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I. EXPERIMENTAL METHODS

Figure 1 schematically describes the experimental setup. A Ti:sapphire laser system operating at a repetition rate of 1 kHz delivers infrared (IR) pulses centered at 800 nm, with a pulse duration of ~ 30 fs and pulse energy of ~ 4.9 mJ. The beam is directed into a 80:20 beamsplitter, where the stronger arm serves as the coherent driving field for high-harmonic generation (HHG), while the weaker arm is used for the generation of frequency non-degenerate two-mode bright squeezed vacuum (TMBSV). Prior to the TMBSV generation stage, the pump beam is shaped using a telescope that reduces the beam diameter by a factor of 3 to increase the peak intensity and improve conversion efficiency.

The TMBSV field is generated by high-gain type-I spontaneous parametric down-conversion (SPDC) in a cascaded series of BiBO (BiB_3O_6 , $\theta = 10.9^\circ$ and $\phi = 0^\circ$) crystals. The beam traverses a 2 mm crystal, propagates through approximately 1.5m of free space, and then through an additional 3 mm of BiBO crystal. During the free-space propagation, higher-order spatial modes diffract more strongly than the fundamental mode. Consequently, the second crystal preferentially amplifies the lowest-order spatial modes, effectively spatially filtering the TMBSV field. After the two stages of amplification the TMBSV contains ~ 1.2 spatial modes. The generated TMBSV exhibits broadband non-degenerate emission with dominant spectral lobes centered around ~ 1375 nm and ~ 1990 nm, with a mean power of ~ 2 mW. The detailed spectral and modal properties of the generated TMBSV are discussed in section II .

Following the TMBSV generation stage, a spectral filter suppresses the residual 800 nm pump and additional parasitic nonlinear radiation generated in the crystals. A beam-expanding telescope subsequently enlarges the TMBSV mode by a factor of ~ 7.5 to optimize focusing conditions and suppress chromatic aberrations for the downstream HHG interaction. In the coherent arm, a motorized wedges based delay line compensates for the optical path difference between the coherent pump and TMBSV beams and enables sub-cycle delay control. A half-wave plate rotates the polarization of the coherent pump beam to match the polarization axis of the generated TMBSV. The coherent pump and TMBSV beams are recombined using a broadband dichroic mirror and focused by a CaF_2 lens with an effective focal length of approximately 1 m into a continuous-flow gas nozzle filled with krypton. The combined field drives the HHG process, producing extreme-ultraviolet (XUV) attosecond radiation. The emitted XUV spectrum is dispersed by a flat-field aberration-corrected concave grating and detected using a microchannel plate (MCP) coupled to a CCD camera. Single-shot acquisitions are synchronized to the laser system, with an integration

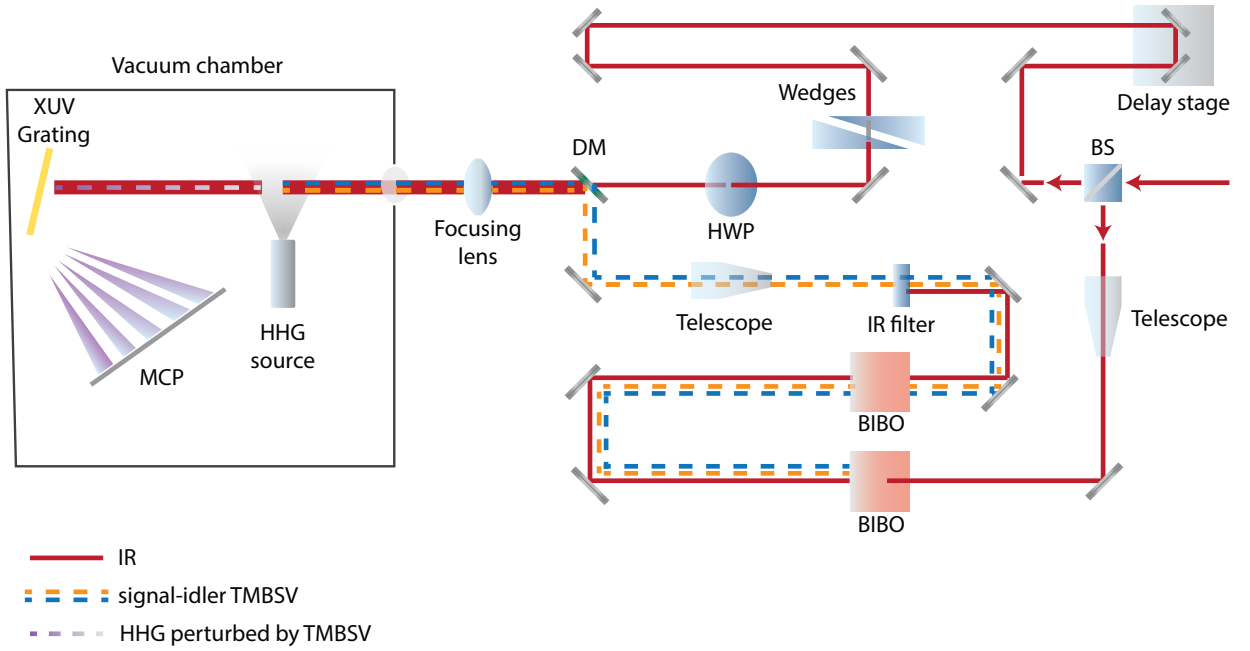


FIG. 1. Schematic of the experimental setup. A frequency-nondegenerate TMBSV field (dashed orange and blue lines) is generated in a cascaded BiBO source, spectrally filtered, and recombined with a strong coherent infrared beam (red line). The delay between the beams is scanned with a pair of wedges in the coherent beam line. The combined field drives the HHG process in krypton. The generated XUV spectrum is dispersed by a grating spectrometer and recorded shot-to-shot with an MCP detector.

time of 1 ms per exposure. With an MCP exponential decay time of 1.2 ms per decade and a 1 kHz repetition rate, the effective number of pulses per shot is ~ 1.15 . This slight deviation from unity has a minor impact on the measured $g^{(2)}$, as discussed below.

II. TMBSV CHARACTERIZATION

A. TMBSV modal characterization

Here we estimate the effective spectral and spatial mode numbers of the TMBSV source, and show how it affects the measured second-order coherence ($g^{(2)}(0)$) of the XUV light. For an ideal frequency-nondegenerate SPDC process, the signal and idler beams individually exhibit thermal photon statistics with $g^{(2)}(0) = 2^1$. Under typical experimental conditions, however, the detected field contains a few spectral and spatial modes, reducing the measured bunching according to

$$g^{(2)}(0) = 1 + \frac{1}{K_{\text{eff}}}, \quad (1)$$

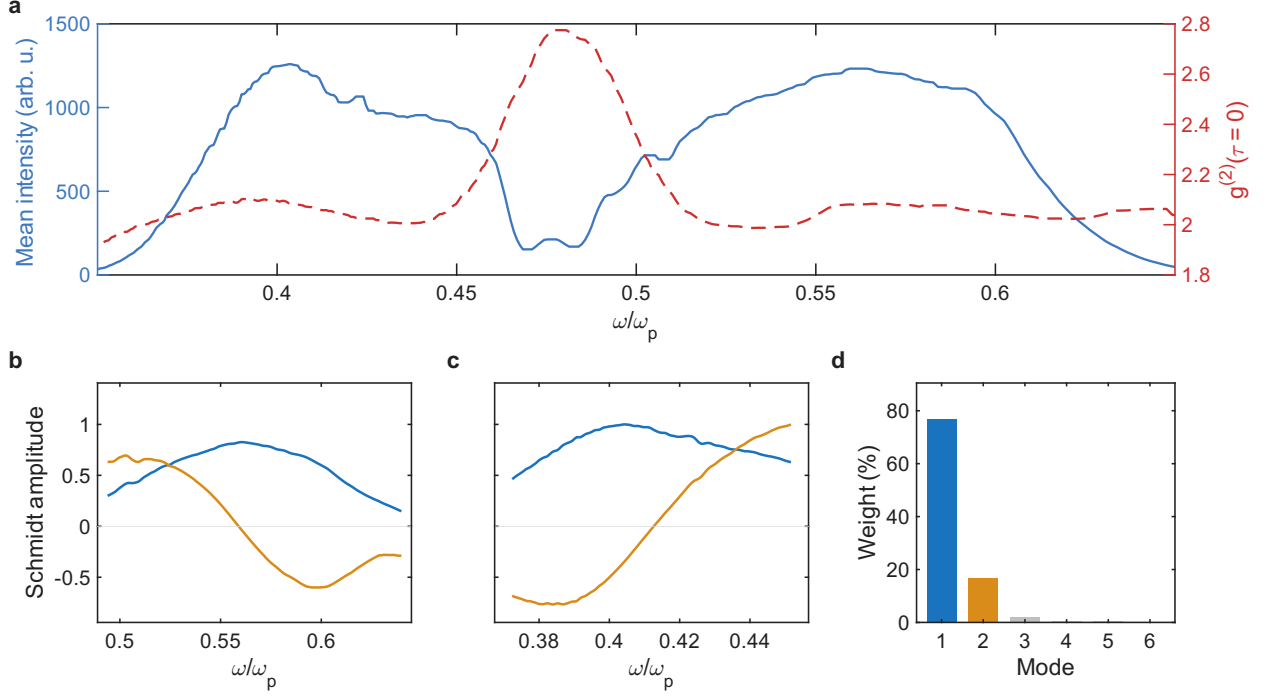


FIG. 2. **Spectral mode characterization of the TMBSV source.** **a**, Mean spectral intensity (blue, left axis) and the frequency-resolved second-order coherence $g^{(2)}(\tau = 0)$ (red, right axis), plotted against the normalized frequency ω/ω_p . Across both spectral wings $g^{(2)} \approx 2$, confirming the thermal photon statistics expected for parametric down-conversion, while the central near-degenerate region ($\omega/\omega_p \approx 0.5$) is strongly suppressed. **b,c**, The two dominant Schmidt modes of the signal (**b**, higher-frequency wing) and idler (**c**, lower-frequency wing) fields, obtained from a singular-value decomposition of the frequency-domain signal-idler covariance matrix (Mode 1, blue; Mode 2, orange). **d**, The corresponding Schmidt-mode weights: the emission is concentrated in the first two modes, giving an effective mode number $K_{\text{spec}} \approx 1.6$.

where K_{eff} is the effective number of detected modes^{2,3}. We note here that in femtosecond degenerate BSV the effective number of modes achieved in experiments is near unity^{4–6}. In contrast, in the frequency-nondegenerate case considered in this paper, the finite bandwidth and the chromatic dependence of the crystal parameters result in few modes emission.

To characterize the modal structure of the TMBSV source, we employ two complementary measurements. A single-shot infrared spectrometer samples the emitted beam through a multimode fiber, providing access to the spectral statistics of the detected light. This measurement yields $g^{(2)}(0, \lambda) \simeq 2$ across the signal and idler spectral bands (Figure 2a), confirming that the detected signal and idler beams retain the thermal statistics expected for SPDC. Since the spectrometer samples the beam through a finite fiber aperture, it may preferentially collect only a subset of the spatial modes. This measurement is therefore primarily used to verify the thermal nature of the

source rather than to determine its total modal content. In contrast, integrating the entire beam on the infrared camera gives $g^{(2)}(0) \approx 1.45$, corresponding to an overall effective mode number of $K_{\text{eff}} \approx 2.2$. Because this measurement includes essentially the entire emitted beam, while retaining approximately the spectral modal content, it provides the most representative estimate of the number of modes participating in the HHG experiment.

To understand the origin of this finite mode number, we separately estimate the spectral and spatial contributions. Spectral SVD of the covariance matrix⁷ (Fig. 2b,c,d) yields an effective spectral mode number of $K_{\text{spec}} \approx 1.6$. An independent estimate obtained from $g^{(2)}$ measured within a single camera pixel gives $K_{\text{spec}} \approx 1.8$. Because residual spatio-spectral correlations can redistribute the spectral modal content across the beam profile, this estimate should be regarded as approximate.

The number of spectral modes arises from two successive mechanisms. In the low-gain regime, the number of occupied spectral Schmidt modes is determined by the ratio of the phase-matching bandwidth of the BiBO crystal to the pump bandwidth⁸. Here, the 30 fs pump provides a spectral bandwidth of order 15–20 THz, narrower than the BiBO phase-matching bandwidth (~ 70 THz for 5 mm crystal length), which already limits the number of independently seeded mode pairs to a few. High parametric gain further reduces the effective number of spectral modes. Because the dominant Schmidt mode experiences the highest single-pass gain, its exponential amplification exceeds that of higher-order modes, concentrating most of the emission into fewer modes⁹.

Spatial SVD yields an effective spatial mode number of $K_{\text{spat}} \approx 1.2$. Together, these estimates provide a consistent interpretation of the measured total effective mode number, $K_{\text{eff}} \approx 2.2$, which contribute to the reduction of the measured $g^{(2)}(0)$ in the XUV regime, from its single-mode value (as discussed in section IV).

B. Characterization of central SPDC frequencies

The spectral covariance matrix in Figure 3a shows four localized diagonal and anti-diagonal lobes, the characteristic signature of frequency-nondegenerate SPDC. The anti-diagonal reflects energy conservation, $\omega_s + \omega_i \simeq \omega_p$, such that each point corresponds to a correlated signal–idler frequency pair within the pump bandwidth.

The spectral covariance matrix reveals that the signal–idler pairs are centered at photon energies $E_s \approx 0.90$ eV ($\lambda_s \approx 1375$ nm) and $E_i \approx 0.62$ eV ($\lambda_i \approx 1990$ nm) (See figure 3). Their energy sum corresponds to an effective pump photon energy of $E_p^{\text{eff}} \approx 1.52$ eV ($\lambda_p^{\text{eff}} \approx 815$ nm), slightly

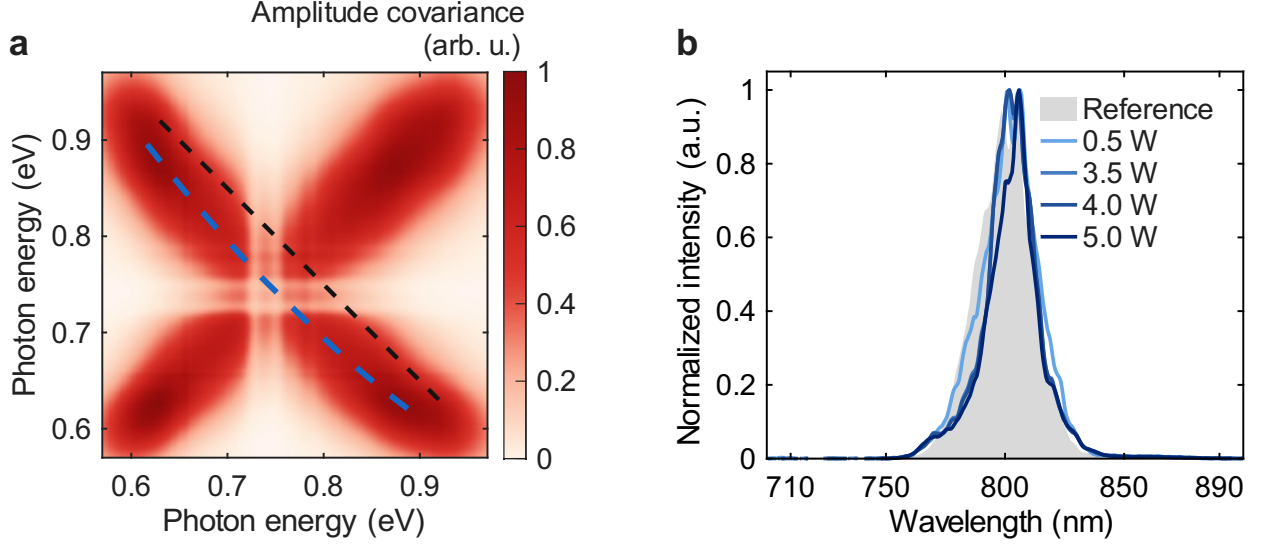


FIG. 3. **Spectral analysis of the TMBSV** (a) Measured spectral amplitude covariance of the TMBSV field as a function of the signal and idler photon energies (eV). The anti-diagonal correlation reflects energy conservation within each pair. The black dashed line marks the anti-diagonal expected for an 800 nm pump ($\omega_s + \omega_i = \omega_p$), i.e. the energy-conservation reference. The blue dashed line is the measured correlation ridge, extracted as the mid-point of the anti-diagonal of the Pearson correlation matrix. Its systematic offset from the black reference reveals the spectral shift: the pairs are generated by an effective pump frequency displaced from 800 nm, consistent with the group-velocity-matching mechanism between the generated pairs and the generation frequency in the pump. (b) Normalized spectra of the 800 nm pump measured after the 5 mm BiBO crystal at increasing input powers (0.5, 3.5, 4.0 and 5.0 W), compared with the reference pump spectrum that did not pass through the crystal (grey filled area). The pump spectrum is essentially unchanged with power, showing no significant shift or reshaping from nonlinear processes. This confirms that the covariance shift in (a) originates from group-velocity-matched pair generation rather than from any modification of the pump spectrum itself.

red-shifted from the nominal pump center at 1.55 eV (800 nm). Although this shift does not alter our core conclusions, a detailed explanation is included below for completeness.

We attribute this shift to the interplay between group-velocity mismatch (GVM) and the chirp the pump pulse acquires through dispersion in the BiBO crystal. Within the relevant generation frequencies in BiBO, corresponding to idler photon energies of $\sim 0.57\text{--}0.73$ eV ($\sim 2160\text{--}1700$ nm) and signal photon energies of $\sim 0.81\text{--}0.96$ eV ($\sim 1540\text{--}1290$ nm), both the idler and signal group velocities are strictly higher than that of the pump central frequency. Owing to the positive dispersion of the 800 nm pump at the crystal, the group velocity mismatch of both signal and idler with the effective pump approaches zero at $E_p^{\text{eff}} \approx 1.52$ eV ($\lambda_p^{\text{eff}} \approx 815$ nm), which indeed

corresponds to the measured centroid of the effective pump frequency. At this zero-mismatch condition, the pump, signal, and idler co-propagate, enabling exponential amplification across the entire crystal length and consequently localizing signal generation predominantly at these frequencies. The signal and idler fields have some dispersion as well, which leads to a minor change in effective pump frequency across different generation frequencies.

To confirm that this shift arises from parametric group-velocity dynamics rather than from nonlinear frequency conversion of the pump itself, we monitored the 800 nm pump spectrum after the crystal across average powers ranging between (0.5-5.0 W) (Figure 3b). The spectra remained symmetric and undistorted, with no evidence of spectral broadening, self-steepening, or Stokes sidebands, directly ruling out self-phase modulation and stimulated Raman scattering as contributing mechanisms. Furthermore, scanning the crystal phase-matching angle over a wide range, corresponding to signal central photon energies of $\sim 0.89\text{--}0.95$ eV ($\sim 1400\text{--}1300$ nm) and idler central photon energies of $\sim 0.57\text{--}0.64$ eV ($\sim 2190\text{--}1950$ nm) did not measurably affect the covariance centroid position. This confirms that the shift is a fixed property of the crystal at the relevant signal and idler frequencies rather than an artifact of the phase-matching geometry or pump nonlinearity.

III. TWO-COLOR PHASE-PERTURBATION MODEL FOR FRACTIONAL AND EVEN HARMONICS

A. Symmetry-based derivation of the fractional and even harmonics fields

In this section, we use a perturbative HHG model to show the dependence of the fractional and even harmonics on the perturbation. We model HHG driven by a strong coherent field at frequency ω_p , weakly perturbed by the signal and idler components of the TMBSV field at $\omega_s = 0.6\omega_p$ and $\omega_i = 0.4\omega_p$, closely resembling the experimental values.

The model relies on three assumptions. First, the perturbative fields are weak compared to the coherent driving field, allowing a perturbative expansion of the HHG response with respect to the TMBSV field. Second, in the absence of the perturbation, the HHG process preserves the half-cycle symmetry of the strong-field interaction, leading to destructive interference of even harmonics and the generation of odd harmonics only. The weak stochastic perturbations break the symmetry between consecutive half cycles, leading to the appearance of fractional and even harmonics. Third, we assume slowly varying envelopes for the fundamental field and the perturbation, which allows

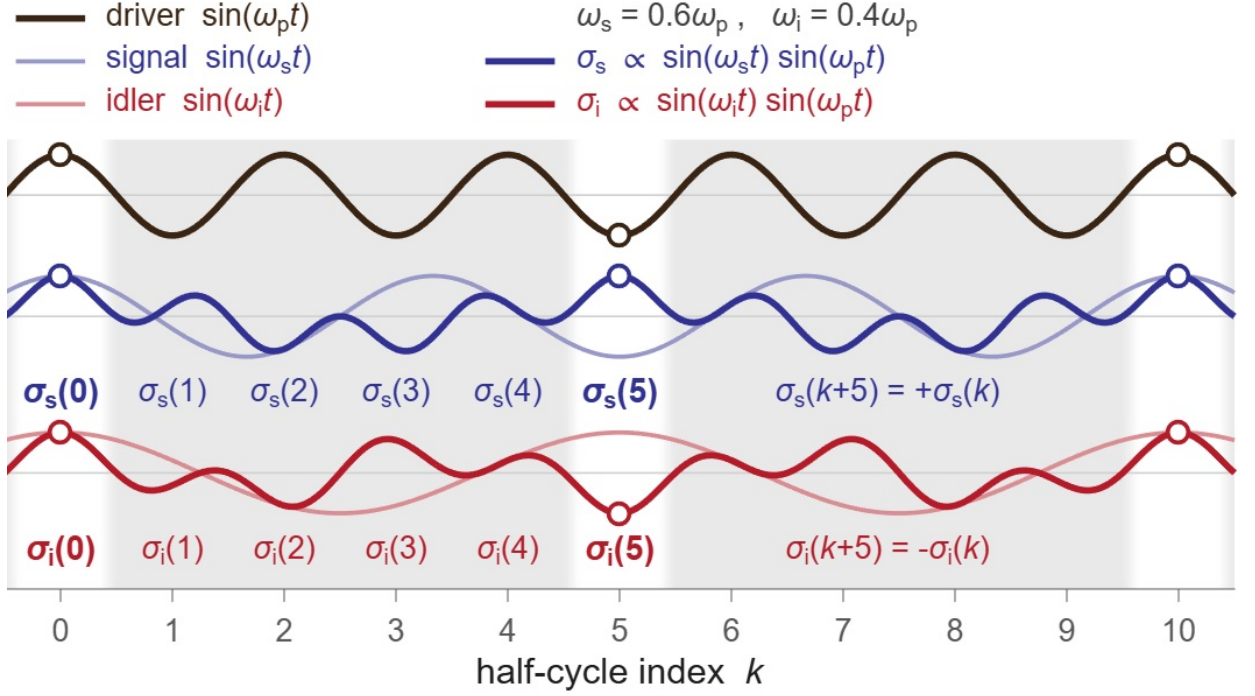


FIG. 4. **Half-cycle symmetry of the signal and idler perturbations.** The driving field at frequency ω_p (black), the signal and idler fields at $\omega_s = 0.6\omega_p$ and $\omega_i = 0.4\omega_p$ (light blue and light red respectively), and their corresponding perturbation-induced phases $\sigma_s(k), \sigma_i(k)$ (dark blue and dark red respectively) are shown over five cycles of the pump. The half cycles $k = 0, 5, 10$ (painted white) explicitly show the five-half-cycles symmetry of the signal perturbation, $\sigma_s(k+5) = \sigma_s(k)$, and the corresponding antisymmetry of the idler perturbation, $\sigma_i(k+5) = -\sigma_i(k)$.

for spectral separation of different photonic processes.

We first derive the emitted HHG field by considering the radiation generated in a two consecutive half cycles of the driving laser. In the absence of the perturbation, the half-cycle symmetry of the HHG interaction guarantees that each half cycle k generates identical optical emission. The emitted bursts at frequency Ω differ only by the phase factor $e^{i\Omega k T/2}$, arising from the temporal delay between half cycles, and by the sign reversal of the driving field, which contributes a factor $(-1)^k$. Hence, the interference between the k subcycles is destructive for even harmonics and constructive for odd harmonics, according to^{10–12}:

$$E(\Omega) \sim x(\Omega) \left(1 - e^{i\Omega T/2}\right) = x(\Omega) \begin{cases} 2, & N \text{ odd} \\ 0, & N \text{ even} \end{cases} \quad (2)$$

where $E(\Omega)$ is the total emitted harmonic field at frequency Ω , $x(\Omega)$ is the complex spectral

amplitude emitted by a single half cycle, T is the period of the driving field, and N denotes the harmonic order, such that $\Omega = N\omega_p$, where ω_p is the driving field's frequency.

We now introduce a weak perturbing field. We assume that the perturbation modifies the complex phase associated with the harmonic field emitted during each half cycle: $x(\Omega) \rightarrow x(\Omega)e^{i\sigma}$. The perturbation-induced phase can be expanded as a power series in the perturbing field, E_{pert} :

$$\sigma \propto \sum_{m=1}^{\infty} a_m (E_{\text{pert}})^m \quad (3)$$

Under the weak-perturbation assumption, $|E_{\text{pert}}| \ll |E_p|$ we neglect nonlinear orders and assume:

$$\sigma \propto E_{\text{pert}} \quad (4)$$

We now coherently sum the contributions from all half cycles to obtain the HHG spectrum:

$$E(\Omega) \sim x(\Omega) \sum_k (-1)^k e^{i\sigma(k)} e^{i\Omega k T/2} \quad (5)$$

where $e^{i\sigma(k)}$ is the complex phase the perturbation imprints on the k -th half cycle. In the absence of the perturbation ($\sigma(k) = 0$), Equation (5) reduces to the unperturbed result of Equation (2).

For two weak perturbations each perturbing field imprints a complex phase on the XUV emission from every half cycle of the driver, σ_s and σ_i , for the signal and idler respectively. Both complex phases are linear in the corresponding perturbation field amplitudes, $\sigma_s \propto E_s$ and $\sigma_i \propto E_i$. The perturbed emission reads:

$$E(\Omega) \sim x(\Omega) \sum_k (-1)^k e^{i\sigma(k)} e^{i\Omega k T/2} = x(\Omega) \sum_k (-1)^k e^{i(\sigma_s(k) + \sigma_i(k))} e^{i\Omega k T/2} \quad (6)$$

The $0.2\omega_p$ detuning between ω_s and ω_i extends the periodicity of the total field to ten fundamental half cycles ($5T$). Under a five-half-cycle shift the signal driven perturbation phase is symmetric, $\sigma_s(k+5) = \sigma_s(k)$, and the idler driven perturbation phase is antisymmetric, $\sigma_i(k+5) = -\sigma_i(k)$, as illustrated in Figure 4. Therefore it is convenient to group the 10 half cycles into five pairs $(k, k+5)$, $k \in \{0, 1, 2, 3, 4\}$. The symmetric signal factor $e^{i\sigma_s(k)}$ is common to both members of a pair, while the antisymmetric idler enters as $e^{i\sigma_i(k)}$ for the k -th half cycle and $e^{-i\sigma_i(k)}$ for its partner, which additionally carries the reversal $(-1)^5 = -1$ and the delay phase $e^{i\Omega 2.5T}$. Using equation (6) we get:

$$X(\Omega) \sim x(\Omega) \sum_{k=0}^4 (-1)^k e^{i\sigma_s(k)} \left[e^{i\sigma_i(k)} e^{i\Omega k T/2} - e^{-i\sigma_i(k)} e^{i\Omega(kT/2 + 2.5T)} \right], \quad (7)$$

The five-half-cycle symmetry of the perturbation leads to harmonic frequencies separated by $0.2\omega_p$, such that the emitted spectrum contains odd, even, and fractional harmonics,

$$\Omega = (2n + q)\omega_p, \quad q \in \{0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8, 1\}. \quad (8)$$

Here n is an integer and q is the fractional offset from even harmonic order, in steps of $0.2\omega_p$. $q = 0$ corresponds to the even harmonics, $q = 1$ corresponds to the odd harmonics, and the intermediate fractional q values correspond to the fractional harmonics. Extracting the field associated with each harmonic order we get:

$$E_{2n+q} \propto \sum_{k=0}^4 e^{i\pi k(1+q)} e^{i\sigma_s(k)} \left[e^{i\sigma_i(k)} - e^{-i\sigma_i(k)} e^{i5\pi q} \right] \quad (9)$$

Here E_{2n+q} is the complex amplitude of the harmonic at frequency $\Omega = (2n + q)\omega_p$.

The parity of $5q$ separates the harmonics into two symmetry classes:

$$E_{2n+q} \propto \sum_k e^{i\pi k(1+q)} e^{i\sigma_s(k)} \begin{cases} \cos[\sigma_i(k)], & 5q \text{ odd } (q = \pm 0.2, \pm 0.6, 1), \\ i \sin[\sigma_i(k)], & 5q \text{ even } (q = 0, \pm 0.4, \pm 0.8). \end{cases} \quad (10)$$

In the perturbative regime, $\sigma_s, \sigma_i \ll 1$. Therefore, up to second order:

$$e^{i\sigma_s} \approx 1 + i\sigma_s - \frac{\sigma_s^2}{2}, \quad \cos \sigma_i \approx 1 - \frac{\sigma_i^2}{2}, \quad i \sin \sigma_i \approx i\sigma_i. \quad (11)$$

In the remainder of this section, we focus on the even and fractional harmonics, which are central to this work. Substituting the perturbative expansion into the harmonic amplitudes yields the following expressions:

$$E_{2n} \propto \sum_k (-1)^k [i\sigma_i(k) - \sigma_i(k)\sigma_s(k)], \quad (12a)$$

$$E_{2n \pm \delta_i} \propto \sum_k (-1)^k e^{\pm i\delta_i \pi k} [i\sigma_i(k) - \sigma_s(k)\sigma_i(k)], \quad \{q = \pm \delta_i = \pm 0.4\} \quad (12b)$$

$$E_{2n \pm \delta_s} \propto \sum_k (-1)^k e^{\pm i\delta_s \pi k} \left[1 + i\sigma_s(k) - \frac{1}{2}\sigma_i^2(k) - \frac{1}{2}\sigma_s^2(k) \right] \quad \{q = \pm \delta_s = \pm 0.6\} \quad (12c)$$

where we assigned $\delta_{i,s} = \omega_{i,s}/\omega_p$. Each of the terms in equation (12) acquires a different phase from one half cycle to the next through the k -dependent phase terms. Since the observed harmonics arise from the coherent sum over five consecutive half cycles, the relevant contributions must be identified by explicitly evaluating this interference.

We will now substitute the expressions for $\sigma_s(k)$ and $\sigma_i(k)$ and perform the summation for each harmonic family separately. The complex perturbative phase σ is linear in the perturbing field and can therefore be written as the action of a linear time dependent operator $\mathcal{F}$ on the field,

$$\sigma(t) = \mathcal{F}[E_{\text{pert}}] = \mathcal{F}[A_{\text{pert}} \sin(\omega_{\text{pert}}t + \phi_{\text{pert}})] \quad (13)$$

where $A_{\text{pert}}, \phi_{\text{pert}}, \omega_{\text{pert}}$ are the amplitude, phase and frequency of the perturbing field, respectively. The operator $\mathcal{F}$ is kept general, subject only to the assumption of linearity. Under the weak-perturbation assumption, it retains only the first-order dependence on both the perturbing field and the fundamental field. A notable specific realization is the strong-field approximation (SFA), in which $\mathcal{F}$ corresponds to a temporal integral along a strong-field trajectory¹³.

We now consider the symmetry of σ under a shift by a half cycle of the fundamental field. The field reverses sign every half cycle, so the perturbation phase σ alternates sign accordingly; over k half cycles this contributes a factor $(-1)^k$. The perturbing field, in turn, acquires an additional phase $\omega_{\text{pert}}kT/2 = \pi k \omega_{\text{pert}}/\omega_p$, hence:

$$\sigma(t + k\frac{T}{2}) = (-1)^k \mathcal{F}\left[A_{\text{pert}} \sin\left(\omega_{\text{pert}}t + (\omega_{\text{pert}}/\omega_p)\pi k + \phi_{\text{pert}}\right)\right] \quad (14)$$

Expressing the sine function and using the linearity of $\mathcal{F}$ we have:

$$\sigma(k) = \frac{A_{\text{pert}}}{2i} \left(e^{i(1+\omega_{\text{pert}}/\omega_p)\pi k} \mathcal{F}\left[e^{i(\omega_{\text{pert}}t+\phi_{\text{pert}})}\right] - e^{i(1-\omega_{\text{pert}}/\omega_p)\pi k} \mathcal{F}\left[e^{-i(\omega_{\text{pert}}t+\phi_{\text{pert}})}\right] \right) \quad (15)$$

We now simplify and explicitly denote the signal and idler perturbations:

$$\sigma_j(k) = C_j^+ e^{i(1+\delta_j)\pi k} + C_j^- e^{i(1-\delta_j)\pi k}, \quad j = s, i \quad (16)$$

where $C_j^\pm \equiv \pm \frac{A_j e^{\pm i\phi_j}}{2i} \mathcal{F}[e^{\pm i\omega_j t}]$ are k -independent coefficients and $\delta_j = \omega_j/\omega_p$.

Substituting $\sigma_j(k)$ into equation (12) determines which terms vanish when coherently summing over the fundamental half cycles.

For the fractional harmonics associated with the idler perturbation in equation (12b), the quadratic cross term reads

$$\begin{aligned} \sigma_s(k)\sigma_i(k) &= C_s^+ C_i^+ e^{i(2+\delta_s+\delta_i)\pi k} + C_s^- C_i^- e^{i(2-\delta_s-\delta_i)\pi k} \\ &\quad + C_s^+ C_i^- e^{i(2+\delta_s-\delta_i)\pi k} + C_s^- C_i^+ e^{i(2-\delta_s+\delta_i)\pi k} \end{aligned} \quad (17)$$

Multiplying by $(-1)^k e^{\pm i\delta_i \pi k}$ and summing over the five half cycles, with $\delta_s + \delta_i = 1$ and $\delta_s - \delta_i = 0.2$,

gives

$$\begin{aligned}
\sum_k (-1)^k e^{+i\delta_i \pi k} \sigma_s(k) \sigma_i(k) &= C_s^+ C_i^+ \sum_k e^{i4.4\pi k} + C_s^- C_i^- \sum_k e^{i2.4\pi k} \\
&\quad + C_s^+ C_i^- \sum_k e^{i3.6\pi k} + C_s^- C_i^+ \sum_k e^{i3.2\pi k} = 0 \\
\sum_k (-1)^k e^{-i\delta_i \pi k} \sigma_s(k) \sigma_i(k) &= C_s^+ C_i^+ \sum_k e^{i3.6\pi k} + C_s^- C_i^- \sum_k e^{i1.6\pi k} \\
&\quad + C_s^+ C_i^- \sum_k e^{i2.8\pi k} + C_s^- C_i^+ \sum_k e^{i2.4\pi k} = 0
\end{aligned} \tag{18}$$

All eight sums vanish: every exponent is of the form $e^{i\pi(2m/5)}$ with m not a multiple of five, so the summands are five equally spaced points on the unit circle. The linear term gives

$$\begin{aligned}
\sum_k (-1)^k e^{+i\delta_i \pi k} \sigma_i(k) &= C_i^+ \sum_k e^{i2.8\pi k} + C_i^- \sum_k e^{i2\pi k} \\
\sum_k (-1)^k e^{-i\delta_i \pi k} \sigma_i(k) &= C_i^+ \sum_k e^{i2\pi k} + C_i^- \sum_k e^{i1.2\pi k}
\end{aligned} \tag{19}$$

Yielding:

$$E_{2n+\delta_i} \propto C_i^- \propto E_i^*, \quad E_{2n-\delta_i} \propto C_i^+ \propto E_i \tag{20}$$

This follows the frequency domain intuition of perturbative parametric processes - absorbing a single idler photon increases the frequency of the output light by ω_i and emitting a single idler photon decreases it by ω_i .

Similarly, for the fractional harmonics associated with the signal perturbation the constant and quadratic terms vanish by the same argument, while the linear term gives:

$$\begin{aligned}
\sum_k (-1)^k e^{+i\delta_s \pi k} \sigma_s(k) &= C_s^+ \sum_k e^{i3.2\pi k} + C_s^- \sum_k e^{i2\pi k} \\
\sum_k (-1)^k e^{-i\delta_s \pi k} \sigma_s(k) &= C_s^+ \sum_k e^{i2\pi k} + C_s^- \sum_k e^{i0.8\pi k}
\end{aligned} \tag{21}$$

Hence

$$E_{2n+\delta_s} \propto C_s^- \propto E_s^*, \quad E_{2n-\delta_s} \propto C_s^+ \propto E_s \tag{22}$$

In similarity to the results of the idler associated fractional harmonic fields.

We now turn to the even harmonics. In equation (12a) the term that depends linearly on σ_i involves only the exponents $e^{i2.4\pi k}$ and $e^{i1.6\pi k}$, and therefore vanishes by the same argument as above. The nonlinear contribution to the even harmonic is (using $\omega_s + \omega_i = \omega_p$):

$$\begin{aligned}
& \sum_k (-1)^k \sigma_s(k) \sigma_i(k) \\
&= \sum_k \left[C_s^+ C_i^+ e^{i(3+\delta_s+\delta_i)\pi k} + C_s^- C_i^- e^{i(3-(\delta_s+\delta_i))\pi k} + C_s^+ C_i^- e^{i(3+\delta_s-\delta_i)\pi k} + C_s^- C_i^+ e^{i(3-(\delta_s-\delta_i))\pi k} \right] \\
&= C_s^+ C_i^+ \sum_k e^{i4\pi k} + C_s^- C_i^- \sum_k e^{i2\pi k} + C_s^+ C_i^- \sum_k e^{i3.2\pi k} + C_s^- C_i^+ \sum_k e^{i2.8\pi k} \\
&\propto C_s^+ C_i^+ + C_s^- C_i^-
\end{aligned} \tag{23}$$

The first two sums represent constructive interference between half cycles and therefore do not vanish, whereas the last two vanish as explained above, due to destructive interference between the 5 half cycles. Consequently,

$$E_{2n} \propto C_s^+ C_i^+ + C_s^- C_i^- \tag{24}$$

We now define $\tilde{C}_j^\pm = \pm \frac{1}{2i} \mathcal{F}[e^{\pm i\omega_j t}]$ which is independent of the perturbation field and get:

$$E_{2n} \propto \tilde{C}_s^+ \tilde{C}_i^+ \cdot E_s E_i + \tilde{C}_s^- \tilde{C}_i^- E_s^* E_i^* \tag{25}$$

This expression describes the two-path interferometer. One path corresponds to the stimulated emission of a signal and an idler photon ($\omega_i + \omega_s = \omega_p$), while the other corresponds to their absorption.

Comparing equation (24) with equations (20) and (22), the products $C_s^+ C_i^+$ and $C_s^- C_i^-$ are exactly the products of the signal- and idler-driven fractional harmonics mapped into the same even harmonic. The even-harmonic field can therefore be written directly as a product of the two neighboring fractional harmonics

$$E_{2n} \propto E_{2n-\delta_s} E_{2n-\delta_i} + E_{2n+\delta_s} E_{2n+\delta_i} \tag{26}$$

up to deterministic prefactors set by the strong-field response, which are stable from shot to shot. Each term multiplies one signal-driven and one idler-driven fractional harmonic, so the even harmonics emission only occurs when both are present within the same sub-cycle emission window, acting as an attosecond coincidence detector between the two fractional-harmonic families. Since equation (26) holds for every even harmonic across the emitted XUV spectrum, the delay-dependent oscillations of the even harmonics probe the joint statistics of the fractional harmonics over the full bandwidth: their contrast reflects the amplitude correlations between the two families, and their oscillation phase reflects the phase correlations.

Together, equations (20) and (22) show that the complex fields of the even and fractional harmonics scale according to the perturbation order of their generating processes.

B. Interferometric form of the even-harmonics

We now expand the model to include both the delay between the fundamental field and the TMBSV perturbation and to allow for a statistical analysis of the resulting interferometric signal. The single shot perturbation is defined as:

$$C_j^{(m)\pm} \equiv \pm \frac{A_j^{(m)} \cdot e^{\pm i\phi_j^{(m)}}}{2i} \mathcal{F}[e^{\pm i\omega_j t}] = \tilde{C}_j^\pm A_j^{(m)} \cdot e^{\pm i\phi_j^{(m)}}, \quad j = i, s \quad (27)$$

with $\tilde{C}_j^\pm$ independent of the perturbation stochastic field and the laser shot m and $A_j^{(m)}, \phi_j^{(m)}$ are the perturbation field amplitude and phase per shot, respectively.

Rewriting equation (15) for a single shot m and fundamental-TMBSV delay τ :

$$\begin{aligned} \sigma_s^{(m)}(k) &= |E_s^{(m)}| \left(\tilde{C}_s^+ e^{i(1+\delta_s)\pi k} e^{i\omega_s \tau} e^{i\phi_s^{(m)}} + \tilde{C}_s^- e^{i(1-\delta_s)\pi k} e^{-i\omega_s \tau} e^{-i\phi_s^{(m)}} \right) \\ \sigma_i^{(m)}(k) &= |E_i^{(m)}| \left(\tilde{C}_i^+ e^{i(1+\delta_i)\pi k} e^{i\omega_i \tau} e^{i\phi_i^{(m)}} + \tilde{C}_i^- e^{i(1-\delta_i)\pi k} e^{-i\omega_i \tau} e^{-i\phi_i^{(m)}} \right) \end{aligned} \quad (28)$$

where we have explicitly denoted the field amplitude for each shot as $A_j^{(m)} = |E_j^{(m)}|$.

Substituting (28) into (24) gives

$$E_{2N}^{(m)} \propto |E_s^{(m)}| |E_i^{(m)}| \left(\tilde{C}_i^+ \tilde{C}_s^+ e^{i[(\omega_s + \omega_i)\tau + \phi_s^{(m)} + \phi_i^{(m)}]} + \tilde{C}_i^- \tilde{C}_s^- e^{-i[(\omega_s + \omega_i)\tau + \phi_s^{(m)} + \phi_i^{(m)}]} \right) \quad (29)$$

Since $\omega_s + \omega_i = \omega_p$, equation(29) describes the absorption or stimulated emission of a correlated signal-idler photon pair. We note:

$$\begin{aligned} (2N-1)\omega_p + \omega_s + \omega_i &= 2N\omega_p \\ (2N+1)\omega_p - \omega_s - \omega_i &= 2N\omega_p \end{aligned} \quad (30)$$

We now use equations (20) and (22) to write the fractional harmonics fields separating amplitude and phase terms:

$$\begin{aligned} |E_{2N\pm\delta_s}^{(m)}| &\propto |E_s^{(m)}|, & |E_{2N\pm\delta_i}^{(m)}| &\propto |E_i^{(m)}|, \\ \phi_s^{(m)} + \phi_i^{(m)} &= \pm \left[\phi_0 - \left(\phi_{2N\pm\delta_s}^{(m)} + \phi_{2N\pm\delta_i}^{(m)} \right) \right] \end{aligned} \quad (31)$$

where ϕ_0 is an approximately deterministic, shot-independent phase offset arising from the complex HHG response that maps the signal and idler perturbation fields onto the corresponding fractional

harmonics. Although ϕ_0 generally depends on frequency (harmonic order), we leave it implicit for simplicity.

Substituting Equation (31) to equation (29) allows us to compactly write :

$$E_{2N}^{(m)} = E_+^{(m)} + E_-^{(m)} \propto \left| E_{2N\pm\delta_s}^{(m)} \right| \left| E_{2N\pm\delta_i}^{(m)} \right| \left[E_{2N+1} e^{i(\omega_p\tau + \Phi_-^{(m)})} + E_{2N-1} e^{-i(\omega_p\tau - \Phi_+^{(m)})} \right] \quad (32)$$

where $\Phi_{\pm}^{(m)} \equiv \phi_{2N\pm\delta_s}^{(m)} + \phi_{2N\pm\delta_i}^{(m)}$ are the single-shot phase sums of the lower- and upper-sideband pairs of fractional harmonics, $E_+^{(m)}$ and $E_-^{(m)}$ denote the two indistinguishable pathways of the interferometer. The complex quantities $E_{2N\pm 1} \equiv \tilde{C}_i^{\pm} \tilde{C}_s^{\pm} e^{\pm i\phi_0}$ denote the effective deterministic amplitudes of the neighboring odd-harmonic pathways, with the HHG phase offset ϕ_0 absorbed into their complex phases.

The even-harmonic intensity is therefore

$$I_{2N}^{(m)} = \left| E_+^{(m)} + E_-^{(m)} \right|^2 = I_+^{(m)} + I_-^{(m)} + 2 \left| E_+^{(m)} \right| \left| E_-^{(m)} \right| \cos \Delta\Phi^{(m)} \quad (33)$$

with the single-shot interference phase

$$\Delta\Phi^{(m)} = \phi_{2N+1} - \phi_{2N-1} + 2\omega_p\tau + \Phi_-^{(m)} - \Phi_+^{(m)} \quad (34)$$

where the effective complex phases of the neighboring odd-harmonic pathways are defined as $\phi_{2N-1} = \arg(\tilde{C}_i^- \tilde{C}_s^-) - \phi_0$, $\phi_{2N+1} = \arg(\tilde{C}_i^+ \tilde{C}_s^+) + \phi_0$. This pathway-resolved form shows that the even harmonic acts as an interferometer between two neighboring odd-harmonic pathways, while its single-shot interference phase is controlled by the fundamental field-TMBSV delay and by the correlated phase sum $\Phi^{(m)}$ of the signal- and idler-driven XUV fractional harmonics.

IV. SECOND-ORDER COHERENCE $g^{(2)}(0)$ OF THE XUV EMISSION

Here, we elucidate the impact of the few-modes emission described in section II on the $g^{(2)}$ of the new XUV frequencies generated in the presence of the TMBSV perturbation. The individual spectral and spatial modes are not resolved in the XUV measurement because of the finite spectral resolution of the XUV spectrometer and the spatial integration required for single-shot detection with the MCP. The spectral resolution of the XUV spectrometer (~ 0.1 eV) is sufficient to distinguish the signal and idler fractional harmonics (separated by 0.28 eV), but insufficient to resolve the individual spectral modes, whose characteristic spacing is approximately 0.05 eV. In addition, the signal is integrated over many detector pixels to achieve a sufficient signal-to-noise ratio for single-shot detection with the MCP. Consequently, the measured $g^{(2)}$ is reduced relative to the single-mode value.

The theoretical framework established in section III demonstrates that fractional harmonics arise from first-order perturbations (involving either a single signal or idler photon), whereas even harmonics stem from second-order perturbations (requiring both a signal and an idler photon). Consequently, as described in section II, assuming the expected $g^{(2)}$ corresponds to a single-photon process with a thermal distribution, the second-order coherence for 2.2 modes as calculated at section II, is given by equation (1): $g_{1\text{ photon}}^{(2)} = 1 + \frac{1}{2.2} \simeq 1.45$.

In the experimental determination of the $g^{(2)}$ of the harmonics, we account for the finite fluorescence decay of the MCP, which results in an effective contribution of 1.15 laser pulses per recorded shot due to residual fluorescence from the previous pulse (arriving 1 ms earlier). To compute the effect of it on the $g^{(2)}(0)$, we model the recorded signal in the n -th shot as

$$M_n = S_n + \eta S_{n-1}, \quad (35)$$

where S_n is the true signal from the n -th laser pulse and $\eta = 0.15$ is the residual contribution from the previous pulse. Assuming that successive shots are independent and identically distributed, with $\langle S_n \rangle = \langle S_{n-1} \rangle \equiv \langle S \rangle$ and $\text{Var}(S_n) = \text{Var}(S_{n-1}) \equiv \text{Var}(S)$ we obtain

$$\langle M_n \rangle = \langle S_n \rangle + \eta \langle S_{n-1} \rangle = (1 + \eta) \langle S \rangle \quad (36)$$

and

$$\text{Var}(M_n) = \text{Var}(S_n) + \eta^2 \text{Var}(S_{n-1}) = (1 + \eta^2) \text{Var}(S) \quad (37)$$

where the cross term $\langle S_n S_{n-1} \rangle$ vanishes because different shots are uncorrelated.

For high photon numbers:

$$g^{(2)} \simeq 1 + \frac{\text{Var}(M_n)}{\langle M_n \rangle^2} \quad (38)$$

substituting equations (36) and (37) into (38) relates the measured and corrected second-order coherences:

$$g_{\text{meas}}^{(2)} - 1 = \frac{1 + \eta^2}{(1 + \eta)^2} (g^{(2)} - 1) \quad (39)$$

or equivalently,

$$g^{(2)} = 1 + \frac{(1 + \eta)^2}{1 + \eta^2} (g_{\text{meas}}^{(2)} - 1) \approx 1 + 1.3 (g_{\text{meas}}^{(2)} - 1) \quad (40)$$

Here, $g_{\text{meas}}^{(2)}$ is the value extracted directly from the detector and $g^{(2)}$ is the corrected value.

After the correction procedure, we obtain a value of $g^{(2)}(0) \simeq 1.4 \pm 0.03$ at the fractional harmonics, corresponding to an effective mode number of $K_{\text{eff}} \simeq 2.5$. This value is slightly larger than the one inferred from the TMBSV characterization ($K_{\text{eff}} \simeq 2.2$), suggesting the presence of additional modes not captured in the IR measurements, possibly due to spatio-spectral mode coupling.

The even harmonics are expected to exhibit statistics analogous to those of sum-frequency generation driven by correlated thermal photons (section III). Accordingly, the resulting $g^{(2)}$ value depends strongly on the degree of correlation between the signal and idler fields. Higher correlations intensify the intensity fluctuations, thereby elevating the $g^{(2)}$ value. For single-mode thermal light, the theoretical limits map to $g_{2 \text{ photons}}^{(2)} = 4$ for completely uncorrelated fields and $g_{2 \text{ photons}}^{(2)} = 6$ for fully correlated fields¹⁴. Scaling these limits for a multi-mode case with $K_{\text{eff}} \sim 2.5$ modes yields:

$$g_{\text{corr}}^{(2)} = 1 + 5/K_{\text{eff}} \simeq 3, \quad g_{\text{uncorr}}^{(2)} = 1 + 3/K_{\text{eff}} \simeq 2.2 \quad (41)$$

The experimentally observed $g^{(2)}$ of the even harmonics closely matches $g^{(2)} \simeq 3.2 \pm 0.2$, taking into account the finite decay of the MCP signal. Given the fluctuations of the individual signal and idler driven fractional harmonics, the even harmonics – which are associated with single signal and idler photons – exhibit maximum possible fluctuations. This behavior corresponds to strongly correlated signal and idler photons, indicating that the HHG pathways associated with these photons are likewise also strongly correlated.

V. TIME-DOMAIN RECONSTRUCTION

A. Reconstruction Methodology

The central objective of the time domain reconstruction analysis, shown in main text Figure 4, is to resolve the quantum-origin correlations of the attosecond pulses in the time domain. For this purpose we introduce the assumption that the individual phases of the signal- and idler-driven fractional harmonics are uniformly distributed over $[0, 2\pi)$ from shot to shot. As shown by the linear field relations in Eqs. (20) and (22), the fractional-harmonic fields are directly proportional to the corresponding TMBSV modes and therefore inherit their phase statistics. The uniform marginal phase distribution follows from the established statistics of SPDC and remains valid beyond the low-gain regime (see later in this section). These phases are nevertheless not independent: their sum remains locked, as demonstrated by the persistent delay-dependent oscillations of the even harmonics, which would be washed out upon ensemble averaging if the phase sum fluctuated from

shot to shot. The rest of the complex information of the attosecond light is either directly measured or deduced from the experimental measurements.

With the stochastic phase statistics of the fractional harmonics established, the remaining ingredient is their deterministic spectral phase (the attochirp), which is governed by the strong-field HHG dynamics. The attochirp was previously shown to exhibit negligible shot-to-shot fluctuations under degenerate BSV perturbation¹⁵. In the present TMBSV experiment, the pronounced even-harmonic oscillations, reflecting phase-locked HHG pathways, likewise indicate that the attochirp remains approximately constant from shot to shot. Otherwise, fluctuations of the attochirp would introduce random relative phases between the HHG pathways at a fixed delay, washing out the interferometric oscillations upon averaging. The average attochirp has been extensively studied over the past two decades using a variety of experimental techniques and is determined by the strong-field electron trajectories underlying high-harmonic generation^{16–19}. We extract this spectral phase using an in situ approach^{10,12}, in which the phase of the delay-dependent even-harmonic oscillations directly encodes the spectral phase (see a detailed description later in this section). We then reconstruct the time-domain field by combining the measured spectral amplitudes with both the in situ extracted deterministic spectral phase and the correlated stochastic spectral phase, followed by an inverse Fourier transform. The amplitudes used in the reconstruction are extracted directly from the single-shot experimental data.

We define three distinct types of spectral regions of the XUV signal, corresponding to three separate attosecond pulse trains:

$$E_{2N\pm\delta_i}^{(m)}(t, \Delta t_0) \equiv \mathcal{F}^{-1} \left\{ E^{(m)}(\omega = 2N\omega_p \pm \omega_i, \Delta t_0) \right\} \quad (42)$$

$$E_{2N\pm\delta_s}^{(m)}(t, \Delta t_0) \equiv \mathcal{F}^{-1} \left\{ E^{(m)}(\omega = 2N\omega_p \pm \omega_s, \Delta t_0) \right\} \quad (43)$$

$$E_{(2N\pm\delta_i), (2N\pm\delta_s)}^{(m)}(t, \Delta t_0) \equiv \mathcal{F}^{-1} \left\{ E^{(m)}(\omega = 2N\omega_p \pm \omega_i, \Delta t_0) + E^{(m)}(\omega = 2N\omega_p \pm \omega_s, \Delta t_0) \right\} \quad (44)$$

Where Δt_0 is some chosen XUV-IR delay and m is the laser shot. $2N\omega_p \pm \omega_i$ and $2N\omega_p \pm \omega_s$ are the frequencies of the fractional harmonics associated with the idler and signal respectively.

We decompose the complex fields into their amplitude and phase components for the signal and idler fields separately:

$$E_j^{(m)}(t, \Delta t_0) = \mathcal{F}^{-1} \left\{ |E_j^{(m)}(\omega_j, \Delta t_0)| \cdot e^{i\phi^{(m)}(\omega_j, \Delta t_0)} \right\} \quad (45)$$

$$j = 2N \pm \delta_i, 2N \pm \delta_s; \omega_j = \omega_p \cdot j$$

where $|E_j^{(m)}(\omega_j, \Delta t_0)|$ is the spectral amplitude which is measured directly, and $\phi^{(m)}(\omega_j, \Delta t_0)$ is

the phase, extracted from the interferometric measurement and the in-situ measurement, discussed above. A similar process is performed on the combined field.

The field's distribution in time for the 3 spectral regions is given in main text figure 4. All regions show an emission event every half cycle. However, the signal-driven ($E_{2N\pm\delta_s}$) and idler-driven ($E_{2N\pm\delta_i}$) fractional harmonics do not individually exhibit structured attosecond-scale noise, because each field has a random phase. By contrast, the combined field, $E_{(2N\pm\delta_s),(2N\pm\delta_i)}$, displays attosecond-scale structure in its fluctuations, arising from phase and amplitude correlations between the signal and idler components.

B. Attochirp extraction from the oscillation phase of the even harmonics

In the in situ scheme, a weak second harmonic field is added to the strong fundamental driving field, introducing a small perturbation to both the electron trajectories and the tunneling probability. By breaking the symmetry between consecutive half cycles, the perturbation gives rise to even harmonics. Scanning the relative delay between the two fields modulates the even-harmonic signal, mapping the sub-cycle perturbation onto delay-dependent oscillations. It has been established that the phase of these oscillations directly encodes the electron recollision time and therefore the harmonic spectral phase (attochirp). This approach has been successfully applied to gases²⁰, liquids²¹, and solids²², revealing the underlying strong-field dynamics and the spectral phase of HHG.

Here we extend this concept by replacing the deterministic SH perturbation with a TMBSV. The TMBSV acts as a weak perturbation of the electron trajectories while simultaneously imprinting quantum correlations onto the emitted fractional harmonics. Consequently, the delay-dependent even-harmonic oscillations carry two independent pieces of information. Their contrast reveals the phase correlations between the fractional harmonics, whereas their oscillation phase encodes the deterministic spectral phase of the emitted field. Figure 5a shows the isolated even-harmonic oscillations. Their systematic phase shift with harmonic order directly reflects the harmonic attochirp. We compare the measured oscillation phases with numerical solutions of the strong-field approximation for krypton under the same multicolour driving conditions used in the experiment. The ionization potential is corrected to account for the Coulomb interaction²³, and the peak intensity is inferred from the HHG cutoff at approximately harmonic 25. We consider the short-trajectory solutions, which provide the dominant contribution to the experimental signal. Figure 5b presents the experimentally resolved two-colour phases together with the calculated phases, showing good agreement. This comparison establishes a direct mapping between the measured oscillation phases

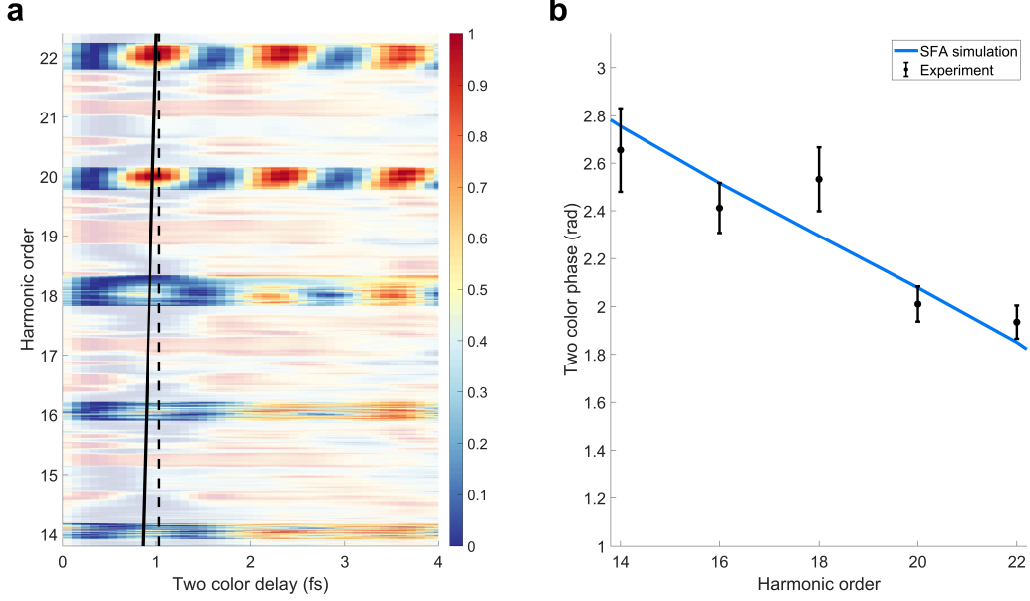


FIG. 5. **Mean spectral phase extraction** (a) Oscillations of the even harmonics as a function of the delay between the driving field and the perturbation, Fourier filtered and normalized per optical frequency. A linear line that passes through the maxima of the different harmonics (solid line), and a constant delay (dashed line) are given for reference. (b) Experimentally measured phases that maximize the yield of each even harmonic with 95% confidence intervals (black) and simulated phases for the same harmonics (light blue line).

and the spectral phases of the harmonics. We then use the corresponding simulated phases of the fractional harmonics to obtain the complete deterministic spectral phase required for the attosecond-field reconstruction.

C. Randomness of the individual signal and idler phases

Here we show that the individual optical phases of the signal and idler fields generated by nondegenerate parametric down-conversion are strictly random, uniformly distributed over $[0, 2\pi)$. This holds at any parametric gain, and in particular in the high-gain TMBSV regime relevant to our experiment. The pump phase constrains only the sum of the two phases, leaving each of them individually undefined. This underlies the assumption of thermal phase statistics used in the time-domain reconstruction.

The two-mode state generated by nondegenerate parametric down-conversion from the vacuum

is²⁴

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle_s |n\rangle_i, \quad c_n = \frac{e^{in\varphi_p} \tanh^n G}{\cosh G} \quad (46)$$

where $|n\rangle_{s,i}$ are Fock states of the signal and idler modes, φ_p is the pump phase and G is the parametric gain. Equation (46) is the exact solution of the two-mode squeezing interaction at arbitrary G : since photons are created only in signal-idler pairs, the state is confined to equal photon numbers at any gain, and increasing G modifies only the weights c_n . The pump phase enters solely through the relative phase $e^{i\varphi_p}$ acquired with each emitted pair, i.e. through the joint signal-idler coherence.

The state of the signal beam alone is described by the reduced density operator $\rho^{(s)} = \text{Tr}_i(|\Psi\rangle\langle\Psi|)$, whose matrix elements in the Fock basis are

$$\begin{aligned} \rho_{nm}^{(s)} &= \langle n | \rho^{(s)} | m \rangle = \sum_l \langle n, l | \Psi \rangle \langle \Psi | m, l \rangle \\ &= \sum_l c_n c_m^* \delta_{ln} \delta_{lm} = |c_n|^2 \delta_{nm} \\ &= \frac{\tanh^{2n} G}{\cosh^2 G} \delta_{nm} \end{aligned} \quad (47)$$

Equation (47) shows explicitly that

$$\rho_{nm}^{(s)} = 0, \quad n \neq m \quad (48)$$

i.e. all off-diagonal coherences vanish identically after tracing over the idler. The reduced state is therefore thermal, with mean photon number $\bar{n} = \sinh^2 G$, and the pump phase is completely absent from the single-mode density operator.

The phase content of a single-mode state is quantified by the canonical phase distribution^{25,26}

$$P(\theta) = \frac{1}{2\pi} \sum_{n,m} e^{i(n-m)\theta} \rho_{mn}, \quad (49)$$

the probability density for the optical phase $\theta \in [0, 2\pi)$. Separating the diagonal and off-diagonal contributions gives

$$P(\theta) = \frac{1}{2\pi} \sum_n \rho_{nn} + \frac{1}{2\pi} \sum_{n \neq m} e^{i(n-m)\theta} \rho_{mn} \quad (50)$$

From Eq. (47), all off-diagonal matrix elements vanish, $\rho_{mn} = 0$ for $m \neq n$, so the second term is identically zero. The first term evaluates to

$$\frac{1}{2\pi} \sum_n \rho_{nn} = \frac{1}{2\pi} \text{Tr}(\rho) = \frac{1}{2\pi} \quad (51)$$

since $\text{Tr}(\rho) = 1$. Therefore

$$P^{(s)}(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta < 2\pi \quad (52)$$

independent of both θ and the parametric gain G . The signal field therefore possesses no preferred optical phase. By symmetry, the same result holds for the idler field. We emphasize that this uniformity is complete, in contrast with *degenerate* bright squeezed vacuum, where both photons of a pair occupy a single mode, the moment $\langle a^2 \rangle = -e^{i\varphi_p} \sinh G \cosh G$ does not vanish, and the phase is locked to the pump up to a random sign, as observed experimentally⁵. The derivation relies only on the pairwise creation process, which confines the state to the equal-photon-number subspace of Eq. (46), a property that holds exactly at arbitrary gain. The result therefore applies equally in the low and high gain regimes. We note that linear absorption does not alter this result. Namely, this result holds also in the lossy high gain regime.

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