. \quad (3.19)$$

Proof. The Jacobi generating function [2,3,21] is

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(X) r^n = \frac{2^{\alpha+\beta}}{\mathcal{D}(1-r+\mathcal{D})^\alpha (1+r+\mathcal{D})^\beta}.$$

For $|r| \leq r_0 < 1$ and X in the compact image of the simplex, the series converges uniformly. Since $P_n(X) = h_n R_n(27\rho uvw)$, termwise integration is valid and gives (3.19).

4. Positive quadratic Jacobi tilts: positivity, log-convexity, and Turán inequalities

Definition 4.1

For real $a, b, c > 0$, $\alpha, \beta > -1$, and $\rho \in \mathbb{R}$, define

$$\mathcal{Q}_n^{(\alpha, \beta)}(\rho; a, b, c) = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} \left[R_n^{(\alpha, \beta)}(27\rho uvw) \right]^2 du dv. \quad (4.1)$$

Theorem 4.1. Strict positivity and finite representations

For every admissible real parameter tuple,

$$\mathcal{Q}_n^{(\alpha, \beta)}(\rho; a, b, c) > 0. \quad (4.2)$$

Moreover,

$$\mathcal{Q}_n = \sum_{k=0}^n \sum_{\ell=0}^n c_{n,k} c_{n,\ell} (27\rho)^{k+\ell} B_3(a+k+\ell, b+k+\ell, c+k+\ell). \quad (4.3)$$

If

$$d_{n,r} = \sum_{k=\max(0,r-n)}^{\min(n,r)} c_{n,k} c_{n,r-k}, \quad (4.4)$$

then the equivalent one-dimensional sum is

$$Q_n = \sum_{r=0}^{2n} d_{n,r} (27\rho)^r B_3(a+r, b+r, c+r). \quad (4.5)$$

Proof. The integrand in (4.1) is nonnegative. The polynomial R_n is not identically zero, so its zero set in the scalar variable is finite. The continuous function $27\rho uvw$ cannot map an open subset of the simplex entirely into that finite set unless the polynomial factor is constant zero, which it is not. Hence the integrand is positive on a set of positive measure and the integral is strictly positive.

Expand each copy of R_n using (2.6), multiply the two finite sums, and integrate each term. This proves (4.3). Grouping terms with equal total degree $r = k + \ell$ gives (4.5).

Corollary 4.1. Classical limit

$$Q_n(0; a, b, c) = Q_0(\rho; a, b, c) = B_3(a, b, c). \quad (4.6)$$

Theorem 4.2. Joint log-convexity

Fix n, ρ, α, β . For $\mathbf{x} = (a, b, c)$ and $\mathbf{y} = (a', b', c')$ in $(0, \infty)^3$ and $0 < \lambda < 1$,

$$Q_n(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq Q_n(\mathbf{x})^\lambda Q_n(\mathbf{y})^{1-\lambda}. \quad (4.7)$$

Proof. Write

$$K(u, v, w) = [R_n(27\rho uvw)]^2 \geq 0.$$

The integrand at the interpolated parameter is

$$K[u^{a-1}v^{b-1}w^{c-1}]^\lambda [u^{a'-1}v^{b'-1}w^{c'-1}]^{1-\lambda}.$$

Absorb $K = K^\lambda K^{1-\lambda}$ and apply Hölder's inequality with conjugate exponents $1/\lambda$ and $1/(1-\lambda)$. The two resulting integrals are $Q_n(\mathbf{x})$ and $Q_n(\mathbf{y})$.

Corollary 4.2. Turán-type inequality

If $\mathbf{x} \pm \mathbf{h} \in (0, \infty)^3$, then

$$Q_n(\mathbf{x})^2 \leq Q_n(\mathbf{x} - \mathbf{h})Q_n(\mathbf{x} + \mathbf{h}). \quad (4.8)$$

Proof. Apply Theorem 4.2 with $\lambda = 1/2$ and endpoints $\mathbf{x} - \mathbf{h}$ and $\mathbf{x} + \mathbf{h}$.

Theorem 4.3. Partition derivatives and strict convexity

Define the probability density

$$f_n(u, v) = \frac{u^{a-1}v^{b-1}w^{c-1}[R_n(27\rho uvw)]^2}{Q_n(\rho; a, b, c)}. \quad (4.9)$$

Then

$$\nabla \log Q_n = \begin{pmatrix} \mathbb{E}(\log U) \\ \mathbb{E}(\log V) \\ \mathbb{E}(\log W) \end{pmatrix}, \quad (4.10)$$

and

$$\nabla^2 \log Q_n = \text{Cov} \begin{pmatrix} \log U \\ \log V \\ \log W \end{pmatrix}. \quad (4.11)$$

The Hessian is positive definite; hence $\log Q_n$ is strictly convex on $(0, \infty)^3$.

Proof. Differentiate Q_n under the integral sign. For example,

$$\partial_a Q_n = \int_{\Delta_2} (\log u) u^{a-1} v^{b-1} w^{c-1} R_n^2 du dv.$$

Division by Q_n gives $\partial_a \log Q_n = \mathbb{E}(\log U)$. A second derivative gives

$$\partial_{ab}^2 \log Q_n = \mathbb{E}(\log U \log V) - \mathbb{E}(\log U) \mathbb{E}(\log V),$$

and similarly for the other entries, proving (4.11).

For $\mathbf{h} \neq 0$,

$$\mathbf{h}^\top \nabla^2 \log Q_n \mathbf{h} = \text{Var}(h_1 \log U + h_2 \log V + h_3 \log W).$$

If this variance were zero, $h_1 \log u + h_2 \log v + h_3 \log w$ would be constant almost everywhere on an open subset of the simplex. Differentiating with respect to u and v would give

$$\frac{h_1}{u} - \frac{h_3}{w} = 0, \quad \frac{h_2}{v} - \frac{h_3}{w} = 0$$

throughout that open set. Varying u, v forces $h_1 = h_2 = h_3 = 0$, a contradiction. Therefore the variance is positive for every nonzero $\mathbf{h}$.

5. The Jacobi–Dirichlet probability model

Definition 5.1

The density (4.9) defines the Jacobi–Dirichlet distribution

$$(U, V, W) \sim \text{JD}_n^{\alpha, \beta}(\rho; a, b, c), \quad U + V + W = 1. \quad (5.1)$$

It is a finite polynomial tilt of the ordinary Dirichlet law. At $\rho = 0$ or $n = 0$, it reduces exactly to the Dirichlet distribution.

Theorem 5.1. Mixed moments

For nonnegative integers r, s, t ,

$$\mathbb{E}(U^r V^s W^t) = \frac{Q_n(\rho; a+r, b+s, c+t)}{Q_n(\rho; a, b, c)}. \quad (5.2)$$

Proof. Multiply the density (4.9) by $u^r v^s w^t$ and integrate. The numerator is precisely the positive normalizer with shifted beta parameters.

Consequently,

$$\mathbb{E}(U) = \frac{Q_n(a+1, b, c)}{Q_n(a, b, c)}, \quad \mathbb{E}(UV) = \frac{Q_n(a+1, b+1, c)}{Q_n(a, b, c)}, \quad (5.3)$$

and

$$\text{Var}(U) = \frac{Q_n(a+2, b, c)}{Q_n(a, b, c)} - \left[\frac{Q_n(a+1, b, c)}{Q_n(a, b, c)} \right]^2. \quad (5.4)$$

The remaining first and second moments follow by permutation.

Theorem 5.2. Entire joint moment-generating function

The function

$$\mathcal{G}_n(x, y) = \mathbb{E}(e^{xU+yV}), \quad (x, y) \in \mathbb{C}^2, \quad (5.5)$$

is entire and has the expansion

$$\mathcal{G}_n(x, y) = \sum_{r, s \geq 0} \frac{x^r y^s}{r! s!} \frac{Q_n(a+r, b+s, c)}{Q_n(a, b, c)}. \quad (5.6)$$

Proof. Since $0 \leq U, V \leq 1$,

$$|e^{xU+yV}| \leq e^{|x|+|y|}.$$

This bound is uniform on compact subsets of $\mathbb{C}^2$, so the exponential series may be integrated term by term. Formula (5.2) identifies each coefficient. Uniform convergence on compact sets proves entireness.

6. Appell and Humbert companions

The simplex construction also accommodates standard two-variable kernels. The complete parameter lists are displayed in the defining formulas; thereafter, fixed Jacobi indices and unchanged arguments are suppressed only when no ambiguity can arise.

Definition 6.1. Jacobi-modulated Appell-type integral

For $|x| < 1$, $|y| < 1$, define

$$\mathcal{A}_n^{(\alpha, \beta)}(x, y; \rho; a, b, c; p, q) = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} (1-xu)^{-p} (1-yv)^{-q} R_n(27puvw) du dv. \quad (6.1)$$

Theorem 6.1. Double series and Appell- F_3 reduction

$$\mathcal{A}_n(x, y) = \sum_{r, s \geq 0} \frac{(p)_r (q)_s}{r! s!} x^r y^s \mathcal{J}_n(\rho; a+r, b+s, c). \quad (6.2)$$

At $\rho = 0$ or $n = 0$,

$$\mathcal{A}_n(x, y) = B_3(a, b, c) F_3(a, b; p, q; \sigma; x, y). \quad (6.3)$$

Proof. Expand the binomial factors:

$$(1-xu)^{-p} = \sum_{r \geq 0} \frac{(p)_r}{r!} x^r u^r, \quad (1-yv)^{-q} = \sum_{s \geq 0} \frac{(q)_s}{s!} y^s v^s.$$

The double series converges uniformly on the compact simplex when $|x|, |y| < 1$. Termwise integration gives (6.2). In the undeformed case, divide the shifted beta factor by $B_3(a, b, c)$:

$$\frac{B_3(a+r, b+s, c)}{B_3(a, b, c)} = \frac{(a)_r (b)_s}{(\sigma)_{r+s}},$$

which is the coefficient of F_3 .

Definition 6.2. Jacobi-modulated Humbert integral

For $x, y \in \mathbb{C}$, define

$$\mathcal{H}_n^{(\alpha, \beta)}(x, y; \rho; a, b, c) = \int_{\Delta_2} u^{a-1} v^{b-1} w^{c-1} e^{xu+yv} R_n(27puvw) du dv. \quad (6.4)$$

Theorem 6.2. Humbert- Φ_2 reduction

$$\mathcal{H}_n(x, y) = \sum_{r, s \geq 0} \frac{x^r y^s}{r! s!} \mathcal{J}_n(\rho; a+r, b+s, c), \quad (6.5)$$

and, at $\rho = 0$ or $n = 0$,

$$\mathcal{H}_n(x, y) = B_3(a, b, c) \Phi_2(a, b; \sigma; x, y). \quad (6.6)$$

Proof. Expand e^{xu+yv} into its uniformly convergent double power series and integrate term by term. The classical coefficient is again $B_3(a+r, b+s, c)/B_3(a, b, c) = (a)_r (b)_s / (\sigma)_{r+s}$.

Theorem 6.3. External-variable and deformation derivatives

For nonnegative integers r, s ,

$$\frac{\partial^{r+s}}{\partial x^r \partial y^s} \mathcal{A}_n^{(\alpha, \beta)}(x, y; \rho; a, b, c; p, q) = (p)_r (q)_s \mathcal{A}_n^{(\alpha, \beta)}(x, y; \rho; a+r, b+s, c; p+r, q+s), \quad (6.7)$$

Here the argument order is (external variables; deformation parameter; beta parameters; binomial parameters). Also,

$$\frac{\partial^m}{\partial \rho^m} \mathcal{A}_n^{(\alpha, \beta)} = 27^m \frac{(-n)_m (n+\alpha+\beta+1)_m}{(\alpha+1)_m} \mathcal{A}_{n-m}^{(\alpha+m, \beta+m)}(x, y; \rho; a+m, b+m, c+m; p, q), \quad (6.8)$$

and the same deformation ladder holds for $\mathcal{H}_n$.

Proof. External differentiation contributes u^r , v^s , $(p)_r$, and $(q)_s$, producing the displayed parameter shifts. Deformation differentiation is identical to the proof of Theorem 3.5 because the Appell and Humbert kernels do not depend on ρ .

7. Classical polynomial specializations

The normalization in (2.5) removes constant conversion factors, so several classical families are obtained without changing any theorem.

7.1 Legendre case

For $\alpha = \beta = 0$,

$$R_n^{(0,0)}(t) = P_n(1 - 2t). \quad (7.1)$$

In particular,

$$\mathcal{J}_1^{(0,0)}(\rho; a, b, c) = B_3(a, b, c) - 54\rho B_3(a + 1, b + 1, c + 1). \quad (7.2)$$

7.2 Gegenbauer case

For $\alpha = \beta = \lambda - 1/2$,

$$R_n^{(\lambda-1/2, \lambda-1/2)}(t) = \frac{c_n^{(\lambda)}(1-2t)}{c_n^{(\lambda)}(1)}. \quad (7.3)$$

7.3 Chebyshev cases

For $\alpha = \beta = -1/2$,

$$R_n^{(-1/2, -1/2)}(t) = T_n(1 - 2t), \quad (7.4)$$

whereas for $\alpha = \beta = 1/2$,

$$R_n^{(1/2, 1/2)}(t) = \frac{U_n(1-2t)}{n+1}. \quad (7.5)$$

Thus the signed, quadratic, probabilistic, and two-variable theories specialize immediately to Legendre-, Gegenbauer-, and Chebyshev-modulated Dirichlet integrals.

8. Meromorphic continuation, degree, zeros, and asymptotics

Theorem 8.1. Meromorphic continuation in the Dirichlet parameters

Fix n, ρ, α, β , with $(\alpha + 1)_k \neq 0$ for $0 \leq k \leq n$. Then $\mathcal{J}_n$ extends meromorphically from $\Re a, \Re b, \Re c > 0$ to $\mathbb{C}^3$. Its candidate poles are contained in

$$\bigcup_{m \in \mathbb{N}_0} \{a = -m\} \cup \bigcup_{m \in \mathbb{N}_0} \{b = -m\} \cup \bigcup_{m \in \mathbb{N}_0} \{c = -m\}. \quad (8.1)$$

The same statement holds for $\mathcal{Q}_n$.

Proof. Each term of (3.3) is

$$c_{n,k}(27\rho)^k \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(\sigma+3k)}.$$

The numerator gamma functions have poles when $a+k$, $b+k$, or $c+k$ is a nonpositive integer. Their union over the finite range of k is contained in (8.1). By contrast, $1/\Gamma(\sigma+3k)$ is an entire function and therefore cannot create a pole. A finite sum of meromorphic terms is meromorphic and cannot acquire singularities outside the union of the singular sets of its terms. Formula (4.3) gives the same argument for $\mathcal{Q}_n$.

Remark 8.1. A necessary correction in pole bookkeeping. It is incorrect to list planes of the form $a+b+c = -3k-m$ as candidate poles of the beta quotient. On those planes the reciprocal gamma factor $1/\Gamma(\sigma+3k)$ vanishes. Cancellations may remove some coordinate poles, but no sum-plane pole is created by the denominator.

Theorem 8.2. Exact degree and zero count

Assume

$$(\alpha + 1)_n \neq 0, \quad (n + \alpha + \beta + 1)_n \neq 0,$$

and the beta parameters avoid the polar set. Then $\mathcal{J}_n$ has exact degree n in ρ . Its leading coefficient is

$$[\rho^n]\mathcal{J}_n = 27^n(-1)^n \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n} B_3(a+n, b+n, c+n). \quad (8.2)$$

Consequently, $\mathcal{J}_n$ has at most n complex zeros counted with multiplicity. For real admissible parameters, $\mathcal{Q}_n$ has exact degree $2n$ and is strictly positive for all real ρ .

Proof. In (3.3), the coefficient of ρ^n comes only from $k = n$. Since $(-n)_n = (-1)^n n!$, (8.2) follows from (2.6). Under the hypotheses, the coefficient is nonzero. The zero count follows from the fundamental theorem of algebra. For $\mathcal{Q}_n$, the top term in (4.3) is obtained only from $k = \ell = n$ and has positive coefficient

$$27^{2n} \left[\frac{(n + \alpha + \beta + 1)_n}{(\alpha + 1)_n} \right]^2 B_3(a + 2n, b + 2n, c + 2n).$$

Strict positivity on the real axis was proved in Theorem 4.1.

Theorem 8.3. Large-deformation asymptotics

Under the hypotheses of Theorem 8.2, as $|\rho| \rightarrow \infty$,

$$\mathcal{J}_n(\rho; a, b, c) = \kappa_n B_3(a + n, b + n, c + n) \rho^n + O(|\rho|^{n-1}), \quad (8.3)$$

where

$$\kappa_n = 27^n (-1)^n \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n}. \quad (8.4)$$

For real ρ ,

$$\mathcal{Q}_n(\rho; a, b, c) = \kappa_n^2 B_3(a + 2n, b + 2n, c + 2n) \rho^{2n} + O(|\rho|^{2n-1}). \quad (8.5)$$

Proof. Both functions are finite polynomials. Isolate the highest-degree term identified in Theorem 8.2 and absorb the remaining finitely many powers into the stated remainder.

9. Numerical analysis and reproducibility

9.1 Parameter set and independent evaluation routes

The numerical illustrations use

$$(a, b, c) = (1.35, 1.70, 2.10), \quad (\alpha, \beta) = (0.60, 1.10). \quad (9.1)$$

The classical normalizer is

$$B_3(1.35, 1.70, 2.10) = 0.028098327636217658. \quad (9.2)$$

Two independent routes were implemented.

Finite route. The signed function uses (3.3), and the positive function uses (4.3). Gamma quotients are evaluated through log-gamma values to avoid overflow and underflow.

Quadrature route. The simplex is mapped to the unit square by

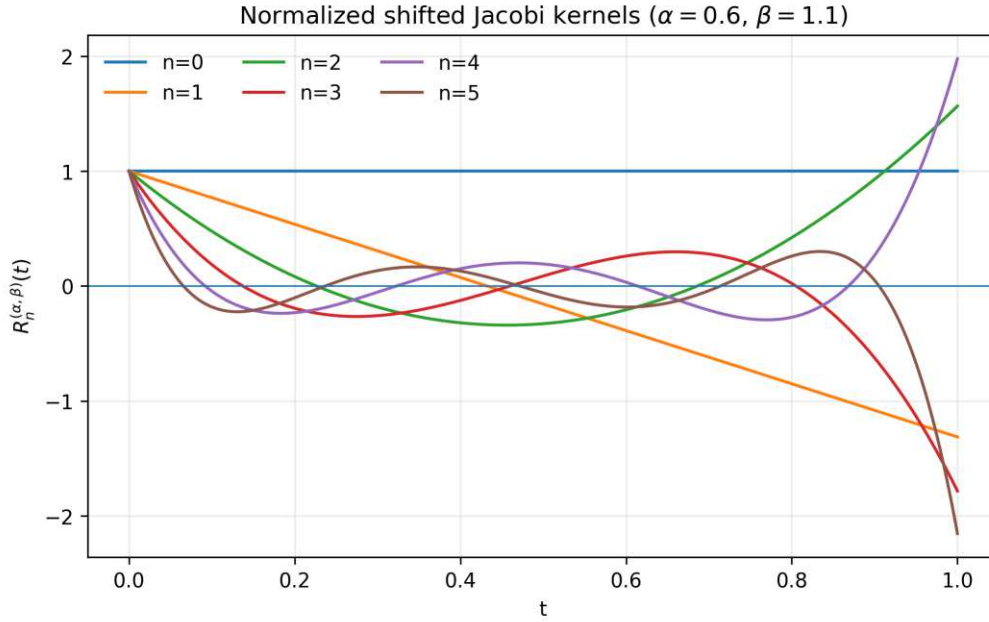
$$u = x, \quad v = (1 - x)y, \quad w = (1 - x)(1 - y), \quad 0 < x, y < 1, \quad (9.3)$$

with Jacobian $1 - x$. Thus

$$\begin{aligned} \mathcal{J}_n &= \int_0^1 \int_0^1 x^{a-1} (1-x)^{b+c-1} y^{b-1} (1-y)^{c-1} \\ &\quad \times R_n(27\rho x(1-x)^2 y(1-y)) dy dx. \end{aligned} \quad (9.4)$$

The same formula with R_n^2 evaluates $\mathcal{Q}_n$. Nested adaptive Gauss–Kronrod quadrature was run with matched absolute and relative tolerances.

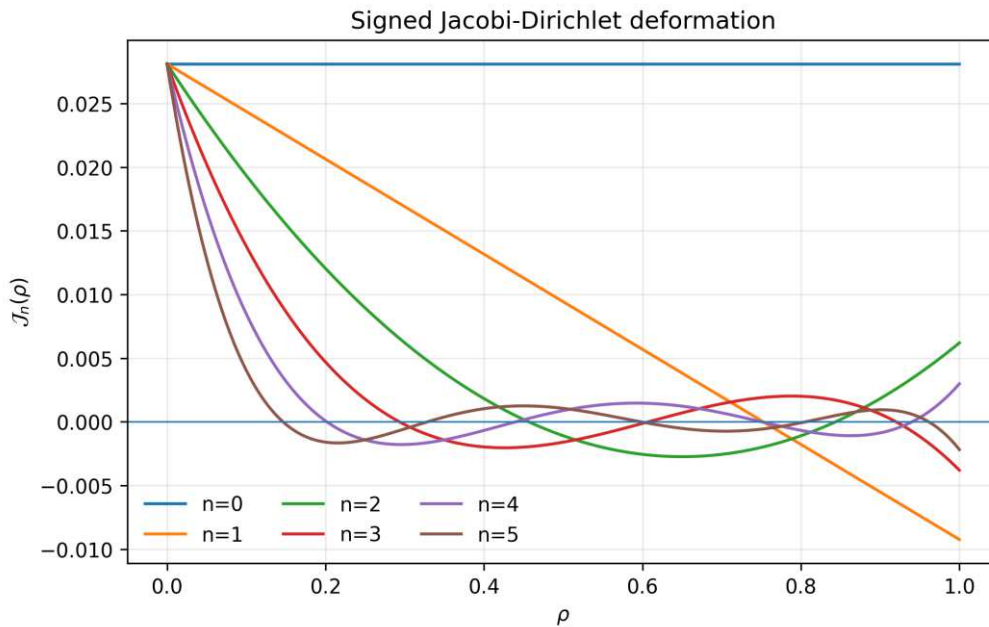
9.2 Kernel profiles



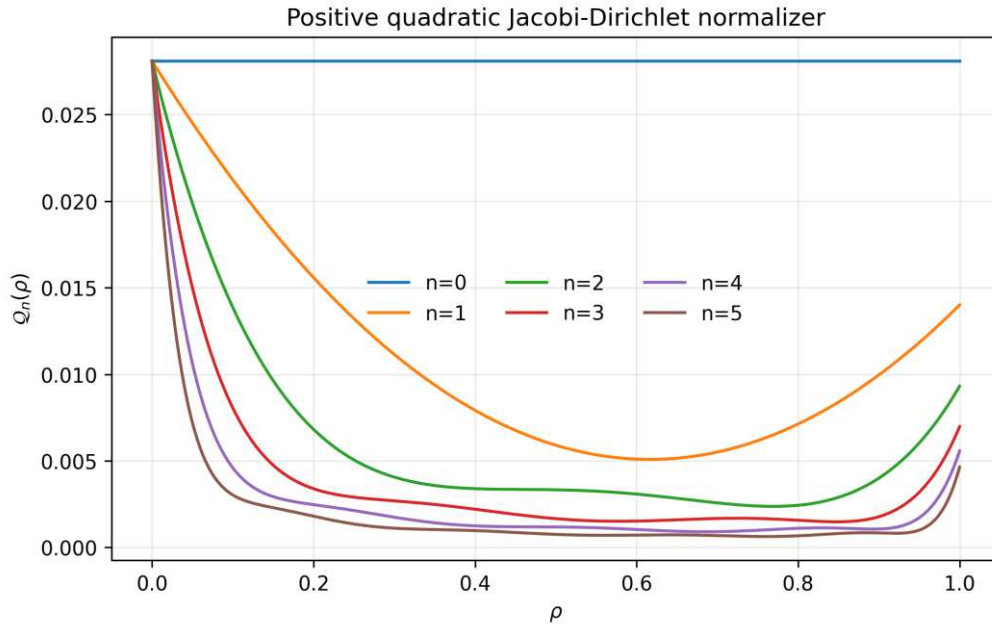
Normalized shifted Jacobi kernels for degrees zero through five. The increasing oscillation is inherited by the signed deformation.

The endpoint normalization $R_n(0) = 1$ is visible in every curve. The increasing number of sign changes explains why a separate squared family is required for probability normalization.

9.3 Dependence on the deformation parameter



Signed Jacobi-Dirichlet deformation for degrees zero through five.

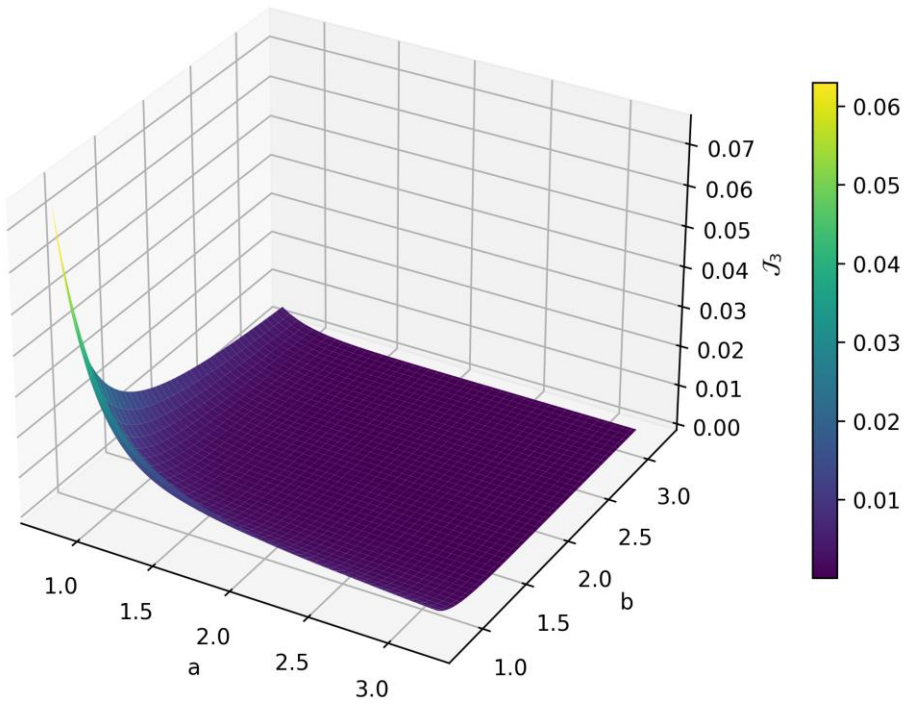


Positive quadratic normalizer for the same parameters.

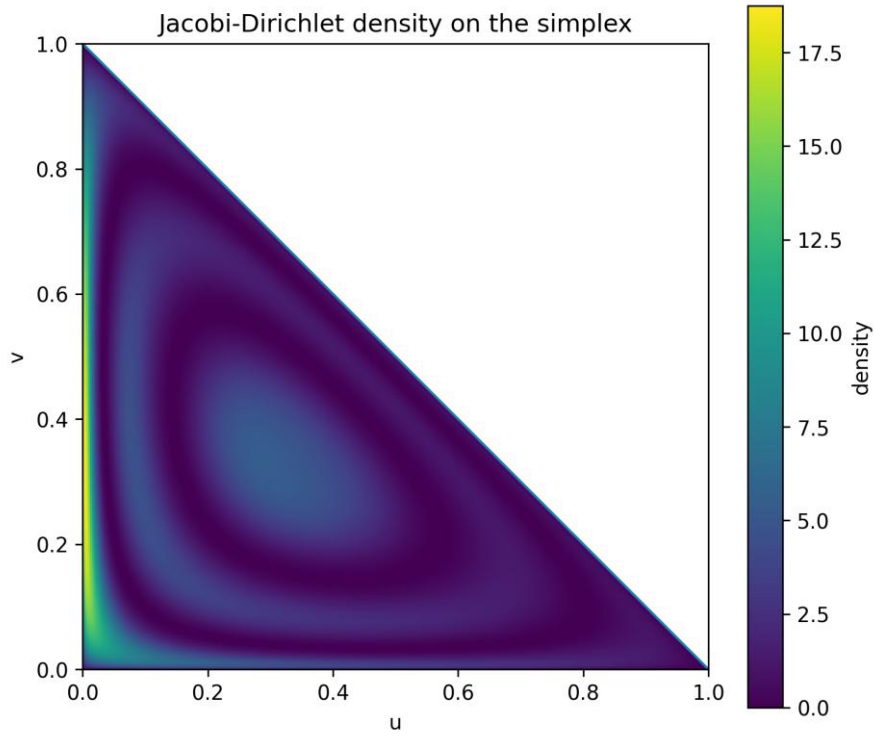
The signed curves cross zero in a degree-dependent manner. The positive curves remain strictly above zero, in agreement with Theorem 4.1.

9.4 Parameter surface and tilted density

Parameter surface ($c = 1.6$, $\rho = 0.65$)



Degree-three signed deformation as a function of a and b , with $c = 1.6$ and $\rho = 0.65$.



Jacobi–Dirichlet density on the simplex for $n = 3$ and $\rho = 0.65$.

The parameter surface is smooth on the positive domain, as predicted by Theorem 3.1. The density plot shows that the squared Jacobi factor can redistribute mass into several interior bands; this behavior is unavailable to a basic unimodal Dirichlet law with the same parameter vector.

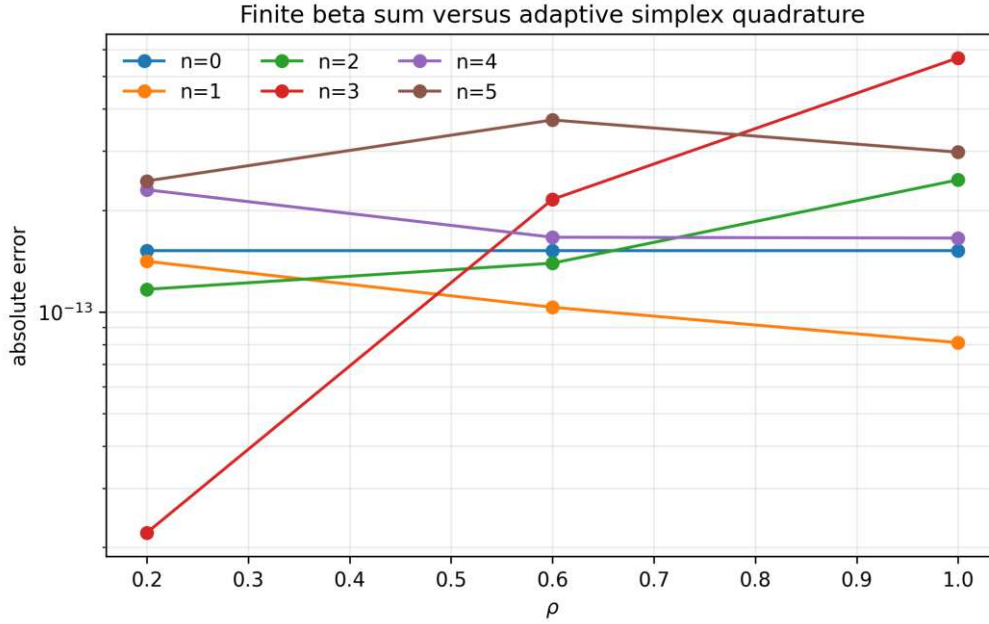
9.5 Representative values

Degree n	rho	Signed J_n	Positive Q_n
0	0.00	0.028098327636	0.028098327636
0	0.50	0.028098327636	0.028098327636
0	1.00	0.028098327636	0.028098327636
1	0.00	0.028098327636	0.028098327636
1	0.50	0.009429820930	0.005906467231
1	1.00	-0.009238685775	0.014004912840
2	0.00	0.028098327636	0.028098327636
2	0.50	-0.001091251672	0.003331703345
2	1.00	0.006196206636	0.009314224084
3	0.00	0.028098327636	0.028098327636
3	0.50	-0.001597853901	0.001665670102
3	1.00	-0.003793566955	0.006975345235
4	0.00	0.028098327636	0.028098327636
4	0.50	0.000885173286	0.001187145453
4	1.00	0.002984339227	0.005575168250
5	0.00	0.028098327636	0.028098327636
5	0.50	0.001057004297	0.000756210830
5	1.00	-0.002177581385	0.004643282245

Table 1. Representative signed and positive values for the parameter set (9.1).

At $\rho = 0$, both families equal the classical Dirichlet value for every degree. Away from zero, the signed values alternate, while every positive value remains strictly positive.

9.6 Validation against adaptive quadrature



Absolute discrepancy between the finite signed beta sum and adaptive simplex quadrature.

Across the reported validation grid, the largest absolute discrepancy for the signed family was $5.674e - 13$, and the largest absolute discrepancy for the positive family was $2.181e - 11$. The relative error can be less informative when a signed value is close to zero; the absolute discrepancy is therefore the primary diagnostic for oscillatory cases.

9.7 Reproducible algorithm

Algorithm 1: finite evaluation of $\mathcal{J}_n$.

1. Compute $c_{n,k}$ for $k = 0, \dots, n$ from (2.6).
2. For each k , evaluate $\log B_3(a+k, b+k, c+k)$ from log-gamma values.
3. Accumulate $c_{n,k}(27\rho)^k B_3(a+k, b+k, c+k)$ using compensated summation when cancellation is severe.
4. Compare with (3.4) only as a secondary check; the finite beta sum is generally the more transparent numerical formula.

Algorithm 2: finite evaluation of $\mathcal{Q}_n$.

1. Form the coefficient vector $c_{n,k}$.
2. Convolve the vector with itself to obtain $d_{n,r}$ in (4.4).
3. Evaluate the one-dimensional finite sum (4.5).
4. Check positivity; a negative computed value indicates numerical cancellation or implementation error, not a property of the exact function.

10. Discussion

The central mechanism is diagonal beta shifting. Every power of $27\rho uvw$ raises a, b, c by the same integer, so the entire deformation is controlled by a finite sequence of shifted Dirichlet normalizers. The terminating ${}_5F_4$ form, the derivative ladder, the Mellin transform, the meromorphic continuation, and the exact asymptotics are different consequences of this one observation.

The signed and positive families are complementary rather than competing definitions. For the present journal context, the essential structural gain occurs in the quadratic family: squaring the Jacobi kernel preserves finite computability while producing a strictly positive partition function. Its logarithm is jointly and strictly convex,

its Hessian is a covariance matrix of logarithmic barycentric coordinates, and midpoint convexity yields Turán-type inequalities. The signed family remains indispensable because it supplies the oscillatory Jacobi information, finite hypergeometric closure, differential equations, recurrences, and zeros from which the positive construction originates.

The proof of strict convexity deserves emphasis. Hölder’s inequality alone gives non-strict log-convexity. The covariance representation identifies the only possible obstruction: a linear combination of $\log U, \log V, \log W$ would have to be almost surely constant. Because the tilted law has full two-dimensional support up to a lower-dimensional zero set of the polynomial, that obstruction occurs only in the zero direction. Thus the Hessian is positive definite, not merely semidefinite.

The meromorphic analysis also illustrates why formula-level auditing is necessary. It is tempting to list every affine argument of a gamma function as a possible pole. That shortcut fails for gamma factors in the denominator: their reciprocals are entire and vanish at nonpositive integers. The corrected polar description is both simpler and stronger.

The present model has limitations. First, because the kernel is polynomial and bounded on the simplex, it does not regularize endpoint singularities or enlarge the primary convergence domain. Second, for large degree and deformation, the signed finite sum can suffer severe cancellation. Third, the scalar statistic uvw cannot represent every multivariate pattern available to a genuine multivariate Jacobi basis. These limitations identify natural directions for further work rather than defects in the internal theory.

10.1 Potential applications and computational relevance

The construction contributes simultaneously to positivity theory, special functions, simplex integration, and probability. The positive quadratic family is particularly suited to applications in which a normalizing constant must remain strictly positive while allowing nontrivial interior modulation of a simplex density. Its Jacobi–Dirichlet law can therefore be investigated for compositional observations whose nonnegative components sum to one, including portfolio weights, market shares, budget proportions, energy-source allocations, mixture compositions, and allocations of risk or resources among competing categories. These are prospective applications rather than empirically established outcomes of the present paper.

The squared Jacobi factor adds an interior polynomial modulation to the classical Dirichlet weight. It can therefore redistribute probability mass while retaining a finite normalizing constant, explicit shifted-parameter moments, and an entire joint moment-generating function. These properties provide an analytic basis for future estimation, uncertainty quantification, sensitivity analysis, and model comparison for simplex-valued data. A statistical use of the family will nevertheless require separate work on identifiability, parameter estimation, degree selection, simulation, and comparison with the Dirichlet, generalized Dirichlet, and logistic-normal families.

The finite beta-sum formulas also have computational relevance. For moderate degree, repeated evaluations reduce to a finite collection of beta or gamma quotients rather than a new adaptive simplex integration at every parameter value. This feature may be useful in likelihood optimization, Bayesian computation, simulation, and repeated sensitivity calculations. The numerical section verifies the formulas by an independent quadrature route, but a formal efficiency claim would require dedicated timing, conditioning, and high-degree stability benchmarks.

Any economic relevance is consequently indirect and potential. It may arise from potential reductions in repeated computational cost or from the possibility of a more flexible representation of allocation data in finance, economics, energy planning, insurance, or resource management. The present article does not establish economic feasibility in a quantitative sense, because it contains neither an empirical economic data set nor a cost-benefit analysis. Such questions belong to future application-oriented studies.

10.2 Future research directions

The theory suggests several natural directions for further research. On the analytic side, natural problems include deformation on higher-dimensional simplices, replacement of the barycentric product by other symmetric simplex statistics, genuinely multivariate Jacobi kernels, q -analogues, exact pole-cancellation criteria, and a systematic study of the zeros of the signed deformation. Large-degree asymptotics and stable

recurrence-based algorithms are especially important because alternating finite sums may become poorly conditioned as the degree grows.

On the probabilistic and statistical side, future work should address identifiability, maximum-likelihood and Bayesian estimation, consistency and asymptotic normality of estimators, sampling algorithms, and real-data comparisons. The Appell- and Humbert-type companions introduced above may also serve as starting points for new multivariable special functions, associated differential systems, and further integral transforms. These directions indicate that the present construction may provide a reusable framework for polynomial deformations of simplex integrals.

11. Conclusions

A normalized Jacobi polynomial composed with the barycentric product yields a finite symmetric deformation of the three-variable beta integral with a natural positivity counterpart. The principal positivity result is the quadratic normalizer obtained by squaring the kernel: it is strictly positive for all admissible real parameters, its logarithm is strictly convex, it satisfies Turán-type inequalities, and its derivatives encode logarithmic moments through covariance identities. After normalization, this kernel defines a Jacobi–Dirichlet probability law with explicit mixed moments and an entire joint moment-generating function. The signed companion retains the exact beta sum, terminating generalized-hypergeometric representation, differential and recurrence structure, finite Mellin transform, meromorphic continuation, and large-deformation behavior. Appell- and Humbert-type companions follow from the same mechanism.

The combination of positivity, finite symbolic closure, and independent quadrature verification provides a tractable framework in which order-related inequalities, convexity, probability normalization, and special-function structure can be studied together. The central point is that a single symmetric simplex statistic simultaneously produces exact computability and a genuinely positive deformation family.

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Data availability. No external dataset was used. All reported numerical values are generated from the formulas stated in the manuscript.

Code availability. The computations require only the finite sums (3.3) and (4.3)/(4.5), standard log-gamma evaluation, and adaptive quadrature after the map (9.3). The algorithmic details are provided in Section 9.7.

Author contribution. The author conceived the deformation, developed the proofs, organized the analytic and probabilistic consequences, and designed the independent numerical checks.

Use of generative AI. Generative AI tools were used as research-assistance tools during drafting, language editing, and manuscript formatting. The author reviewed and verified the mathematical derivations, proofs, computations, references, and final text and takes full responsibility for the manuscript.

References

- [1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [2] G. Szegő, *Orthogonal Polynomials*, 4th ed., American Mathematical Society Colloquium Publications, Vol. 23, Providence, RI, 1975.
- [3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [4] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, 2nd ed., Cambridge University Press, Cambridge, 2014.
- [5] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

- [6] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, Extension of Euler's beta function, *Journal of Computational and Applied Mathematics* 78 (1997), 19–32. [https://doi.org/10.1016/S0377-0427\(96\)00102-1](https://doi.org/10.1016/S0377-0427(96)00102-1).
- [7] M. Raïssouli and S. El-Soubhy, On some generalizations of the beta function in several variables, *Turkish Journal of Mathematics* 45 (2021), 820–842. <https://doi.org/10.3906/mat-2010-57>.
- [8] N. H. Al-harbi and M. Raïssouli, Extension of the Beta function in several variables and applications for multivariate orthogonal polynomials, *Journal of Inequalities and Applications* 2025 (2025), Article 124. <https://doi.org/10.1186/s13660-025-03292-9>.
- [9] R. C. Griffiths and D. Spanò, Multivariate Jacobi and Laguerre polynomials, infinite-dimensional extensions, and their probabilistic connections with multivariate Hahn and Meixner polynomials, *Bernoulli* 17 (2011), 1095–1125. <https://doi.org/10.3150/10-BEJ305>.
- [10] R. C. Griffiths and D. Spanò, Orthogonal polynomial kernels and canonical correlations for Dirichlet measures, *Bernoulli* 19 (2013), 548–598. <https://doi.org/10.3150/11-BEJ403>.
- [11] S. Beuchler, T. Haubold, and V. Pillwein, Recursion formulas for integrated products of Jacobi polynomials, *Constructive Approximation* 59 (2024), 583–618. <https://doi.org/10.1007/s00365-023-09655-z>.
- [12] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q -Analogues*, Springer, Berlin, 2010.
- [13] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, Cambridge, 2005.
- [14] T. H. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, in R. A. Askey (ed.), *Theory and Application of Special Functions*, Academic Press, New York, 1975, pp. 435–495.
- [15] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
- [16] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, Chichester, 1985.
- [17] H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwood, Chichester, 1976.
- [18] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood, Chichester, 1984.
- [19] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Multivariate Distributions, Volume 1: Models and Applications*, 2nd ed., Wiley, New York, 2000.
- [20] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton, 2000.
- [21] W. N. Bailey, The generating function of Jacobi polynomials, *Journal of the London Mathematical Society* 13 (1938), 8–12.
- [22] E. De Micheli, On the integral representation of Jacobi polynomials, *Mathematics* 13 (2025), Article 483. <https://doi.org/10.3390/math13030483>.
- [23] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, DC, 1964.
- [24] L. C. Andrews, *Special Functions of Mathematics for Engineers*, 2nd ed., SPIE Press, Bellingham, WA, 1998.
- [25] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman & Hall/CRC, Boca Raton, 2002.