

Feasible-Space Quantum-Walk Optimization for Atomic Congestion Games Incentive Feasibility and an Eight-Instance Ideal-Statevector Study

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Research Article

Keywords: atomic congestion game, correlated equilibrium, traffic assignment, quantum walk, QAOA, statevector simulation, hybrid optimization

Posted Date: October 16th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11382114/v1>

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Additional Declarations: The authors declare no competing interests.

Feasible-Space Quantum-Walk Optimization for Atomic Congestion Games

Incentive Feasibility and an Eight-Instance Ideal-Statevector Study

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14 October 2026

Abstract

Quantum optimization for routing must be evaluated against both feasibility and strategic incentives. We study a finite atomic congestion game in which four vehicles choose among three parallel roads with affine, increasing latencies. The experiment enumerates all 81 assignments in each of eight seeded synthetic instances, computes pure Nash equilibria and tests by linear programming whether any social-optimal correlated equilibrium exists. We compare a 12-qubit, depth-one one-hot penalty-QAOA statevector ansatz, a depth-one continuous-time quantum-walk (CTQW) ansatz represented directly on the 81 feasible profiles, and classical local search under the same maximum of 40 objective calls per start. Across 24 starts per method, local search reached the exact optimum in all starts. The mean relative gap was 23.22% (SD 11.21 percentage points) for QAOA’s expected cost conditioned on feasible outcomes and 8.34% (SD 3.42 points) for the feasible-profile CTQW expected cost. QAOA assigned an average 17.05% of its final ideal-state probability mass to feasible assignments; feasibility was 100% for CTQW by construction. An optimal correlated equilibrium existed in five of the eight games. These are descriptive results for a very small, noiseless statevector benchmark. The CTQW calculation uses classical dense diagonalization, the representations are not resource-matched, and no hardware advantage or quantum speedup is established.

Keywords: atomic congestion game; correlated equilibrium; traffic assignment; quantum walk; QAOA; statevector simulation; hybrid optimization

1 Introduction

Traffic assignment combines an optimization problem with a strategic interaction. A system planner may minimize aggregate travel time, whereas each driver chooses a route to minimize their own travel time. An assignment that is best for the system need not be stable against unilateral deviations. This distinction underlies congestion games, potential functions, and the price of anarchy [1, 2].

Variational quantum optimization provides a family of parameterized circuits that alternate cost-dependent phases and mixing operations. QAOA uses a transverse-field mixer, while the quantum alternating-operator framework allows mixers tailored to constrained feasible spaces [4, 5]. Quantum-walk-based optimization has been studied for combinatorial problems and, specifically, for capacitated vehicle routing [6, 7, 8]. The latter work is important context: the present study does not introduce quantum-walk optimization for vehicle routing. It instead examines a different objective with congestion externalities and adds an explicit test of whether social optima are incentive compatible.

This paper reports a narrow, reproducible pilot, not a general algorithmic or complexity result. It makes three contributions. First, it states the exact finite-game condition for a distribution

supported on social optima to be a correlated equilibrium (CE), and gives a small congestion-game counterexample to universal optimal-CE claims. Second, it specifies two depth-one ideal-statevector ansätze on the same toy games: penalty-QAOA on a one-hot binary encoding and a CTQW ansatz on the feasible-profile graph. Third, it reports their results, alongside exact enumeration and a capped local-search baseline, on eight seeded instances.

The scope is a static, atomic routing snapshot with four unit-demand vehicles, three parallel roads, and affine latencies. It contains no departure times, time-varying demand, incidents, vehicle capacities, or route recourse. Accordingly, the results do not establish performance for dynamic urban traffic, large vehicle-routing problems, or physical quantum hardware. They also do not constitute quantum machine learning: there is no learned model, training dataset, or generalization analysis.

2 Routing game and incentive condition

2.1 Finite atomic congestion model

Let $K = \{1, \dots, k\}$ be a set of vehicles and let $\mathcal{R}_i$ be the finite set of routes available to vehicle i . A profile is $x = (x_i)_{i \in K} \in \mathcal{X} = \prod_i \mathcal{R}_i$. Each route uses a subset of road resources E . For road e , its load under profile x is

$$n_e(x) = \sum_{i \in K} \mathbf{1}\{e \in x_i\}.$$

For a nonnegative latency function $\ell_e(n)$, vehicle i incurs private cost

$$c_i(x) = \sum_{e \in x_i} \ell_e(n_e(x)),$$

and total system cost is

$$C(x) = \sum_{i \in K} c_i(x) = \sum_{e \in E} n_e(x) \ell_e(n_e(x)). \quad (1)$$

The system-optimal value and its minimizer set are

$$C^* = \min_{x \in \mathcal{X}} C(x), \quad \mathcal{X}^* = \operatorname{argmin}_{x \in \mathcal{X}} C(x).$$

For finite congestion games with resource-specific costs, a pure Nash equilibrium exists by the congestion-game potential argument of Rosenthal [1]. The same general statement need not hold for arbitrary finite games outside this class.

Definition 2.1 (Correlated equilibrium for costs). A probability distribution μ on $\mathcal{X}$ is a correlated equilibrium if, for each player i and each pair of actions $a_i, b_i \in \mathcal{R}_i$,

$$\sum_{x_{-i}} \mu(a_i, x_{-i}) [c_i(a_i, x_{-i}) - c_i(b_i, x_{-i})] \leq 0. \quad (2)$$

The inequality says that conditional on being recommended a_i , player i cannot reduce expected private cost by switching to b_i . It is linear in μ , so CE feasibility is a linear program. A distribution's aggregate cost and its incentive properties are separate quantities.

2.2 When can a social optimum be a correlated equilibrium?

Proposition 2.2 (Support condition for an optimal distribution). *For a finite profile space, $\mathbb{E}_{x \sim \mu}[C(x)] = C^*$ if and only if $\operatorname{supp}(\mu) \subseteq \mathcal{X}^*$.*

Proof. The difference is $\mathbb{E}[C] - C^* = \sum_x \mu(x)(C(x) - C^*)$, a sum of nonnegative terms. It vanishes exactly when every profile receiving positive mass has cost C^* . $\square$

Theorem 2.3 (Exact optimal-CE feasibility condition). *A correlated equilibrium with expected system cost C^* exists if and only if there is a probability distribution supported on $\mathcal{X}^*$ satisfying all inequalities in (2). Equivalently, the CE polytope intersects the simplex over the social-optimal profiles.*

Proof. By [proposition 2.2](#), equality in expected system cost holds exactly for distributions supported on $\mathcal{X}^*$. Such a distribution is a CE exactly when it obeys (2). This proves both directions. $\square$

Corollary 2.4 (Unique social optimum). *If x^* is the unique system-optimal profile, an optimal CE exists if and only if x^* is a pure Nash equilibrium.*

Proof. The only distribution supported on the singleton $\{x^*\}$ is the point mass at x^* . Its CE inequalities reduce to the unilateral-deviation inequalities for a pure Nash equilibrium. $\square$

Theorem 2.3 is a feasibility characterization, not a claim that a quantum device produces a CE. For each finite instance it can be checked by a linear program over optimal profiles. A measured joint distribution generated by a quantum circuit is incentive compatible only if that distribution itself satisfies the relevant obedience inequalities. Non-signaling or entanglement alone does not certify those inequalities. Quantum-strategy equilibrium concepts in which players act directly on quantum systems would require a separate model and are outside this paper.

Example 2.5 (A social optimum need not be a CE). Two players choose one of two parallel resources, A or B . Let $\ell_A(1) = 0$, $\ell_A(2) = 3/5$, and $\ell_B(n) = 1$ for either load. The profile costs are shown in [table 1](#). The split profiles are exactly the social optima. In either split, the player assigned

Table 1: A two-player congestion-game counterexample.

Profile	(c_1, c_2)	C
(A, A)	$(3/5, 3/5)$	$6/5$
(A, B)	$(0, 1)$	1
(B, A)	$(1, 0)$	1
(B, B)	$(1, 1)$	2

to B can move to A and lower their cost from 1 to $3/5$. Every distribution attaining cost $C^* = 1$ must be supported on the two split profiles; any such distribution assigns positive probability to a recommendation of B to at least one player and violates that player’s obedience inequality. Hence no CE attains the social optimum. By contrast, (A, A) is a pure Nash equilibrium: a unilateral move to B raises the deviator’s cost from $3/5$ to 1.

3 Optimization ansätze and resource model

3.1 One-hot penalty-QAOA

For the pilot, each of $K = 4$ players selects one of $R = 3$ roads. The binary variables $z_{ir} \in \{0, 1\}$ encode player i selecting road r , using $q = KR = 12$ qubits. A bit string is one-hot feasible when $\sum_r z_{ir} = 1$ for every i . Define $n_r(z) = \sum_i z_{ir}$ and

$$C_{\text{bin}}(z) = \sum_{r=1}^R n_r(z)(a_r + b_r n_r(z)), \quad V(z) = \sum_{i=1}^K \left(\sum_{r=1}^R z_{ir} - 1 \right)^2.$$

Let C_{max} be the largest cost among feasible profiles. The diagonal penalty objective is

$$H_P(z) = \frac{C_{\text{bin}}(z)}{C_{\text{max}}} + 2V(z). \quad (3)$$

All latencies and binary loads are nonnegative. Thus every infeasible bit string has $H_P(z) \geq 2$, whereas each feasible profile has $H_P(z) \leq 1$. The penalty is therefore sufficient to keep every global ground state feasible in this toy encoding. This does not guarantee that a finite-depth variational state has high feasible probability.

The depth-one circuit is

$$|\psi_Q(\gamma, \beta)\rangle = e^{-i\beta H_M} e^{-i\gamma H_P} |+\rangle^{\otimes 12}, \quad H_M = \sum_{j=1}^{12} X_j. \quad (4)$$

The unnormalized transverse-field mixer in (4) has spectral gap 2. Its spectrum is $\{-q, -q+2, \dots, q\}$ with $q = 12$ here. The gap is $2/12$ only if the mixer is normalized as $H_M/12$. Neither endpoint gap implies a lower bound on a minimum gap along an interpolating path or a convergence guarantee for the variational optimizer.

3.2 Feasible-profile CTQW ansatz

The feasible profiles form the vertex set $\mathcal{X} = \{1, 2, 3\}^4$, of size $M = 81$. Connect two profiles when exactly one player changes roads. This graph is the Cartesian product $K_3^{\square 4}$; it is 8-regular and has 324 undirected edges. Let A be its adjacency matrix, normalized in the simulation as $\tilde{A} = A/8$. Define normalized profile cost

$$h(x) = \frac{C(x) - C_{\min}}{C_{\max} - C_{\min}}, \quad |s\rangle = \frac{1}{\sqrt{81}} \sum_{x \in \mathcal{X}} |x\rangle.$$

The depth-one feasible-space ansatz is

$$|\psi_W(\gamma, \beta)\rangle = e^{-i\beta \tilde{A}} e^{-i\gamma H_C} |s\rangle, \quad H_C |x\rangle = h(x) |x\rangle. \quad (5)$$

Because every basis vector in this representation is feasible and the walk changes one player's road at a time, the ideal CTQW output is feasible by construction. In the experiment, however, the 81-component state vector and dense eigendecomposition of the 81×81 adjacency matrix are classical numerical operations. The code does not compile a quantum circuit for the walk. Seven is only the information-theoretic lower bound $\lceil \log_2 81 \rceil$ on binary address qubits; indexing, padding, ancillas, and gates are not implemented or counted.

3.3 Classical optimization and accounting

The expectation of the stated objective was optimized by SciPy COBYLA from independently sampled initial parameters in $[-\pi, \pi]^2$, with at most 40 objective calls per start, initial trust-region radius 0.7, and tolerance 10^{-4} . Expectations were evaluated exactly by ideal statevector arithmetic, without shot noise. After optimization, 5,000 samples were drawn from each final state for empirical diagnostics. Therefore the optimizer is not a finite-shot or noisy-hardware optimizer; the samples are used only to estimate output frequencies and sampled costs.

The classical comparator is best-improvement single-player local search with random restarts under a cap of 40 objective calls per restart. It returns the best profile visited. There were three starts per method per instance. Quantum methods can terminate before exhausting the cap; actual objective-call counts are reported. Equal call caps do not equalize reversible oracle work, gate counts, shots used inside optimization, or wall-clock cost. The experiment also uses different state-space representations: a 4,096-amplitude binary QAOA statevector versus an 81-amplitude feasible-profile CTQW vector. Their wall times and raw success metrics should not be interpreted as a hardware-resource comparison.

4 Experimental design

4.1 Synthetic instances and exact diagnostics

For each of eight fixed seeds, the four-vehicle/three-road game used road latencies

$$\ell_r(n) = a_r + b_r n, \quad a_r \sim \text{Uniform}[2, 8], \quad b_r \sim \text{Uniform}[0.5, 3], \quad (6)$$

independently across roads. The quantities are synthetic, dimensionless time units, not calibrated travel-time observations. The profile space has $3^4 = 81$ assignments, all of which were enumerated to compute C^* and the complete set of minimizers. Pure Nash equilibria were identified by checking all unilateral road changes. The price of anarchy was defined for this cost-minimization game as

$$\text{PoA} = \frac{\max_{x \in \text{NE}} C(x)}{C^*}.$$

Optimal-CE existence was tested by linear programming over distributions supported only on $\mathcal{X}^*$, imposing all CE obedience inequalities from (2). This diagnostic concerns existence of some optimal CE; it does not test whether either variational ansatz’s output distribution is a CE.

The random-generation master seed was 20261016; instance $j \in \{0, \dots, 7\}$ used seed 20262016 + j . Each instance was run with three starts per optimizer, at most 40 optimizer objective evaluations per start, and 5,000 final-state samples. The workflow is summarized in [figure 1](#). The complete 80-row run record is provided as supplementary data. The reported Colab run finished in 16.6 seconds; the runtime hardware details were not recorded, so this measurement is not used to compare algorithms.

4.2 Reported metrics

For local search, the relative gap is the gap of the best visited feasible profile. For CTQW it is the gap of the final exact expectation, since its support is feasible. For penalty-QAOA, the reported conditional cost is the exact expectation over the feasible part of the final state divided by its feasible probability; the relative gap is this postselected conditional expectation relative to C^* . QAOA’s feasibility probability is reported separately and must be considered alongside this postselected gap. The exact optimal-state probability is the full-state mass on optimal feasible bit strings; the QAOA conditional optimal probability divides this mass by its feasibility probability. The empirical optimal rate is estimated from feasible final samples only.

Means and sample standard deviations summarize 24 optimizer starts per method (eight games times three starts). Starts are nested within games and are not treated as 24 independent traffic instances. We report descriptive summaries, not hypothesis tests or population-level confidence intervals. For each game, per-instance values are summarized separately in [table 2](#).

5 Results

5.1 Game-theoretic diagnostics

An optimal correlated equilibrium existed in five of eight instances and did not exist in three. The number of pure Nash equilibria ranged from one to twelve. Five games had $\text{PoA} = 1$; the other three had modest inefficiency, with values from 1.0342 to 1.0532. Across the eight instances, mean PoA was 1.0171. Per-instance diagnostics and optimizer gaps are listed in [table 2](#). These outcomes illustrate that an optimal CE is instance-dependent even in this small congestion model.

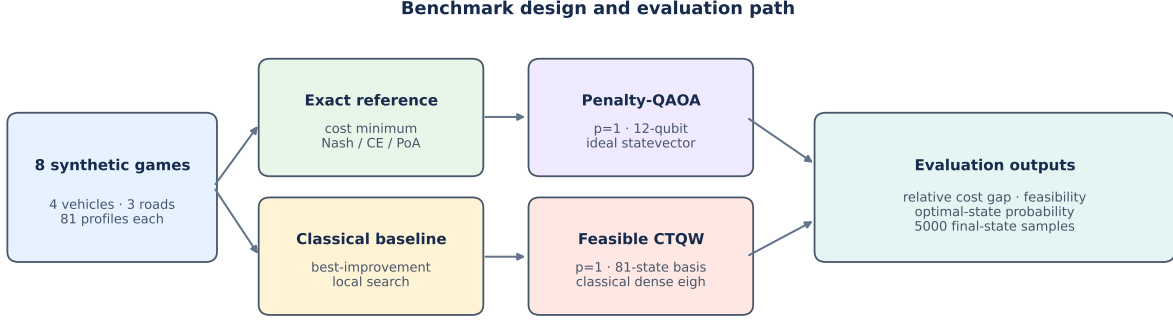


Figure 1: Benchmark workflow. The exact solver is used only as a small-instance reference and for game diagnostics. The CTQW box describes a classical statevector simulation over feasible profiles, not a compiled circuit.

Table 2: Exact game diagnostics and mean optimizer gaps by instance. Gaps average the three starts of the indicated method.

Instance	C^*	Pure NE	PoA	Optimal CE?	QAOA gap	CTQW gap
1	22.9370	4	1.0000	Yes	31.52%	15.16%
2	32.9693	4	1.0342	No	12.04%	5.56%
3	32.0594	12	1.0000	Yes	21.05%	8.91%
4	19.1416	1	1.0532	No	47.09%	12.12%
5	26.3521	12	1.0000	Yes	21.04%	7.79%
6	37.5560	12	1.0000	Yes	18.16%	5.67%
7	35.9350	6	1.0494	No	17.81%	5.44%
8	30.0281	12	1.0000	Yes	17.08%	6.08%

5.2 Optimization outcomes

Local search reached the exact optimum in all 24 starts using its 40-call budget. The penalty-QAOA relative gap averaged 23.22% (SD 11.21 points), while its exact feasible probability averaged 17.05% (SD 4.05 points). Its full-state optimal-profile probability averaged 2.30% (SD 1.08 points); conditioned on feasibility, this was 13.58%. The sampled optimal fraction among feasible QAOA samples averaged 13.38% (SD 5.32 points), close to the exact conditional probability.

The feasible-profile CTQW ansatz had an average expected-cost gap of 8.34% (SD 3.42 points). Feasibility was 100% because the simulated state space contained feasible assignments only. Its exact optimal-profile probability averaged 38.79% (SD 8.36 points), and the sampled optimal fraction averaged 38.68% (SD 8.49 points). In every instance, the mean CTQW gap across three starts was below the corresponding mean QAOA gap. This is a descriptive result for these encodings and seeds, not evidence of an algorithmic or quantum advantage.

The objective-call cap was used exactly by local search and CTQW in all runs. QAOA used 37.54 calls on average (SD 2.72; range 33–40) because COBYLA sometimes stopped early. The aggregate comparison is given in [table 3](#), and the per-instance pattern is visualized in [figure 2](#). The 24-start gap summaries thus compare equal maximum call caps, not equal realized calls.

Table 3: Descriptive results across 24 starts per method. Parentheses show sample SD in percentage points. The QAOA gap is a feasible-conditioned expected-cost gap; the CTQW gap is an expected-cost gap on its feasible-profile basis. The local-search gap uses the best visited profile.

Method	Cost gap (mean, SD)	Feasible mass	Optimal outcome
Local search	0.00%	100%	24/24 best profiles optimal
Penalty-QAOA, $p = 1$	23.22% (11.21 pp)	17.05% (4.05 pp)	2.30% (1.08 pp), unconditional
Feasible CTQW, $p = 1$	8.34% (3.42 pp)	100%	38.79% (8.36 pp)

6 Discussion

6.1 Interpretation

The most direct empirical finding is that the CTQW ansatz concentrated more probability on optimal profiles and had a lower expected-cost gap than the tested penalty-QAOA ansatz on this eight-instance sample. The feasible-profile representation avoids one-hot constraint leakage by construction. That difference is a design feature of the simulated representation, not an isolated causal effect of quantum walking: the number of amplitudes, state preparation, mixer access, and classical preprocessing differ. The CTQW result also depends on a dense classical diagonalization performed before variational optimization.

The local-search outcome is an important scale check. A simple classical heuristic reached the exact solution in every run on a space of only 81 profiles. Thus the experiment establishes that the small simulation is executable and that the diagnostics behave as specified; it does not show that a quantum-inspired or quantum statevector method improves on a capable classical solver. Larger instances and stronger classical baselines are required before making comparative claims.

The game-theoretic result is separate from optimization performance. The linear program tests whether an optimal CE exists for the game; it does not certify that either ansatz’s output law is an equilibrium. To test that, one would need to retain the full probability distribution over profiles, compute all recommendation-conditioned deviation residuals, and report their maximum positive violation. An optimal expected system cost and a correlated equilibrium are different requirements, as the counterexample demonstrates.

6.2 Complexity and implementation limits

At $p = 1$, both variational procedures are small numerical experiments. They do not imply rapid convergence at logarithmic depth, polynomial-time optimization, or a solution to general NP-hard routing. For the standard unnormalized mixer $\sum_j X_j$, the spectral gap is 2 (or $2/q$ under division by q); this endpoint fact does not lower-bound the minimum gap of an interpolating Hamiltonian or bound outer-optimizer convergence. For the walk ansatz, graph sparsity alone does not provide an efficient quantum implementation. A gate-level study would need to account for coherent indexing, reversible cost evaluation, neighbor access, state preparation, Hamiltonian-simulation error, circuit depth, shots, and noise. Efficient quantum walks have been constructed for particular indexed combinatorial spaces under explicit assumptions; those constructions do not automatically compile the present 81-node simulation into an efficient circuit [7, 8].

6.3 Limitations and next experiments

The experiment has several explicit limitations: (i) only eight synthetic games with four players and three roads were tested; (ii) all feasible profiles were exactly enumerable; (iii) latencies are randomly

Eight-instance ideal-statevector pilot

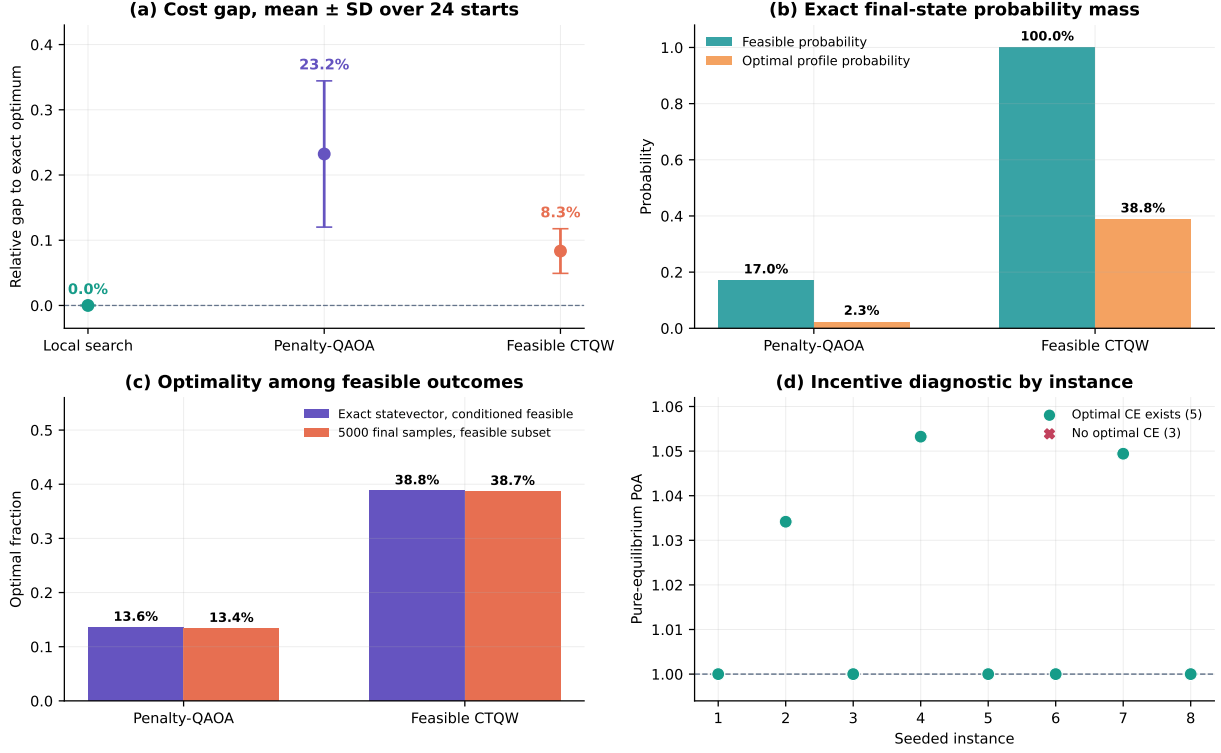


Figure 2: Descriptive pilot results. (a) Mean relative gap with standard deviation across 24 starts; starts are nested in eight instances. (b) Exact final-state feasible and optimal-profile probability mass. (c) Exact versus sampled optimal fraction among feasible outcomes. (d) Pure-equilibrium price of anarchy per instance; marker shape/color denotes whether the optimal-CE feasibility LP was feasible. The QAOA cost gap is postselected on feasibility, so panel (b) is essential context.

generated and not calibrated to observed traffic; (iv) only depth one was used; (v) optimization used exact statevector expectations rather than shot-based noisy evaluations; (vi) CTQW was evaluated in a classical 81-dimensional basis with dense eigendecomposition; (vii) no gate counts, quantum queries, or physical-device runs were measured; and (viii) only one local-search baseline was tested. The 24 optimizer starts are nested within eight games, so the standard deviations describe restart variability rather than uncertainty over a broad traffic population.

A useful next study would scale the game size while retaining exact references only where feasible, include stronger classical methods, and report per-instance seed-level results. The same game instances and objective should be used across methods; resource comparisons should match either wall-clock budgets or circuit/shot budgets and account for preprocessing and access oracles. If the strategic component is retained, the complete output distribution should be tested against CE inequalities. A dynamic-traffic extension would additionally require time-indexed demand, travel-time updates, a recourse policy, and held-out temporal evaluation.

7 Conclusion

We formulated a finite atomic congestion game, stated the exact condition under which a system-optimal distribution can also be a CE, and demonstrated that the condition can fail. We then compared exact enumeration, a capped local search, penalty-QAOA, and a feasible-profile CTQW ansatz on eight small synthetic games. Local search found the exact solution in all 24 starts; CTQW

had a lower descriptive expected-cost gap than the tested QAOA encoding, while QAOA assigned limited probability mass to feasible strings. Five of eight games admitted an optimal CE.

The results support a reproducible small-instance pilot and a conditional incentive analysis. They do not establish quantum advantage, large-scale routing utility, dynamic traffic control, or equilibrium behavior of the variational output distributions. The next contribution should be a scale-up with stronger classical baselines and explicit circuit-level access and incentive diagnostics.

Declarations

Funding: No external funding is declared.

Competing interests: The author declares no competing interests.

Ethics approval: Not applicable; the study uses synthetic computational instances and no human participants or animals.

Data and code availability: The 80-row run-level CSV, benchmark Python source, and figure-generation source are supplied with this manuscript as companion materials. No external traffic dataset was used.

Author contributions: Parham Ghayour conceived and authored the original draft and supplied the benchmark run and its interpretation.

Use of generative AI: An AI system assisted with code preparation, figure generation, and manuscript editing. The author reviewed the final manuscript and remains responsible for the reported methods, data, and conclusions.

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