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Structure-motivated phantom-crossing dark energy: joint constraints from DESI DR2, supernovae and growth

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ABSTRACT

The observed dark-energy density can be written as $\epsilon_{\text{DE}} = K\langle a_p^2 \rangle / G$ with $K \approx 4$, where $\langle a_p^2 \rangle^{1/2} \approx 10^{-10} \text{ m s}^{-2}$ is a provisional, simulation-based estimate of the mass-weighted rms dark-matter particle acceleration. We study the background law $\rho_{\text{DE}} \propto \langle a_p^2 \rangle (D/D_0)^n$, with D the linear growth factor, motivated by a self-tuned q -theory vacuum carrying a postulated structural interaction energy. Matching the linear bulk flow to the nonlinear virial flow at their crossover, $a_* \simeq 0.64$, gives a halo acceleration $a_{c0} \simeq 0.16 cH_0$, close to Milgrom’s a_0 . The cosmic coincidence then requires only K of order a few, and three epochs follow one another closely: the crossover, the crossing of $w = -1$ and dark-energy–matter equality, $a_* < a_x < a_\Lambda$. In its scale-free limit the law becomes a power-law density driven by a steady cascade, $\rho_{\text{DE}} \propto a^{-nf} D^n$, equivalent to an equation of state linear in the growth rate $f \equiv d \ln D / d \ln a$, $w = -1 - \frac{n}{3}(f - f_x)$; this w stays bounded and crosses -1 where $f = f_x$. Against DESI DR2 BAO, the DESI compressed CMB likelihood, Pantheon+ supernovae and SDSS growth rates, the structural law improves on Λ CDM by $\Delta\chi^2 = -8.4$ to -9.7 and on CPL by 0.6 – 1.9 . The linear- $w(f)$ model fits like CPL, yet both structure-growth-tied families have higher Bayesian evidence than CPL and better predict supernovae withheld from the fit. The structural law differs from the linear- $w(f)$ model only through its halo-assembly slope, which curves $w(f)$ and improves the Pantheon+ fit; most of its empirical content is therefore a power-law response to growth near the crossing. The data constrain the history rather than its decomposition: 97–100% of posterior histories have $w_0 > -1$ and a single density maximum at $z \simeq 0.4$ – 0.8 , whereas n is degenerate with the acceleration clock and the assembly timing. The coupling is postulated, and all results are conditional on the compressed CMB likelihood and the acceleration calibration.

Key words: dark energy – dark matter – cosmology: theory – cosmological parameters – large-scale structure

1 INTRODUCTION

The accelerated expansion of the Universe (Riess *et al.* 1998; Perlmutter *et al.* 1999), confirmed by the CMB (Planck Collaboration *et al.* 2020) and BAO (DESI Collaboration *et al.* 2025a,b), is described in the standard model by a cosmological constant with $\epsilon_{\text{DE}} \simeq 5 \times 10^{-10} \text{ J m}^{-3}$. This leaves two problems open: why the vacuum energy is so small, the cosmological-constant problem (Zel’dovich 1968; Weinberg 1989; Martin 2012), and why it is comparable to the matter density today, the coincidence problem (Zlatev *et al.* 1999; Velten *et al.* 2014). The data now also question whether ϵ_{DE} is constant. DESI DR2 BAO combined with the CMB prefers the CPL form $w = w_0 + w_a(1 - a)$ (Chevallier and Polarski 2001; Linder 2003) at 3.1σ , and at 2.8 – 4.2σ with supernovae, with $w_0 > -1$ and $w_a < 0$ (DESI Collaboration *et al.* 2025b). The preferred solutions cross the phantom divide, which a single minimally coupled scalar with Lagrangian $p(\phi, X)$ cannot do smoothly (Vikman 2005), but broader theories can (Cremellini *et al.* 2009; Deffayet *et al.* 2010).

Many explanations have been proposed. Each reproduces part of the phenomenology, but none fixes both the amplitude and the time dependence of ϵ_{DE} from known physics. Among vacuum-energy approaches, q -theory describes the vacuum as a self-sustained medium carrying a conserved variable q , giving a gravitating vacuum energy $\tilde{\epsilon} = \epsilon - q d\epsilon/dq$ that vanishes in equilibrium. Its main merit is that Planck-scale vacuum energy need not gravitate (Klinkhamer and Volovik 2008a, 2010; Klinkhamer *et al.* 2017; Volovik 2006). Later work added dynamics, propagating q -fields and relaxation towards equilibrium (Klinkhamer and Volovik 2008b, 2011, 2017), but the equilibrium relation is not an evolution equation. A nonzero residual needs additional dynamics, an interaction energy, or a departure from equilibrium, and q -theory alone specifies neither its amplitude nor its response to the late Universe.

We seek this missing response in late-time structure, which becomes important at the same epoch as dark energy. Related proposals let structure formation itself set the timing of the acceleration: stochastic, emergent dark energy in the η CDM framework (Lapi *et al.* 2025), whose viability has been questioned (Adamek 2025), and backreaction and light-

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propagation effects of the cosmic web, proposed to explain the DESI signal (Ginat and Ferreira 2026) and tested in relativistic simulations (Macpherson and Valogiannis 2026). In these, inhomogeneities mimic dark energy. Here, instead, a separately conserved, minimally coupled dark-energy component responds to the formation and evolution of structure.

The most direct structural interpretations fail. Ordinary weak-field peculiar kinetic and binding energies are too small: $\frac{3}{2}\rho_m u_0^2 \approx 4 \times 10^{-16} \text{ J m}^{-3}$ lies a factor $\sim (u_0/c)^2 \sim 10^{-6}$ below ϵ_{DE} , where $u_0 \simeq 330 \text{ km s}^{-1}$ is the one-component virial dispersion of halo particles today. Nor can they drive acceleration in combination: kinetic and potential terms exchange energy through the Layzer–Irvine equation (Layzer 1963; Irvine 1961; Peebles 1980), and their combined effective pressure vanishes in virial equilibrium at leading order.

Structure nevertheless carries a relevant scale, set by accelerations rather than energies. Galaxy dynamics exhibits an acceleration scale $a_0 \approx 1.2 \times 10^{-10} \text{ m s}^{-2}$ (Milgrom 1983; McGaugh *et al.* 2000, 2016; Lelli *et al.* 2019), numerically close to $cH_0/2\pi$ (Milgrom 1999) and to $2\pi G\Sigma_0$ for the universal halo surface density Σ_0 (Donato *et al.* 2009; Milgrom 2009). Without a mechanism linking galactic and cosmic scales, these proximities remain coincidences. Emergent gravity supplies one such link: treating dark energy as an elastic medium whose entropy is displaced by baryons, it recovers $a_0 \simeq cH_0/6$ and an apparent dark-matter component without particle dark matter (Verlinde 2017).

We take the opposite route and retain particle dark matter, whose collisionless flow carries its own critical acceleration (Xu 2025a). Dark-matter particles are continuously accelerated by halo-scale gravitational forces, and we characterize this acceleration state by the mass-weighted mean square of the acceleration vector, $\langle a_p^2 \rangle \equiv \langle |\mathbf{a}|^2 \rangle_M$. Numerically,

$$\epsilon_{\text{DE}} \simeq K \frac{\langle a_p^2 \rangle}{G} \Phi\left(\frac{D}{D_0}\right), \quad \Omega_{\text{DE}} = \frac{8\pi K}{3} \frac{\langle a_p^2 \rangle}{(cH)^2} \Phi\left(\frac{D}{D_0}\right), \quad (1)$$

where $K \approx 4$, $\Phi(1) = 1$, and $D(a)$ is the linear growth factor. We assume the simplest form, $\Phi = (D/D_0)^n$, and fit n . Today $\epsilon_{\text{DE}} \simeq 5.3 \times 10^{-10} \text{ J m}^{-3}$ and $\langle a_p^2 \rangle^{1/2} \simeq 0.93 \times 10^{-10} \text{ m s}^{-2}$.

As a phenomenological law, relation (1) supplies what q -theory lacks. Its scale is set by $\langle a_p^2 \rangle/G$ rather than by a free constant, leaving a fitted coefficient K of order a few, and its time dependence follows structure growth, so that dark energy becomes dynamically relevant only once structures form. If it holds, the cosmic coincidence, $\Omega_{\text{DE}} \sim \Omega_m$, and the galactic coincidence, $a_0 \sim cH_0$, refer to the same epoch (Section 4). Because the mean acceleration vanishes, Eq. (1) is driven by the acceleration variance, multiplied by a configuration factor, a structure resembling the Kapitza pendulum, held in dynamic equilibrium by rapid driving (Appendix A).

Coupling to structure would complement vacuum self-tuning: the bulk vacuum energy cancels, and its late-time residual tracks the dark-matter acceleration field. The coupling then acts both ways. Structure sources dark energy through the competition between decaying $\langle a_p^2 \rangle$ and rising D , and dark energy in turn suppresses growth. For some clocks and positive n , the residual can rise while structure forms and decline once growth slows; demonstrating a regulating feedback, however, requires coupled dynamics not supplied here. Moreover, an evolving residual cannot be both exactly vacuum-like and separately conserved: with minimally coupled matter, additional dynamics must carry its enthalpy.

We therefore study the background law $\rho_{\text{DE}} \propto \langle a_p^2 \rangle D^n$ and keep three questions separate:

- (E) *Empirical*: does the law fit the expansion history?
- (M) *Mechanism*: does structure source the dark-energy dynamics, producing the dependence on $\langle a_p^2 \rangle$ and D ?
- (R) *Realization*: does that background admit a dynamical theory with healthy linear perturbations?

We address (E) conditionally, fitting the law to DESI DR2 BAO with the DESI compressed CMB likelihood, alone and jointly with Pantheon+ supernovae and SDSS growth rates; the fits improve on CPL by 0.6–1.9 in χ^2 at equal parameter count. For (M) we give only a background-level effective model in q -theory (Section 6): a self-tuned vacuum carrying a postulated structural interaction reproduces the law, and its positive residual is the explicit interaction energy, not an effect of vacuum displacement. The interaction is postulated, the strain D/D_0 is prescribed, the enthalpy is assigned to a carrier sector, and a covariant source remains to be constructed. For (R), Horndeski theories with kinetic braiding can cross the phantom divide without ghost or gradient instabilities (Defay et al. 2010); constructing such a realization for the fitted backgrounds is left for future work.

Finally, with a constant structure cascade rate and an acceleration–dispersion relation, matching the velocity of the linear bulk flow to that of the nonlinear virial flow in halos at the crossover a_* gives a halo acceleration of the observed order, $a_{c0} \sim 0.16 cH_0$, comparable to Milgrom’s $a_0 \simeq 0.18 cH_0$. If the density law (1) holds, the linear–nonlinear crossover a_* , the phantom crossing $a_\times$ and the cosmic coincidence a_Λ fall in the same epoch, ordered as $a_* \lesssim a_\times < a_\Lambda$.

Section 2 introduces the $x_0 x_a$ parametrization, with $w(f)$ linear in the growth rate f , together with the power-law dark energy density, and its dynamic interpretation. Section 3 presents the simulated acceleration statistics and their calibration, relates them to the Milgrom scale a_0 , and derives the relations among a_* , $a_\times$ and a_Λ , detailed in Section 4. Section 5 describes the steady cascade and its self-similarity, which motivate the power-law source. Section 6 defines the response law and its conservation requirements. Section 7 sets out the model, priors and inference, and Sections 8 and 9 test the law against BAO alone and the joint data. Section 10 discusses the results, and Section 11 concludes.

2 AN $x_0 x_a$ MODEL AND POWER-LAW DENSITY

CPL makes w linear in the scale factor, $w = w_0 + w_a(1-a)$. A structure-motivated alternative makes it linear in the growth rate $f \equiv d \ln D / d \ln a$, which lies in $0 < f \leq 1$ and is the natural independent variable for a structure-driven model:

$$w = x_0 + x_a(1-f). \quad (2)$$

Because f is a nonlinear function of a , a history curved in a may become linear in f . The $x_0 x_a$ parametrization also clarifies the density law, Eq. (1).

Linear $w(f)$. With $s \equiv d \ln \rho_{\text{DE}} / d \ln a = -3(1+w)$, we write the parametrization of phantom-crossing dark energy in terms of the crossing point $f_\times$, where $w = -1$, and the slope n , setting $x_a = n/3$ and $x_0 = -1 + n(f_\times - 1)/3$:

$$w = -1 - \frac{n}{3}(f - f_\times) \iff s = n(f - f_\times). \quad (3)$$

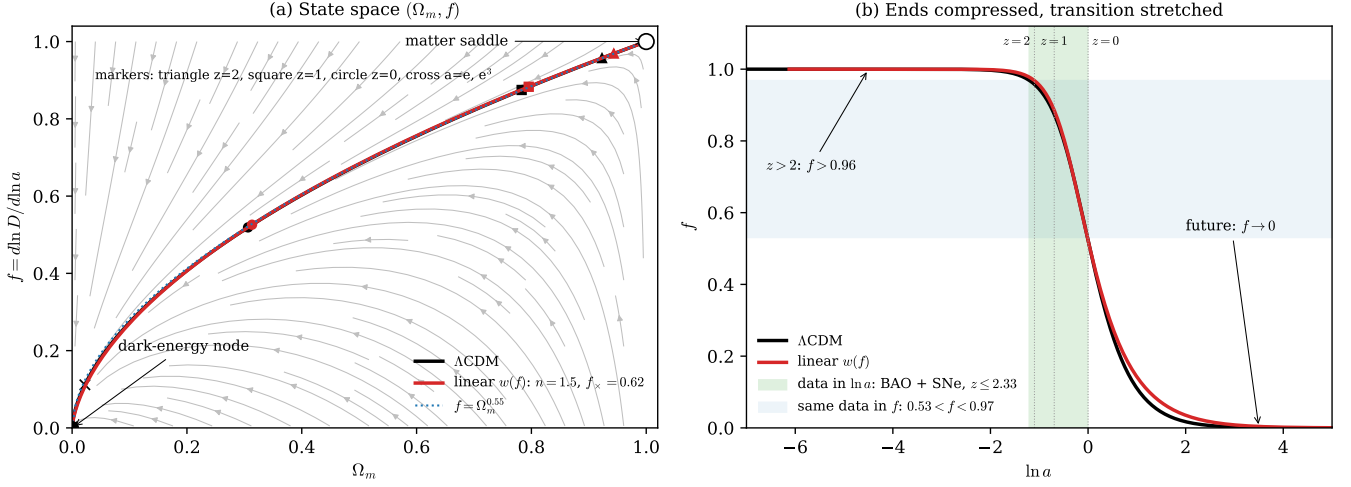


Figure 1. The growth rate f and Ω_m as state variables and radiation neglected. (a) Flow of Eq. (6) for Λ CDM (grey streamlines), with the trajectories of Λ CDM (black) and of a linear $w(f)$ with $n = 1.5$, $f_\times = 0.62$ (red) from the matter saddle, $(1, 1)$, to the dark-energy fixed point, $(0, 0)$; markers show $z = 2, 1, 0$ and the future epochs $a = e$ and e^3 . Both lie close to $f = \Omega_m^{0.55}$ (dotted). (b) The same histories against $\ln a$: the data, $z \leq 2.33$, occupy a narrow interval in $\ln a$ (green) but $0.53 < f < 0.97$ in f (blue), while $z > 2$ is compressed into $f > 0.96$ and the future into $f < 0.52$. The map $a \mapsto f$ compresses both ends of the history and stretches the transition between them.

We use $(n, f_\times)$ from now on. Since $f = d \ln D / d \ln a$, integrating s shows that a w linear in f on an interval, equivalently $d^2 w / df^2 = 0$ there, is a power-law density on that interval,

$$\rho_{\text{DE}} \propto a^{-n f_\times} D^n \iff \mathcal{I}_{\text{DE}} \equiv \rho_{\text{DE}} a^{n f_\times} D^{-n} = \text{const.} \quad (4)$$

Linearity in f thus means a constant invariant $\mathcal{I}_{\text{DE}}$, set by a balance between the factor $a^{-n f_\times}$, which decays for $n f_\times > 0$ and plays the role of the drive, and the structure-enhancing factor D^n , the configuration. CPL has no such interval: with primes denoting $d/d \ln a$, its curvature is

$$\frac{d^2 w}{df^2} = w_a a \frac{f'' - f'}{f'^3}, \quad (5)$$

which for $w_a \neq 0$ vanishes only where $f'' = f'$, the inflection point of $f(a)$, so its density cannot be a power law on any interval. Conversely, a w linear in f with $n \neq 0$ cannot be linear in a on any interval, since $d^2 w / da^2 = -(n/3) a^{-2} (f'' - f')$ vanishes only at the same point.

The growth rate f as a state variable. For a flat universe in general relativity containing pressureless matter and a separately conserved, smooth dark energy, with radiation neglected and $x \equiv \ln a$, the background Ω_m of matter and linear sub-horizon growth rate f obey

$$\begin{aligned} \frac{d\Omega_m}{dx} &= 3w\Omega_m(1 - \Omega_m), \\ \frac{df}{dx} &= -f^2 - \left(\frac{1}{2} - \frac{3}{2}w(1 - \Omega_m)\right)f + \frac{3}{2}\Omega_m. \end{aligned} \quad (6)$$

These equations are exact under the stated assumptions. Only the fit $f \simeq \Omega_m^\gamma$ is approximate, with $\gamma = 6/11$ for $w = -1$ and $\gamma \simeq 0.55$ for smooth dark energy in general relativity (Linder 2005; Calderon *et al.* 2020). Dynamical-system analyses in (Ω_m, f) or (Ω_m, γ) identify the physical solution as the heteroclinic orbit from a matter saddle at $\Omega_m = 1$ to a dark-energy attractor at $\Omega_m = 0$ (Calderon *et al.* 2020; Sharma and Sur 2023).

Equation (6) gives two equations for three functions, Ω_m , f and w , and therefore needs a closure for w . A w that depends explicitly on a , as in CPL, makes the system non-autonomous. In such closures the expansion drives the growth, but the growth does not act back on it. The first equation can be integrate separately once $w(a)$ is fixed. We instead close the system with a w linear in f , which makes $d\Omega_m/dx$ depend on f , so that expansion and growth form a genuinely coupled system at the phenomenological level. This closure is also universal: at fixed $(n, f_\times)$, w is the same function for every solution, whatever its initial conditions or present matter density. The vector field is therefore time-translation invariant: if $(\Omega_m(x), f(x))$ is a solution, so is $(\Omega_m(x + C), f(x + C))$ for any constant C .

With the past and future limits $w_{-\infty} \equiv w(1) = -1 - \frac{n}{3}(1 - f_\times)$ and $w_\infty \equiv w(0) = -1 + \frac{n}{3}f_\times$, the eigenvalues at the matter point, $(\Omega_m, f) = (1, 1)$, are $-3w_{-\infty}$ and $-\frac{5}{2}$, so it is a saddle for $w_{-\infty} < 0$; at the dark-energy point, $(0, 0)$, they are $3w_\infty$ and $\frac{1}{2}(3w_\infty - 1)$, so it is an attractor for $w_\infty < 0$. The trajectory thus runs from the matter saddle to the dark-energy attractor, which accelerates for $w_\infty < -1/3$ and is de Sitter only for $n f_\times = 0$ (Fig. 1a).

On this orbit f labels the state rather than the time: it is invariant under shifts of x and under $D \rightarrow \mu D$, and near the matter saddle $1 - f \simeq \gamma_{-\infty} \Omega_{\text{DE}}$, so the distance from $f = 1$ measures the dynamical importance of dark energy. Near either fixed point the universe lingers in time but barely moves in state, so the map $a \mapsto f$ compresses both ends of the history and stretches the transition between them (Fig. 1b): all of $z > 2$ lies in $f > 0.96$, the infinite future in $0 < f < f_0 \simeq 0.52$, and the BAO and supernova range, $z \leq 2.33$, in $0.53 < f < 0.97$. A w linear in f therefore remains bounded over the whole history, between $w(1)$ and $w(0)$, whereas CPL diverges as $a \rightarrow \infty$.

Why a power law: intermediate asymptotics. Let the density depend on the expansion, on the structure amplitude, and on any characteristic scales the physics contains, such as the onset D_* of statistically steady, self-similar structure formation (Section 5) and the saturation scale D_Λ ,

$$\rho_{\text{DE}} = a^{-nf} D^n \Phi_{\text{DE}}\left(\frac{D}{D_*}, \frac{D}{D_\Lambda}\right). \quad (7)$$

The power law follows from two symmetries. Autonomy in $x = \ln a$ is invariance under $a \rightarrow \lambda a$, so a response with no preferred epoch is homogeneous in a ; the linear growth equation is invariant under $D \rightarrow \mu D$, so a response with no preferred amplitude is homogeneous in D . By Euler's theorem in both variables, ρ_{DE} is then a product of powers of a and D , the form of Eq. (4). CPL breaks the first symmetry by privileging $a = 1$, while D_* and D_Λ break them at the ends of the interval: for $D_* < D < D_\Lambda$, Φ_{DE} tends to a constant and the curvature of $w(f)$ is suppressed. The physical origin is the self-similarity of gravitational clustering, exact for an Einstein-de Sitter universe with a power-law spectrum and broken by the curvature of the spectrum, which sets D_* , and by dark energy, which sets D_Λ . Kolmogorov's inertial range in turbulence has the same structure: with $\ln k$ in the role of $\ln a$, scale invariance and a constant energy flux between the injection scale L and the dissipation scale η_K force $E(k) \propto k^{-5/3}$, as the constant cascade rate ε_u does across halo masses here (Eq. (21)). One difference matters: Kolmogorov's exponent follows from dimensional analysis, whereas here only the amplitude, $\epsilon_{\text{DE}} \sim \langle a_p^2 \rangle / G$, does, and n is an exponent that must be computed from the dynamics or fitted (Eq. (1)).

Why an invariant. Invariants of the form $\mathcal{I}_{\text{DE}}$ arise in two ways. The first is a conservation law, as in freely decaying turbulence, where the energy is tied to the integral scale $L(t)$ by a large-scale invariant, $\mathcal{I} = u'^2 L^3$ (Saffman) or $u'^2 L^5$ (Loitsyansky), following from conservation of linear and angular momentum, so that the total energy

$$\frac{d \ln \mathcal{I}}{d \ln t} = -(\sigma + 1) \frac{d \ln L}{d \ln t}, \quad \sigma + 1 = 3 \text{ or } 5. \quad (8)$$

The analogy maps the largest eddy scale L to D , the eddy growth rate $d \ln L / d \ln t$ to f , the exponent $-(\sigma + 1)$ to n , and $\mathcal{I}$ to $\mathcal{I}_{\text{DE}}$. The second is attractor slaving. Tracker quintessence converges to a scale-free attractor on which $\rho_\phi \propto \rho_B^\kappa$ exactly (Ratra and Peebles 1988; Steinhardt *et al.* 1999), so $\rho_\phi \rho_B^{-\kappa}$ is an invariant of the attractor rather than of a conservation law: its value is set by the dynamics, and the initial state is forgotten. Within the linear- $w(f)$ closure, $\mathcal{I}_{\text{DE}}$ is instead an exact integral of the first Eq. (6), a constant of motion whose value is fixed by the initial conditions,

$$\mathcal{I}_{\text{DE}} \equiv \rho_{\text{DE}} a^{nf} D^{-n} = C \rho_{m0} \Rightarrow C = \frac{\rho_{\text{DE}0}}{\rho_{m0}} D(1)^{-n}. \quad (9)$$

Conditions. A density that responds to structure satisfies Eq. (4) on an interval if three conditions hold. First, slaving: the dark energy follows its instantaneous equilibrium faster than that equilibrium moves. Second, scale-free dependence: no characteristic amplitude lies within the interval. Third, stability: deviations from the power-law attractor decay, so that the history is independent of initial conditions. In the model of Section 6, slaving follows from the stiffness of the

vacuum, $\omega_{\text{vac}} \gg H$, with a lag $\sim H^2 / \omega_{\text{vac}}^2$, so that the residual is a state function of the current configuration. Scale-free dependence holds during the steady cascade and self-similarity of the collisionless flow. Stability holds only weakly: a misalignment decays in amplitude as $a^{-3/2}$, but its energy dilutes like matter rather than being dissipated, so it must start small. The density law, Eq. (1), has this power-law form during the steady cascade, before dark energy dominates. We start with the driving: dark matter acceleration $\langle a_p^2 \rangle$.

3 LINEAR FLOW AND NONLINEAR CASCADE

Velocity and acceleration fluctuations. In a collisional gas, molecules interact only at short range. Between collisions they move freely, so the relevant fluctuations are those of velocity, while accelerations are confined to brief collisions. Self-gravitating dark matter differs because gravity is long-range: every particle is accelerated by all others, and a small displacement of any particle perturbs the acceleration of every other. Dark matter therefore has fluctuations of acceleration as well as of velocity, each with a redshift-dependent distribution (Xu 2023a, 2025a). The rms of the acceleration distribution defines a critical acceleration a_c , the counterpart for accelerations of the velocity dispersion. Throughout, a_c and the other accelerations denote rms magnitudes of the acceleration vector, whereas u denotes the one-component velocity dispersion. Today $a_{c0} \simeq 1.2 \times 10^{-10} \text{ ms}^{-2}$, close to the empirical MOND scale a_0 of the baryonic Tully-Fisher relation (BTFR) (Milgrom 1983; McGaugh *et al.* 2000; Xu 2025a). We illustrate both fluctuations with simulations: the Virgo SCDM (CDM only) (Jenkins *et al.* 1998) and Illustris-1-Dark (Λ CDM) (Vogelsberger *et al.* 2014). Their parameters are listed in Table B1 (Appendix B).

Velocity decomposition. The velocity of a halo particle splits into the mean velocity of its halo and a fluctuation about it, $\mathbf{v}_p = \mathbf{v}_h + \mathbf{v}'_p$. Pointwise, $\frac{1}{2}|\mathbf{v}_p|^2 = \frac{1}{2}v_h^2 + \frac{1}{2}v_p'^2 + \mathbf{v}_h \cdot \mathbf{v}'_p$; the cross term vanishes only after the mass-weighted average over each halo, where $\langle \mathbf{v}'_p \rangle_h = 0$. Hence

$$\langle K_p \rangle_h = K_{ph} + \langle K_{pv} \rangle_h, \quad K_{ph} = \frac{1}{2}v_h^2, \quad K_{pv} = \frac{1}{2}v_p'^2. \quad (10)$$

The halo kinetic energy K_{ph} is the bulk motion of halos, driven by large-scale, linear gravitational interactions. The virial kinetic energy K_{pv} is the random motion inside halos, driven by small-scale nonlinear intra-halo interactions. Writing $V_\parallel$ for one component of the bulk flow and u for the one-component virial dispersion, $\langle K_{ph} \rangle = \frac{3}{2}V_\parallel^2$ and $\langle K_{pv} \rangle = \frac{3}{2}u^2$, where both averages are per unit mass of particles.

Figure 2 shows both energies. In both simulations the bulk energy follows linear theory, $\langle K_{ph} \rangle \propto a$, flattening in Illustris after a_Λ as dark energy suppresses growth, while the virial energy grows as $\langle K_{pv} \rangle \propto t$, the signature of a steady cascade. Because the virial energy grows faster, the two cross, at $a_* \simeq 0.5$ in Illustris-1-Dark and $a_* \simeq 0.89$ in Virgo SCDM. After this bulk-virial crossover, the kinetic energy of halo particles is dominated by random multistream motion rather than by the coherent linear flow. Because the bulk flow is linear and the virial flow is nonlinear, the crossover at a_* also marks the epoch at which the kinetic energy of the dark matter passes from linear to nonlinear domination, and the epoch at which the self-similar inertial range comes to dominate and contain

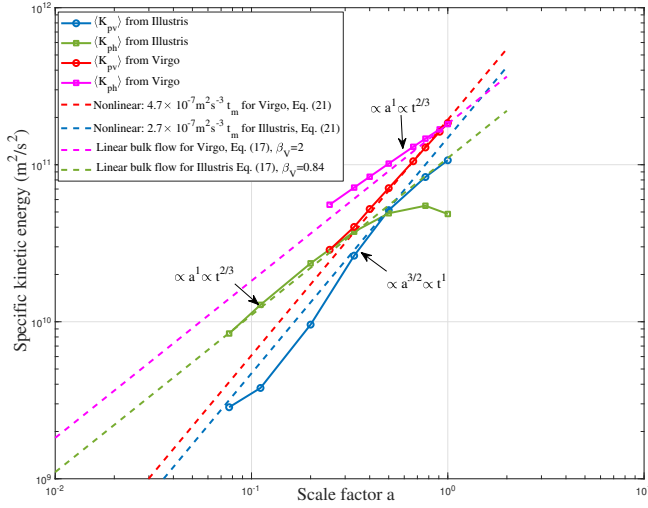


Figure 2. Specific kinetic energy of dark-matter particles versus scale factor for Virgo SCDM and Illustris-1-Dark. Squares: halo bulk energy $\langle K_{ph} \rangle$, which grows as $a \propto t^{2/3}$ in the linear regime. Circles: virial energy $\langle K_{pv} \rangle$, which grows as t in the steady cascade. Straight dashed lines: $\langle K_{pv} \rangle = |\varepsilon_u|t$, Eq. (21), with $|\varepsilon_u| = 4.7 \times 10^{-7}$ (Virgo) and $2.7 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (Illustris). Other dashed lines: linear bulk flow, Eq. (17). The curves cross, marking the bulk–virial crossover, at $a_* \simeq 0.5$ (Illustris) and $a_* \simeq 0.89$ (Virgo).

most of the halo mass, characterized by a steady cascade at a scale-independent rate ε_u (Section 5).

Acceleration decomposition. Accelerations separate in the same way, between halo particles, dominated by intra-halo forces, and out-of-halo particles, which respond to the large-scale field. The natural measure of this acceleration state is the mass-weighted mean square $\langle a_p^2 \rangle$ over all dark matter. As a particle average it is intensive, independent of system size, and global, depending only on time, which makes it a suitable source for the homogeneous background. With halo mass fraction f_h ,

$$\langle a_p^2 \rangle = f_h a_c^2 + (1 - f_h) a_{op}^2, \quad (11)$$

where $a_c^2 \equiv \langle |\mathbf{a}|^2 \rangle_{hp}$ and $a_{op}^2 \equiv \langle |\mathbf{a}|^2 \rangle_{op}$ are the mean squares over halo and out-of-halo particles; the identity holds for any partition. The target quantity is the physical peculiar gravitational acceleration, $\mathbf{g} = \dot{\mathbf{v}}_{\text{pec}} + H\mathbf{v}_{\text{pec}} = -a^{-1}\nabla_{\mathbf{x}}\Phi$, in SI units; Appendix C lists the conventions of the simulation measurements. This statistic drives the vacuum in Section 6.

The halo-particle mean square splits exactly into the halo bulk term and the halo-relative fluctuation, $\langle |\mathbf{g}|^2 \rangle_{hp} = \langle |\mathbf{g} - \mathbf{g}_h|^2 \rangle_{hp} + \langle |\mathbf{g}_h|^2 \rangle_{hp}$. We define the critical acceleration a_c as the rms magnitude of the halo-relative fluctuation and identify a_c^2 with the halo-particle mean square in Eq. (11); this is an approximation, accurate because the bulk term, $\sim a_{op}^2$, is four orders of magnitude smaller. From the Virgo SCDM acceleration analysis (Xu 2025a),

$$a_c = a_{c0} a^{-p_c}, \quad p_c \simeq \frac{3}{4}, \quad a_{op} \simeq a_{op0} a^{-1}, \quad (12)$$

with $a_{c0} \simeq 1.2 \times 10^{-10}$ and $a_{op0} \simeq 1.2 \times 10^{-12} \text{ m s}^{-2}$. The out-of-halo scaling is that of linear theory, $g \propto D/a^2$, which reduces to a^{-1} in matter domination. Figure 3 shows these accelerations for Virgo SCDM. Halo-particle accelerations fol-

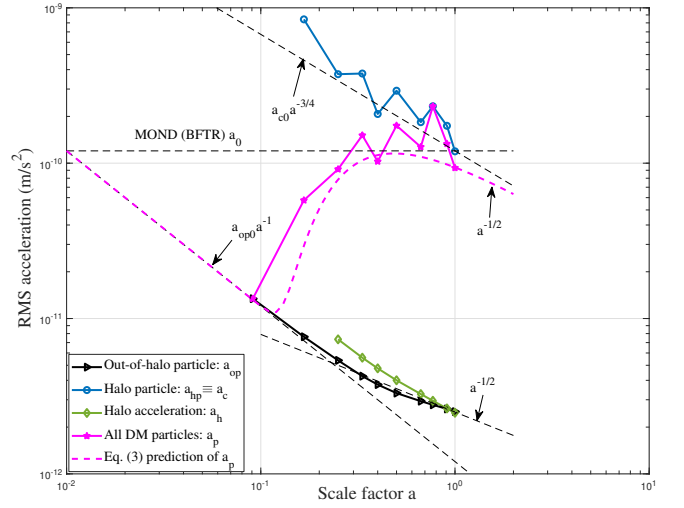


Figure 3. RMS acceleration magnitude of dark-matter particles versus scale factor for Virgo SCDM. Circles: halo particles, $a_{hp} \equiv a_c \propto a^{-3/4}$. Triangles: out-of-halo particles, $a_{op} \propto a^{-1}$ while the field is linear, flattening toward $a^{-1/2}$ as surrounding structure becomes nonlinear. Diamonds: halo bulk acceleration a_h , which matches a_{op} . Stars: all particles, $\langle a_p^2 \rangle^{1/2}$. Black dashed: $a_{c0}a^{-3/4}$. Horizontal: the Tully–Fisher (MOND) scale a_0 . Magenta dashed: Eq. (11) with the halo fraction of Fig. 4. It follows a_{op} early, approaches $f_h^{1/2}a_c$ once halos form, and then falls as $a^{-1/2}$ once $f_h \propto a^{1/2}$.

low $a_{c0}a^{-3/4}$ and exceed out-of-halo accelerations by about two orders of magnitude. The acceleration of halos as a whole matches that of out-of-halo particles, just as the halo bulk velocity matches the out-of-halo velocity. The all-particle acceleration $a_p \equiv \langle a_p^2 \rangle^{1/2}$ therefore follows a_{op} at early times and approaches $f_h^{1/2}a_c$ as halos form, so that $\langle a_p^2 \rangle \simeq f_h a_c^2$ at late times. Once the cascade is steady, $f_h \propto a^{1/2}$ (Fig. 4), and hence $\langle a_p^2 \rangle \propto a^{1/2}a^{-3/2} = a^{-1}$, i.e. $a_p \propto a^{-1/2}$, which provides the decaying driving in power-law density (1).

Halo mass fraction. Figure 4 shows the halo mass fraction. At late times both simulations approach $f_h \propto a^{1/2}$, after reaching a statistically steady state with a scale-independent cascade. We describe the full evolution with the excursion-set collapsed fraction (Bond *et al.* 1991), with an amplitude β_f :

$$f_h(a) = \beta_f \operatorname{erfc} \left[\frac{\delta_{\text{col}}}{\sqrt{2} \sigma_{\text{min}}(D(a)/D(1))} \right], \quad (13)$$

where $\delta_{\text{col}} = 1.686$ is the spherical-collapse threshold, σ_{min} is the rms linear density fluctuation today at the minimum halo mass, and $D(1) \equiv D_0$ is the present growth factor of the cosmology in question, so that only $D(a)/D(1)$ enters. Excursion-set theory, like Press–Schechter theory with its factor of two (Press and Schechter 1974), gives $\beta_f = 1$. Fitted in $\ln f_h$, Virgo SCDM gives $\beta_f = 0.97$ and $\sigma_{\text{min}} = 3.51$, consistent with $\beta_f = 1$, and Illustris-1-Dark gives $\beta_f = 0.68$ and $\sigma_{\text{min}} = 5.73$. In Illustris, $\beta_f < 1$ is an effective correction: its lower minimum mass reaches the regime where the excursion-set form is known to overpredict halo abundance. Section 7 fixes $f_h(1) = 0.6$, between the two simulations, and fits σ_{min} to the observational data, with and without the bound $\beta_f \leq 1$, which requires $\sigma_{\text{min}} \geq 3.215$.

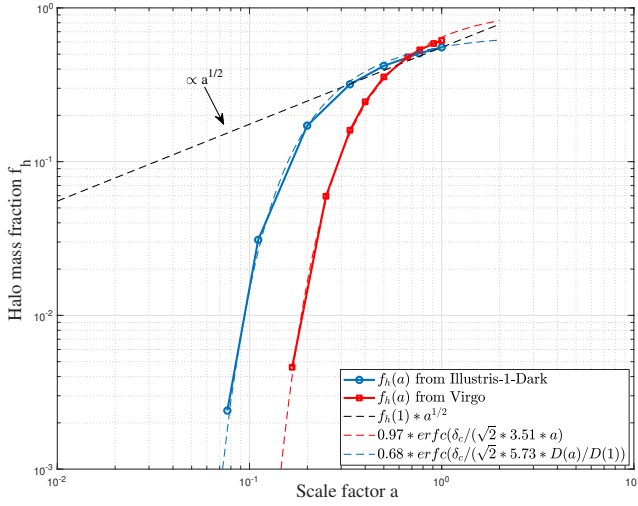


Figure 4. Halo mass fraction versus scale factor for Illustris-1-Dark (circles) and Virgo SCDM (squares). Black dashed: $f_h(1) a^{1/2}$ (Xu 2021), which both simulations approach at steady cascade ($a > a_*$). Blue and red dashed: best fits of Eq. (13).

We now derive both regimes analytically, the large-scale linear bulk flow and the small-scale nonlinear virial motion inside halos, for comparison with the simulations.

Linear regime: bulk flow. We evaluate the bulk flow in matter domination, using matter-era time and Hubble rate

$$t_m = \frac{2}{3} \frac{a^{3/2}}{H_0 \Omega_m^{1/2}}, \quad H_m = H_0 \Omega_m^{1/2} a^{-3/2}, \quad (14)$$

with growth rate $f = 1$ and $D \propto a$. For the growing mode the peculiar velocity is $v = afH\Psi_{\parallel}$, with $\Psi_{\parallel}$ the comoving displacement. The three-dimensional rms velocity is then (Strauss and Willick 1995; Xu 2025b)

$$V^2(a) = \frac{(aH_m)^2}{2\pi^2} \int_0^\infty P(k, a) dk. \quad (15)$$

The integral is dominated by the peak of $P(k)$ at the equality scale. Since $H^2 = 2H_m^2$ at equality, and modes entering then grow as $D \propto a$ thereafter,

$$k_{\text{eq}} = \frac{H_0}{c} \left(\frac{2\Omega_m}{a_{\text{eq}}} \right)^{1/2}, \quad P(k_{\text{eq}}, a) \simeq \frac{2\pi^2 A_s}{k_{\text{eq}}^3} \left(\frac{a}{a_{\text{eq}}} \right)^2, \quad (16)$$

with $a_{\text{eq}} = \Omega_r/\Omega_m$ and the scalar amplitude of the primordial power spectrum A_s . Taking $\int P dk \simeq \beta_V^2 P(k_{\text{eq}}, a) k_{\text{eq}}$, all dependence on H_0 and Ω_m cancels. One velocity component, and the corresponding bulk energy, are

$$V_{\parallel}(a) = V_{\text{eq}} \left(\frac{a}{a_{\text{eq}}} \right)^{1/2}, \quad V_{\text{eq}} = \beta_V \left(\frac{A_s}{6} \right)^{1/2} c, \quad (17)$$

$$\langle K_{ph} \rangle = \frac{3}{2} V_{\parallel}^2 = \frac{1}{4} \beta_V^2 A_s c^2 \frac{a}{a_{\text{eq}}}.$$

The factor $\beta_V = O(1)$ absorbs what the peak approximation omits: the width of the spectrum and the transfer function. It is calibrated, not predicted: Figure 2 requires $\beta_V \simeq 0.84$ for Illustris-1-Dark and $\beta_V \simeq 2$ for Virgo SCDM. Both values are affected by the finite boxes, which truncate the long-wavelength modes. For single-field slow-roll inflation, with

Hubble rate H_I and slow-roll parameter ϵ_{sr} ,

$$A_s = \frac{H_I^2}{8\pi^2 \epsilon_{\text{sr}} M_{\text{pl}}^2} \Rightarrow V_{\text{eq}} = \beta_V \frac{H_I}{4\pi M_{\text{pl}}} \cdot \frac{c}{\sqrt{3\epsilon_{\text{sr}}}}, \quad (18)$$

the speed of light enters V_{eq} through the equality horizon, $k_{\text{eq}}^{-1} \propto c$, and the magnitude through inflation H_I ; the equality value V_{eq} depends only on A_s . The bulk flow velocity $V_{\parallel} = V_{\text{eq}}$ is constant and independent of the dark-matter particle mass in the radiation era (Xu 2025b). Equation (17) holds for $a_{\text{eq}} \ll a < a_{\Lambda}$. Thereafter dark energy suppresses the bulk flow $V_{\parallel}$ by $S(a) = f g(a)/\Omega_m(a)^{1/2} \simeq \Omega_m(a)^{0.05} g(a)$, with $g \equiv D(a)/a$ and approximation

$$g \simeq \frac{\frac{5}{2} \Omega_m(a)}{\Omega_m(a)^{\frac{4}{7}} - \Omega_{\Lambda}(a) + [1 + \frac{1}{2} \Omega_m(a)][1 + \frac{1}{70} \Omega_{\Lambda}(a)]}. \quad (19)$$

With $f \simeq \Omega_m^{0.55}$ and, for $\Omega_m \simeq 0.31$, $g_0 \equiv g(a=1) \simeq 0.787$ (Carroll et al. 1992), $S_0 \simeq 0.74$ today (Fig. 2), and $g_{\Lambda} \equiv g(a_{\Lambda}) \simeq 0.873$ at dark-energy-matter equality.

Linear regime: accelerations. The same linear modes set the out-of-halo accelerations. In the matter-era growing mode the peculiar acceleration is parallel to the velocity, $\mathbf{g} = \frac{3}{2} H_m \mathbf{v}$, so its power per $\ln k$ also scales as $kP(k)$ and peaks near k_{eq} . The linear acceleration is therefore coherent over $\sim 25\text{--}40 h^{-1}\text{Mpc}$ comoving (the separations at which $\mathbf{g}$ correlation falls to $1/2$ and $1/e$ for a Planck spectrum), about half the Hubble radius at equality. With $V = \sqrt{3} V_{\parallel}$ and Eq. (17), its three-dimensional rms magnitude is

$$g(a) = g_{\text{eq}} \left(\frac{a}{a_{\text{eq}}} \right)^{-1}, \quad g_{\text{eq}} = \frac{3}{2\sqrt{2}} \beta_V A_s^{1/2} c H_{m,\text{eq}}, \quad (20)$$

$$\frac{g}{cH_m} = \frac{3}{2\sqrt{2}} \beta_V \left(\frac{A_s a}{a_{\text{eq}}} \right)^{1/2},$$

so that $g \propto D/a^2$. For the Planck parameters, $g_{\text{eq}} \simeq 3.5 \beta_V \times 10^{-9} \text{ m s}^{-2}$, falling to $\sim 10^{-12} \text{ m s}^{-2}$ today, of the same order as the simulated out-of-halo acceleration (Fig. 3), although that simulation has a different spectrum. On halo scales this field is nearly uniform, and hence largely removable by a change of frame; this bears on the pre-halo behaviour of the law (Section 6). Today the linear acceleration is only $g \sim 10^{-3} cH_0$, far below the galactic scale $a_0 \simeq 0.18 cH_0$, so a_0 must arise from nonlinear structure (Section 4).

Nonlinear regime: virial motion. Inside halos, motion is multistream and random. Mergers and accretion transfer energy across mass scales at a scale-independent rate $|\varepsilon_u|$ (Xu 2021); because dark matter is collisionless, the transfer proceeds through gravitational merging rather than viscosity in turbulence. The specific virial kinetic energy then grows linearly in cosmic time,

$$\langle K_{pv} \rangle = \frac{3}{2} u^2 = |\varepsilon_u| t, \quad (21)$$

with $\varepsilon_u < 0$ denoting transfer toward larger masses. The simulations give $\varepsilon_u = -4.7 \times 10^{-7}$ (Virgo SCDM) and $-2.7 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ (Illustris-1-Dark); we adopt their mean, $|\varepsilon_u| \simeq 3.7 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$. The law holds in cosmic time, not in a : Illustris-1-Dark gives $\langle K_{pv} \rangle/t = (2.6 \pm 0.1) \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ for $0.33 \leq a \leq 1$, whereas $\langle K_{pv} \rangle/t_m$ falls by 25% because $t/t_m = 0.78$ today, so $u^2 \propto a^{3/2}$ only in matter domination. This constant-rate cascade is the basis of the t -clock of Section 7.

The rate is consistent with the cosmic energy equation (Irvine 1961; Layzer 1963),

$$\frac{\partial E_y}{\partial t} + H(2K_p + P_y) = 0, \quad (22)$$

which expresses exact energy conservation for Newtonian peculiar motion, with K_p and P_y the specific peculiar kinetic and potential energies of all particles and $E_y = K_p + P_y$. It admits a linear solution $K_p \propto a \propto t^{2/3}$ for matter domination (Xu 2023b). If virial motion dominated the kinetic energy of all particles ($a > a_*$), $K_p \simeq \frac{3}{2}u^2$ and effective spectrum index $n_{\text{eff}} = P_y/K_p = -7/5$, the cascade rate would follow as $\varepsilon_u \simeq -K_p/t$; with $u \simeq 300 \text{ km s}^{-1}$ today this gives $|\varepsilon_u| \simeq 3 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, of the order found from the linear–nonlinear decomposition of the simulations. The estimate is approximate, because only a fraction $f_h \simeq 0.6$ of the mass today is in halos and the bulk flow still contributes to K_p .

Nonlinear regime: accelerations. The cascade also links the velocity and acceleration fluctuations (Xu 2025a). Consider a merger at the boundary of a halo of mass $m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho}$, with $\Delta_c = 18\pi^2$ and $H^2 = 8\pi G\bar{\rho}/3$. The circular speed there is $u_{\text{cir}} = Hr_h(\Delta_c/2)^{1/2} = 3\pi Hr_h$, while stable clustering gives an infall speed $u_r = Hr_h$, so the infall angle satisfies $\cos^2 \theta \simeq (u_r/u_{\text{cir}})^2 = 2/\Delta_c$. A merger that starts at rest relative to the halo and reaches u_r after a time τ delivers the specific energy $\frac{1}{2}u_r^2$ at the mean rate

$$|\varepsilon_u| = \frac{u_r^2}{2\tau} = \frac{1}{2} \bar{a}_r u_r, \quad (23)$$

with $\bar{a}_r = u_r/\tau$ its mean radial acceleration, for any infall from rest. Identifying $\bar{a}_r = a_c \cos \theta$ and $u_r = u_c \cos \theta$ with the radial projections of the rms fluctuations gives

$$a_c = \Delta_c \frac{|\varepsilon_u|}{u_c} = \beta H u, \quad \beta \equiv \frac{\sqrt{3}\Delta_c}{2Ht}, \quad (24)$$

where a_c and $u_c = \sqrt{3}u$ are the rms magnitudes of the acceleration and velocity fluctuations of halo particles, and the second form uses $|\varepsilon_u| = u_c^2/2t$. The relation assumes infall from rest and identifies the rms acceleration with the mean infall acceleration; its order-unity accuracy is set by the simulations, which it reproduces: $|\varepsilon_u| = 3.7 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$ and $u_0 = (\frac{2}{3}|\varepsilon_u|t_0)^{1/2} \simeq 330 \text{ km s}^{-1}$ give $a_{c0} \simeq 1.2 \times 10^{-10} \text{ m s}^{-2}$. Thus $\beta \simeq 231$ in matter domination ($Ht = \frac{2}{3}$) and $\beta_0 \simeq 162$ today ($H_0 t_0 = 0.951$). Since $u_c \propto t^{1/2}$, $a_c \propto t^{-1/2}$, which reduces to $a^{-3/4}$ in matter domination, reproducing $p_c = \frac{3}{4}$ in Eq. (12). Equation (24) contains no c since it is in the small-scale nonlinear regime: it recasts the Milgrom coincidence as the question why $u_c/c \sim 10^{-3}$, a question addressed next.

4 ACCELERATION & COSMIC COINCIDENCES

Three coincidences frame this work.

- *Milgrom’s coincidence.* The simulated rms halo-particle acceleration today, $a_{c0} \simeq 1.2 \times 10^{-10} \text{ m s}^{-2} \simeq 0.18 cH_0$, matches the Tully–Fisher and MOND scale a_0 , although a_c arises from Newtonian dark-matter dynamics, in which c does not appear.

- *The cosmic coincidence.* Why is $\rho_{\text{DE}} \sim \rho_m$ today?

- *The phantom-crossing coincidence.* If dark energy evolves, why does the phantom crossing $a_\times$ occur at the present epoch?

We explain the first through the bulk and virial flows, the cascade between them and their crossover. This uses only the cascade rate $|\varepsilon_u|$ and the acceleration–dispersion relation (24), not the dark-energy density law (1). Matching the two flows at the crossover gives $a_{c0} \sim 0.16 cH_0$. If the density law holds, the observed cosmic coincidence corresponds to a coefficient K of order a few. The fitted histories place the phantom crossing between the crossover and the cosmic coincidence, $a_* \lesssim a_\times < a_\Lambda$, as expected for a power-law source driven by a steady cascade.

The bulk–virial crossover. In matter domination, the large-scale linear bulk flow grows as $V_\parallel \propto a^{1/2}$, while the small-scale virial dispersion grows as $u \propto t^{1/2} \propto a^{3/4}$. Since $u/V_\parallel \propto a^{1/4}$, coherent bulk motion dominates early and random motion late (Fig. 2). We define the crossover a_* by $u(a_*) = V_\parallel(a_*)$, equivalently $\langle K_{pv} \rangle = \langle K_{ph} \rangle$, which is the crossing seen in Fig. 2. With Eqs. (17) and (21),

$$a_* = a_{\text{eq}} \left(\frac{3V_{\text{eq}}^2}{2|\varepsilon_u|t_{m,\text{eq}}} \right)^2 = \left[\frac{3}{8} \beta_V^2 \Omega_m^{3/2} \frac{A_s}{\Omega_r} \frac{c^2 H_0}{|\varepsilon_u|} \right]^2, \quad (25)$$

with $t_{m,\text{eq}} = t_m(a_{\text{eq}})$. At the crossover,

$$V_* = \frac{3\beta_V^2 \Omega_m^2}{8\sqrt{6}} \left(\frac{A_s}{\Omega_r} \right)^{3/2} \frac{c^2 H_0}{|\varepsilon_u|} c, \quad (26)$$

$$t_* = \frac{9\beta_V^6 \Omega_m^4}{256 H_0} \left(\frac{A_s}{\Omega_r} \right)^3 \left(\frac{c^2 H_0}{|\varepsilon_u|} \right)^3,$$

and the matching condition can be read in reverse,

$$|\varepsilon_u| = \frac{3}{2} \frac{V_*^2}{t_*} : \quad (27)$$

the matching relates the cascade rate to the linear flow at the crossover. For the Planck spectrum with $\beta_V = 1$ and $|\varepsilon_u| = 3.7 \times 10^{-7} \text{ m}^2 \text{ s}^{-3}$, $a_* \simeq 0.64$, $V_* \simeq 260 \text{ km s}^{-1}$ and $t_* \simeq 8.9 \text{ Gyr}$. Before a_* , linear bulk flow dominates kinetic energy. After a_* , nonlinear flow dominates and a steady cascade is established for the entire system (Section 5).

The Milgrom coincidence. In matter domination the virial dispersion grows from its crossover value as $u = V_*(a/a_*)^{3/4}$, with $V_* = V_{\text{eq}}(a_*/a_{\text{eq}})^{1/2}$ from Eq. (17). Equation (24), with $\beta = 3\sqrt{3}\Delta_c/4$ and $H = H_m$ evaluated consistently ($H_m t_m = \frac{2}{3}$), then gives the halo-particle acceleration in units of the matter-era Hubble acceleration:

$$\gamma_m \equiv \frac{a_c}{cH_m} = \beta\beta_V \left(\frac{A_s}{6} \right)^{1/2} \left(\frac{a}{a_{\text{eq}}} \right)^{1/2} \left(\frac{a}{a_*} \right)^{1/4} \quad (28)$$

$$\simeq 0.252 \beta_V a^{1/2} (a/a_*)^{1/4}.$$

The ratio is a product of three factors: the cascade coefficient β involving $\Delta_c = 18\pi^2$ (Eq. (24)); the primordial amplitude, $A_s \sim 2.1 \times 10^{-9}$; and the equality, $a_{\text{eq}} \sim 1/3400$. The large coefficient β and small a_{eq} offset the small primordial amplitude, which is why a_c/cH is of order 0.1, and the speed of light enters only through the equality horizon (Eq. (16)). For $\beta_V = 1$ and $a_* = 0.64$, Eq. (28) gives $\gamma_* = 0.2$ at the crossover and $\gamma_m = 0.28$ extrapolated to $a = 1$. Dark energy raises the actual Hubble rate above the matter-era value,

$H_0 = H_m(1)/\Omega_m^{1/2}$, so the present ratio is ($\Omega_m \sim 0.314$)

$$\gamma_0 \equiv \frac{a_{c0}}{cH_0} = \Omega_m^{1/2} \gamma_m(1) = 0.14 \beta_V a_*^{-1/4}. \quad (29)$$

This compares with Milgrom's $a_0 \approx 1.2 \times 10^{-10} \text{ m s}^{-2}$, which is $0.18 cH_0$ for $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and close to $cH_0/2\pi = 0.16 cH_0$. For $a_* \simeq 0.64$, the dark-matter acceleration scale and a_0 therefore agree (Eq. (29)). The acceleration scale is thus tied to cH through the equality horizon k_{eq} and the primordial amplitude A_s , amplified by the cascade coefficient β . Equation (28) accounts for the variation of a_c/cH , since $\gamma_m \propto a^{3/4}$ in matter domination. The argument explains the magnitude of a global rms dark-matter acceleration and its proximity to cH_0 ; it does not derive the galactic acceleration relation itself.

The cosmic coincidence. The crossover cannot by itself fix dark-energy–matter equality: a_* depends on A_s and ε_u , whereas the equality epoch a_Λ , defined by $\rho_{DE}(a_\Lambda) = \rho_m(a_\Lambda)$, depends on ρ_{DE} . If dark energy were independent of structure, $a_\Lambda \sim a_*$ would be a coincidence. One direction of influence is clear: dark energy freezes the bulk flow while the virial dispersion keeps growing, so the crossover cannot occur much after dark-energy domination, $a_* \lesssim a_\Lambda$, a dynamical counterpart of the anthropic bound of Weinberg (1987). The opposite direction, of structure on dark energy, is supplied by the density law (1), which we now assume in order to relate the two epochs. With it, the coincidence follows for a coefficient K of order a few.

With $\langle a_p^2 \rangle \simeq f_h a_c^2$ (Eq. (11)) and the matter-era rate $H_m^2 = 8\pi G \rho_m/3$, the density ratio is exactly (Eq. (1))

$$\frac{\rho_{DE}}{\rho_m} = \frac{8\pi K}{3} f_h \left(\frac{D}{D_0} \right)^n \gamma_m^2, \quad \gamma_m \equiv \frac{a_c}{cH_m}. \quad (30)$$

Inserting the matter-era γ_m of Eq. (28),

$$\frac{\rho_{DE}}{\rho_m} = \frac{3\pi}{4} K \Delta_c^2 \beta_V^2 \frac{A_s \Omega_m}{\Omega_r} f_h a \left(\frac{a}{a_*} \right)^{1/2} \left(\frac{D}{D_0} \right)^n. \quad (31)$$

At the linear–nonlinear crossover about half of the mass is in halos, and thereafter the halo fraction grows as $f_h \simeq \frac{1}{2}(a/a_*)^{1/2}$ (Fig. 4). Setting $\rho_{DE}/\rho_m = 1$ at a_Λ , with $D_\Lambda/D_0 = a_\Lambda g_\Lambda/g_0$ from Eq. (19), gives

$$a_* = C_* \beta_V^2 a_\Lambda^{2+n} \left(\frac{g_\Lambda}{g_0} \right)^n, \quad C_* \equiv \frac{3\pi}{8} K \Delta_c^2 \frac{A_s \Omega_m}{\Omega_r} \simeq 1.04, \quad (32)$$

for the fitted $K \simeq 4$. For $\beta_V = 1$, $a_* \simeq a_\Lambda^{2+n} < a_\Lambda$. The two epochs are positively correlated for $n > -2$, with $a_\Lambda/a_* = (C_* \beta_V^2 a_\Lambda^{1+n})^{-1} (g_0/g_\Lambda)^n \simeq 1.3\text{--}1.8$: equality follows the crossover, as the anthropic bound $a_* \lesssim a_\Lambda$ requires.

The phantom crossing coincidence. The same relations fix the epoch $a_\times$ at which ρ_{DE} peaks and $w = -1$. After the linear–nonlinear crossover a_* , with $f_h \simeq \frac{1}{2}(a/a_*)^{1/2}$, Eq. (31) gives $\rho_{DE}/\rho_m \propto a^2(D/D_0)^n$; normalized at equality,

$$\Omega_m(a) = \left[1 + \left(\frac{a}{a_\Lambda} \right)^2 \left(\frac{D}{D_\Lambda} \right)^n \right]^{-1}. \quad (33)$$

Since $\rho_m \propto a^{-3}$, $\rho_{DE} \propto a^{-1}(D/D_0)^n$. This is a power-law density for $a_* < a < a_\Lambda$ (Section 2) with logarithmic slope

$$s \equiv \frac{d \ln \rho_{DE}}{d \ln a} = nf - 1, \quad f = \frac{d \ln D}{d \ln a} \simeq \Omega_m(a)^{0.55}. \quad (34)$$

In matter domination $f = 1$ and $s = n - 1$: for $n > 1$ the density rises and $w < -1$. As dark energy grows, Ω_m and hence f fall, and $w = -1$ is crossed where $nf_\times = 1$, i.e. at $\Omega_m(a_\times) = n^{-\frac{1}{0.55}}$. With Eq. (33) and $D \propto a g(a)$, where g varies slowly between $a_\times$ and a_Λ ,

$$\left(\frac{a_\times}{a_\Lambda} \right)^{2+n} \left(\frac{g_\times}{g_\Lambda} \right)^n = n^{\frac{1}{0.55}} - 1, \quad (35)$$

$$a_\times = a_\Lambda \left(n^{\frac{1}{0.55}} - 1 \right)^{\frac{1}{2+n}} \left(\frac{g_\Lambda}{g_\times} \right)^{\frac{n}{2+n}}.$$

Combined with Eq. (32), the same power $2 + n$ relates the crossing directly to the crossover,

$$a_* \simeq C_* \beta_V^2 a_\times^{2+n} \left(n^{\frac{1}{0.55}} - 1 \right)^{-1} \left(\frac{g_\times}{g_0} \right)^n \quad (36)$$

Equation (35) holds for $a_* < a_\times < a_\Lambda$, during steady cascade and before dark energy dominates. For $n \leq 1$, or when it gives $a_\times < a_*$, the density peaks at the crossover itself, because the halo fraction rises steeply before it and $s < 0$ after it; hence $a_\times \simeq \max\{a_*, a_\Lambda(n^{1/0.55} - 1)^{1/(2+n)}\}$. The crossing precedes equality ($a_\times \leq a_\Lambda$) only if $n < 2^{0.55} = 1.46$, so the three epochs are ordered as: steady cascade starts at the linear–nonlinear crossover a_* ; the dark-energy density then peaks at $a_\times$ once its own suppression of growth has reduced f to $f_\times = 1/n$; and equality a_Λ follows.

5 STEADY CASCADE AFTER CROSSOVER a_*

This section describes the statistically steady cascade of the halo population and its self-similarity, which motivate the power-law structural source around phantom crossing $a_\times$.

Two mass scales. Let $\sigma_v^2(m_h, t)$ and $\sigma_h^2(m_h, t)$ be the one-component virial and bulk dispersions of halos of mass m_h . Their mass-weighted averages are $\langle u^2 \rangle = \frac{2}{3} \langle K_{pv} \rangle$ and $V_\parallel^2 = \frac{2}{3} \langle K_{ph} \rangle$. Two mass scales organize the halo population.

The characteristic mass m_h^* , defined by $\sigma_v(m_h^*) = \sigma_h(m_h^*)$ (Fig. 5), marks kinematic decoupling. Smaller halos ($\sigma_v < \sigma_h$) move with the surrounding bulk flow and retain an imprint of the primordial spectrum. Larger halos grow by gravitational merging alone and lose this memory.

The nonlinear mass M^* , defined by $\sigma(R_L, a) = \delta_{col}$, separates collapsed scales ($m_h < M^*$) from linear ones and sets the cutoff of the mass function. Because the halo mass fraction rises as $m_h^{1-\lambda_1}$ with $\lambda_1 < 1$, halos near M^* dominate the virial energy: $\langle u^2 \rangle \sim \sigma_v^2(M^*)$.

The ordering of the two scales distinguishes the two regimes. Before a_* , $M^* < m_h^*$: the cutoff lies below m_h^* , and halos at m_h^* are rare objects in the linear regime. After a_* , $M^* > m_h^*$, and three ranges appear. Below m_h^* , halos form a deeply nonlinear, time-invariant power law in mass function, which retains the primordial imprint. Between m_h^* and M^* , gravity dominates and memory of the initial spectrum is lost: the universal index $n = -1$ appears as the curvature of the cutoff, through which the cascaded mass is passed on and deposited toward M^* . Above M^* , scales remain linear and halos are exponentially rare.

Relations at all epochs. At fixed time $\sigma_v^2 \propto m_h^{2/3}$, while the crossover values follow $\sigma_v^2(m_h^*) \propto m_h^{*1/3}$ (Fig. 5). Write

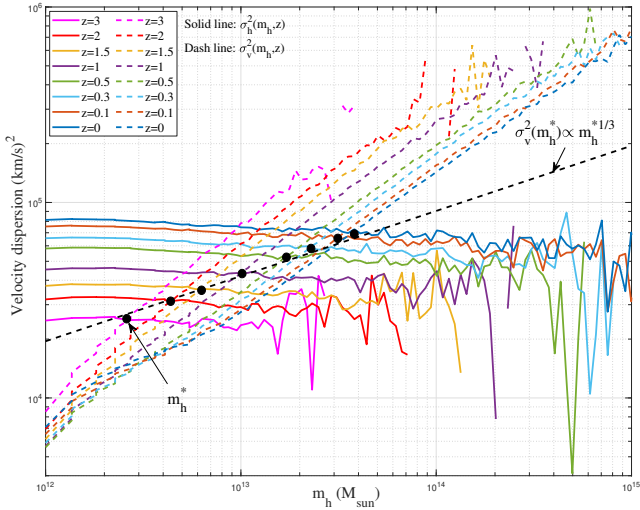


Figure 5. Virial dispersion $\sigma_v^2(m_h)$ (dashed) and bulk dispersion $\sigma_h^2(m_h)$ (solid) versus halo mass at several redshifts in Virgo SCDM. Their intersections (black dots) define $m_h^*(a)$ and follow $\sigma_v^2(m_h^*) \propto m_h^{*1/3}$.

$\sigma_h^2(m_h^*) \propto a^p$, and $\sigma_v^2 \propto m_h^{2/3} a^{-q}$ at fixed mass. The crossover gives $m_h^* \propto a^{3(p+q)/2}$, and the $m_h^{*1/3}$ law then requires

$$p = q, \quad m_h^* \propto a^{3p}, \quad \rho_h^* m_h^* = \text{const}, \quad (37)$$

where ρ_h^* is the dynamical density of halos at m_h^* . Because halos near M^* dominate, $\langle K_{pv} \rangle \sim \sigma_v^2(M^*) \propto M^{*2/3} a^{-q}$. The virial energy grows linearly in time at all epochs, $\langle K_{pv} \rangle = |\varepsilon_u|t$: in Illustris-1-Dark, $\langle K_{pv} \rangle/t$ is constant to within 25% for $0.075 \leq a \leq 1$ (Fig. 2). With $M^* \propto a^{6/(n+3)}$, this gives

$$M^* \propto (a^q t)^{3/2} \propto t^{3/2} m_h^{*1/2}, \quad q = \frac{4}{n+3} - \frac{3}{2}, \quad (38)$$

so $q = 1$ for $n = -7/5$ and $q = 1/2$ for $n = -1$ (universal spectrum) (Xu 2021). The two regimes thus differ only through q , which reflects how halos grow, or equivalently the effective index n . Linear growth of $\langle K_{pv} \rangle$ alone therefore does not imply a system-wide steady cascade. That also requires a steady partition of mass between halos and the linear flow, i.e. a self-similar halo fraction f_h .

Before crossover a_* . Multistream regions are isolated within a single-stream flow, and the nonlinear scale is smaller than the coherent displacement, $R_L(M^*) < \Psi_{\parallel}$. Here $R_L(M) = (3M/4\pi\bar{\rho}_0)^{1/3}$ is the Lagrangian radius of mass M at mean density, and $\Psi_{\parallel}$ is the rms displacement per component by the coherent, curl-free linear flow, $\Psi_{\parallel}^2 = \frac{1}{6\pi^2} \int P_L(k) dk$, with velocity $V_{\parallel} = aHf\Psi_{\parallel}$. Collapsed regions are smaller than the distance the flow moves them, so the linear flow carries them as wholes.

Halos near m_h^* therefore have a linear bulk flow, $\sigma_h^2(m_h^*) \simeq V_{\parallel}^2 = \sigma_v^2(m_h^*)$. They accrete cold single-stream matter and virialize at fixed contrast, $\sigma_v \sim aHR_L$. Hence $p = q = 1$ and $R_L(m_h^*) \sim \Psi_{\parallel}$, and Eqs. (37) and (38) give $m_h^* \propto t^2$ and $M^* \propto t^{5/2}$, so that

$$\frac{\langle K_{pv} \rangle}{\langle K_{ph} \rangle} \simeq \left(\frac{M^*}{m_h^*} \right)^{2/3} \propto t^{1/3}. \quad (39)$$

Table 1. The transition at a_* . Scalings are in cosmic time t .

	$a < a_*$	$a > a_*$
<i>Unchanged</i>		
Virial energy	$\langle K_{pv} \rangle = \varepsilon_u t$, same ε_u	
Structural relations	$\sigma_v^2 \propto m_h^{2/3}$, $p = q$, $\rho_h^* m_h^* = \text{const}$	
Mass injection	$S = \dot{\rho}_h \delta_D(m - m_p)$	
<i>The switch</i>		
Multistream regions	isolated	connected
Scale ordering	$M^* < m_h^*$	$m_h^* < M^*$
Accreted matter	cold, single-stream	pre-processed
$\rho_h^*/\bar{\rho}$	const	$t [a^{3/2}]$
$p = q$ (n_{eff})	1 $(-7/5)$	1/2 (-1)
<i>Consequences</i>		
m_h^*	$t^2 [a^3]$	$t [a^{3/2}]$
M^*	$t^{5/2} [a^{15/4}]$	$t^2 [a^3]$
$\sigma_v^2(m_h^*)$	$t^{2/3} [a]$	$t^{1/3} [a^{1/2}]$
σ_v^2 at fixed m_h	$m_h^{2/3} a^{-1}$	$m_h^{2/3} a^{-1/2}$
Dominant kinetic energy	$\langle K_{ph} \rangle$	$\langle K_{pv} \rangle$
$\partial_t \rho_g$ below M^*	$\neq 0$	0
Mass flux Π_m	decreases with m	$\varepsilon_m = \dot{\rho}_h \propto a^{-1}$
Halo fraction f_h	faster than $a^{1/2}$	$\propto a^{1/2}$

To describe how mass moves between halo masses, we write mass conservation in halo-mass space,

$$\partial_t \rho_g + \partial_m \Pi = S, \quad \rho_g(m, t) \equiv m \frac{dn_h}{dm}. \quad (40)$$

Here dn_h/dm is the halo mass function, so $\rho_g dm$ is the co-moving mass density in halos with masses in $[m, m + dm]$. $\Pi(m, t)$ is the net mass flux from halos lighter than m to heavier ones (Xu 2021). S is the rate at which unbound matter enters halos. We count each unbound particle as an object of the smallest mass m_p , so unbound matter enters only at m_p , $S = \dot{\rho}_h \delta_D(m - m_p)$ with $\rho_h \equiv f_h \bar{\rho}_0$, and accreting a particle onto a halo of mass m carries mass m_p across every scale between m_p and m . Integrating Eq. (40) gives

$$\Pi(m, t) = \dot{\rho}_h - \partial_t \int_{m_p}^m \rho_g dm'. \quad (41)$$

Before a_* , the coherent flow keeps raising the abundance of halos of every mass, so $\partial_t \rho_g \neq 0$ and Π decreases with m . The distribution is not steady, and f_h grows faster than $a^{1/2}$.

The crossover. The condition $\langle K_{pv} \rangle = \langle K_{ph} \rangle$ defines a_* . By Eq. (39) it is equivalent to $M^* \simeq m_h^*$, or $R_L(M^*) \sim \Psi_{\parallel}$: the nonlinear scale reaches the coherent displacement, and multistream regions come to span it. About half of the mass is then in halos, $f_h(a_*) \simeq 1/2$ (Fig. 4).

After crossover a_* . Halos near m_h^* now move and accrete within multistream regions. The matter they accrete is pre-processed and already dense, so their density no longer tracks the expansion: they lose their fixed contrast, and their bulk flow departs from linear theory, $p = q = 1/2$. Equations (37) and (38) then give

$$m_h^* \propto a^{3/2} \propto t, \quad M^* \propto a^3 \propto t^2, \quad (42)$$

where $M^* \propto a^3$ is the nonlinear mass of the universal $n_{\text{eff}} = -1$ spectrum. The range below the cutoff therefore widens as $M^*/m_h^* \propto t$.

Unbound matter, still about half of the total at a_* , continues to be accreted, now by halos of all masses at once and

mainly as smooth accretion of individual particles. It still enters only at m_p , but the distribution below the cutoff is now statistically steady, $\partial_t \rho_g = 0$ (*small-scale permanence*). By Eq. (41), each scale then passes on all the mass it receives,

$$\Pi = \varepsilon_m \equiv \dot{\rho}_h \quad (m_p < m \lesssim M^*), \quad (43)$$

like a leak-free pipe at constant level (Xu 2021). The cascaded mass accumulates only near the cutoff M^* , in the exponential tail whose shape is set by $\lambda_2 = 2/3$. With $f_h \propto a^{1/2}$, this gives $\varepsilon_m \propto a^{-1}$ in the matter era. The same argument applies to the energy flux $|\varepsilon_u|$.

The cascade rate from cosmological parameters. Before a_* , halos of M^* has a fixed contrast, so linear theory gives the rate. For a locally power-law spectrum of index n_{eff} ,

$$R_L = R_8 \left(\frac{\sigma_8 D}{\delta_{\text{col}}} \right)^{2/(n_{\text{eff}}+3)}, \quad GM^* = \frac{1}{2} \Omega_m H_0^2 R_L^3, \quad (44)$$

with $R_8 = 8 h^{-1} \text{Mpc}$. A halo of mean density $\Delta_c \bar{\rho}$ has $R_{\text{vir}} = a R_L \Delta_c^{-1/3}$ and $\sigma_v^2 = \kappa_v GM/R_{\text{vir}}$, with $\kappa_v \simeq 1/3$ for NFW profiles. The Press-Schechter average gives $W = 2^{p_n/2} \Gamma(1 + p_n/2)$, with $p_n = 4/(n_{\text{eff}} + 3)$. Hence

$$|\varepsilon_u| = \frac{\langle K_{pv} \rangle}{t} = \frac{3\kappa_v W}{4} \Delta_c^{1/3} \Omega_m \frac{(H_0 R_L)^2}{a t} \propto \frac{D^{p_n}}{a t} \sigma_8^{p_n}, \quad (45)$$

where $H_0 R_8 = 800 \text{ km s}^{-1}$.

In matter domination this rate scales as $a^{p_n-5/2}$. It is constant for $n_{\text{eff}} = -7/5$, consistent with the linear solution of Eq. (22). The rate fixed before a_* therefore carries over to the steady cascade. With $\kappa_v = 1/3$, $\Delta_c = 178$ and $W \simeq 2.7$, Eq. (45) gives $|\varepsilon_u| \simeq 3.0$ and $2.4 \times 10^{-7} \text{ m}^2 \text{s}^{-3}$ for Virgo SCDM and Illustris-1-Dark, against the measured 4.7 and 2.7×10^{-7} . For the same background and $\sigma_8 \propto A_s^{1/2}$, the rate scales as $|\varepsilon_u| \propto A_s^{p_n/2} \simeq A_s^{1.25}$.

6 STRUCTURALLY DRIVEN DARK ENERGY

We address a narrower question: given a vacuum tuned to zero, why does a small residual dark energy appear today, just as structure forms? We show that the density law, Eq. (1), is compatible with a conditional effective model: a self-tuned q -theory vacuum, a postulated interaction energy from structure, and additional dynamics that carry the pressure and energy exchange its evolution requires. We derive the conservation structure any realization must satisfy; a covariant action remains open.

Conservation requirements. Three constraints shape the construction. First, an exactly vacuum-like component, $T_{\text{vac}}^{\mu\nu} = -\epsilon_{\text{vac}} g^{\mu\nu}$, has $\nabla_\mu T_{\text{vac}}^{\mu\nu} = -\nabla^\nu \epsilon_{\text{vac}}$, so an evolving dark energy cannot be an isolated vacuum. Second, a separately conserved dark-energy sector obeys

$$\epsilon_{\text{DE}} + p_{\text{DE}} = -\frac{\dot{\epsilon}_{\text{DE}}}{3H}, \quad w = -1 - \frac{1}{3} \frac{d \ln \epsilon_{\text{DE}}}{d \ln a}, \quad (46)$$

so it needs stress beyond a varying vacuum term; it is phantom exactly while its density grows. Third, a strictly vacuum-like sector following the fitted history would require $d(\rho_m a^3)/da = -a^3 d\rho_{\text{DE}}/da$, exchanging 6–14% of the present matter density ρ_{m0} with the dark energy between $z = 3$ and today. We therefore take matter to be minimally

coupled, $\nabla_\mu T_m^{\mu\nu} = 0$, so that $\rho_m \propto a^{-3}$ and structure acts only through the metric. We choose two dark-sector components: the vacuum variable q and a carrier scalar φ . The kinetic enthalpy of q , $I\dot{q}^2 \geq 0$, is small and has the wrong sign for the phantom phase. The carrier therefore supplies Eq. (46), defines the vacuum frame, and exchanges energy with q , as a varying vacuum must (Wands *et al.* 2012); kinetic braiding can cross the phantom divide (Deffayet *et al.* 2010), whereas a single k-essence field cannot (Vikman 2005). The total energy is conserved, separately from matter.

Model assumptions.

- (A1) *Self-tuned vacuum.* The vacuum variable q has energy density $\epsilon_0(q)$ (Klinkhamer and Volovik 2008a). Its chemical potential $\mu_q = d\epsilon_0/dq$ is a constant μ_0 , fixed as an integration constant, and its gravitating energy is $\tilde{\epsilon} = \epsilon_0 - q d\epsilon_0/dq$. The perfect vacuum satisfies $\epsilon'_0(q_0) = \mu_0$ and $\tilde{\epsilon}(q_0) = 0$, and is stable, $E_2 \equiv \epsilon''_0(q_0) > 0$, with compressibility $\chi_v = (E_2 q_0^2)^{-1}$. We assume this self-tuned branch. Since a shift $\epsilon_0 \rightarrow \epsilon_0 + C$ at fixed μ_0 shifts the residual by C .
- (A2) *Background truncation.* We describe only the homogeneous vacuum mode, the background truncation of a complete theory coarse-grained over scales larger than halos.
- (A3) *Scale separation.* Halo-scale motion has the rate $\omega_{\text{DM}} \equiv a_c/u = \beta H \simeq 162 H_0$ today (Eq. (24)). We assume the halo force field decorrelates on times $\sim \omega_{\text{DM}}^{-1} \ll H^{-1}$, so that the drive $\langle a_p^2 \rangle$ is a well-defined statistic evolving on the Hubble time. The vacuum responds at $\omega_{\text{vac}} = (E_2/I)^{1/2}$, where I is the coefficient of $\frac{1}{2} I \dot{q}^2$. Adiabatic tracking requires only $\omega_{\text{vac}} \gg H$, the rate at which the drive varies; $\omega_{\text{vac}} \gg \omega_{\text{DM}}$ would matter only for an unaveraged coupling. q then follows the minimum set by the current drive, and the response is a static susceptibility; whether q stays vacuum-like is estimated below. A Planckian ω_{vac} requires both a Planck-scale E_2 and a corresponding I , an additional assumption.

A limited analogy. The interaction resembles the Kapitza pendulum (Appendix A): a rapidly shaken pivot leaves the static effective potential of Eq. (A1), whose drive term is a variance a_{piv}^2 times a configuration factor that vanishes at $\theta = 0$, where the rod is aligned with the drive. The postulates share this form, with $\langle a_p^2 \rangle$ as the variance and $\kappa(S)$ as the configuration factor; in a mechanical reading, the carrier plays the rod that defines the frame. The analogy is limited: $\langle a_p^2 \rangle$ is a coarse-grained statistic, not an oscillating force; the vacuum's response is a direct interaction energy, not an induced one; and the carrier's roles are requirements on a completion.

Model postulates.

- (P1) *Interaction.* Coupling to structure changes the vacuum energy to

$$\epsilon(q) = \epsilon_0(q) + \epsilon_i, \quad \epsilon_i \equiv \frac{1}{2} k(q) B_M, \quad B_M \equiv \kappa(S) \langle a_p^2 \rangle, \quad (47)$$

with gravitating energy $\tilde{\epsilon} = \epsilon - q \partial_q \epsilon$. Here $\langle a_p^2 \rangle$ is the statistic of Section 3, $\kappa(S)$ the configuration factor, and $k(q)$ is normalized with units of $1/G$, a dimensional choice rather than a derived susceptibility. Since $\sum_i m_i \mathbf{a}_i = 0$, a coupling linear in the acceleration vanishes, and for an analytic, rotationally invariant coupling the leading term is quadratic. No direct coupling to the matter density is assumed. Built from Newtonian peculiar accelerations, $\langle a_p^2 \rangle$ is defined in the

cosmic rest frame, which suffices for the density law, a background relation. Since a freely falling particle has zero proper acceleration, a local covariant source would require a physical frame, an invariant built relative to it, and a coarse-graining scale; we leave this formulation to future work.

(P2) *Configuration factor.* We take $\kappa = (S/S_{\text{ref}})^n$, where S is the rms largest principal strain of the matter displacement and S_{ref} a cosmology-independent constant; $n = 2$ is the analogue of $\sin^2 \theta \simeq \theta^2$. For $n > 0$ an undeformed configuration does not couple, $\kappa(0) = 0$, and deformation raises the energy for $k > 0$; for $n = 0$ the coupling depends on structure only through $\langle a_p^2 \rangle$; for $n < 0$ it diverges as $S \rightarrow 0$ and is admitted only phenomenologically.

(P3) *Strain as a background clock.* In the background truncation, S enters only as a function of time, which we prescribe by the linear growing mode: there $\partial_j \Psi_i = D \partial_j \Psi_i^{(0)}$, so every principal strain grows as D , and at fixed smoothing $S = S_1 D(a)$ with $S_1 \propto \sigma_{\text{min}}$. This is a constitutive choice for the background, not a derived inheritance of the matter strain by a local carrier. The work associated with the changing strain enters the source term of Eq. (52) and is exchanged with the carrier; matter is affected only through the metric. Two remarks apply. First, the scaling need not be mechanical: in a covariant model, a power law of ϵ_{DE} in a and D could hold as an emergent relation along a joint attractor (Section 2), to be predicted rather than imposed. Second, the frame matters: a source built from the carrier's own acceleration vanishes for a geodesic carrier, and so requires the carrier not to follow matter inside halos, whereas a tidal source survives free fall. In the growing mode the deformation tensor equals the tidal tensor, so $S \propto D$ can also be read as the slowly growing large-scale tidal amplitude.

(P4) *Closure.* The carrier supplies the enthalpy of Eq. (46). A completion must derive the carrier's kinetic coefficient and response, and show that a single field φ can supply this enthalpy, exchange energy with q , define the frame that a covariant drive requires, reproduce the fitted density and remain stable, with acceptable matter growth; it must also derive the coupling's dependence on the strain or tidal amplitude.

Static residual. Without the drive, $\mathcal{U}_0(q) \equiv \epsilon_0(q) - \mu_0 q$ satisfies $\mathcal{U}_0(q_0) = \mathcal{U}'_0(q_0) = 0$ and $\mathcal{U}''_0(q_0) = E_2$: a bowl with its bottom at zero height. With the drive, $\mathcal{U}(q) \equiv \epsilon(q) - \mu_0 q$, and we hold B_M fixed under variations of q . On the stationary branch, $\mathcal{U}'(q) = 0$, so the gravitating vacuum energy is $\tilde{\epsilon} = \mathcal{U}$. Expanding at q_0 , $q = q_0 + \delta q$, $k_0 \equiv k(q_0)$ and $k'_0 \equiv k'(q_0)$,

$$\mathcal{U} \simeq \underbrace{\frac{1}{2} E_2 \delta q^2}_{\text{bowl}} + \underbrace{\frac{1}{2} k_0 B_M}_{\text{lift}} + \underbrace{\frac{1}{2} k'_0 B_M \delta q}_{\text{tilt}}. \quad (48)$$

The drive lifts the whole bowl and, because k depends on q , tilts it. Minimizing ($d\mathcal{U}/d\delta q = 0$),

$$\delta q = -\frac{k'_0 B_M}{2E_2}, \quad \tilde{\epsilon}_{\text{min}} = \underbrace{\frac{1}{2} k_0 B_M}_{\text{direct}} - \underbrace{\frac{k_0'^2 B_M^2}{8E_2}}_{\text{induced}} + O(B_M^3), \quad (49)$$

valid for $|\delta q/q_0| \ll 1$ and $\chi_v \tilde{\epsilon}_{\text{min}} \ll 1$. The induced term is negative, because tilting lowers the minimum, and smaller than the direct term by $\frac{1}{4} \chi_v (q_0 k'_0/k_0)^2 k_0 B_M \sim \epsilon_{\text{DE}}/\epsilon_{\text{Pl}}$ for $\chi_v \sim \epsilon_{\text{Pl}}^{-1}$ and $q_0 k'_0/k_0 = O(1)$, where ϵ_{Pl} is the Planck energy density. For $k'_0 = 0$ the vacuum is not displaced, yet $\frac{1}{2} k_0 B_M$ survives. The leading residual is therefore the direct interaction energy. On the stationary branch, the drive enters the

gravitating energy in two parts,

$$\tilde{\epsilon} = (\epsilon_0 - q\epsilon'_0) + \underbrace{\frac{1}{2} k B_M}_{\text{explicit}} - \underbrace{\frac{1}{2} q k' B_M}_{q\text{-derivative}}. \quad (50)$$

Because μ_0 is an integration constant, the constraint $\partial_q \epsilon = \mu_0$ holds before and after structure forms. The drive adds $\frac{1}{2} k'_0 B_M$ to $\partial_q \epsilon$, so q relaxes until $E_2 \delta q = -\frac{1}{2} k'_0 B_M$. This cancels the q -derivative part, leaving $\tilde{\epsilon}_{\text{min}} = \frac{1}{2} k_0 B_M$ to first order. The explicit part could be removed only by changing μ_0 . For the same reason, a constant shift $\epsilon_0 \rightarrow \epsilon_0 + C$ is not cancelled (A1). It is positive if $k_0 > 0$, which is an input. The construction therefore shows that a postulated structural energy term is compatible with a self-tuned vacuum; it does not derive that energy from self-tuning or from a driven displacement of the vacuum.

Adiabatic dynamics. The static residual assumes that the vacuum sits at the minimum while B_M evolves. To justify this, we give the homogeneous mode an inertia $I = E_2/\omega_{\text{vac}}^2$, as in the propagating- q formulation (Klinkhamer and Volovik 2017), with effective action per comoving volume $\int dt a^3 [\frac{1}{2} I \dot{q}^2 - \mathcal{U}]$. The equation of motion and the energy balance are

$$I(\ddot{q} + 3H\dot{q}) = -\mathcal{U}'(q) = \mu_0 - \partial_q \epsilon, \quad (51)$$

$$\dot{\epsilon}_V + 3H(\epsilon_V + p_V) = \frac{1}{2} k(q) \dot{B}_M, \quad (52)$$

with $\epsilon_V = \frac{1}{2} I \dot{q}^2 + \mathcal{U}$ and $p_V = \frac{1}{2} I \dot{q}^2 - \mathcal{U}$. The second equation follows from differentiating ϵ_V and eliminating $I\ddot{q}$ with the first. Write $q = q_* + \zeta$ about the moving minimum, $\mathcal{U}'(q_*, t) = 0$. The minimum moves at $\dot{q}_* \sim H \delta q$, because B_M changes on the Hubble time. Expanding $\mathcal{U}'$ to first order in ζ gives

$$\ddot{\zeta} + 3H\dot{\zeta} + \omega_{\text{vac}}^2 \zeta = -(\ddot{q}_* + 3H\dot{q}_*), \quad \zeta \simeq -\frac{\ddot{q}_* + 3H\dot{q}_*}{\omega_{\text{vac}}^2}, \quad (53)$$

so the vacuum lags the minimum only by $\zeta/\delta q \sim H^2/\omega_{\text{vac}}^2$ and moves with it, $\dot{q} \simeq \dot{q}_*$. Its kinetic energy is negligible,

$$\frac{\epsilon_V + p_V}{\epsilon_V} = \frac{I\dot{q}^2}{\epsilon_V} \sim \frac{H^2}{\omega_{\text{vac}}^2} \cdot \frac{k_0'^2 B_M}{k_0 E_2}, \quad (54)$$

so $p_V \simeq -\epsilon_V$ and $\epsilon_V \simeq \tilde{\epsilon}_{\text{min}}$. An initial misalignment oscillates at ω_{vac} with energy $\propto a^{-3}$; Hubble friction dilutes it like matter but does not dissipate it, so it must start small and negligible. Since $H \ll \omega_{\text{vac}}$, the residual is then a state function of the current configuration (static residual), which predicts an instantaneous response.

The density law. Combining Eq. (49) and (P2), (P3),

$$\epsilon_{\text{DE}} = K \frac{\langle a_p^2 \rangle}{G} \left(\frac{D}{D_0} \right)^n, \quad K \equiv \frac{1}{2} G k_0 \left(\frac{S_1 D_0}{S_{\text{ref}}} \right)^n, \quad (55)$$

which is Eq. (1) with $\Phi = (D/D_0)^n$ and contains no explicit dependence on a , t or H . Because S_{ref} is fixed, $K \propto (S_1 D_0)^n$; at fixed D_0 , $K \propto A_s^{n/2}$.

The exponent is set by the configuration factor. A strain-independent coupling gives $n = 0$. A coupling linear in the magnitude of the largest principal strain gives $n = 1$, natural because λ_1 controls Zel'dovich collapse (Zel'dovich 1970) and is the strain that emergent gravity assigns to the dark-energy medium (Verlinde 2017); but it requires a non-analytic dependence on $|\lambda_1|$. An analytic coupling that is positive and even in the strain, such as $\kappa \propto \text{tr} S^2$, gives $n = 2$, the analogue

of the pendulum's $\sin^2 \theta \simeq \theta^2$ (Appendix A). Since neither choice is derived, we treat n as free (Section 7).

Consequences of the law. When halo accelerations dominate in the matter era, with $a_c \propto a^{-p_c}$,

$$\frac{d \ln \epsilon_{\text{DE}}}{d \ln a} = -2p_c + (n + \eta)f, \quad \eta \equiv \frac{d \ln f_h}{d \ln D}, \quad (56)$$

where η is the halo-assembly slope. The expansion data constrain a combination of driving p_c , the halo-fraction history η , and configuration n , not n alone. In the steady cascade ($p_c = \frac{3}{4}$, $f = 1$, $f_h \propto a^{1/2}$) the right-hand side is $n - 1$. For $n = 1$ the density is nearly constant, $w \simeq -1$ during late matter domination, and $w = -1$ is crossed near the linear-nonlinear crossover a_* ; for $n > 1$ the steady cascade is phantom, and for $n < 1$ the density already declines.

Before halos form, $f_h \rightarrow 0$ and $\langle a_p^2 \rangle \rightarrow a_{op}^2 \propto (D/a^2)^2$, so in matter domination $\epsilon_{\text{DE}} \propto a^{n-2}$ and

$$w \rightarrow -\frac{n+1}{3}. \quad (57)$$

Only $n = 2$ gives a vacuum-like plateau. Since $\epsilon_{\text{DE}}/\rho_m \propto a^{n+1}$, early dark energy is negligible for $n > -1$, a fraction $\lesssim 4 \times 10^{-6}$ of the total near recombination at the best fits, and grows toward the past for $n < -1$. The fits apply the law only for $a \geq 10^{-3}$ and hold the density constant below.

7 MODEL, PARAMETERS AND INFERENCE

This section defines the quantities common to all confrontations with data: the BAO and CMB likelihood, the model and its parameters, the priors, the degeneracies of the density law and the inference. Section 8 applies them to DESI DR2 BAO with CMB acoustic-scale information, and Section 9 adds supernovae and growth.

BAO and CMB likelihood. We use the 13 DESI DR2 BAO measurements with their D_M – D_H correlations (DESI Collaboration *et al.* 2025b,c), from the official machine-readable mean vector and covariance (Appendix D). For a flat background with $E(z) = H/H_0$, the transverse, Hubble and angle-averaged distances are

$$D_M = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}, \quad (58)$$

$$D_H = \frac{c}{H_0 E}, \quad D_V = (z D_M^2 D_H)^{1/3}.$$

The background contains photons, three neutrino species with $N_{\text{eff}} = 3.044$ and one massive eigenstate with $\Sigma m_\nu = 0.06$ eV, baryons, CDM and dark energy, as in the DESI DR2 analysis; Ω_m is the total matter density today, including the massive neutrinos, and $\omega_{bc} = \Omega_m h^2 - \omega_\nu$. CMB information enters through the compressed likelihood adopted by DESI DR2 (DESI Collaboration *et al.* 2025b): a correlated Gaussian prior on $\mathbf{x} = (\theta_*, \omega_b, \omega_{bc})$, derived from Planck CamSpec chains after marginalizing over the late-time ISW and lensing contributions, with mean $\boldsymbol{\mu} = (0.0104110, 0.02223, 0.14208)$ and covariance

$$\mathbf{C} = 10^{-9} \begin{pmatrix} 0.006621 & 0.12444 & -1.1929 \\ 0.12444 & 21.344 & -94.001 \\ -1.1929 & -94.001 & 1488.4 \end{pmatrix}. \quad (59)$$

Here $\theta_* = r_*/D_M(z_*)$, and the sound horizons at recombination and at the drag epoch, r_* and r_d , and the redshift z_* are computed with CAMB (Lewis *et al.* 2000) as functions of (ω_b, ω_{bc}) ; they are independent of the late-time dark energy, whose fraction at $z \gtrsim 10^3$ is $\lesssim 4 \times 10^{-6}$ at the best fits and below 10^{-3} in all accepted models (Table 3). The BAO and CMB contribution is

$$\chi_{\text{BAO+CMB}}^2 = \sum_i \mathbf{d}_i^T C_i^{-1} \mathbf{d}_i + \left(\frac{D_V/r_d - 7.942}{0.076} \right)^2 + (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x} - \boldsymbol{\mu}), \quad (60)$$

where $\mathbf{d}_i$ holds the residuals of $(D_M/r_d, D_H/r_d)$ for pair i and C_i their 2×2 covariance; it has $N = 16$ terms. For each trial late-time model, h and ω_b are fitted by minimizing Eq. (60), so that the CMB fixes the high-redshift anchor while the late-time history is free.

The model. The structural density law is

$$\epsilon_{\text{DE}} \propto \langle a_p^2 \rangle \left(\frac{D(a)}{D(1)} \right)^n, \quad \langle a_p^2 \rangle = f_h a_c^2 + (1 - f_h) a_{op}^2, \quad (61)$$

normalized by flatness, $\Omega_{\text{DE}} = 1 - \Omega_m - \Omega_\gamma$, with $a_{op} = a_{op0}(D/D_0)/a^2$. We compare three clocks for the halo-particle acceleration,

$$a_c = a_{c0} \times \begin{cases} a^{-p_c}, & p_c = \frac{3}{4} \quad (a\text{-clock}), \\ (t/t_0)^{-p_c}, & p_c = \frac{1}{2} \quad (t\text{-clock}), \\ (D(a)/D(1))^{-p_c}, & p_c = \frac{3}{4} \quad (D\text{-clock}). \end{cases} \quad (62)$$

The clocks share the matter-era exponent, $a_c \propto a^{-3/4}$, but because each is normalized today their early-time amplitudes differ: with $t \simeq t_{m,1} a^{3/2}$ and $D \simeq a$, $a_c^{(t)} \simeq a_{c0}(t_0/t_{m,1})^{1/2} a^{-3/4}$ and $a_c^{(D)} \simeq a_{c0} D_0^{3/4} a^{-3/4}$, against $a_{c0} a^{-3/4}$ for the a -clock. The t -clock assumes a cascade at constant rate in cosmic time, as Illustris-1-Dark supports (Section 3); the a -clock extends the matter-era power law, and the D -clock freezes when growth stops. These alternatives are phenomenological. The halo mass fraction is

$$f_h(a) = \beta_f \operatorname{erfc} \left[\frac{\delta_{\text{col}}}{\sqrt{2} \sigma_{\text{min}} D(a)/D(1)} \right], \quad (63)$$

$$\beta_f = \frac{f_h(1)}{\operatorname{erfc}(\delta_{\text{col}}/(\sqrt{2} \sigma_{\text{min}}))},$$

with $\delta_{\text{col}} = 1.686$ and $f_h(1) = 0.6$ fixed. The parameters are summarized in Table 2 and the priors in Table 3.

Two interpretations of the halo term. In the *effective* interpretation, our baseline, σ_{min} is a timescale parameter of halo assembly: at fixed $f_h(1)$ a smaller value gives later, steeper assembly, and a larger one earlier saturation. In the *excursion-set* interpretation, $\sigma_{\text{min}} = \sigma(M_{\text{min}}, 0)$ is the rms linear fluctuation today at a minimum halo mass and $\beta_f \leq 1$, i.e. $\sigma_{\text{min}} \geq 3.215$. A larger σ_{min} corresponds to a smaller nominal minimum mass: $\sigma_{\text{min}} = 2.0, 2.9, 5.0$ and 5.7 correspond in a Planck cosmology to $M_{\text{min}} \simeq 10^{12.1}, 10^{11.0}, 10^{8.7}$ and $10^{8.0} h^{-1} M_\odot$.

The main text reports the effective interpretation; the excursion-set results are given in Appendix F. Imposing the bound worsens χ^2 by at most 0.9, so no conclusion depends on the choice. In either interpretation the threshold model is an approximation: changing σ_{min} changes both the halo mass

Table 2. Model parameters. Fixed inputs are calibrated on simulations or taken from theory and held constant; fitted parameters are varied in the likelihood; derived and nuisance quantities follow from the others. Parameters marked † enter only when growth or supernova data are included. The last column gives each parameter’s effect on the normalized history $\rho_{\text{DE}}(z)/\rho_{\text{DE},0}$.

Parameter	Value	Status	Effect on $\rho_{\text{DE}}(z)/\rho_{\text{DE},0}$
<i>Background cosmology</i>			
Ω_m	Table 3	fitted	background; also sets $D(a)$ and $t(a)$
h, ω_b	Table 3	fitted (profiled)	anchored by the compressed CMB likelihood; $\Sigma m_\nu = 0.06$ eV, $N_{\text{eff}} = 3.044$ fixed
<i>Structural source (calibrated on simulations; ideally independent of the dark-energy fit)</i>			
a_{c0}, a_{op0}	$1.2 \times 10^{-10}, 1.2 \times 10^{-12} \text{ m s}^{-2}$ (rms magnitudes)	fixed	normalization; ratio matters only before halos form
p_c	$\frac{3}{4}(a, D), \frac{1}{2}(t)$	fixed per clock	shape: decay of halo accelerations
$\delta_{\text{col}}, f_h(1)$	1.686, 0.6	fixed	fix β_f for each σ_{min}
σ_{min}	Table 3	fitted	shape: halo-assembly slope (structural in form, but fitted to the expansion history)
β_f	$f_h(1)/\text{erfc}(\delta_{\text{col}}/\sqrt{2}\sigma_{\text{min}})$	derived	cancels (effect below 0.1%)
<i>Response law</i>			
n	Table 3	fitted	extra growth dependence; degenerate with σ_{min}, p_c and the clock
K	$\simeq 4$	derived	normalization only, from flatness
<i>Nuisance</i>			
$\sigma_8^\dagger$	uniform in $[0.4, 1.2]$	fitted (analytic)	growth amplitude only
$M_B^\dagger$	uniform, broad	nuisance (analytic)	supernova absolute magnitude

Table 3. Priors and numerical support, common to optimization and posteriors. All priors are uniform in the listed variable. The grid has 19 values of n , 19 values of σ_{min} (logarithmic, including 3.215) and 11 values of Ω_m ; profiles in Ω_m extend beyond the grid analytically, and h and ω_b are profiled at each point. Exclusion rule: models with $\Omega_{\text{DE}} > 10^{-3}$ at $a = 10^{-3}$ are excluded, because the compressed CMB information would not apply.

Parameter	Prior range	Grid support
Ω_m	[0.25, 0.45]	[0.27, 0.37]
h, ω_b	[0.4, 1.0], [0.0206, 0.0239]	profiled
n	[-1.5, 3]	[-1.5, 3]
$\ln \sigma_{\text{min}}$ (effective)	[ln 1.2, ln 25]	[ln 1.2, ln 25]
$\ln \sigma_{\text{min}}$ (excursion set)	[ln 3.215, ln 25]	[ln 3.215, ln 25]
w_0, w_a (CPL)	[-3, 1], [-3, 2], $w_0 + w_a < 0$	–
$n, f_\times$ (linear $w(f)$)	[-3, 6], [0.3, 1]	–
σ_8	[0.4, 1.2]	analytic

fraction and the rms acceleration of the selected population, whereas we hold a_c fixed.

What the data constrain. With the halo-assembly slope

$$\eta \equiv \frac{d \ln f_h}{d \ln D} = \frac{2\xi e^{-\xi^2}}{\sqrt{\pi} \text{erfc} \xi}, \quad \xi = \frac{\delta_{\text{col}}}{\sqrt{2} \sigma_{\text{min}} D/D_0}, \quad (64)$$

the logarithmic slope of the density, $s \equiv d \ln \epsilon_{\text{DE}}/d \ln a = -3(1+w)$, is, when halo accelerations dominate,

$$s_a = -\frac{3}{2} + (\eta + n)f, \quad s_t = -\frac{1}{Ht} + (\eta + n)f, \quad (65)$$

$$s_D = (\eta + n - \frac{3}{2})f$$

for the three clocks. Expansion data constrain $s(a)$, not its decomposition into clock decay, halo assembly and an explicit response exponent. Since today $\frac{3}{2} > 1/(H_0 t_0) > \frac{3}{2} f_0$, a clock whose accelerations decay more slowly needs less growth from $\eta + n$, and a later, steeper assembly (smaller σ_{min} , larger η) needs less explicit n . These two compensations are the principal degeneracies of the model. The fits test only the normalized shape: with $y = D/D_0$ and $\nu = \delta_{\text{col}}/(\sqrt{2}\sigma_{\text{min}})$,

$$F(a) \equiv \frac{\rho_{\text{DE}}}{\rho_{\text{DE},0}} = y^n \frac{\text{erfc}(\nu/y)}{\text{erfc}(\nu)} \times \begin{cases} a^{-3/2} & (a\text{-clock}), \\ t_0/t & (t\text{-clock}), \\ y^{-3/2} & (D\text{-clock}), \end{cases} \quad (66)$$

so that K and the absolute acceleration scale are not constrained by the background fits, and the source acts as a

two-parameter shape family, (ν, n) . For the D -clock, η decreases as D grows, so a halo-dominated density has at most one maximum during growth, at $\eta(D_\times) = \frac{3}{2} - n$, and none if $n \geq \frac{3}{2}$; the absence of multiple late crossings is therefore largely a property of the parametrization, not of the data.

A linear- $w(f)$ comparator. If the assembly slope η and the clock exponent n are constant (or $\eta \propto f^{-1}$, Eq. (65) is linear in f , and the density is the power law of Eq. (4).

$$w = -1 - \frac{n}{3}(f - f_\times) \iff \rho_{\text{DE}} \propto a^{-nf_\times} D^n \quad (67)$$

We fit this two-parameter form, $(n, f_\times)$, as a comparator that keeps the response to growth but drops the clock and the halo-assembly model. Like CPL it has two dark-energy parameters, but it is linear in f rather than in a , contains Λ CDM at $n = 0$, and closes the background and growth equations self-consistently and autonomously (Section 2).

Fitting procedure. For each trial cosmology the background, the growth equation and the source are solved together by iteration, until the source changes by less than 10^{-6} , with h and ω_b refitted at each iteration. The growth factor of CDM and baryons obeys

$$D'' + \left(2 + \frac{H'}{H}\right) D' - \frac{3}{2} \Omega_{bc}(a) D = 0, \quad (68)$$

with primes denoting $d/d \ln a$ and the dark energy treated as smooth, integrated from $a = 10^{-3}$ with $D = D' = a$; only $D(a)/D(1)$ enters the source of density law, so the initial amplitude cancels. Below $a = 10^{-3}$ the dark-energy density is held constant. Appendix E shows that the continuation, the initial growth mode and the starting history of the iteration change χ^2 negligibly.

For each data combination, clock and interpretation we (i) minimize χ^2 over $(\Omega_m, \sigma_{\text{min}}, n)$ from several starting points; (ii) refit $(\Omega_m, \sigma_{\text{min}})$ at fixed $n = 0, 1$ and 2 (Appendix G); (iii) evaluate the likelihood on the grid of Table 3, integrate over Ω_m in the Laplace approximation and interpolate in $(n, \ln \sigma_{\text{min}})$, to obtain marginalized constraints; and (iv) draw histories from the posterior to obtain bands of $F(a)$ and $s(a)$ and event probabilities. When growth data are included, σ_8 is marginalized analytically: for a prediction $\sigma_8 \mathbf{q}$ and data $\mathbf{y}$ with covariance C , the marginal likelihood is $\propto A^{-1/2} \exp(-\chi_{\text{prof}}^2/2)$ with $A = \mathbf{q}^\top C^{-1} \mathbf{q}$, and we keep

the model-dependent factor $A^{-1/2}$. The supernova magnitude M_B is eliminated analytically, with a model-independent factor. Best fits and information criteria use the profile likelihood, posteriors the likelihood marginalized over σ_8 .

Information criteria. We compare models with the corrected Akaike criterion, $\text{AIC}_c = \chi^2 + 2kN/(N - k - 1)$ (Hurvich and Tsai 1989; Liddle 2007), and $\text{BIC} = \chi^2 + k \ln N$. With BAO and CMB information alone, each clock has $k = 5$ ($\Omega_m, h, \omega_b, \sigma_{\min}, n$), as have CPL and the linear- $w(f)$ model, against $k = 3$ for ΛCDM ; growth data add σ_8 to every model, and M_B is common and not counted. At equal k and N , differences between models equal their χ^2 differences; removing one parameter lowers the AIC_c penalty by 4.36 for $N = 16$ and 2.02 for $N = 1610$, and the BIC penalty by 2.77 and 7.38. The structural models are not nested in ΛCDM , so a conversion of $\Delta\chi^2$ into a significance is only indicative. Code, data and scripts that reproduce every table and figure are supplied as supplementary material.

8 CONSTRAINTS: DESI DR2 BAO AND CMB

Validation. With the same data and compressed CMB likelihood, we reproduce the DESI DR2 analysis: ΛCDM and $w_0w_a\text{CDM}$, fitted directly with CAMB, give $\chi^2 = 15.01$ and 7.10, so that $\Delta\chi^2 = -7.91$, against -8.0 reported by DESI. The $w_0w_a\text{CDM}$ best fit, $\Omega_m = 0.351$, $H_0 = 63.7 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $w_0 = -0.45$ and $w_a = -1.63$, agrees with the DESI posterior, $\Omega_m = 0.353 \pm 0.022$, $H_0 = 63.7^{+1.7}_{-2.2}$, $w_0 = -0.43 \pm 0.22$ and $w_a = -1.72 \pm 0.64$ (DESI Collaboration et al. 2025b).

Best fits. BAO and CMB information alone favour a strong late decline of the density ($w \rightarrow 0$), which CPL supplies most easily. The clocks improve on ΛCDM by $\Delta\chi^2 = -6.2$ to -7.0 , less than CPL by 0.9–1.7 at equal parameter count (Table 4, first column; full fits in Table F1). They move to small $\sigma_{\min} \simeq 2.1$ –2.7, i.e. late, steep assembly with $\eta_0 = 0.75$ –1.09, and to exponents $n = 0.81, 0.13$ and -0.27 , so that $\eta_0 + n = 1.57, 1.09$ and 0.82 : the fastest-decaying clock, the a -clock, fits best, because it supplies the steepest late decline. The linear- $w(f)$ model matches CPL only with n at the edge of its prior ($n = 6$, $f_x = 0.74$), reaching $w_0 = -0.61$ where CPL reaches -0.45 .

Dark-energy histories. The best-fit histories rise during halo formation, peak and decline, crossing $w = -1$ at the peak, at $a_x \simeq 0.61$ –0.63, before equality at $a_\Lambda \simeq 0.75$, with peaks of 1.12–1.18 $\rho_{\text{DE},0}$ (Fig. F1). These are best-fit properties: the posterior probability of $w_0 > -1$, and of a past crossing, is ≥ 0.98 for the a - and t -clocks but only 0.83 for the D -clock (Table F5). Adding supernovae reverses the ranking of the clocks, as shown next.

9 JOINT CONSTRAINTS AND COMPARISON

Additional data. We add (i) the Pantheon+ supernovae (Scolnic et al. 2022; Brout et al. 2022): the 1590 light curves with $z_{\text{HD}} > 0.01$, with the full statistical and systematic covariance and distance modulus $\mu = 5 \log_{10}[(1 +$

$z_{\text{hel}})D_M(z_{\text{HD}})/10 \text{ pc}]$; and (ii) the SDSS growth rates $f\sigma_8$ at $z = 0.38, 0.51, 0.70$ and 1.48 (Alam et al. 2017, 2021), modelled as $\sigma_8 f(z)D(z)/D(1)$ (Appendix D). The total,

$$\chi^2 = \chi_{\text{BAO+CMB}}^2 + \chi_{\text{SN}}^2 + \chi_{f\sigma_8}^2, \quad (69)$$

has $N = 1610$ terms. We neglect correlations between probes and the Alcock–Paczynski dependence of $f\sigma_8$ on the fiducial cosmology. The growth sector is phenomenological: σ_8 multiplies GR growth independently of the structural source, so the growth data do not test the response of a fixed theory to A_s . With supernovae, our CPL fit gives $(w_0, w_a) = (-0.85, -0.52)$ and $\Delta\chi^2 = -7.50$ relative to ΛCDM , close to the DESI DR2 result with the full CMB likelihood, $(-0.838 \pm 0.055, -0.62^{+0.22}_{-0.19})$ and $\Delta\chi^2 = -10.7$ (DESI Collaboration et al. 2025b); the smaller improvement reflects the absence of CMB lensing from the compression.

Fit quality. All models with evolving dark energy fit the joint data almost equally well; the structural law fits best, by a small margin due to halo assembly history. The clocks improve on ΛCDM by $\Delta\chi^2 = -8.4$ to -9.7 and on CPL by 0.6–1.9 at equal parameter count (Table 4). The improvement is shared between BAO (-3.0 to -3.3), the compressed CMB likelihood (-2.3 to -2.9), supernovae (-2.3 to -3.2) and growth (-0.3 to -0.4), and no probe is fitted worse (Table F2); without the growth data the margins are unchanged to within 0.4. Supernovae reverse the ranking found with BAO alone: the clocks whose cascade slows (t) or freezes (D) now fit better than the a -clock, by 1.0–1.3, because the supernovae prefer a milder decline today.

Two comparators locate the source of the improvement. The linear- $w(f)$ model ($n = 2.27$, $f_x = 0.71$) and the mirage relation $w_a = -3.66(1 + w_0)$ (Linder 2007; Lodha et al. 2025), which crosses $w = -1$ at $a \simeq 0.73$ with one shape parameter, both reproduce CPL; the fixed-exponent clocks, with the same parameter count as the mirage model, improve on it by 0.4–1.9. A response to growth with constant slope therefore does not by itself explain the structural improvement: the curvature of the history supplied by the halo assembly term does. Without CMB information, the clocks improve on ΛCDM by 4.6–5.7 and CPL and the linear- $w(f)$ model by 4.9, so about 40% of the joint preference comes from reconciling the late-time geometry with the CMB anchor, whose own χ^2 falls from 3.0 to 0.1–0.3.

The dark-energy history. The data select a single late density maximum, robustly across models. Figure 6 shows the best-fit histories and Fig. 7 the posterior bands. The bands of the three clocks overlap closely although their n and $\sigma_{\min}$ differ widely, and they differ from CPL mainly in curvature: they steepen to $w \simeq -1.5$ to -2.3 at $z = 3$ and decline more gently today, whereas the linear- $w(f)$ model is a straight line in $w(f)$ that flattens to $w \rightarrow -1 - n(1 - f_x)/3 \simeq -1.22$ in matter domination, as CPL does. In the joint posterior, 98–100% of histories have $w_0 > -1$ and exactly one crossing of $w = -1$ at $0 < z < 3$, a density maximum, against 22–49% of prior histories (Table 5); multiple crossings are rare under both, reflecting the family rather than the data. The maximum lies at $a_x = 0.63$ –0.70, against 0.72 for CPL and 0.71 for the linear- $w(f)$ model, and precedes equality, $a_\Lambda = 0.749$ –0.752, in 81–95% of posterior histories, against 20–33% under the prior. The closed-form crossover, $a_* \simeq 0.64$ (Section 4),

Table 4. Fit quality: $\Delta\chi^2$ relative to Λ CDM for each data combination (effective interpretation), with the number of fitted parameters k for BAO+CMB and BAO+CMB+SN; the joint fit adds σ_8 to every model. Geometry-only: BAO and Pantheon+ without CMB information, with $c/(H_0 r_d)$ and M_B profiled. The last two columns give the joint best-fit exponent and w_0 . Full fits, the probe decomposition and the excursion-set interpretation are given in Appendix F, and the fixed-exponent fits in Appendix G.

Model	k	BAO+CMB	+SN	+SN+ $f\sigma_8$ (joint)	geometry-only	n (joint)	w_0 (joint)
Λ CDM	3	0	0	0	0	—	−1.00
CPL	5	−7.91	−7.50	−7.72	−4.90	—	−0.85
linear $w(f)$	5	−7.91	−7.55	−7.79	−4.94	2.27 ($f_\times = 0.71$)	−0.86
mirage	4	−5.18	—	−7.69	—	—	—
a -clock	5	−6.96	−8.04	−8.35	−4.55	1.73	−0.85
t -clock	5	−6.54	−9.05	−9.40	−5.30	0.75	−0.89
D -clock	5	−6.24	−9.26	−9.65	−5.74	0.14	−0.92
a -clock, $n = 2$	4	−4.98	—	−8.04	—	2	—
t -clock, $n = 1$	4	−5.54	—	−9.08	—	1	—
D -clock, $n = 0$	4	−6.16	—	−9.56	—	0	—

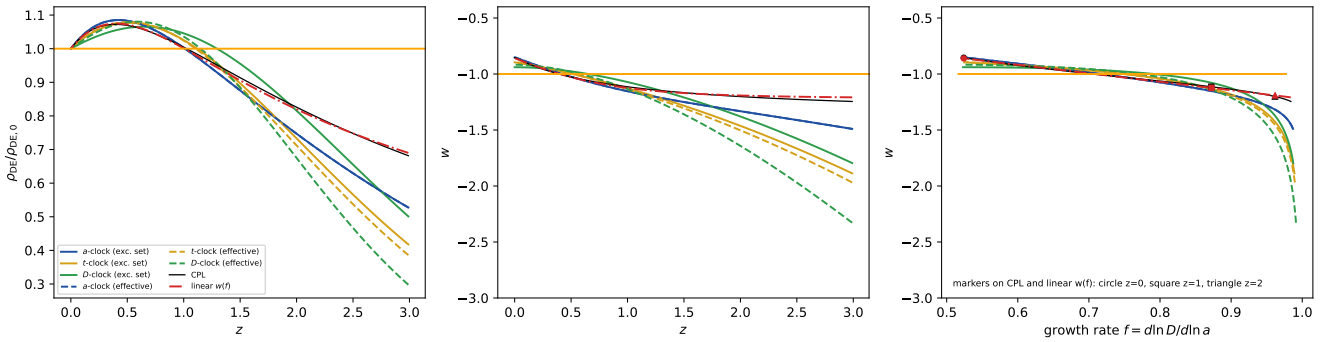


Figure 6. Dark-energy histories at the joint best fits: $\rho_{\text{DE}}/\rho_{\text{DE},0}$ (left) and w (middle) against redshift, and w against the growth rate $f = d \ln D / d \ln a$ (right), for the effective (dashed) and excursion-set (solid) clocks, CPL (black), the linear- $w(f)$ model (red dash-dotted) and Λ CDM (orange). In the right panel f increases toward the past; markers on the CPL and linear- $w(f)$ curves indicate $z = 0, 1$ and 2 . The linear- $w(f)$ model is a straight line crossing $w = -1$ at $f_\times \simeq 0.71$, and CPL is close to it. The clocks share the low-redshift segment but curve steeply downward as $f \rightarrow 1$, where halo assembly makes the source rise rapidly toward the present.

Table 5. Joint posterior event probabilities (effective interpretation), from 300 posterior draws, with 95% Wilson intervals in brackets or as upper limits, and prior probabilities (300 draws from the priors of Table 3) in parentheses. Events: $w_0 > -1$; exactly one crossing of $w = -1$ at $0 < z < 3$, which is a density maximum; more than one crossing; and a maximum before dark-energy-matter equality. $S_8 \equiv \sigma_8(\Omega_m/0.3)^{1/2}$. All probabilities are conditional on the model family and the fixed calibration inputs; the excursion-set and BAO+CMB values are in Table F5.

Clock	$P(w_0 > -1)$	$P(\text{one max.})$	$P(\text{multiple})$	$P(a_\times < a_\Lambda)$	S_8
a	1.00 [0.99,1.00]	1.00 (0.49)	< 0.013	0.81 (0.33)	0.872 ± 0.044
t	1.00 [0.99,1.00]	1.00 (0.33)	< 0.013	0.87 (0.25)	0.875 ± 0.045
D	0.98 [0.96,0.99]	0.98 (0.22)	< 0.013	0.95 (0.20)	0.873 ± 0.050

lies in this range, so the ordering $a_* \lesssim a_\times < a_\Lambda$ of Section 4 is realized by the fitted histories. Before halos form the histories approach $w \rightarrow -(n+1)/3$, with dark energy a fraction $\lesssim 3 \times 10^{-7}$ of the total at $z = 10^3$ (Fig. F3).

Beyond maximum likelihood. *Tying the evolution of w to structure growth restricts the histories to those the data prefer, which raises the evidence and improves prediction.* Three further comparisons use BAO, the compressed CMB likelihood and each supernova compilation in turn (Table 6).

Nested-sampling evidences (Python dynesty, priors of Table 3, uncertainty $\simeq 0.35$) favour the t -clock and the linear- $w(f)$ model over CPL by $\Delta \ln B \simeq 1.7$ –3.8, the linear- $w(f)$ model having the highest evidence for three of the four compilations; part of this is a prior-volume effect, since a narrower CPL prior, $w_0 \in [-1.5, -0.3]$ and $w_a \in [-2.5, 0.5]$, chosen after inspecting the posterior, raises CPL by $\simeq 1.7$ –1.8 and reduces the advantage to 0.4–2.1. Fitted to BAO and CMB information alone, the growth-tied posteriors predict all four supernova compilations better than CPL, by 1.4–6.5 in the posterior-averaged score, because the CPL posterior extends to steeply declining histories ($w_0 \simeq -0.45$). The two highest-redshift BAO measurements, predicted from the remaining data, are fitted best by Λ CDM; CPL and the linear- $w(f)$ model are worse by 0.4–1.7, and the clocks by 0.9–3.2.

Converted into a Gaussian-equivalent significance for the number of additional parameters, as DESI does, the structural law is preferred over Λ CDM at 1.8 – 2.2σ with BAO and CMB information alone and 2.4 – 2.7σ in the joint fit, against 2.3σ for CPL and the linear- $w(f)$ model; the fixed-exponent versions reach 2.8 – 3.1σ , but their exponents were selected on the same data. With the Cobaya implementations used by DESI, which reproduce the published DESI DR2 CPL values to 0.1 in $\Delta\chi^2$, the three families reach 2.3 – 3.8σ depending on

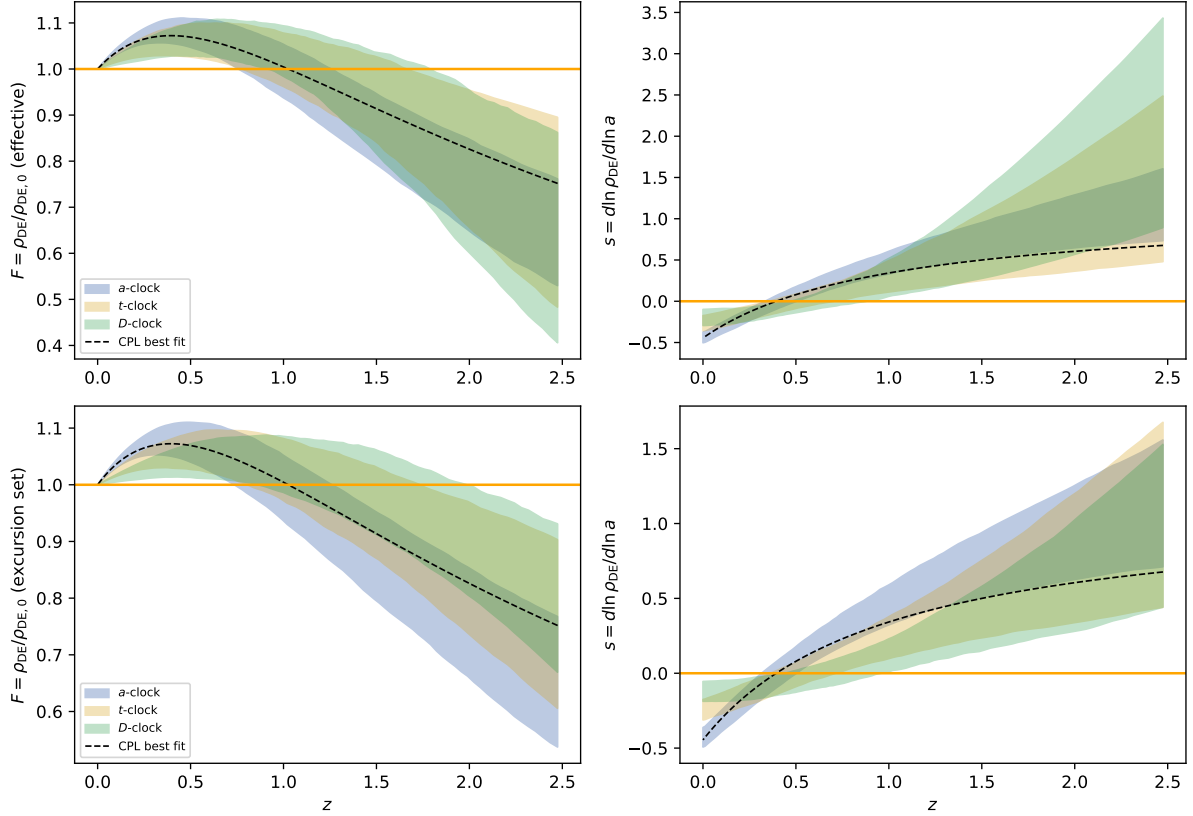


Figure 7. 68% posterior bands of $F(a) = \rho_{\text{DE}}/\rho_{\text{DE},0}$ (left) and $s(a) = d \ln \rho_{\text{DE}}/d \ln a$ (right) from the joint fit, for the three clocks; the dashed line is the CPL best fit. Top: effective interpretation. Bottom: excursion-set interpretation.

Table 6. Comparisons beyond maximum likelihood, using BAO, the compressed CMB likelihood, and each supernova compilation. $\ln B$ is the Bayesian evidence of each model relative to ΛCDM . Positive values favour the model over ΛCDM , and negative values favour ΛCDM . Each evidence has an uncertainty of about 0.35, so differences smaller than about 0.5 should not be interpreted. Prediction is the posterior-averaged supernova score, $-2 \ln(\mathcal{L}_{\text{SN}})$, relative to ΛCDM . The posterior is fitted to BAO and CMB alone, without supernovae, so this is an out-of-sample test. Negative values mean the model predicts the supernova data better than ΛCDM does. Significance is the Gaussian-equivalent preference over ΛCDM , computed with the Cobaya implementations used by DESI; the three models give similar values.

	Model	Pantheon+	Union3	DES-SN5YR	DES-Dovekie
$\ln B$	CPL	-2.87	+1.21	+2.60	-1.02
	<i>t</i> -clock	+0.92	+2.93	+4.72	+1.44
	linear $w(f)$	+0.52	+3.28	+5.28	+1.69
Prediction	CPL	+4.8	-3.6	-6.3	+1.5
	<i>t</i> -clock	-1.7	-5.0	-9.2	-3.3
	linear $w(f)$	+0.2	-5.7	-9.4	-2.3
Significance	all	2.3–2.6 σ	3.3 σ	3.8 σ	2.8–2.9 σ

the supernova compilation (Table 6). Mock data drawn from the ΛCDM best fit confirm the $\chi^2(2)$ calibration for CPL and the linear- $w(f)$ model, which contain ΛCDM ; the structural law does not, so its significance is only indicative. All values are lower than the headline DESI values, which use the full CMB likelihood.

What the parameters mean. *The decomposition of the history is degenerate, so n is not yet a physical exponent.* Table 7 gives, for each clock, the present assembly slope, the best-fit and marginalized exponents and the derived epochs. The effective best fits have $\eta_0 = 0.28, 0.65$ and 0.88 , so that $\eta_0 + n = 2.01, 1.40$ and 1.02 : the *a*-clock chooses early, nearly saturated assembly and compensates with a large n , whereas the *D*-clock chooses late, steep assembly and needs almost none. The posterior follows a narrow valley in the $\sigma_{\text{min}}-n$ plane (Fig. 8), and the marginalized exponents, $n = 1.78 \pm 0.28, 1.05 \pm 0.40$ and 0.46 ± 0.45 , are ordered by the clock decay. The calibration enters the same way: changing p_c by ± 0.05 changes χ^2 by at most 0.2, w_0 by 0.005 and H_0 by $0.03 \text{ km s}^{-1} \text{ Mpc}^{-1}$, but shifts n by ± 0.21 (*a*- and *t*-clocks) and ± 0.10 (*D*-clock), close to $\delta n = 2\delta p_c/f_0$ and $2\delta p_c$ as Eq. (65) requires. The threshold σ_{min} is not identified: its posterior is broad and prior-dependent. In contrast, $\Omega_m = 0.310\text{--}0.312 \pm 0.007$, $H_0 = 67.5\text{--}67.7 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $\sigma_8 = 0.86 \pm 0.04$ are common to all fits; all quoted errors are statistical and conditional on the fixed calibration inputs.

10 DISCUSSION

Three parametrizations compared. CPL, linear in a ; the linear- $w(f)$ model ($x_0 x_a$ model), linear in the growth rate f and equivalent to $\rho_{\text{DE}} \propto a^{-n_f \times D^n}$; and the structural law, which adds a clock and a halo-assembly term to the growth response, fit the current data with nearly the same quality,

Table 7. Decomposition of the joint best fits: present assembly slope η_0 , best-fit exponent and their sum; marginalized exponent with its 95% interval; late crossing of $w = -1$, $a_\times$, and dark-energy–matter equality, a_Λ ; and marginalized w_0 and H_0 [km s^{−1} Mpc^{−1}]. The excursion-set rows are included for comparison. For CPL, $a_\times = 1 + (1 + w_0)/w_a = 0.72$; for the linear- $w(f)$ model, $a_\times = 0.71$.

Prior	Clock	η_0	n	$\eta_0 + n$	n (marg.)	n (95%)	$a_\times$	a_Λ	w_0 (marg.)	H_0 (marg.)
effective	a	0.28	1.73	2.01	1.78 ± 0.28	[1.07, 2.17]	0.70	0.749	-0.854 ± 0.022	67.5 ± 0.5
	t	0.65	0.75	1.40	1.05 ± 0.40	[0.15, 1.66]	0.66	0.751	-0.909 ± 0.029	67.6 ± 0.5
	D	0.88	0.14	1.02	0.46 ± 0.45	[−0.51, 1.27]	0.63	0.752	-0.935 ± 0.031	67.7 ± 0.5
excursion set	a	0.28	1.73	2.01	1.82 ± 0.23	[1.27, 2.17]	0.70	0.749	-0.856 ± 0.020	67.5 ± 0.5
	t	0.61	0.81	1.42	1.22 ± 0.26	[0.73, 1.68]	0.65	0.751	-0.919 ± 0.024	67.7 ± 0.5
	D	0.61	0.54	1.15	0.84 ± 0.25	[0.43, 1.35]	0.60	0.753	-0.956 ± 0.023	67.8 ± 0.5

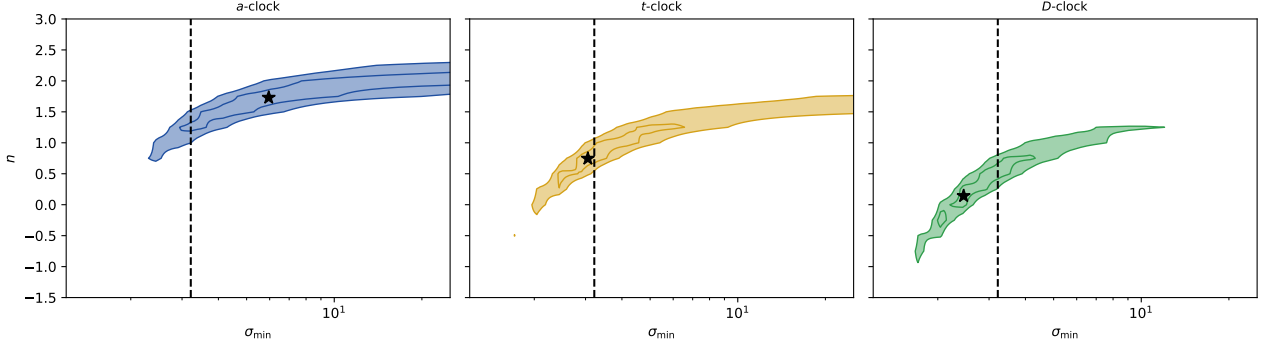


Figure 8. The $\sigma_{\min}$ – n degeneracy in the joint fit (effective interpretation): $\Delta\chi^2 = 2.30$ and 6.18 contours, profiled over Ω_m , with the best fits (stars). The dashed line marks $\beta_f = 1$; the excursion-set interpretation retains the region to its right.

the best $\Delta\chi^2$ differing by at most 0.8 without CMB information and 1.9 with it. They differ in what parameters control.

CPL sets the present value and slope of w independently, so its prior and its BAO+CMB posterior extend to arbitrary crossings and to steeply declining histories that predict the supernovae poorly. The linear- $w(f)$ model admits only histories that cross $w = -1$ once, where the density peaks at $f = f_\times$, and that remain bounded, between $w \rightarrow -1 - n(1 - f_\times)/3$ in matter domination and $w \rightarrow -1 + n f_\times/3$ in the future (Section 2). This concentrates its prior on the histories the data prefer, which explains its higher evidence and better prediction at equal fit; its weaknesses are that it needs n at its prior edge to follow BAO and CMB information alone and that it has no physical scale for n or $f_\times$.

The structural law differs from it only through the assembly slope: because the collapse threshold makes η grow without bound into the past and fall as assembly saturates, its histories curve in $w(f)$, steepen at $z \gtrsim 1$ and decline more gently today. This curvature improves the Pantheon+ fit by up to 1.9 but is not required by the data, slightly degrades the prediction of the highest-redshift BAO, and comes with parameters degenerate with n and with no Λ CDM limit. The near-equality of the linear- $w(f)$ model and the structural law in prediction and evidence indicates that most of the structural law’s empirical content is a power-law response to growth near the crossing.

Interpretation of the exponent. The exponent n describes the response to structure growth beyond that carried by $\langle a_p^2 \rangle$; with a background strain $S \propto D$ it corresponds to a coupling $\propto S^n$ (Section 6). Table 8 summarizes the three physically distinguished values: $n = 0$, the minimal

acceleration-only hypothesis; $n = 1$, a linear structural response; and $n = 2$, a quadratic, elastic-like one. None is derived, and the preferred fixed value depends on the clock, $n = 2, 1$ and 0 for the a -, t - and D -clocks (Appendix G). Among these, the t -clock with $n = 1$ is the most consistent combination: its clock follows the constant cascade rate in cosmic time found in Illustris-1-Dark, and its exponent matches a coupling linear in the largest principal strain; the steady-cascade estimate of Section 4 likewise links the fitted crossings to $n \simeq 1.25$ – 1.4 . The exponent is anomalous and not fixed by dimension in the sense of Section 2: only the amplitude, $\epsilon_{\text{DE}} \sim \langle a_p^2 \rangle / G$, follows from dimensional analysis.

What the data determine. The data constrain the logarithmic slope of the density, not its decomposition into clock decay, halo assembly and an explicit response exponent. The fitted n is exactly degenerate with any missing growth dependence of the source and is best read as the growth dependence not captured by the adopted source model, which may include genuine vacuum strain, an incorrect clock, halo concentrations, mass selection, departures from the assumed halo fraction, averaging, or the feedback of modified gravity on growth; $\sigma_{\min}$ likewise behaves as an effective assembly timescale. An independent calibration of $\langle a_p^2 \rangle(a)$ is needed before n can be read as a vacuum-response exponent. What the data determine, robustly across clocks and halo-assembly parametrizations, is the history: a single density maximum at $z \simeq 0.4$ – 0.8 , preceding equality. Its presence, not its uniqueness, is what the data select.

Table 8. The three physically distinguished values of the response exponent.

Exponent	Motivation	Advantages	Weaknesses
$n = 0$	Response to the acceleration variance alone, $\epsilon_{\text{DE}} \propto \langle a_p^2 \rangle$.	Simplest closure; easiest to complete into a theory; no separate structure variable; most directly testable once $\langle a_p^2 \rangle(a)$ is measured.	Does not suppress the coupling when the strain vanishes. Its success depends entirely on the acceleration history: it fits the effective D -clock well (loss 0.09) but is disfavoured with the excursion-set bound (losses 24–131).
$n = 1$	Response linear in a positive amplitude, motivated by displacement growth and the largest principal strain in collapse.	Simple stress-strain-like dependence. With the matter-era cascade, $\langle a_p^2 \rangle \propto a^{-1}$ and $D \propto a$, it gives a nearly constant density (Eq. (56)). Preferred by the joint t -clock fit (loss 0.3, $\Delta\text{AIC}_c = -1.7$).	A linear dependence on a strain magnitude is non-analytic at zero strain. Neither the coupling nor the continued growth $S \propto D$ on nonlinear scales is derived.
$n = 2$	Quadratic strain invariant, such as $S_{ij}S^{ij}$, analogous to an elastic deformation energy.	Natural positive analytic dependence, vanishing smoothly at zero strain. With the pre-halo $\langle a_p^2 \rangle \propto a^{-2}$ it gives a vacuum-like plateau in matter domination (Eq. (57)).	Analyticity alone does not require $n = 2$; a signed trace can enter linearly if physics allows it. The quadratic growth is too strong for most clocks: it fits the joint a -clock (loss 0.3) but is disfavoured for the t - and D -clocks (losses 17–42).

11 CONCLUSIONS

We have studied a structure-motivated background law in which the dark-energy density responds to the accelerations of dark matter, $\rho_{\text{DE}} \propto \langle a_p^2 \rangle (D/D_0)^n$, its connection to three cosmic coincidences, an effective vacuum model that reproduces it, and its confrontation with DESI DR2 BAO, the DESI compressed CMB likelihood, Pantheon+ supernovae and SDSS growth rates. The main findings are as follows.

(i) *Power-law density and the growth rate as a state variable.* An equation of state linear in the growth rate, the $x_0 x_a$ form $w = -1 - \frac{n}{3}(f - f_\times)$, is equivalent to a power-law density, $\rho_{\text{DE}} \propto a^{-nf_\times} D^n$, and to a constant invariant $\mathcal{I}_{\text{DE}}$; CPL admits no such interval. Because f labels the state of the expansion-growth system rather than the time, this closure makes the equations for (Ω_m, f) autonomous with a time-translation-invariant vector field, couples growth back to expansion, and gives a single heteroclinic orbit from the matter saddle to the dark-energy attractor. Along it the density peaks exactly where $f = f_\times$, w stays bounded over the whole history, and the map $a \mapsto f$ compresses $z > 2$ into $f > 0.96$ while stretching the epoch probed by the data. The power law follows from two symmetries, no preferred epoch and no preferred amplitude of growth, as an intermediate asymptotic between the onset of self-similar clustering (a_*) and dark-energy domination (a_Λ). The power law amplitude is fixed dimensionally, but exponent n is anomalous, departing from a self-similar argument.

(ii) *Three coincidences.* The mass-weighted rms halo acceleration follows from a constant cascade rate and an acceleration-dispersion relation, $a_c = \beta H u$. Matching the linear bulk flow, set by A_s and the equality horizon, to the virial flow at their crossover gives $a_* \simeq 0.64$ and $a_{c0} \simeq 0.14 \beta_V a_*^{-1/4} cH_0 \simeq 0.16 cH_0$, close to Milgrom’s $a_0 \simeq 0.18 cH_0$: the speed of light enters only through the equality horizon. With the density law (1), the cosmic coincidence requires only K of order a few, $a_* \simeq C_* \beta_V^2 a_\Lambda^{2+n} (g_\Lambda/g_0)^n$ with $C_* \simeq 1.04$, and in the steady cascade the phantom crossing and equality fall in one epoch, $a_* \lesssim a_\times < a_\Lambda$.

(iii) *An effective vacuum model.* A self-tuned q -theory vacuum carrying a postulated structural interaction energy reproduces the law at background level. Its positive residual

is the direct interaction energy, not an effect of vacuum displacement; the vacuum tracks the slowly varying drive adiabatically, so the residual is a state function of the current configuration. Conservation requires minimally coupled matter and a carrier sector that supplies the enthalpy of the phantom phase.

(iv) *Confrontation with data.* With the compressed CMB likelihood our analysis reproduces the DESI DR2 result for $w_0 w_a$ CDM. The structural law improves on Λ CDM by $\Delta\chi^2 = -8.4$ to -9.7 and on CPL by 0.6–1.9 at equal parameter count; the linear- $w(f)$ model matches CPL, but both growth-tied families have higher evidence and predict held-out supernovae better, by restricting the histories to those the data prefer. About 40% of the preference comes from reconciling the late-time geometry with the CMB anchor.

(v) *What is robust.* The history is constrained more robustly than its decomposition: 98–100% of posterior histories have $w_0 > -1$ and a single density maximum at $z \simeq 0.4$ –0.8, mostly before equality, whereas n is degenerate with the clock and the assembly timing.

These are conditional phenomenological constraints. The structural coupling is postulated, not derived from a covariant action; the source rests on a provisional calibration from two simulations, whose uncertainties are not propagated; and the compressed CMB likelihood omits lensing.

Future tests and data

Geometry–structure closure. The law ties the dark-energy density to the structure that sources it. Dividing Eq. (55) by its value at a reference epoch a_r removes the unknown coupling:

$$\ln \frac{\epsilon_{\text{DE}}(a)}{\epsilon_{\text{DE}}(a_r)} = \ln \frac{\langle a_p^2 \rangle(a)}{\langle a_p^2 \rangle(a_r)} + n \ln \frac{S(a)}{S(a_r)}. \quad (70)$$

Geometry measures the left-hand side: BAO and supernovae, with an Ω_m anchor from the CMB. Structure measures the right-hand side: halo abundances, lensing and clustering, interpreted through simulation-calibrated models of $\langle a_p^2 \rangle$ and of the structural amplitude S . The law passes if a single n , fixed in advance, satisfies Eq. (70) at all redshifts; CPL and uncoupled smooth dark energy impose no such relation.

Supporting tests. (i) *Held-out prediction*: models fitted to the same data predict independent observations without re-tuning, for example supernova distances from BAO and CMB information, or future survey data. (ii) *The w - f trajectory*: it is straight only for a source of constant logarithmic slope, and evolving halo assembly curves it; since neither w nor f is measured directly, this is a consistency check. (iii) *Primordial amplitude*: matched-phase, self-consistent simulations that vary only A_s , at fixed constants K_* , n and S_{ref} , give

$$\frac{d \ln \epsilon_{\text{DE}}}{d \ln A_s} = \frac{d \ln \langle a_p^2 \rangle}{d \ln A_s} + n \frac{d \ln S}{d \ln A_s}, \quad (71)$$

and thus predict, before any comparison, how the dark-energy history must co-vary with a late-time structure amplitude that departs from the CMB expectation. A dark energy unrelated to structure predicts no such co-variation.

Data required. High-redshift expansion data separate the model histories at $1 < z < 3$; better-calibrated low-redshift supernovae fix the density peak and its present decline; and structure data (cluster counts, weak and CMB lensing, redshift-space distortions), interpreted with calibrated simulations, make the closure test possible.

AUTHOR CONTRIBUTIONS

Z.J. Xu performed the conception, data analysis, and writing.

DATA AVAILABILITY

The BAO, supernova and growth data are public (DESI Collaboration *et al.* 2025b; Scolnic *et al.* 2022; Alam *et al.* 2021). The values and files used are listed in Appendix D.

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APPENDIX A: MECHANICAL ANALOGIES

The Kapitza pendulum illustrates the factorized form of Eq. (1). It does not derive it: as shown below, the leading residual of Section 6 is a direct interaction energy, not the induced energy that stabilizes the pendulum.

The Kapitza pendulum. A rigid pendulum of mass m and length l has its pivot vibrated vertically with acceleration $a(t) = a_{\text{piv}} \cos \omega_{\text{piv}} t$, fast compared with its natural frequency, $\omega_{\text{piv}} \gg \sqrt{g/l}$ (Kapitza 1951; Landau and Lifshitz 1976). The drive averages to zero at first order, but the pendulum's fast quiver leaves a contribution of second order in

a_{piv} . Averaging over the drive gives the effective potential

$$U_{\text{eff}}(\theta) = -mgl \cos \theta + \frac{m a_{\text{piv}}^2}{4\omega_{\text{piv}}^2} \sin^2 \theta. \quad (\text{A1})$$

The drive term, a variance a_{piv}^2 times the configuration factor $\sin^2 \theta$, vanishes at $\theta = 0$, where vertical shaking exerts no torque. For $a_{\text{piv}}^2 > 2gl\omega_{\text{piv}}^2$ it stabilizes the inverted position, about which the pendulum oscillates at $\omega = (a_{\text{piv}}^2/2l^2\omega_{\text{piv}}^2 - g/l)^{1/2}$: a zero-mean fast drive creates a static potential without net work.

Correspondence. The fast drive corresponds to the dark-matter force field, which has zero mass-weighted mean, $\sum_i m_i \mathbf{a}_i = 0$, and a rate $\omega_{\text{DM}} = \beta H$, with $\beta \simeq 231$ in matter domination and 162 today (Eq. (24)), far above H . The drive variance a_{piv}^2 corresponds to $\langle a_p^2 \rangle$, the angle θ to the strain S , and $\sin^2 \theta$ to the configuration factor $\kappa(S)$ of Eq. (47). For $n > 0$ an undeformed configuration does not couple, as the pendulum at $\theta = 0$ does not.

Induced and direct response. The Kapitza pendulum is driven far above its natural frequency, $\omega_{\text{piv}} \gg \Omega = (g/l)^{1/2}$, so its response is inertial and scales as m/ω_{piv}^2 . The vacuum is in the opposite regime: its intrinsic rate exceeds the rate at which its drive varies (A3), so it tracks the drive adiabatically. A pendulum in that regime, driven at $\omega \ll \Omega$, would follow the drive quasi-statically, and its cycle-averaged correction, $\Delta U \simeq -(ma_{\text{piv}}^2/4\Omega^2)\sin^2 \theta$, would be negative, set by the stiffness rather than the drive frequency. Correspondingly, the vacuum's induced term, the second term of Eq. (49), is negative and negligible. A soft vacuum, driven above its own rate, would give a positive but inertial shift, $\sim (k_0' B_M)^2/I\omega_{\text{DM}}^2$, of second order in the drive and fixed by an unknown inertia rather than by the scale $\langle a_p^2 \rangle/G$. The positive residual must therefore come from the explicit dependence of the vacuum energy on the drive, the direct term of Eq. (49). It shares the factorized form of the Kapitza energy, but its scale is a static susceptibility: for a vacuum coupled only through gravity, $1/G$ is the natural dimensional scale, which motivates $\epsilon_{\text{DE}} \propto \langle a_p^2 \rangle/G$ but does not derive it.

The exponent. In the pendulum the drive exerts a torque $-mla_{\text{piv}} \cos \omega_{\text{piv}} t \sin \theta$, whose factor $\sin \theta$ is fixed by the geometry of the rod. For any force $f(x) \cos \omega t$, $U_{\text{eff}} = f^2/4m\omega^2$: geometry supplies f , and the induced dynamics squares it. Equation (55) shares this factorized form, but its exponent is not fixed in the same way, for two reasons. The strain dependence of the coupling, the analogue of the rod's geometry, requires a covariant action; and the leading residual is direct rather than induced, so $\kappa(S)$ is the coupling itself rather than the square of a force. A positive coupling that is analytic and even in the strain gives $n = 2$, the analogue of $\sin^2 \theta \simeq \theta^2$; the data do not select a single exponent, the preferred value depending on the clock and the halo term (Section 9).

Configuration. The pendulum's angle responds freely to the drive-induced torque, and that response is what stabilizes it. The strain, by contrast, is prescribed: in the background truncation $S = S_1 D(a)$ follows the linear growing mode (P3) (Zel'dovich 1970), which is why the law depends on D . In this reading $\langle a_p^2 \rangle/G$ plays the role of a gravitational stress of the

Table B1. Parameters of the two dark-matter-only simulations. Γ is the effective shape parameter of the initial spectrum, including the baryon correction for Illustris. A_s is implied by σ_8 for a BBKS spectrum, and $a_{\text{eq}} = \Omega_r/\Omega_m$ with $\Omega_r h^2 = 4.2 \times 10^{-5}$.

	Virgo Λ CDM	Illustris-1-Dark
Ω_m (collisionless)	1.0	0.2726
Ω_Λ	0	0.7274
Ω_b (initial spectrum only)	0	0.0456
h	0.5	0.704
σ_8	0.51	0.809
n_s	1.0	0.963
Γ	0.50	0.15
Box L [$h^{-1}\text{Mpc}$]	239.5	75
Particles	256^3	1820^3
Particle mass	$2.3 \times 10^{11} h^{-1} M_\odot$	$7.5 \times 10^6 M_\odot$
$k_{\text{min}} = 2\pi/L$ [$h \text{Mpc}^{-1}$]	0.026	0.084
A_s (implied)	3.4×10^{-10}	2.0×10^{-9}
a_{eq}	1.7×10^{-4}	3.1×10^{-4}

dark-matter distribution and $\kappa(S)$ that of a strain-dependent coupling; because the vacuum tracks adiabatically, the residual is a state function of the current configuration, with an instantaneous response (Eq. (54)).

APPENDIX B: N-BODY SIMULATIONS

We illustrate both kinds of fluctuation with two dark-matter-only simulations: the Virgo Λ CDM run (Jenkins *et al.* 1998) and Illustris-1-Dark (Vogelsberger *et al.* 2014). Their parameters are listed in Table B1. In Illustris-1-Dark all matter is collisionless; baryons enter only through the initial power spectrum. The two runs differ in cosmology, volume, resolution and spectrum, so they illustrate the statistics but cannot establish their universality.

APPENDIX C: ACCELERATION STATISTICS

The accelerations of Section 3 were computed in post-processing from particle positions (Xu 2025a). For both simulations, halos are identified with the friends-of-friends algorithm with linking length $b(N/V)^{-1/3}$, $b = 0.2$, and the particles are divided into halo particles and out-of-halo particles. The acceleration of particle i is the pairwise sum

$$\mathbf{a}_i = \frac{Gm_p}{a^2} \sum_{j \neq i} \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|^3}, \quad (\text{C1})$$

with $\mathbf{x}$ comoving coordinates, so that $|\mathbf{a}_i|$ is a physical peculiar acceleration. Periodicity is approximated by the 26 neighbouring replicas of the box. No Hubble-drag or expansion term is included: the quantity is the gravitational acceleration, not the time derivative of a code momentum. The statistic includes all particles, and halo-particle accelerations are decomposed into the mean acceleration of their halo, $\mathbf{a}_h$, and the fluctuation about it, $\mathbf{a}_{hp} - \mathbf{a}_h$; a_c is the rms magnitude of this fluctuation, $\sqrt{3}$ times the standard deviation of one Cartesian component, which dominates a_{hp} because $a_h \sim a_{op} \ll a_c$. All accelerations in this paper are rms magnitudes in this sense. The acceleration analysis shown in Fig. 3 is that of Virgo Λ CDM; the corresponding Illustris-1-Dark acceleration measurement is not presented here, because of its computational cost.

Table D1. DESI DR2 BAO measurements used in this work, derived from the official mean vector and covariance.

Tracer	z_{eff}	D_V/r_d	D_M/r_d	D_H/r_d	ρ
BGS	0.295	7.942 ± 0.076	—	—	—
LRG1	0.510	—	13.588 ± 0.168	21.863 ± 0.429	-0.452
LRG2	0.706	—	17.351 ± 0.180	19.455 ± 0.334	-0.395
LRG3+ELG1	0.934	—	21.576 ± 0.162	17.641 ± 0.201	-0.347
ELG2	1.321	—	27.601 ± 0.325	14.176 ± 0.225	-0.398
QSO	1.484	—	30.512 ± 0.764	12.817 ± 0.518	-0.494
Ly α	2.330	—	38.989 ± 0.532	8.632 ± 0.101	-0.431

APPENDIX D: OBSERVATION DATA

BAO data. We use the DESI DR2 BAO measurements from galaxies and quasars (DESI Collaboration *et al.* 2025b) and from the Lyman- α forest (DESI Collaboration *et al.* 2025c), in the machine-readable form distributed with the public Cobaya BAO likelihood: the mean vector `desi_gaussian_bao_ALL_GCcomb_mean.txt` and the covariance `desi_gaussian_bao_ALL_GCcomb_cov.txt`¹. They are Gaussian compressions of the BAO measurements at seven effective redshifts: one isotropic distance D_V/r_d at $z = 0.295$, and six anisotropic pairs $(D_M/r_d, D_H/r_d)$. The covariance is block-diagonal, so measurements at different redshifts are uncorrelated and each pair carries a single correlation coefficient. Table D1 lists the values, the uncertainties $\sigma_i = \sqrt{C_{ii}}$ and the correlations $\rho = C_{ij}/(\sigma_i \sigma_j)$, which we use as provided. As a check, the Lyman- α uncertainties equal the statistical and systematic errors of DESI Collaboration *et al.* (2025c), $D_H/r_d = 8.632 \pm 0.098 \pm 0.026$ and $D_M/r_d = 38.99 \pm 0.52 \pm 0.12$, combined in quadrature.

CMB inputs. We use the DESI DR2 compressed CMB likelihood (DESI Collaboration *et al.* 2025b, Appendix A), a Gaussian prior on $(\theta_*, \omega_b, \omega_{bc})$ with the mean and covariance of equation (59), derived from the Planck PR4 CamSpec early-Universe chains. The sound horizons r_* and r_d and the redshift z_* are tabulated with CAMB (Lewis *et al.* 2000) on a 15×15 grid in (ω_b, ω_{bc}) and interpolated with bicubic splines, for $T_{\text{CMB}} = 2.7255 \text{ K}$, $N_{\text{eff}} = 3.044$ and one massive neutrino with $\Sigma m_\nu = 0.06 \text{ eV}$, as in DESI DR2; at fixed physical densities they vary by less than 5×10^{-7} with h . The photon and neutrino densities as functions of a are taken from the same CAMB background. With h and ω_b fitted, these inputs give 16 of the $N = 1610$ terms of equation (69) together with the BAO measurements.

Supernovae. We use the Pantheon+ distance file `Pantheon+SH0ES.dat` and the covariance `Pantheon+SH0ES_STAT+SYS.cov` of the public data release² (Scolnic *et al.* 2022; Brout *et al.* 2022), keeping the 1590 light curves (not all independent supernovae) with $z_{\text{HD}} > 0.01$ and the corresponding block of the covariance. The corrected magnitudes $m_{b,\text{corr}}$ are compared with $\mu + M_B$, and M_B is marginalized analytically: $\chi_{\text{SN}}^2 = \mathbf{d}^T C^{-1} \mathbf{d} - (\mathbf{1}^T C^{-1} \mathbf{d})^2 / (\mathbf{1}^T C^{-1} \mathbf{1})$, with residual $\mathbf{d} = m_{b,\text{corr}} - \mu$. No SH0ES calibration is used.

¹ https://github.com/CobayaSampler/bao_data, commit bb0c1c9, directory `desi_bao_dr2`.

² <https://github.com/PantheonPlusSH0ES/DataRelease>, commit c447f0f.

Table D2. SDSS growth-rate measurements used in this work.

Sample	z_{eff}	$f\sigma_8$	correlation
BOSS DR12	0.38	0.497 ± 0.045	0.48 with $z = 0.51$
BOSS DR12	0.51	0.459 ± 0.038	
eBOSS DR16 LRG	0.70	0.473 ± 0.044	–
eBOSS DR16 QSO	1.48	0.462 ± 0.045	–

Growth. We use the $f\sigma_8$ components of the BOSS DR12 and eBOSS DR16 consensus measurements, in the files distributed with the Cobaya BAO likelihood (same commit): `sdss_DR16_BAOplus_LRG_FSBAO_DMDHfs8.dat` and its `_covtot.txt` covariance, which contain the BOSS DR12 values at $z = 0.38$ and 0.51 together with the eBOSS DR16 LRG value at $z = 0.70$, and `sdss_DR16_BAOplus_QSO_FSBAO_DMDHfs8.dat` with its covariance (Alam *et al.* 2017, 2021), listed in Table D2. We take the $f\sigma_8$ block of each covariance, which marginalizes over the correlated distances, and do not use the SDSS distances, which overlap with DESI.

Reproducibility. The file `desi_dr2_bao.csv`, supplied with the code, contains Table D1 in the format read by the fitting scripts; the CAMB tables (`camb_tables.npz`), the supernova and growth files are supplied unchanged.

APPENDIX E: NUMERICAL CHECKS

Background and growth. Holding the dark-energy density constant below $a = 10^{-3}$, or replacing this continuation by zero or by the pre-halo power law a^{n-2} , changes χ^2 by less than 2×10^{-5} for all accepted grid points with $\Omega_{\text{DE}}(10^{-3}) > 10^{-6}$ that fit better than ΛCDM . Starting the growth equation in the matter–radiation growing mode, $D \propto 1 + \frac{3}{2}a/a_{\text{eq}}$, instead of $D = D' = a$, changes χ^2 by less than 10^{-6} : the difference excites only the decaying mode, suppressed as $a^{-5/2}$.

Iteration. Starting the iteration from a constant, a rising and a falling initial history converges to the same solution, with $|\Delta\chi^2| < 10^{-4}$, at the best fits and at test points across the prior box. The iteration fails to be unique only at extreme corners ($\Omega_m \geq 0.36$, $\sigma_{\text{min}} = 24$, $n = 2.9$), which lie more than 450 in χ^2 above the best fits. Profiling h and ω_b on the BAO and CMB terms alone, rather than on the full joint likelihood, changes χ^2 by less than 10^{-7} .

Grid and nuisance parameters. Profiles in n are obtained by direct minimization over $(\Omega_m, \sigma_{\text{min}})$ at fixed n ; profiles from the grid, limited by the resolution of the narrow $\sigma_{\text{min}}-n$ valley, are accurate only to ~ 2 in χ^2 and are not used. Halving the grid resolution in n and σ_{min} shifts the marginal means of n by at most 0.35 standard deviations in the effective interpretation and 0.2 in the excursion-set one; the effective lower 95% limits of n move by up to 0.6 and are therefore approximate. The truncation of σ_8 at $[0.4, 1.2]$ is negligible, the conditional posterior, $\sigma_8 \simeq 0.86 \pm 0.04$, lying more than eight standard deviations inside the bounds. Because the CMB fixes h and ω_b almost independently of the late-time model, profiling over them is equivalent to marginalizing up to a nearly constant normalization.

APPENDIX F: SUPPLEMENTARY FITS

This appendix collects the full best-fit tables for both interpretations of the halo term (Tables F1, F2 and F3), the marginalized constraints (Table F4), the remaining event probabilities (Table F5), the BAO+CMB histories (Fig. F1), the joint posteriors (Fig. F2) and the early behaviour (Fig. F3).

Excursion-set interpretation. With BAO and CMB information alone, the excursion-set fits lie on the bound, $\sigma_{\text{min}} = 3.215$, where all three clocks share $\eta_0 = 0.61$; they select $n = 1.06, 0.75$ and 0.58 , with $\Delta\chi^2 = -6.87, -6.04$ and -5.32 , and their exponents are ordered by the clock decay alone, as Eq. (65) requires at common assembly slope. In the joint fit the a -clock optimum satisfies the bound, so its two fits coincide; the bound costs the t - and D -clocks 0.02 and 0.8 in χ^2 , and the ranking becomes $t > D > a$, all within 1.1. The excursion-set histories cross at $a_{\times} = 0.60\text{--}0.70$ and peak at $1.05\text{--}1.16 \rho_{\text{DE},0}$ with BAO and CMB information.

APPENDIX G: FIXED-EXPONENT FITS

Table G1 lists the fits at fixed exponents $n = 0, 1$ and 2 for both data combinations and both interpretations of the halo term. With BAO and CMB information alone (upper half), $n = 1$ costs 0.04–2.2 in χ^2 relative to free n ; since it removes one parameter, it is preferred by AIC_c in every case ($\Delta\text{AIC}_c = -2.2$ to -4.3). The strain-independent coupling, $n = 0$, is also preferred by AIC_c in the effective interpretation, but is disfavoured by $\Delta\chi^2 = 13\text{--}52$ with the excursion-set bound; $n = 2$ costs 1.9–2.0 for the a -clock and 6.7–12.6 for the others. In the joint data (lower half) each clock selects a different exponent: $n = 2$ for the a -clock, $n = 1$ for the t -clock and $n = 0$ for the effective D -clock, each preferred over free n by AIC_c . With one parameter fewer than CPL, these restricted models reach $\Delta\chi^2 = -8.04, -9.08$ and -9.56 relative to ΛCDM , against -7.72 for CPL, so that $\Delta\text{AIC}_c = -2.3, -3.4$ and -3.9 relative to CPL; they were, however, selected after examining several clocks and exponents on the same data. The other exponents are disfavoured. For $n = 0$ the cost is $\Delta\chi^2 = 24\text{--}131$ with the excursion-set bound and 11.1 for the a -clock. For $n = 1$ it is 2.3–6.1 for the a - and D -clocks. For $n = 2$ it is 17–42 for the t - and D -clocks. No single exponent is selected as a universal law: the preferred value depends on the clock. Among the combinations, the t -clock with $n = 1$ is the most consistent: its clock follows the constant cascade rate in cosmic time supported by the simulations, and its exponent matches a coupling linear in the largest principal strain.

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Table F1. Best fits to DESI DR2 BAO with the compressed CMB likelihood on $(\theta_*, \omega_b, \omega_{bc})$ ($N = 16$; k includes h and ω_b), with Ω_m , $\sigma_{\min}$ and n fitted, for the effective ($\sigma_{\min} \geq 1.2$) and excursion-set ($\sigma_{\min} \geq 3.215$) interpretations. All excursion-set fits lie on the bound. $M_{\min} [h^{-1} M_{\odot}]$ is the nominal threshold mass in a Planck cosmology; values in parentheses have $\beta_f > 1$, for which this mapping is not a physical mass. The linear- $w(f)$ model, Eq. (3), has n and $f_{\times}$ fitted (both given in the n column).

Prior	Model	χ^2	$\Delta\chi^2$	AIC _c / BIC	Ω_m	$\sigma_{\min}$	$\log_{10} M_{\min}$	β_f	n	w_0
	Λ CDM ($k = 3$)	15.01	0	23.01 / 23.32	0.301	–	–	–	–	–1.00
	CPL ($k = 5$)	7.10	–7.91	23.10 / 20.96	0.351	–	–	–	–	–0.45
	linear $w(f)$ ($k = 5$)	7.09	–7.91	23.09 / 20.95	0.341	–	–	–	6.00 ($f_{\times} = 0.74$)	–0.61
effective	a -clock	8.05	–6.96	24.05 / 21.91	0.325	2.74	(11.2)	1.11	0.81	–0.78
	t -clock	8.47	–6.54	24.47 / 22.33	0.321	2.3	(11.7)	1.29	0.13	–0.84
	D -clock	8.77	–6.24	24.77 / 22.63	0.318	2.12	(12.0)	1.41	–0.27	–0.88
excursion set	a -clock	8.13	–6.87	24.13 / 21.99	0.323	3.215	10.6	1.00	1.06	–0.80
	t -clock	8.97	–6.04	24.97 / 22.83	0.314	3.215	10.6	1.00	0.75	–0.89
	D -clock	9.69	–5.32	25.69 / 23.55	0.308	3.215	10.6	1.00	0.58	–0.95

Table F2. Joint best fits ($N = 1610$; k includes h , ω_b and σ_8). The a -clock optimum satisfies the excursion-set bound, so its two fits coincide. The last four columns decompose χ^2 into BAO, the compressed CMB likelihood, supernovae and growth. $M_{\min}$ as in Table F1. The linear- $w(f)$ model has n and $f_{\times}$ fitted (both given in the n column).

Prior	Model	χ^2	$\Delta\chi^2$	AIC _c / BIC	Ω_m	$\sigma_{\min}$	$\log_{10} M_{\min}$	β_f	n	w_0	BAO	CMB	SN	$f\sigma_8$
	Λ CDM ($k = 4$)	1424.09	0	1432.11 / 1453.62	0.302	–	–	–	–	–1.00	12.11	3.00	1405.67	3.30
	CPL ($k = 6$)	1416.36	–7.72	1428.41 / 1460.67	0.311	–	–	–	–	–0.85	9.47	0.83	1402.99	3.08
	linear $w(f)$ ($k = 6$)	1416.30	–7.79	1428.35 / 1460.60	0.311	–	–	–	2.27 ($f_{\times} = 0.71$)	–0.86	9.40	0.82	1403.02	3.06
effective	a -clock	1415.74	–8.35	1427.79 / 1460.04	0.312	5.95	8.0	0.77	1.73	–0.85	9.11	0.30	1403.34	2.99
	t -clock	1414.69	–9.40	1426.74 / 1459.00	0.312	3.06	(10.8)	1.03	0.75	–0.89	8.84	0.19	1402.71	2.95
	D -clock	1414.44	–9.65	1426.49 / 1458.74	0.311	2.45	(11.5)	1.22	0.14	–0.92	8.89	0.12	1402.52	2.91
excursion set	a -clock	1415.74	–8.35	1427.79 / 1460.04	0.312	5.95	8.0	0.77	1.73	–0.85	9.11	0.30	1403.34	2.99
	t -clock	1414.71	–9.38	1426.76 / 1459.01	0.311	3.215	10.6	1.00	0.81	–0.90	8.83	0.27	1402.65	2.96
	D -clock	1415.24	–8.84	1427.30 / 1459.55	0.309	3.215	10.6	1.00	0.54	–0.94	9.03	0.70	1402.48	3.03

Table F3. Best fits without the growth data (BAO, CMB information and supernovae; $N = 1606$). The linear- $w(f)$ model has n and $f_{\times}$ fitted (both given in the n column).

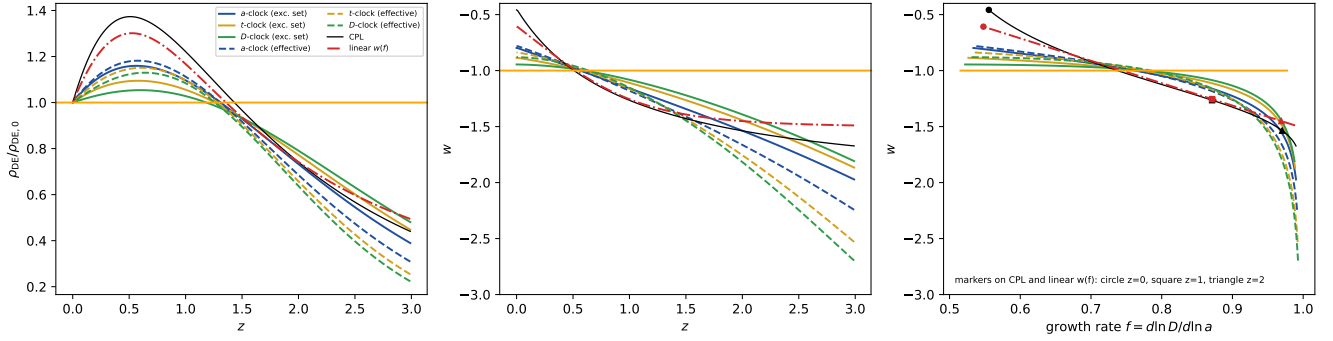
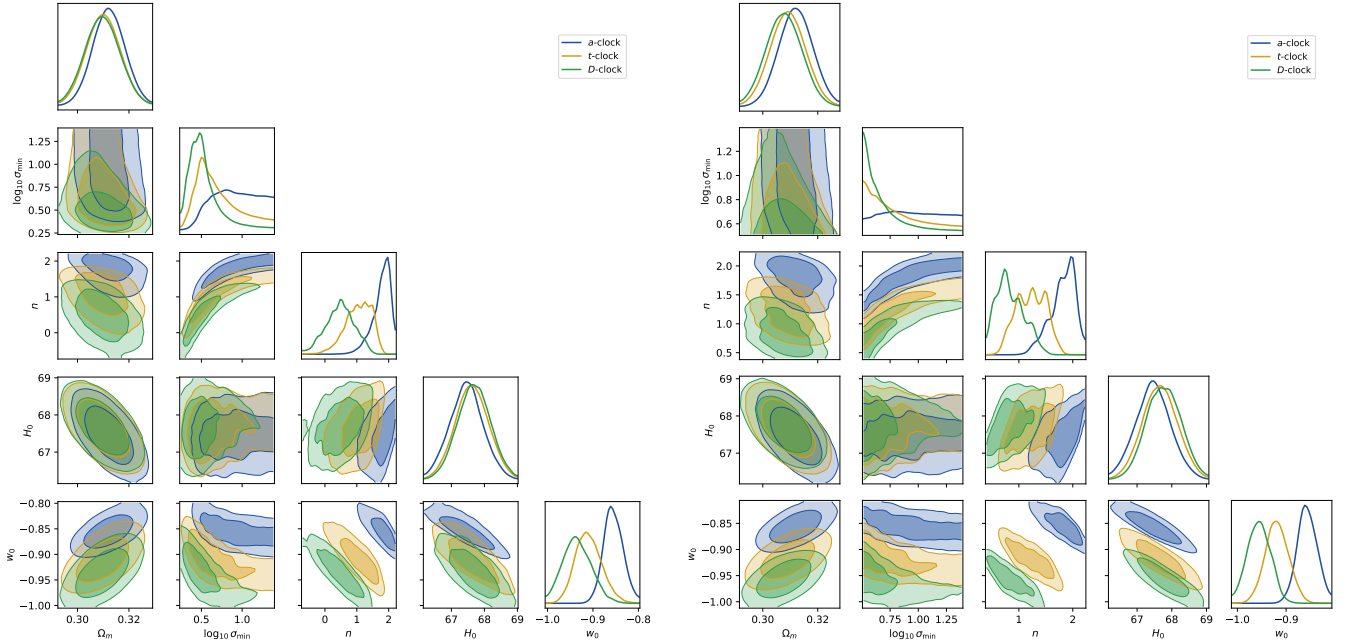
Prior	Model	χ^2	$\Delta\chi^2$	AIC _c / BIC	Ω_m	$\sigma_{\min}$	$\log_{10} M_{\min}$	β_f	n	w_0
	Λ CDM ($k = 3$)	1420.78	0	1426.80 / 1442.93	0.302	–	–	–	–	–1.00
	CPL ($k = 5$)	1413.28	–7.50	1423.32 / 1450.19	0.311	–	–	–	–	–0.85
	linear $w(f)$ ($k = 5$)	1413.23	–7.55	1423.27 / 1450.14	0.311	–	–	–	2.24 ($f_{\times} = 0.71$)	–0.86
effective	a -clock	1412.74	–8.04	1422.78 / 1449.65	0.312	6.29	8.0	0.76	1.76	–0.85
	t -clock	1411.74	–9.05	1421.78 / 1448.65	0.312	3.12	(10.7)	1.02	0.78	–0.89
	D -clock	1411.52	–9.26	1421.56 / 1448.43	0.311	2.5	(11.5)	1.20	0.18	–0.92
excursion set	a -clock	1412.74	–8.04	1422.78 / 1449.65	0.312	6.29	8.0	0.76	1.76	–0.85
	t -clock	1411.74	–9.04	1421.78 / 1448.65	0.311	3.215	10.6	1.00	0.81	–0.90
	D -clock	1412.21	–8.57	1422.25 / 1449.12	0.309	3.215	10.6	1.00	0.54	–0.94

Table F4. Marginalized constraints from the joint fit. σ_8 is marginalized analytically.

Prior	Clock	Ω_m	$\sigma_{\min}$	n	n (95%)	H_0	w_0
effective	a	0.312 ± 0.006	10.2 ± 6.1	1.78 ± 0.28	[1.07, 2.17]	67.5 ± 0.5	-0.854 ± 0.022
	t	0.310 ± 0.007	6.2 ± 4.8	1.05 ± 0.40	[0.15, 1.66]	67.6 ± 0.5	-0.909 ± 0.029
	D	0.310 ± 0.007	3.9 ± 2.8	0.46 ± 0.45	[–0.51, 1.27]	67.7 ± 0.5	-0.935 ± 0.031
excursion set	a	0.312 ± 0.006	10.6 ± 6.0	1.82 ± 0.23	[1.27, 2.17]	67.5 ± 0.5	-0.856 ± 0.020
	t	0.309 ± 0.006	7.4 ± 5.0	1.22 ± 0.26	[0.73, 1.68]	67.7 ± 0.5	-0.919 ± 0.024
	D	0.308 ± 0.007	5.5 ± 3.6	0.84 ± 0.25	[0.43, 1.35]	67.8 ± 0.5	-0.956 ± 0.023

Table F5. Posterior event probabilities not shown in Table 5: the joint fit in the excursion-set interpretation (300 draws) and BAO+CMB information alone (150 draws), with 95% Wilson intervals and prior probabilities in parentheses, as in Table 5.

Data	Prior	Clock	$P(w_0 > -1)$	$P(\text{one maximum})$	$P(\text{multiple})$	$P(a_\times < a_\Lambda)$	S_8
joint	excursion set	a	1.00 [0.99,1.00]	1.00 (0.41)	< 0.013	0.77 (0.29)	0.876 ± 0.045
		t	1.00 [0.99,1.00]	1.00 (0.23)	< 0.013	0.87 (0.17)	0.874 ± 0.046
		D	0.97 [0.95,0.99]	0.97 (0.19)	< 0.013	0.88 (0.18)	0.871 ± 0.045
BAO+CMB	effective	a	1.00 [0.98,1.00]	1.00 (0.49)	< 0.025	0.91 (0.33)	—
		t	0.98 [0.94,0.99]	0.98 (0.33)	< 0.025	0.80 (0.25)	—
		D	0.83 [0.76,0.88]	0.82 (0.22)	< 0.025	0.67 (0.20)	—
	excursion set	a	1.00 [0.98,1.00]	1.00 (0.41)	< 0.025	0.90 (0.29)	—
		t	0.96 [0.92,0.98]	0.95 (0.23)	< 0.025	0.69 (0.17)	—
		D	0.66 [0.58,0.73]	0.65 (0.19)	< 0.025	0.60 (0.18)	—

**Figure F1.** Dark-energy histories at the BAO+CMB best fits: $\rho_{\text{DE}}/\rho_{\text{DE},0}$ (left) and w (middle) against z , and w against the growth rate f (right), for the effective (dashed) and excursion-set (solid) clocks, CPL (black), the linear- $w(f)$ model (red) and Λ CDM (orange).**Figure F2.** Marginalized 68% and 95% constraints from the joint fit. Left: effective interpretation. Right: excursion-set interpretation.

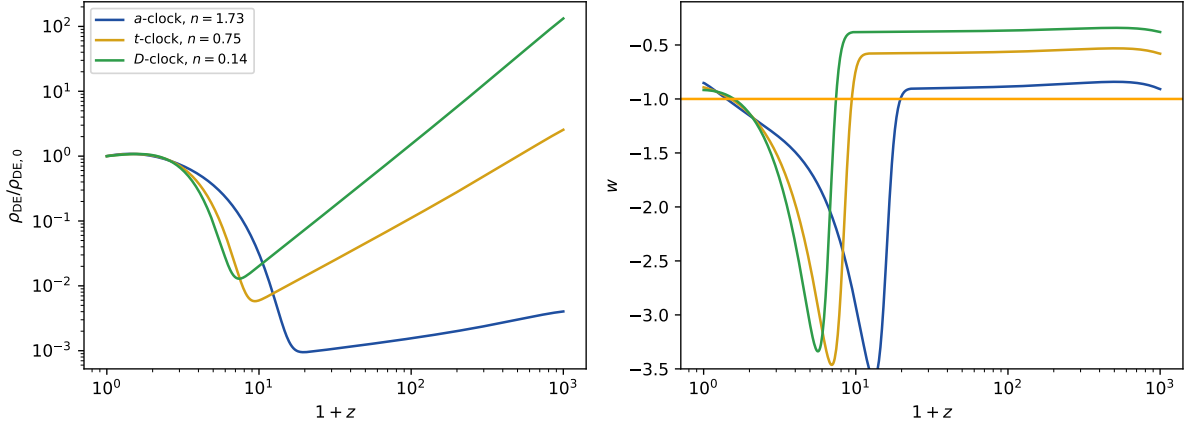


Figure F3. Early-time behaviour of the effective joint best fits: $\rho_{\text{DE}}/\rho_{\text{DE},0}$ (left) and w (right) versus $1+z$. Before halos form, $w \rightarrow -(n+1)/3$, i.e. -0.91 , -0.58 and -0.38 , and an early crossing of $w = -1$ occurs at $z \simeq 6$ – 19 as halo formation begins.

Table G1. $\Delta\chi^2$ relative to ΛCDM at fixed exponent n , with Ω_m and σ_{min} refitted; in brackets (loss; ΔAIC_c), the loss relative to the free- n optimum of the same clock, data set and prior, and ΔAIC_c of the restricted model relative to free n (negative values favour the restricted model). ΔBIC follows by subtracting 2.77 (BAO+CMB) or 7.38 (joint) from the loss. A constrained fit cannot be better than the unconstrained one; equal values mean that the bound is inactive.

Data	Prior	Clock	free n	$n = 0$	$n = 1$	$n = 2$
BAO+CMB	effective	a	-6.96	-6.33 (0.6; -3.7)	-6.91 (0.0; -4.3)	-4.98 (2.0; -2.4)
		t	-6.54	-6.52 (0.0; -4.3)	-5.54 (1.0; -3.4)	+0.68 (7.2; +2.9)
		D	-6.24	-6.16 (0.1; -4.3)	-4.06 (2.2; -2.2)	+6.39 (12.6; +8.3)
	excursion set	a	-6.87	+44.65 (51.5; +47.2)	-6.73 (0.1; -4.2)	-4.98 (1.9; -2.5)
		t	-6.04	+19.16 (25.2; +20.8)	-5.54 (0.5; -3.9)	+0.68 (6.7; +2.4)
		D	-5.32	+8.00 (13.3; +9.0)	-4.06 (1.3; -3.1)	+6.39 (11.7; +7.3)
joint	effective	a	-8.35	+2.72 (11.1; +9.0)	-6.05 (2.3; +0.3)	-8.04 (0.3; -1.7)
		t	-9.40	-7.13 (2.3; +0.2)	-9.08 (0.3; -1.7)	+8.05 (17.4; +15.4)
		D	-9.65	-9.56 (0.1; -1.9)	-5.87 (3.8; +1.8)	+32.51 (42.2; +40.1)
	excursion set	a	-8.35	+122.50 (130.9; +128.8)	-2.23 (6.1; +4.1)	-8.04 (0.3; -1.7)
		t	-9.38	+48.93 (58.3; +56.3)	-9.08 (0.3; -1.7)	+8.05 (17.4; +15.4)
		D	-8.84	+15.32 (24.2; +22.1)	-5.87 (3.0; +0.9)	+32.51 (41.4; +39.3)