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Posted Date: October 27th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11371408/v1>

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Additional Declarations: No competing interests reported.

Nastrom–Gage-like Energy Spectral Transition in a Minimal Divergent Two-Layer Model

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Abstract

The atmospheric horizontal kinetic energy spectrum exhibits a well-known transition from a steep k^{-3} power law at synoptic scales to a shallower $k^{-5/3}$ power law in the mesoscale range (where k is the horizontal wavenumber), as established by aircraft observations. While recent global high-resolution models reproduce this feature, identifying the minimal dynamical ingredients responsible for the transition remains challenging due to full-model complexity. Here, we perform ultra-high-resolution numerical simulations (up to T3413) using a minimal two-layer baroclinic model on a rotating sphere that retains vertical baroclinicity and inertia-gravity waves while filtering out Lamb waves. The model self-consistently reproduces the spectral transition to a shallow high-wavenumber regime (k^{-1} to $k^{-5/3}$). Hough mode decomposition and cumulative energy flux analyses reveal that the k^{-3} subrange is governed by the quasi-geostrophic forward enstrophy cascade of Rossby modes, whereas the shallow spectrum is driven by a forward energy cascade into inertia-gravity modes. Spa-

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tiotemporal analyses show that these gravity waves are excited via spontaneous imbalance in the exit region of the upper-tropospheric jet streak. The resulting k^{-1} spectrum is explained by a straining of small-scale gravity waves by the large-scale barotropic strain field, analogous to passive scalar dynamics. Sensitivity experiments demonstrate that the transition wavenumber is modulated by static stability and planetary rotation rate via variations in eddy scale and Rossby number. Furthermore, numerical experiments with wavenumber-dependent damping on divergent components suggest that vertical wave escape provides an effective dissipation that bridges the theoretical k^{-1} spectrum to the observed $k^{-5/3}$ power law.

keywords: Nastrom–Gage spectrum, Mesoscale turbulence, Spontaneous imbalance, Inertia-gravity waves, Baroclinic two-layer model, Energy cascade

1 Introduction

Nastrom and Gage (1985) extensively demonstrated, using commercial aircraft observation data (GASP), that the horizontal energy spectrum of atmospheric motions exhibits a characteristic bent structure from synoptic scales (wavelengths around 2000 km) down to the mesoscale (around 500 km) and smaller scales. They reported that the spectrum follows a -3 power law consistent with the quasi-geostrophic (QG) turbulence theory of Charney (1971) in the low-wavenumber region (scales larger than 500 km), and transitions to a shallower $-5/3$ power law in the high-wavenumber mesoscale region. Hereafter, this change in the spectral slope is referred to as the “bend” (or spectral transition), and the resulting structure is widely known as the Nastrom–Gage spectrum.

12 Since the landmark study by Nastrom and Gage (1985), the elucidation of
 13 the dynamical mechanisms governing the spectral bend in the mesoscale has ad-
 14 vanced alongside the accumulation of observational data and the refinement of
 15 analytical techniques. In particular, Callies et al. (2016) analyzed a more recent
 16 and extensive dataset from commercial aircraft (MOZAIC/IAGOS) and applied
 17 Helmholtz decomposition to separate the wind velocity field into rotational and
 18 divergent components. They demonstrated that while the rotational compo-
 19 nent overwhelmingly dominates at low wavenumbers, the contributions of the
 20 divergent component and inertia-gravity waves with short vertical wavelengths
 21 increase rapidly in the mesoscale $-5/3$ regime. This definitively established the
 22 importance of non-geostrophic and wave effects from an observational perspec-
 23 tive.

24 Meanwhile, driven by dramatic improvements in computational power, nu-
 25 merical simulations using global high-resolution general circulation models (GCMs)
 26 have made significant progress in reproducing the Nastrom–Gage spectrum and
 27 clarifying its underlying mechanisms. Hamilton et al. (2008) showed through
 28 aquaplanet experiments and high-resolution simulations with a dry dynamical
 29 core (horizontal resolution of ~ 10 km) that this spectral structure is robustly
 30 reproduced solely by dry atmospheric dynamics, independent of complex to-
 31 pography or moist physical processes. To identify the physical modes gener-
 32 ating this spectral bend, modal decomposition techniques were subsequently
 33 introduced. Augier and Lindborg (2013) applied Helmholtz decomposition to
 34 demonstrate that the dominance of the $-5/3$ spectrum at high wavenumbers
 35 stems from a forward kinetic energy cascade, where the divergent component
 36 plays an essential role. Furthermore, Morfa and Stephan (2025) performed
 37 global high-resolution simulations and employed both Helmholtz decomposition
 38 and normal-mode decomposition via Hough mode expansion based on Laplace’s

39 tidal equations. They directly identified the forward energy cascade into inertia-
40 gravity wave modes at high wavenumbers as the primary driver of the spectral
41 bend.

42 However, in complex multi-layer GCMs, numerous physical processes and
43 higher-order vertical modes operate simultaneously, making it difficult to iso-
44 late the minimal essential dynamical factors responsible for the spectral transi-
45 tion due to computational cost and variable complexity. Therefore, a simplified
46 “toy model” approach that omits non-essential physical parameterizations is
47 crucial. Tung and Orlando (2003) used a two-layer QG model (which excludes
48 inertia-gravity waves) to argue that a mesoscale $-5/3$ spectrum can be gener-
49 ated within QG dynamics via a forward energy cascade from large-scale forcing.
50 However, Smith (2004) pointed out that such a $-5/3$ spectrum within the QG
51 framework may be a numerical artifact dependent on resolution or numerical
52 dissipation, such as hyperviscosity. As a toy model incorporating divergent ef-
53 fects, Lindborg and Mohanan (2017) conducted forced turbulence experiments
54 in a single-layer f -plane shallow water system. However, their model failed
55 to reproduce a clear Nastrom–Gage-like bent structure due to the absence of
56 vertical baroclinic effects. Regarding turbulence in a two-layer model on a ro-
57 tating sphere, Kitamura and Matsuda (2004) systematically discussed energy
58 conversion processes between barotropic and baroclinic modes and wave-mean
59 flow interactions, while Wirth (2013) demonstrated the continuous radiation
60 of gravity waves associated with spontaneous imbalance in decaying two-layer
61 turbulence on a β -plane.

62 Therefore, in this study, we conduct high-resolution forced turbulence exper-
63 iments using the baroclinic two-layer toy model proposed by Ishioka et al. (2022),
64 which eliminates Lamb waves while maintaining symmetry between barotropic
65 and baroclinic modes. This model provides a minimal dynamical framework ca-

66 pable of generating the Nastrom–Gage spectrum while incorporating divergence,
67 gravity waves, and vertical baroclinicity. By performing numerical integrations
68 at ultra-high horizontal resolutions (resolving multi-scale interactions) alongside
69 detailed inter-modal energy flux analyses via Helmholtz and Hough decompo-
70 sitions, we update classical understanding (e.g., Kitamura and Matsuda, 2004)
71 from a modern ultra-high-resolution perspective and clarify a robust spectral
72 transition mechanism independent of viscous properties or numerical resolution.

73 The remainder of this paper is organized as follows. Section 2 describes
74 the formulation and assumptions of the baroclinic toy model equations (Ishioka
75 et al., 2022), the forcing-dissipation setup, and the formulation of inter-modal
76 energy transfer functions via Hough mode expansion. Section 3 presents the
77 energy spectra, energy fluxes, and temporal evolution of field variables obtained
78 from our high-resolution experiments. In Section 4, we discuss the physical
79 mechanisms generating the spectral bend by comparing our findings with previ-
80 ous observational, GCM, and toy-model studies. Finally, Section 5 summarizes
81 the main conclusions of this study.

82 **2 Model Formulation, Numerical Experimental** 83 **Setup, and Analytical Methods**

84 **2.1 Governing Equations**

85 In this study, we employ the baroclinic toy model equations on a rotating sphere
86 derived from the non-dimensionalized σ -coordinate primitive equation system
87 proposed by Ishioka et al. (2022). While the detailed derivation follows that of
88 Ishioka et al. (2022), an outline is presented below for the reader’s convenience.

89 We first summarize the physical variables and the non-dimensionalization
90 scheme. Horizontal position is represented by longitude λ and latitude ϕ (or sine

91 latitude $\mu = \sin \phi$), and time by t . For the vertical coordinate, a σ -coordinate
 92 system ($\sigma = p_*/p_{s*}$) defined by the ratio of pressure p_* to surface pressure
 93 p_{s*} is used, where the asterisk subscript (*) denotes a dimensional quantity.
 94 The reference physical parameters used in this model are the planetary radius
 95 $a_* = 6.371 \times 10^6$ m, the planetary rotation rate $\Omega_* = 7.292 \times 10^{-5}$ s $^{-1}$, the
 96 gas constant of dry air $R_* = 287$ J kg $^{-1}$ K $^{-1}$, the specific heat at constant
 97 pressure C_{p*} ($\kappa = R_*/C_{p*} = 2/7$), the reference temperature $T_{0*} = 280$ K, and
 98 the reference surface pressure $p_{0*} = 1000$ hPa. Length, time, temperature, and
 99 pressure are non-dimensionalized by a_* , $a_*/\sqrt{R_*T_{0*}}$, T_{0*} , and p_{0*} , respectively.
 100 Under this choice, the non-dimensional planetary rotation rate is given by $\Omega =$
 101 $\Omega_* a_*/\sqrt{R_*T_{0*}}$. Furthermore, the logarithmic deviation of surface pressure is
 102 defined as $s = \ln(p_{s*}/p_{0*})$.

103 The non-dimensionalized σ -coordinate primitive equation system, which serves
 104 as the starting point, is written as:

$$\frac{\partial \delta}{\partial t} = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial B}{\partial \lambda} - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} A \right) - \nabla^2 \left(\Phi' + \frac{1}{2}(u^2 + v^2) \right), \quad (1)$$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{\sqrt{1-\mu^2}} \frac{\partial A}{\partial \lambda} - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} B \right), \quad (2)$$

$$\begin{aligned} \frac{\partial \tau}{\partial t} = & -u \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \tau}{\partial \lambda} - v \sqrt{1-\mu^2} \frac{\partial \tau}{\partial \mu} - \dot{\sigma} \frac{\partial}{\partial \sigma} (\bar{T} + \tau) \\ & + \left(C + \frac{\dot{\sigma}}{\sigma} + \int_1^0 (C + \delta) d\sigma' \right) \kappa (\bar{T} + \tau), \end{aligned} \quad (3)$$

$$\frac{\partial s}{\partial t} = \int_1^0 (C + \delta) d\sigma'. \quad (4)$$

105 Here, $\delta = \nabla^2 \chi$ is the divergence and $\zeta = \nabla^2 \psi$ is the vorticity. Using the velocity
 106 potential χ and the streamfunction ψ , the zonal wind u and meridional wind v

are given by:

$$u = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \chi}{\partial \lambda} - \sqrt{1-\mu^2} \frac{\partial \psi}{\partial \mu}, \quad (5)$$

$$v = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \psi}{\partial \lambda} + \sqrt{1-\mu^2} \frac{\partial \chi}{\partial \mu}. \quad (6)$$

The non-dimensional temperature field T is decomposed into a basic state $\bar{T}(\sigma)$ and a deviation $\tau(\lambda, \mu, \sigma, t)$. Additionally, the terms A, B, C , absolute vorticity ξ , vertical velocity in σ -coordinates $\dot{\sigma}$, geopotential deviation Φ' , and horizontal Laplacian operator ∇^2 are defined as follows:

$$A = u\xi + \dot{\sigma} \frac{\partial v}{\partial \sigma} + \tau \sqrt{1-\mu^2} \frac{\partial s}{\partial \mu}, \quad (7)$$

$$B = v\xi - \dot{\sigma} \frac{\partial u}{\partial \sigma} - \tau \frac{1}{\sqrt{1-\mu^2}} \frac{\partial s}{\partial \lambda}, \quad (8)$$

$$C = u \frac{1}{\sqrt{1-\mu^2}} \frac{\partial s}{\partial \lambda} + v \sqrt{1-\mu^2} \frac{\partial s}{\partial \mu}, \quad (9)$$

$$\xi = 2\Omega\mu + \zeta, \quad (10)$$

$$\dot{\sigma} = \int_{\sigma}^0 (C + \delta) d\sigma' - \sigma \int_1^0 (C + \delta) d\sigma', \quad (11)$$

$$\Phi' = \Phi'_s - \int_1^{\sigma} \frac{\tau}{\sigma'} d\sigma', \quad (12)$$

$$\nabla^2 = \frac{1}{1-\mu^2} \frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \mu} \left[(1-\mu^2) \frac{\partial}{\partial \mu} \right]. \quad (13)$$

To construct the toy model, we introduce two physical assumptions. First, surface pressure is assumed to be constantly equal to the reference pressure ($p_{s*} = p_{0*}$, i.e., $s = 0$). Second, the barotropic (vertical 0th-order) component of the divergence field δ is set to zero at all times. Through these assumptions, while inheriting the essential characteristics of the two-layer model of Kitamura and Matsuda (2004), Lamb waves are dynamically filtered out, yielding a simplified model with excellent vertical symmetry between barotropic and baroclinic

119 modes.

120 The dependent variables are expanded vertically using Legendre polynomials
 121 $P_l(1 - 2\sigma)$, whose square integral over $\sigma \in [0, 1]$ is normalized to 1. Using
 122 $P_1(1 - 2\sigma) = \sqrt{3}(1 - 2\sigma)$ and under the aforementioned constraints ($s = 0$
 123 and the absence of the barotropic divergence mode $\delta_t = 0$), the divergence and
 124 vorticity fields are truncated horizontally at a triangular truncation wavenumber
 125 M and expanded at the first vertical mode as:

$$\delta(\lambda, \mu, \sigma, t) = \delta_c(\lambda, \mu, t)\sqrt{3}(1 - 2\sigma), \quad (14)$$

$$\delta_c(\lambda, \mu, t) = \sum_{n=1}^M \sum_{m=-n}^n \delta_{c,n}^m(t) Y_n^m(\lambda, \mu), \quad (15)$$

$$\zeta(\lambda, \mu, \sigma, t) = \zeta_t(\lambda, \mu, t) + \zeta_c(\lambda, \mu, t)\sqrt{3}(1 - 2\sigma), \quad (16)$$

$$\zeta_t(\lambda, \mu, t) = \sum_{n=1}^M \sum_{m=-n}^n \zeta_{t,n}^m(t) Y_n^m(\lambda, \mu), \quad (17)$$

$$\zeta_c(\lambda, \mu, t) = \sum_{n=1}^M \sum_{m=-n}^n \zeta_{c,n}^m(t) Y_n^m(\lambda, \mu). \quad (18)$$

126 Here, subscripts t and c denote the barotropic mode and the first baroclinic
 127 mode, respectively. Spherical harmonics are defined as $Y_n^m(\lambda, \mu) = P_n^m(\mu)e^{im\lambda}$,
 128 where $P_n^m(\mu)$ is the associated Legendre polynomial normalized such that its
 129 square integral over $-1 \leq \mu \leq 1$ equals 2.

130 For the temperature deviation τ , assuming a quadratic vertical profile $4\sqrt{3}\sigma(1 -$
 131 $\sigma)$ with maximum amplitude in the mid-troposphere, it is expanded as:

$$\tau(\lambda, \mu, \sigma, t) = \tau_c(\lambda, \mu, t)4\sqrt{3}\sigma(1 - \sigma), \quad (19)$$

$$\tau_c(\lambda, \mu, t) = \sum_{n=0}^M \sum_{m=-n}^n \tau_{c,n}^m(t) Y_n^m(\lambda, \mu), \quad (20)$$

132 where the numerical coefficient $4\sqrt{3}$ is chosen to simplify the resulting evolution

133 equations.

134 Substituting these expansions into the primitive equations and applying the
 135 Galerkin projection ultimately yields the governing time-evolution equations for
 136 the two-layer baroclinic toy model:

$$\begin{aligned} \frac{\partial \delta_c}{\partial t} = & \frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (v_c \xi_t + v_t \zeta_c) - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (u_c \xi_t + u_t \zeta_c) \right) \\ & - \nabla^2 (\tau_c + u_t u_c + v_t v_c), \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial \zeta_t}{\partial t} = & -\frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (u_t \xi_t + u_c \zeta_c + \delta_c v_c) \\ & - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (v_t \xi_t + v_c \zeta_c - \delta_c u_c) \right), \end{aligned} \quad (22)$$

$$\frac{\partial \zeta_c}{\partial t} = -\frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (u_c \xi_t + u_t \zeta_c) - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (v_c \xi_t + v_t \zeta_c) \right), \quad (23)$$

$$\frac{\partial \tau_c}{\partial t} = -u_t \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \tau_c}{\partial \lambda} - v_t \sqrt{1-\mu^2} \frac{\partial \tau_c}{\partial \mu} - S \delta_c, \quad (24)$$

137 where $\xi_t = \zeta_t + 2\Omega\mu$. The barotropic and baroclinic wind components (u_t, v_t)
 138 and (u_c, v_c) are determined from the streamfunctions (ψ_t, ψ_c) and velocity po-
 139 tential χ_c (obtained via $\zeta_t = \nabla^2 \psi_t$, $\zeta_c = \nabla^2 \psi_c$, and $\delta_c = \nabla^2 \chi_c$) as:

$$u_t = -\sqrt{1-\mu^2} \frac{\partial \psi_t}{\partial \mu}, \quad v_t = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \psi_t}{\partial \lambda}, \quad (25)$$

$$u_c = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \chi_c}{\partial \lambda} - \sqrt{1-\mu^2} \frac{\partial \psi_c}{\partial \mu}, \quad v_c = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \psi_c}{\partial \lambda} + \sqrt{1-\mu^2} \frac{\partial \chi_c}{\partial \mu}. \quad (26)$$

140 The static stability parameter S is originally defined as:

$$S = \frac{15}{2} \int_0^1 \sigma (1-\sigma)^2 \left(\kappa \bar{T} - \sigma \frac{d\bar{T}}{d\sigma} \right) d\sigma. \quad (27)$$

141 Following Stevens et al. (1977), assuming a vertical basic-state profile such that

142 $\kappa\bar{T} - \sigma\frac{d\bar{T}}{d\sigma} = \Gamma = \text{const}$, the static stability simplifies to

$$S = \frac{5}{8}\Gamma, \quad (28)$$

143 which is adopted throughout this study.

144 2.2 Forcing and Dissipation

145 In the numerical experiments of this study, the baroclinic toy model equations
146 incorporating thermal forcing and dissipation are expressed as follows:

$$\frac{\partial\delta_c}{\partial t} = \mathcal{N}_{\delta_c} + D(\delta_c), \quad (29)$$

$$\frac{\partial\zeta_t}{\partial t} = \mathcal{N}_{\zeta_t} + D(\zeta_t), \quad (30)$$

$$\frac{\partial\zeta_c}{\partial t} = \mathcal{N}_{\zeta_c} + D(\zeta_c), \quad (31)$$

$$\frac{\partial\tau_c}{\partial t} = \mathcal{N}_{\tau_c} - \gamma_{\text{rad}}(\tau_c - \hat{T}'_{\text{eq}}) + D(\tau_c). \quad (32)$$

147 Here, $\mathcal{N}_{\delta_c}$, $\mathcal{N}_{\zeta_t}$, $\mathcal{N}_{\zeta_c}$, and $\mathcal{N}_{\tau_c}$ represent the dynamical terms (encompassing both
148 non-linear and linear dynamics) appearing on the right-hand sides of Eqs. (21)–
149 (24), respectively.

150 As thermal forcing, a Newtonian cooling term $-\gamma_{\text{rad}}(\tau_c - \hat{T}'_{\text{eq}})$ that relaxes the
151 temperature field toward an equilibrium state $\hat{T}'_{\text{eq}}$ is introduced following Held
152 and Suarez (1994). Held and Suarez (1994) proposed an equilibrium tempera-
153 ture distribution at the bottom of the atmosphere as $T_{\text{eq}*} = (315 - 60\mu^2)$ K,
154 which can be decomposed into a global mean component $\bar{T}_{\text{eq}*} = 295$ K and
155 a latitude-dependent perturbation component $T'_{\text{eq}*} = (20 - 60\mu^2)$ K. In our
156 toy model setup, this temperature distribution is applied at $\sigma = 1/2$, where
157 the temperature perturbation reaches its vertical maximum. Since the verti-
158 cal profile function $4\sqrt{3}\sigma(1 - \sigma)$ evaluates to $\sqrt{3}$ at $\sigma = 1/2$, the equilibrium

159 temperature deviation before non-dimensionalization is obtained as $\tau_{c,eq*}(\mu) =$
160 $\frac{20}{\sqrt{3}}(1 - 3\mu^2)$ K. In the numerical experiments, we use the non-dimensionalized
161 form $\hat{T}'_{eq} = \tau_{c,eq*}/T_{0*}$. For the Newtonian cooling rate γ_{rad} (corresponding to k_2
162 in Held and Suarez, 1994), we adopt a non-dimensionalized value corresponding
163 to 0.05 day^{-1} , which is representative of the upper-layer damping rate in Held
164 and Suarez (1994).

165 The dissipation operator $D(\cdot)$ consists of a high-wavenumber damping term
166 (harmonic viscosity or higher-order hyperviscosity) and a low-wavenumber hy-
167 podiffusion term:

$$D(\cdot) = - [\nu_h(-\nabla^2)^{\alpha_h} + \nu_l(-\nabla^2)^{\alpha_l}] (\cdot). \quad (33)$$

168 Here, ν_h and ν_l denote the viscosity/diffusion coefficients, and α_h and α_l rep-
169 resent their corresponding operator orders. For the high-wavenumber dissipa-
170 tion, $\alpha_h = 1$ corresponds to standard harmonic (Laplacian) viscosity ($-\nu_h \nabla^2$),
171 whereas $\alpha_h > 1$ represents hyperviscosity (e.g., biharmonic viscosity for $\alpha_h = 2$).
172 Hypodiffusion ($\alpha_l < 0$) is introduced specifically to prevent the unphysical ac-
173 cumulation of energy in the low-wavenumber regime (i.e., the condensation or
174 arrest of the inverse energy cascade) characteristic of two-dimensional barotropic
175 turbulence; its order is set to $\alpha_l = -1$, representing an inverse Laplacian damp-
176 ing operator $(-\nabla^2)^{-1}$.

177 2.3 Mode Expansion

178 Unlike on an f -plane, modes on a rotating sphere are not independent for each
179 total wavenumber; therefore, we perform a normal mode expansion using Hough
180 functions, which are the eigenfunctions of Laplace's tidal equations in the ab-
181 sence of forcing and dissipation.

182 First, linearizing the equations for the baroclinic component around a resting

183 basic state (zero background wind) yields the following system:

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\lambda}\right)\chi_c - 2\Omega\left(\mu\nabla^2 + (1-\mu^2)\frac{\partial}{\partial\mu}\right)\psi_c = -\nabla^2\tau_c, \quad (34)$$

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\lambda}\right)\psi_c + 2\Omega\left(\mu\nabla^2 + (1-\mu^2)\frac{\partial}{\partial\mu}\right)\chi_c = 0, \quad (35)$$

$$\frac{\partial\tau_c}{\partial t} + S\nabla^2\chi_c = 0. \quad (36)$$

184 Assuming wave solutions of the form $\chi_c = \text{Re}\{\tilde{X}(\mu)e^{i(m\lambda-\omega t)}\}$, $\psi_c = -\text{Re}\{i\tilde{\Psi}(\mu)e^{i(m\lambda-\omega t)}\}$,
 185 and $\tau_c = -2\Omega\text{Re}\{i\tilde{T}(\mu)e^{i(m\lambda-\omega t)}\}$ (where the complex phase coefficients for ψ_c
 186 and τ_c are devised to simplify the resulting equations), the problem reduces to
 187 the following eigenvalue system using the normalized non-dimensional frequency
 188 $\nu = \frac{\omega}{2\Omega}$ and the Lamb parameter $\epsilon = \frac{4\Omega^2}{S}$:

$$(\nu\Delta - m)\tilde{X} = \left\{\mu\Delta + (1-\mu^2)\frac{d}{d\mu}\right\}\tilde{\Psi} - \Delta\tilde{T}, \quad (37)$$

$$(\nu\Delta - m)\tilde{\Psi} = \left\{\mu\Delta + (1-\mu^2)\frac{d}{d\mu}\right\}\tilde{X}, \quad (38)$$

$$\epsilon\nu\tilde{T} = \Delta\tilde{X}. \quad (39)$$

189 Here, $\Delta = \frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\right] - \frac{m^2}{1-\mu^2}$. Letting A_n^m, B_n^m, C_n^m be the expansion
 190 coefficients when $\tilde{X}, \tilde{\Psi}, \tilde{T}$ are expanded in terms of associated Legendre polyno-
 191 mials, we define the following transformations:

$$a_n^m = \sqrt{\frac{n(n+1)}{\epsilon}}A_n^m, \quad b_n^m = \sqrt{\frac{n(n+1)}{\epsilon}}B_n^m, \quad c_n^m = C_n^m, \quad (40)$$

$$k_n^m = -\frac{m}{n(n+1)}, \quad l_n = -\sqrt{\frac{n(n+1)}{\epsilon}}, \quad (41)$$

$$s_n^m = \sqrt{\frac{(n-1)(n+1)(n^2-m^2)}{n^2(4n^2-1)}}. \quad (42)$$

192 Thus, the algebraic system is reduced to a matrix eigenvalue problem for the

193 normalized frequency eigenvalue ν :

$$\nu a_n^m = k_n^m a_n^m + s_n^m b_{n-1}^m + s_{n+1}^m b_{n+1}^m + l_n c_n^m, \quad (43)$$

$$\nu b_n^m = k_n^m b_n^m + s_n^m a_{n-1}^m + s_{n+1}^m a_{n+1}^m, \quad (44)$$

$$\nu c_n^m = l_n a_n^m. \quad (45)$$

194 Here, $n = m, m+1, \dots$, with $a_{m-1}^m = b_{m-1}^m = 0$. By taking the inner
 195 product of the resulting eigenvectors (Hough functions) and the physical fields,
 196 the expansion coefficients for each mode are uniquely determined. In this study,
 197 we classify and analyze the eastward-propagating Kelvin waves, eastward mixed
 198 Rossby-gravity waves, and the oscillatory components with $m = 0$ under the
 199 category of “inertia-gravity wave modes (W)”, while Rossby waves, westward
 200 mixed Rossby-gravity waves, and the stationary components with $m = 0$ are
 201 grouped into the category of “Rossby wave modes (V)”.

202 On the other hand, since the barotropic vorticity field ($\zeta_t = \nabla^2 \psi_t$) has no
 203 divergent component, the linearized equation simply reduces to the governing
 204 equation for Rossby waves in a two-dimensional non-divergent system:

$$\frac{\partial}{\partial t} \nabla^2 \psi_t + 2\Omega \frac{\partial \psi_t}{\partial \lambda} = 0. \quad (46)$$

205 Substituting the wave solution $\psi_t \propto P_n^m(\mu) e^{i(m\lambda - \omega t)}$ yields the dispersion
 206 relation $\nu = -\frac{m}{n(n+1)}$, indicating that the barotropic field is always composed
 207 purely of Rossby wave modes (V).

2.4 Energy Spectrum

The globally integrated kinetic energy E_K and available potential energy E_A in the baroclinic toy model are defined as follows:

$$E_K = -\frac{1}{2} \langle \psi_t \zeta_t + \psi_c \zeta_c + \chi_c \delta_c \rangle = \frac{1}{2} \sum_{n=1}^M \frac{1}{n(n+1)} \sum_{m=-n}^n (|\zeta_{t,n}^m|^2 + |\zeta_{c,n}^m|^2 + |\delta_{c,n}^m|^2), \quad (47)$$

$$E_A = \frac{1}{2S} \langle \tau_c^2 \rangle = \frac{1}{2S} \sum_{n=0}^M \sum_{m=-n}^n |\tau_{c,n}^m|^2. \quad (48)$$

Here, $\langle \cdot \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 (\cdot) d\mu d\lambda$ represents the global area average. The total energy $E = E_K + E_A$ is a conserved quantity in the absence of forcing and dissipation.

The energy spectrum component at each total wavenumber n is given by:

$$E_K(n) = \frac{1}{2n(n+1)} \sum_{m=-n}^n (|\zeta_{t,n}^m|^2 + |\zeta_{c,n}^m|^2 + |\delta_{c,n}^m|^2), \quad (49)$$

$$E_A(n) = \frac{1}{2S} \sum_{m=-n}^n |\tau_{c,n}^m|^2. \quad (50)$$

By utilizing the physical fields decomposed via the Hough mode expansion, the energy spectrum for each normal mode category can be evaluated. However, it should be noted that while Hough modes are orthogonal to each other under the global energy norm, individual modes do not strictly correspond to a single total wavenumber n . Consequently, the scale-dependent energy spectra derived from the Hough mode decomposition should be interpreted as an approximate indicator at each wavenumber n .

2.5 Energy Flux

To analyze the inter-wavenumber and inter-modal energy transfer processes resulting from nonlinear triad interactions, we formulate the energy transfer functions based on Augier and Lindborg (2013) and Lindborg and Mohanan (2017). The kinetic energy transfer function $T_K(n)$ and the potential energy transfer function $T_A(n)$ at total wavenumber n are given by:

$$T_K(n) = \frac{1}{n(n+1)} \sum_{m=-n}^n \operatorname{Re} (F_n^m \delta_{c,n}^{m*} + G_n^m \zeta_{t,n}^{m*} + H_n^m \zeta_{c,n}^{m*}), \quad (51)$$

$$T_A(n) = \frac{1}{S} \sum_{m=-n}^n \operatorname{Re} (I_n^m \tau_{c,n}^{m*}). \quad (52)$$

Here, the superscript $*$ denotes the complex conjugate. Furthermore, F_n^m, G_n^m, H_n^m , and I_n^m are the expansion coefficients obtained by expanding the purely nonlinear terms F, G, H , and I (the nonlinear parts on the right-hand sides of Eqs. (21)–(24)) into spherical harmonics:

$$F = \frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (v_c \zeta_t + v_t \zeta_c) - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (u_c \zeta_t + u_t \zeta_c) \right) - \nabla^2 (u_t u_c + v_t v_c), \quad (53)$$

$$G = -\frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (u_t \zeta_t + u_c \zeta_c + \delta_c v_c) - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (v_t \zeta_t + v_c \zeta_c - \delta_c u_c) \right), \quad (54)$$

$$H = -\frac{1}{\sqrt{1-\mu^2}} \frac{\partial}{\partial \lambda} (u_c \zeta_t + u_t \zeta_c) - \frac{\partial}{\partial \mu} \left(\sqrt{1-\mu^2} (v_c \zeta_t + v_t \zeta_c) \right), \quad (55)$$

$$I = -u_t \frac{1}{\sqrt{1-\mu^2}} \frac{\partial \tau_c}{\partial \lambda} - v_t \sqrt{1-\mu^2} \frac{\partial \tau_c}{\partial \mu}. \quad (56)$$

Note that the planetary rotation effect $2\Omega\mu$ is explicitly excluded from the absolute vorticity ξ_t in Eqs. (53)–(55) so that F, G , and H isolate the pure three-wave nonlinear interactions from linear Rossby wave propagation.

The cumulative energy flux $\Pi(n)$, which measures the net rate of energy

transfer across total wavenumber n , is defined as the cumulative sum of the energy transfer functions from total wavenumber n to the truncation wavenumber M :

$$\Pi(n) = \sum_{n'=n}^M \left(T_K(n') + T_A(n') \right). \quad (57)$$

Under this convention, a positive value of $\Pi(n)$ represents a forward energy cascade toward higher wavenumbers.

When evaluating energy transfers between specific mode categories, the input velocity and thermodynamic fields used to evaluate the nonlinear terms F, G, H , and I are restricted to designated modes. For instance, by evaluating the nonlinear terms using exclusively the Rossby wave mode fields (V) and projecting the resulting nonlinear forcing onto the inertia-gravity wave modes (W), one can isolate the energy flux transferred into the inertia-gravity modes via interactions strictly among Rossby modes ($VV \rightarrow W$).

2.6 Numerical Experimental Setup

To examine parameter dependencies, we conduct nine numerical experiments (RUN01–RUN09) as summarized in Table 1. Triangular spectral truncation (T) is employed at two horizontal resolutions: T1999 (6000×3008 longitude-latitude Gaussian grid) and T3413 (10240×5120 grid). The non-dimensional time step Δt is specified by the number of time steps per day (ndv). Standard settings are $\text{ndv} = 1440$ ($\Delta t = 60$ s) for T1999 and $\text{ndv} = 2520$ ($\Delta t \approx 34.3$ s) for T3413. For sensitivity runs with increased flow speeds or reduced stability (RUN05–RUN07), ndv is increased to 2520.

To prevent unphysical energy accumulation at the largest scales, a hypodiffusion operator ($\alpha_1 = -1$) is applied. The dimensional hypodiffusion coefficient ν_{1*} is set such that the damping time constant for the lowest total wavenumber

260 ($n = 1$) is 10 days:

$$\nu_{1*} = \frac{1}{10 \text{ days}} \frac{1}{2^{\alpha_1}}. \quad (58)$$

261 To prevent spectral blocking near the truncation wavenumber M , high-wavenumber
 262 dissipation is provided by either standard harmonic viscosity ($\alpha_h = 1$) or hyper-
 263 viscosity ($\alpha_h = 2$). The coefficient ν_{h*} is determined by prescribing the damping
 264 time constant τ_{M*} at the maximum wavenumber M :

$$\nu_{h*} = \frac{1}{\tau_{M*}} \frac{1}{[M(M+1)]^{\alpha_h}}. \quad (59)$$

265 For the static stability S , the standard non-dimensional value S_0 is evaluated
 266 from Eq. (28) using the reference vertical temperature gradient $\Gamma_* = 21.17$ K
 267 (Stevens et al., 1977). Sensitivity experiments explore doubled ($S/S_0 = 2.0$,
 268 RUN04) and halved ($S/S_0 = 0.5$, RUN05) stability conditions. In addition,
 269 the planetary rotation rate Ω is varied by factors of $\sqrt{2}$ (RUN06) and $1/\sqrt{2}$
 270 (RUN07). In these sensitivity runs, to prevent spectral blocking caused by en-
 271 hanced nonlinear cascades or reduced wave speeds, the maximum-wavenumber
 272 dissipation timescale τ_{M*} is adjusted accordingly: $\tau_{M*} = (1/7)$ day for RUN04
 273 and RUN06, while much stronger dissipation is applied in RUN07 ($\tau_{M*} =$
 274 $(1/448)$ day) and RUN05 ($\tau_{M*} = (1/7168)$ day). Finally, RUN08 and RUN09
 275 incorporate k_x -dependent dissipation acting exclusively on the divergent com-
 276 ponent to simulate the vertical escape of inertia-gravity wave energy, as detailed
 277 in Section 4.5.

278 The initial condition consists of a localized Gaussian temperature perturba-
 279 tion (Ishioka and Yoden, 1994):

$$\tau_c(\lambda, \mu) = A \left[\exp \left(B(\cos \theta - 1) \right) - \frac{1 - \exp(-2B)}{2B} \right], \quad (60)$$

280 where $A = 0.001$, $B = 100$, and θ denotes the great-circle angular distance from

the center $(\lambda_0, \phi_0) = (\pi, \pi/6)$:

$$\cos \theta = \mu \mu_0 + \sqrt{1 - \mu^2} \sqrt{1 - \mu_0^2} \cos(\lambda - \lambda_0), \quad (61)$$

with $\mu_0 = \sin \phi_0 = 1/2$.

Table 1: List of parameter settings for the numerical experiments. M denotes the horizontal triangular truncation wavenumber, ndv is the number of time steps per day (1 day), α_h is the order of high-wavenumber dissipation ($\alpha_h = 1$ for harmonic viscosity, $\alpha_h = 2$ for standard hyperviscosity), τ_{M*} is the damping time constant at the maximum wavenumber M , S/S_0 is the ratio of static stability to the reference value S_0 , and Ω/Ω_0 is the ratio of the planetary rotation rate to the standard Earth rotation rate Ω_0 . RUN08 and RUN09 represent hypothesis verification experiments incorporating selective wavenumber-dependent dissipation on the divergent component to mimic vertical wave escape.

RUN	M	ndv	α_h	τ_{M*}	S/S_0	Ω/Ω_0	Remarks
01	3413	2520	1	1/7	1.0	1.0	Standard (Harmonic viscosity)
02	3413	2520	2	1/28	1.0	1.0	Standard (Hyperviscosity)
03	1999	1440	2	1/7	1.0	1.0	T1999 Standard
04	1999	1440	2	1/7	2.0	1.0	Stability $\times 2.0$
05	1999	2520	2	1/7168	0.5	1.0	Stability $\times 0.5$
06	1999	2520	2	1/7	1.0	$\sqrt{2}$	Rotation rate $\times \sqrt{2}$
07	1999	2520	2	1/448	1.0	$1/\sqrt{2}$	Rotation rate $\times 1/\sqrt{2}$
08	1999	1440	2	1/7	1.0	1.0	Divergent dissipation ($\lambda_z = 1$ km)
09	1999	1440	2	1/7	1.0	1.0	Divergent dissipation ($\lambda_z = 2$ km)

3 Results

For the two-layer baroclinic model formulated in Section 2, we conducted long-term numerical integrations by introducing the localized Gaussian temperature perturbation given by Eq. (60) as the initial condition. In this section, focusing primarily on the highest-resolution standard experiment with harmonic viscosity (RUN01: T3413), we first present the temporal evolution of physical fields as baroclinic instability develops from the initial perturbation and transitions

290 into a fully turbulent statistical equilibrium state. Subsequently, we clarify the
 291 structural changes in energy spectra and energy fluxes accompanying this tran-
 292 sition, and investigate the relationship between the excitation of inertia-gravity
 293 waves and the spectral bend. Furthermore, by comparing these results with
 294 sensitivity experiments in which viscosity parameters and physical parameters
 295 (static stability, planetary rotation rate, and dissipation form) are systemati-
 296 cally varied, we demonstrate the robustness and parameter dependence of the
 297 spectral transition structure.

298 For clarity, all physical time instances in this and subsequent sections are
 299 reported in dimensional time t_* (day) rather than non-dimensional time t . For
 300 instance, $t_* = 30$ day represents the 30th day from the start of integration,
 301 which corresponds to the 75600th time step in RUN01 (where the number of
 302 steps per day is set to $\text{ndv} = 2520$).

303 **3.1 Time Evolution of Physical Fields and Excitation of** 304 **Inertia-Gravity Waves**

305 We first document the spatial structure and initial time evolution of the physical
 306 fields obtained from the numerical experiment. Figure 1 presents snapshots of
 307 the baroclinic vorticity field at $t_* = 30$ day and $t_* = 60$ day in RUN01. At
 308 $t_* = 30$ day, near the early stage of the integration, the field primarily retains
 309 a zonal structure, though small-amplitude disturbances associated with initial
 310 baroclinic growth begin to emerge. By $t_* = 60$ day, these baroclinic instability
 311 waves fully develop, giving rise to prominent wave-like structures, particularly
 312 near the equatorial region.

313 [Figure 1 about here.]

314 The spatial structure of the flow is further characterized by the distinct
 315 time evolutions of the barotropic and baroclinic components of the zonal mean

316 zonal wind (Fig. 2). In this model, the barotropic component represents the
317 vertical mean flow, whereas the baroclinic component represents the vertical
318 shear (defined here as half of the difference between the layers). The upper-
319 and lower-layer fields are reconstructed as the sum and difference of these two
320 components, respectively. Considering the vertical expansions (14) and (16) for
321 divergence and vorticity, the upper and lower layers correspond to levels where
322 $\sqrt{3}(1-2\sigma) = \pm 1$ ($\sigma \approx 0.21$ and 0.79 , respectively). Assuming a surface pressure
323 of 1000 hPa, these levels roughly correspond to the 210 hPa and 790 hPa pressure
324 surfaces in Earth's atmosphere.

325 For the barotropic component (Figs. 2a,b), while the wind speed is nearly
326 zero at $t_* = 30$ day, momentum transport driven by growing baroclinic instabil-
327 ity waves and radiated barotropic Rossby waves forms a mid-latitude westerly
328 jet and an equatorial easterly jet by $t_* = 60$ day. In contrast, for the baroclinic
329 component (Figs. 2c,d), because thermal forcing with a meridional temperature
330 gradient is continuously applied, vertical shear is maintained via thermal wind
331 balance. Consequently, a vertical westerly shear in the mid-latitudes and a ver-
332 tical easterly shear in the equatorial region are present consistently from the
333 initial stage.

334 [Figure 2 about here.]

335 As the integration progresses beyond $t_* = 70$ day, localized intensification
336 of the divergence field emerges without major changes in the quasi-geostrophic
337 flow. Figure 3 shows snapshots of the baroclinic vorticity field (top row), di-
338 vergence overlaid with upper-layer streamfunction (middle row), and divergence
339 overlaid with lower-layer streamfunction (bottom row) at $t_* = 70$ day, 100 day,
340 and 130 day. At $t_* = 70$ day (left column), large-scale wave structures dominate
341 the equatorial region while fine coherent vortices appear at high latitudes. At
342 this stage, the streamfunctions exhibit gentle meandering corresponding to the

high-latitude jet, and the magnitude of the divergence field remains extremely small globally. However, by $t_* = 100$ day (middle column), localized intensification of divergence begins around the high-latitude jet, despite the macroscopic baroclinic vorticity field remaining largely unchanged. This local divergence enhancement becomes increasingly pronounced toward $t_* = 130$ day (right column).

[Figure 3 about here.]

[Figure 4 about here.]

To capture the detailed process of divergence field intensification, Figure 4 provides magnified snapshots of the high-latitude region shortly after $t_* = 70$ day. At $t_* = 78.13$ day (left column of Fig. 4), local amplification of the divergence field initiates around $\mu = 0.50$ – 0.75 and $\lambda = 0.50$ – 0.75 . By $t_* = 79.33$ day (right column), the absolute value of divergence in this localized area rapidly increases relative to its surroundings. Streamfunction contours reveal that in the region where divergence is enhanced, contour spacing widens in both the upper and lower layers, indicating a “jet exit” region characterized by flow deceleration. These observations suggest that spontaneous emission of intense inertia-gravity waves occurs as the local balance of the background flow breaks down within regions of flow deceleration, manifesting as prominent localized divergence patterns.

3.2 Time Evolution of the Energy Spectrum

We next examine how the structural transitions in the physical fields described above are reflected in the energy spectrum. Figure 5 shows the time evolution of the total energy, inertia-gravity wave, and Rossby wave energy spectra in RUN01.

[Figure 5 about here.]

During the initial stage ($t_* = 60$ –70 day; dotted line), the total energy spectrum is dominated by Rossby wave modes and follows a clear -3 power-law slope across most wavenumbers up to the high-wavenumber truncation (Fig. 5a). At this stage, Rossby wave modes govern the overall spectrum (Fig. 5c), whereas the energy of inertia-gravity wave modes is significantly lower at low wavenumbers and falls off even more steeply toward higher wavenumbers (Fig. 5b).

As the system enters the transition stage ($t_* = 90$ –100 day; dashed line), corresponding to the localized excitation of inertia-gravity waves observed in Fig. 4, the energy of the inertia-gravity wave modes begins to rapidly increase at high wavenumbers ($n \geq 50$).

Once statistical equilibrium is reached ($t_* = 120$ –130 day; solid line), the energy density of the inertia-gravity wave modes eventually surpasses that of the Rossby wave modes at high total wavenumbers ($n \geq 50$). Consequently, a prominent spectral break forms in the total energy spectrum. The high-wavenumber spectral slope shifts to a $-5/3$ slope or an even shallower slope around -1 , whereas the Rossby wave modes consistently maintain a -3 slope throughout the entire integration.

3.3 Spectral Structure and Energy Flux in the Statistical Equilibrium State

To quantitatively evaluate how the excitation of inertia-gravity waves affects the overall energy distribution and spectral transport, we examine the energy spectrum and cumulative energy flux for each mode in the statistical equilibrium state ($t_* = 120$ –130 day). Figure 6 presents these time-averaged spectral profiles.

[Figure 6 about here.]

394 The energy spectrum clearly demonstrates that the spectral break in the
 395 total energy is governed by a transition in the dominant physical wave modes
 396 (Fig. 6a). In the low-wavenumber range ($n < 50$), Rossby wave modes following
 397 a -3 power law dominate the system. Conversely, in the high-wavenumber range
 398 ($n \geq 50$), inertia-gravity wave modes with a shallower slope near -1 become
 399 dominant. The distinct bend in the total energy spectrum (red curve) occurs
 400 precisely at the wavenumber band where the energy densities of these two mode
 401 categories intersect.

402 The cumulative energy flux reveals the detailed pathways of nonlinear energy
 403 transfer across wavenumber space (Fig. 6b). Energy injected by thermal forcing
 404 at low wavenumbers ($n \sim 3$) is primarily transferred within $n \leq 10$ through
 405 Rossby–Rossby wave interactions ($VV \rightarrow V$; red curve). Concurrently, within
 406 this low-wavenumber range, Rossby wave interactions also excite inertia-gravity
 407 wave modes ($VV \rightarrow W$; blue curve). At higher total wavenumbers ($n \geq 10$),
 408 the primary driver of nonlinear interaction shifts to Rossby–gravity wave inter-
 409 actions ($VW \rightarrow W$; green curve), establishing a nearly constant forward energy
 410 flux toward higher wavenumbers. This indicates that a forward energy cas-
 411 cade mediated by inertia-gravity wave modes dominates the high-wavenumber
 412 dynamics, the detailed mechanism of which will be discussed in Section 4.

413 To verify the physical manifestation of the low-wavenumber gravity wave ex-
 414 citation ($VV \rightarrow W$) indicated by the energy flux peak, we examine the spatial
 415 structure of the divergent flow at an earlier stage. Figure 7 displays the diver-
 416 gence field and the associated kinetic energy density distribution at $t_* = 50$ day.

417 [Figure 7 about here.]

418 Although the divergence field itself at $t_* = 50$ day shows no extreme local
 419 values (Fig. 7a), similar to $t_* = 70$ day in Figs. 3d and 3g, the corresponding
 420 divergent kinetic energy density exhibits a clear enhancement along the mean-

421 dering jet (Fig. 7b). Notably, the spatial scale of this early divergent energy
 422 structure is substantially larger than the fine-scale gravity wave signatures that
 423 appear after $t_* = 100$ day. This large-scale divergent structure matches the
 424 spectral peak of the $VV \rightarrow W$ flux at low wavenumbers ($n \leq 10$), confirming
 425 that large-scale Rossby wave interactions continuously feed energy into low-
 426 wavenumber divergent modes prior to the onset of localized small-scale gravity
 427 wave emission.

428 **3.4 Parameter Sensitivity Experiments**

429 To evaluate the robustness and parameter dependence of the spectral transi-
 430 tion structure, we conducted sensitivity experiments by systematically varying
 431 the viscosity formulation, spatial resolution, static stability S , and planetary
 432 rotation rate Ω (relative to its standard value Ω_0).

433 We first address the effects of numerical dissipation order and resolution
 434 (Fig. 8). The comparison between harmonic viscosity ($\alpha_h = 1, \tau_{M*} = (1/7)$ day)
 435 and hyperviscosity ($\alpha_h = 2, \tau_{M*} = (1/28)$ day) demonstrates that hypervis-
 436 cosity, when appropriately scaled, shifts the dissipation range toward higher
 437 wavenumbers, thereby resolving a broad inertial subrange extending from the
 438 spectral break to the truncation wavenumber (Fig. 8a). Furthermore, compar-
 439 ing resolutions T3413 ($\alpha_h = 2, \tau_{M*} = (1/28)$ day) and T1999 ($\alpha_h = 2, \tau_{M*} =$
 440 $(1/7)$ day) confirms that the lower-resolution T1999 experiment reproduces a
 441 spectral transition structure equivalent to that in T3413 (Fig. 8b).

442 [Figure 8 about here.]

443 Next, we evaluate the sensitivity to physical parameters (Fig. 9). In these
 444 experiments, as noted earlier, the hyperviscosity coefficient in each run was
 445 tuned to prevent artificial energy accumulation (spectral blocking) near the

446 truncation wavenumber. The dissipation timescale τ_{M*} for each sensitivity run
 447 was set as described in Section 2.2 and Table 1.

448 [Figure 9 about here.]

449 The response of the energy spectrum exhibits distinct dependencies on static
 450 stability and rotation rate. When the static stability is doubled (RUN04;
 451 Fig. 9b), the spectral break leading to a shallow, Nastrom–Gage-like profile dis-
 452 appears; instead, the spectrum displays a -3 slope at high wavenumbers and a
 453 steeper -4 slope at lower wavenumbers. Conversely, when the static stability is
 454 halved (RUN05; Fig. 9c), a shallow spectrum close to a $-5/3$ power law emerges
 455 broadly from the low-wavenumber region, flattening further toward a -1 slope
 456 at high wavenumbers. Varying the planetary rotation rate yields contrasting
 457 behaviors: for $\sqrt{2}\Omega_0$ (RUN06; Fig. 9d), the spectral transition closely mirrors
 458 that of the standard case (RUN03), whereas for $\Omega_0/\sqrt{2}$ (RUN07; Fig. 9e), the
 459 break wavenumber shifts toward lower wavenumbers.

460 To understand the dynamical origin of the spectral shift under different
 461 rotation rates, we examine the response of the mean zonal flow. Figure 10
 462 illustrates the upper-layer zonal mean zonal wind for each rotation rate, and
 463 Table 2 summarizes the corresponding maximum wind speeds ($U_{\max}$).

464 [Figure 10 about here.]

Table 2: Ratio of planetary rotation rate to the standard value (Ω/Ω_0) and the corresponding maximum upper-layer zonal mean zonal wind speed ($U_{\max}$) at statistical equilibrium ($t_* = 130$ day).

Ω/Ω_0	$U_{\max}$
$1/\sqrt{2}$	0.231
1.0	0.180
$\sqrt{2}$	0.163

465 The maximum wind speed increases to approximately 1.3 times the standard
466 value in the slower-rotating case (RUN07; $\Omega_0/\sqrt{2}$) and decreases to roughly 0.9
467 times in the faster-rotating case (RUN06; $\sqrt{2}\Omega_0$). The dynamical impact of
468 these wind field changes on the energy spectrum, along with the broader physical
469 implications of parameter sensitivity, will be analyzed in detail in Section 4.

470 4 Discussion

471 In this section, we provide a physical interpretation of the numerical findings
472 presented in Section 3, focusing on the generation mechanisms of inertia-gravity
473 waves, their spectral energy transfer, and their sensitivity to physical parame-
474 ters. Note that while asterisks were used to denote dimensional quantities up
475 to Section 3, they are omitted for dimensional quantities in this section unless
476 otherwise noted.

477 4.1 Formation of the k^{-3} Spectrum in the Background 478 Vorticity Field and Gravity Wave Excitation by Spon- 479 taneous Imbalance at the Jet Exit

480 We discuss the generation mechanism of the localized divergence field observed
481 near the high-latitude jet stream in Section 3 (Fig. 4) and the associated exci-
482 tation of inertia-gravity waves.

483 First, we focus on the spectral structure of the background vorticity field,
484 comprising barotropic and baroclinic Rossby wave modes (V), prior to the
485 growth and emergence of prominent gravity waves. In this model, kinetic en-
486 ergy is injected into the low- to mid-wavenumber range ($n \lesssim O(10)$) via baro-
487 clinic instability driven by the meridional temperature gradient. According to
488 two-dimensional and quasi-geostrophic turbulence theories, a forward enstro-

489 phy cascade, wherein enstrophy (vorticity squared) is transferred from low to
 490 high wavenumbers, dominates the higher-wavenumber range beyond the energy
 491 injection range. Through this cascade, the energy spectrum of the balanced
 492 barotropic and baroclinic vortex components forms a steep slope proportional
 493 to k^{-3} (or n^{-3} , where k denotes the dimensional horizontal wavenumber), pre-
 494 cisely matching theoretical predictions.

495 Next, we examine how gravity waves are excited from this background vortic-
 496 ity field with the k^{-3} spectrum. Associated with the forward energy cascade of
 497 the vortex modes (jet streaks) amplified by baroclinic instability, strong nonlin-
 498 ear interactions between barotropic and baroclinic components occur at the exit
 499 of the high-latitude jet stream, producing intense localized acceleration. Under
 500 these conditions, the geostrophic adjustment process cannot keep pace with the
 501 rapid variations in flow velocity and strain, leading to a local breakdown of bal-
 502 ance known as spontaneous imbalance (Zhang, 2004; Plougonven and Zhang,
 503 2014). Consequently, an ageostrophic divergence field (inertia-gravity waves
 504 W) is continuously and spontaneously excited and emitted from the balanced
 505 background vortical flows.

506 The steady-state cumulative energy flux provides quantitative support for
 507 this emission process (Fig. 6b). Specifically, the positive $VV \rightarrow W$ flux (blue
 508 curve) in the low-wavenumber range ($n \leq 10$) demonstrates that energy is con-
 509 tinuously transferred from the k^{-3} vorticity field, maintained by baroclinic in-
 510 stability and the enstrophy cascade, into ageostrophic gravity wave modes via
 511 spontaneous imbalance at the jet exit.

512 4.2 Formation Mechanism of the k^{-1} Spectrum due to the 513 Strain Effect of the Barotropic Flow

514 As shown in Section 3, the inertia-gravity waves radiated through the geostrophic
515 adjustment process form an energy spectrum shallower than the $-5/3$ power
516 law, approaching a -1 power law in the high-wavenumber region. In this re-
517 gion, the forward energy cascade from Rossby wave modes (barotropic flow) to
518 inertia-gravity wave modes ($VW \rightarrow W$) becomes dominant (Fig. 6b). In this
519 section, we discuss the formation mechanism of this shallow high-wavenumber
520 slope from the perspective of the strain (deformation) effect exerted by the large-
521 scale barotropic flow within the framework of statistical turbulence theory.

522 First, consider the background velocity field of the dominant barotropic
523 Rossby wave modes, denoted by $\mathbf{U}$, and let $E_{\text{BT}}(k)$ be its energy spectrum.
524 When the barotropic flow follows a k^{-3} spectrum, the characteristic velocity
525 $U(k)$ associated with each wavenumber band (octave band) can be estimated
526 by dimensional analysis as:

$$U(k) \sim \sqrt{k E_{\text{BT}}(k)} \propto \sqrt{k \cdot k^{-3}} = k^{-1}. \quad (62)$$

527 The strain rate (deformation rate) $S_d(k)$ exerted by this velocity field on motion
528 at equal or smaller scales is on the order of the spatial gradient of the background
529 velocity field, $\nabla \mathbf{U}$. Consequently, we obtain:

$$S_d(k) \sim k U(k) \propto k \cdot k^{-1} = k^0 = \text{const.} \quad (63)$$

530 In other words, in the wavenumber range where the barotropic flow maintains
531 a k^{-3} spectrum, the strain rate $S_d(k)$ imparted by each wavenumber band k
532 takes a constant value, S_d , independent of the wavenumber k .

533 Next, consider inertia-gravity waves whose wavelengths are sufficiently short

534 (high wavenumbers) relative to the spatial scale of the background barotropic
 535 flow $\mathbf{U}$. Within the framework of WKB ray-tracing theory in balanced back-
 536 ground flows (e.g., Plougonven and Zhang, 2014), the time evolution of the
 537 wavevector $\mathbf{k}$ of a localized wave packet under the strain effect of this back-
 538 ground flow is governed by:

$$\frac{d\mathbf{k}}{dt} = -(\nabla\mathbf{U}) \cdot \mathbf{k}. \quad (64)$$

539 Dynamically, this process is equivalent to the advection and stretching of a
 540 conserved passive scalar gradient field by a background strain field. Through this
 541 straining mechanism, the wave vector of the inertia-gravity waves is continuously
 542 stretched toward higher wavenumbers at a rate of $\frac{d\mathbf{k}}{dt} \sim S_d \mathbf{k}$.

543 Now, let $E(k)$ be the energy spectrum of the inertia-gravity waves, and let
 544 $\tau_S \sim S_d^{-1}$ represent the characteristic time scale of energy transfer mediated by
 545 the strain of the barotropic flow. The forward energy flux Π at wavenumber k
 546 within the steady inertial subrange is formulated via dimensional analysis as:

$$\Pi \sim \frac{kE(k)}{\tau_S} \sim S_d k E(k). \quad (65)$$

547 In the energy cascade subrange, the energy flux Π must be constant independent
 548 of wavenumber k ($\Pi = \text{const}$). Because the strain rate S_d is also scale-invariant
 549 ($S_d = \text{const}$), Eq. (65) immediately yields the spectral distribution:

$$E(k) \propto \frac{\Pi}{S_d} k^{-1} \propto k^{-1}. \quad (66)$$

550 This physical derivation is completely analogous to the classical theory for
 551 the k^{-1} subrange spectrum of a passive scalar field developed by Batchelor
 552 (1959) and Kraichnan (1971, 1974), where a passive scalar gradient is stretched

by the strain field in the enstrophy cascade range (k^{-3}) of two-dimensional turbulence. Therefore, the shallow spectrum approaching the -1 power law observed in the high-wavenumber regime of our model is naturally explained as the result of a scale-invariant strain field ($S_d = \text{const}$) created by the barotropic flow (V). This strain continuously stretches the wavenumbers of small-scale inertia-gravity waves (W), which behave like a passive scalar, driving a continuous forward energy cascade toward higher wavenumbers ($VW \rightarrow W$).

4.3 Sensitivity to Static Stability and the Emergence Mechanism of the Zonostrophic Regime

In this section, we discuss the emergence mechanism of the steep spectrum (a k^{-4} -like slope) in the high-wavenumber region and the distinct zonal jet structures (the zonostrophic regime) observed in the sensitivity experiment where the static stability S was set larger than in the standard experiment.

An increase in static stability S produces two primary effects in geophysical fluid dynamics. First, with the expansion of the Rossby radius of deformation $L_D \propto \sqrt{S}$, the growth rate of baroclinic instability driven by thermal forcing and the associated energy conversion are suppressed. Consequently, the kinetic energy and the characteristic eddy velocity U_{eddy} of the disturbance field decrease significantly compared to the standard experiment.

Here, we focus on the Rhines scale (Rhines, 1975), L_β , which marks the threshold where nonlinear energy cascade balances the anisotropic propagation of Rossby waves due to the β -effect, and its corresponding Rhines wavenumber k_β :

$$L_\beta \sim \sqrt{\frac{U_{\text{eddy}}}{\beta}}, \quad k_\beta \sim \sqrt{\frac{\beta}{U_{\text{eddy}}}}. \quad (67)$$

When U_{eddy} decreases, the Rhines scale L_β shrinks, shifting the Rhines wavenumber k_β toward higher wavenumbers (larger total wavenumber n). In the regime

below the Rhines wavenumber ($k < k_\beta$), the β -effect induces the concentration of nonlinear energy into the zonal component ($m = 0$). This triggers a transition to the zonostrophic regime (Galperin et al, 2008), characterized by multiple zonal jets and an extremely steep spectral slope (k^{-4} to k^{-5}). Thus, the high-wavenumber shift of k_β caused by the reduced U_{eddy} expands the wavenumber band occupied by the zonostrophic regime within the resolvable domain of the model.

Second, the increase in S increases the phase speed of inertia-gravity waves, strongly suppressing the efficiency of gravity wave excitation via spontaneous imbalance at the jet exit (discussed in Section 4.1). In the standard experiment, the k^{-1} spectrum of excited inertia-gravity waves dominated the high-wavenumber region, masking the underlying spectral structure of the background vorticity field. In contrast, in this high-stability experiment, the weakening of gravity wave generation allows the inherently steep spectrum of the expanded zonostrophic regime to emerge clearly and extend into the high-wavenumber range.

Conversely, when the static stability S is halved relative to the standard experiment, the reduced stability enhances the maximum growth rate of baroclinic instability, promoting disturbance growth (increasing U_{eddy}) while simultaneously decreasing the phase speed of inertia-gravity waves. As noted by Plougonven and Zhang (2014), the intensity of radiated waves depends heavily on how effectively advective processes align the forcing timescale of the balanced flow with the intrinsic frequency of gravity waves (i.e., through an increased Froude number). The concurrent increase in U_{eddy} and decrease in gravity wave phase speed narrow the timescale gap between the slow balanced flow and the fast inertia-gravity waves. Consequently, localized flow variations at the jet exit region resonate more directly with gravity waves, leading to exceptionally active

605 wave excitation ($VV \rightarrow W$). The resulting surge of inertia-gravity wave energy
606 completely masks the enstrophy cascade range of the Rossby waves. As a result,
607 shallow spectral slopes ($-5/3$ and -1) originating from the gravity wave com-
608 ponents dominate the total energy spectrum across a broad wavenumber range
609 starting from relatively low wavenumbers.

610 4.4 Sensitivity to Planetary Rotation Rate

611 In this section, we examine how changes in the planetary rotation rate Ω af-
612 fect the excitation of inertia-gravity waves and the resulting energy spectrum.
613 Vanneste (2013) demonstrated that in primitive equation systems at sufficiently
614 small Rossby numbers (Ro), the amplitude of inertia-gravity waves excited by a
615 time-varying balanced flow can be estimated as $\exp(-\alpha/Ro)$, where α is a con-
616 stant determined by the temporal evolution of the background balanced flow.
617 Using the characteristic velocity scale U , length scale L , and Coriolis parameter
618 $f = 2\Omega\mu$, the Rossby number is evaluated as $Ro = U/(2\Omega\mu L)$, scaling approx-
619 imately as $Ro \propto U/\Omega$. Although the two-layer model used in this study is a
620 vertically discretized primitive equation system, which means the exact scal-
621 ing of Vanneste (2013) may not apply directly, it is reasonable to expect that
622 the amplitude of the generated inertia-gravity waves depends strongly on the
623 Rossby number in a similar exponential fashion.

624 As confirmed in Section 3.4 (Fig. 10, Table 2), altering the planetary rota-
625 tion rate Ω leads to corresponding changes in the maximum wind speed $U_{\max}$.
626 Assuming that L and α do not vary substantially across experiments, the overall
627 impact on Ro when changing Ω results from a competition between the change
628 in Ω itself and the induced change in U . In the reduced-rotation experiment
629 (RUN07, scaled by $1/\sqrt{2}$), Ro increases because Ω decreases while U increases
630 by a factor of approximately 1.3. Conversely, in the increased-rotation exper-

iment (RUN06, scaled by $\sqrt{2}$), Ro decreases due to the combined effect of an increased Ω and a reduced U (approximately 0.9 times the standard value).

This variation in Ro alters the energy level of the excited inertia-gravity waves, thereby shifting the spectral transition wavenumber where inertia-gravity waves begin to dominate over Rossby wave modes. This mechanism naturally explains the prominent shift of the spectral bend toward lower wavenumbers in RUN07 and its slight displacement in RUN06.

4.5 Discussion on the Gap Between the k^{-1} and $k^{-5/3}$ Spectra

In this section, we address the spectral discrepancy between our two-dimensional (or two-layer) toy model and the observed real atmosphere. In the horizontal two-dimensional framework used here, inertia-gravity waves excited by the strain effect of the barotropic flow are cascade-transported to higher horizontal wavenumbers, forming an energy spectrum proportional to k^{-1} . However, observational studies of the real atmosphere (e.g., Callies et al., 2016) and high-resolution three-dimensional numerical simulations typically exhibit slopes closer to $k^{-5/3}$ or steeper decay rates at higher wavenumbers. To reconcile this spectral gap, we propose a physical mechanism based on effective dissipation induced by the vertical escape of inertia-gravity waves, and quantitatively evaluate its order of magnitude.

Under the hydrostatic approximation, the dispersion relation of inertia-gravity waves in dimensional form is given by:

$$\omega = \pm \frac{\sqrt{f^2 k_z^2 + N^2 k_x^2}}{k_z}, \quad (68)$$

where f is the Coriolis parameter, N is the Brunt–Väisälä frequency, k_z is the vertical wavenumber, and k_x is the horizontal wavenumber. From Eq. (68), the

655 magnitude of the vertical group velocity $|c_{gz}|$ is derived as:

$$|c_{gz}| = \left| \frac{\partial \omega}{\partial k_z} \right| = \frac{N^2 k_x^2}{k_z^2 \sqrt{f^2 k_z^2 + N^2 k_x^2}}. \quad (69)$$

656 Assuming a vertical depth H for the domain of interest (such as the tropo-
 657 sphere), the rate at which these waves escape vertically outside the layer can be
 658 defined as $|c_{gz}|/H$. This ratio acts as an effective dissipation rate (the inverse
 659 of the decay time constant) for inertia-gravity waves within the domain.

660 Analyzing the wavenumber dependence of Eq. (69) reveals two distinct asymp-
 661 totic regimes. In the long-wave limit ($|Nk_x| \ll fk_z$), the vertical group veloc-
 662 ity simplifies to $|c_{gz}| \propto k_x^2$, making the vertical escape rate proportional to
 663 the square of the horizontal wavenumber. This dependence matches standard
 664 harmonic viscosity ($\alpha_h = 1$, corresponding to a ∇^2 -type dissipation) used in
 665 our numerical experiments. As discussed in Section 4.2, when the horizontal
 666 wavelength of gravity waves is shortened by the background strain field, their
 667 vertical group velocity increases rapidly, accelerating their vertical escape. This
 668 mechanism, where horizontal strain shortens the wave scale and accelerates the
 669 vertical escape of wave energy from the layer, is physically analogous to the
 670 ocean inertia-gravity wave dynamics described by Asselin et al. (2020), where
 671 strain-induced wave-steepening accelerates energy departure from the surface
 672 boundary layer. In the short-wave limit ($|Nk_x| \gg fk_z$), the vertical group ve-
 673 locity approaches $|c_{gz}| \approx Nk_x/k_z^2$, which weakens the wavenumber dependence
 674 of the effective dissipation rate to k_x^1 .

675 We quantitatively estimate the timescale over which this vertical escape
 676 operates using realistic mid-latitude parameters ($f = 1.0 \times 10^{-4} \text{ s}^{-1}$, $N =$
 677 $1.0 \times 10^{-2} \text{ s}^{-1}$, $H = 10 \text{ km}$, and a vertical wavelength $\lambda_z = 2\pi/k_z = 2 \text{ km}$).
 678 For a horizontal wavelength $\lambda_x = 2\pi/k_x = 200 \text{ km}$, the damping time constant
 679 is $\tau = H/|c_{gz}| \approx 5.14 \text{ day}$, whereas for a shorter horizontal wavelength $\lambda_x =$

680 20 km, the timescale decreases substantially to $\tau \approx 0.37$ day (approximately 9
681 hours). This damping intensity (ranging from hours to one day) aligns closely
682 with the magnitude of harmonic viscosity ($\alpha_h = 1$) applied in our high-resolution
683 T3413 experiment.

684 While an n^{-1} spectrum naturally arises in the theoretical limit of a two-
685 dimensional toy model where wave energy is strictly confined within the layer,
686 vertical wave escape in the real atmosphere strengthens toward higher wavenum-
687 bers. This loss of energy suppresses the high-wavenumber spectrum, causing
688 it to steepen toward an $n^{-5/3}$ or steeper slope. To test this hypothesis, we
689 conducted additional T1999 numerical experiments (RUN08 and RUN09) that
690 mimic vertical wave escape. Under Helmholtz decomposition, the balanced
691 quasi-geostrophic motion (Rossby waves) is dominated by the rotational veloc-
692 ity component, whereas the divergent velocity component is primarily associ-
693 ated with inertia-gravity wave modes. Exploiting this dynamical separation,
694 we selectively applied k_x -dependent dissipation solely to the divergent compo-
695 nent (represented by the velocity potential) to represent energy loss via vertical
696 wave escape. Assuming vertical wavelengths λ_z of 1 km (Fig. 11a) and 2 km
697 (Fig. 11b), the resulting energy spectra of the divergent component deviate
698 from the theoretical n^{-1} slope and steepen toward a realistic $n^{-5/3}$ power law,
699 confirming the qualitative validity of our argument.

700 [Figure 11 about here.]

701 Finally, we emphasize that this parameterization of vertical escape remains a
702 conceptual interpretation within the boundaries of a horizontal two-dimensional
703 (or low-order vertical mode) model. Furthermore, at small horizontal scales
704 where the horizontal wavelength approaches the vertical wavelength ($\lambda_x \sim \lambda_z$),
705 the hydrostatic approximation breaks down and alters the underlying dispersion
706 relation. Nevertheless, this vertical escape mechanism offers a plausible physi-

cal bridge between the theoretical k^{-1} spectrum of simplified two-dimensional systems and the observed mesoscale spectra of the real atmosphere.

5 Conclusions

In this study, to elucidate the transition mechanism of the Nastrom–Gage-like energy spectrum observed in the atmospheric mesoscale, we conducted high-resolution numerical experiments of forced turbulence using a baroclinic two-layer model (Ishioka et al., 2022) that maintains the symmetry between barotropic and baroclinic modes while filtering out Lamb waves. Within this two-layer framework, we successfully and self-consistently reproduced the spectral transition (bend) structure, characterized by a transition from a k^{-3} power law in the low-wavenumber regime to a shallow slope (k^{-1} to $k^{-5/3}$) in the high-wavenumber regime, consistent with high-resolution global circulation models (GCMs) and atmospheric observations. Hough mode decomposition and cumulative energy flux analyses clarified that the low-wavenumber k^{-3} spectrum is dominated by Rossby wave modes undergoing a quasi-geostrophic forward enstrophy cascade, whereas the shallow high-wavenumber spectrum stems from a forward energy cascade driven by inertia-gravity wave modes ($VW \rightarrow W$).

Spatiotemporal analysis of the streamfunction and divergence fields revealed that the principal source of these inertia-gravity waves is spontaneous imbalance ($VV \rightarrow W$) localized at the deceleration and exit region of the high-latitude jet. Furthermore, the shallow spectral slope close to k^{-1} is maintained at high wavenumbers because the excited small-scale inertia-gravity waves are stretched toward higher wavenumbers by the scale-invariant strain (deformation) field of the large-scale barotropic flow, a physical mechanism analogous to Batchelor/Kraichnan-type passive scalar turbulence.

Sensitivity experiments provided further insight into how environmental pa-

rameters govern these spectral characteristics. Altering the static stability S demonstrates that higher stability strongly suppresses wave excitation via spontaneous imbalance, shifting the Rhines wavenumber to higher wavenumbers and giving rise to zonostrophic turbulence structures with steep (k^{-4} to k^{-5}) slopes. Conversely, lower stability enhances inertia-gravity wave excitation, allowing the gravity wave component to dominate the total energy spectrum across broader, lower wavenumber ranges. Meanwhile, varying the planetary rotation rate Ω induces changes in the zonal mean wind speed and modifies the Rossby number (Ro) of the system. This variation in Ro modulates the energy of inertia-gravity waves excited by spontaneous imbalance, thereby shifting the transition wavenumber (spectral bend position) of the energy spectrum.

Overall, these findings demonstrate that the Nastrom–Gage spectral transition can be generated within a minimal dynamical model incorporating divergence, vertical baroclinicity, and inertia-gravity waves, without requiring complex GCM formulations or a high number of vertical layers. We believe this work provides a concise, fundamental dynamical framework for understanding atmospheric mesoscale turbulence.

Acknowledgments

The authors express their sincere gratitude to Dr. Hitoshi Mukougawa for providing and maintaining the generous computing environment, which was essential for completing this study. The authors also thank Dr. Wataru Ofuchi for helpful discussions.

This work was supported by JSPS KAKENHI Grant Numbers 24K07136, 23K25941, and 24K00706.

757 **Author contributions**

758 K. I. conceived and designed the study, provided experimental guidance, and
759 participated in manuscript writing. S. A. led the experiments and data anal-
760 ysis, and took the initiative in drafting the manuscript. M. Y. performed the
761 preliminary experiments and contributed to the review of the final manuscript.
762 All authors have read and approved the final version.

763 **Data availability**

764 The datasets generated during and/or analyzed during the current study are
765 available from the authors on reasonable request.

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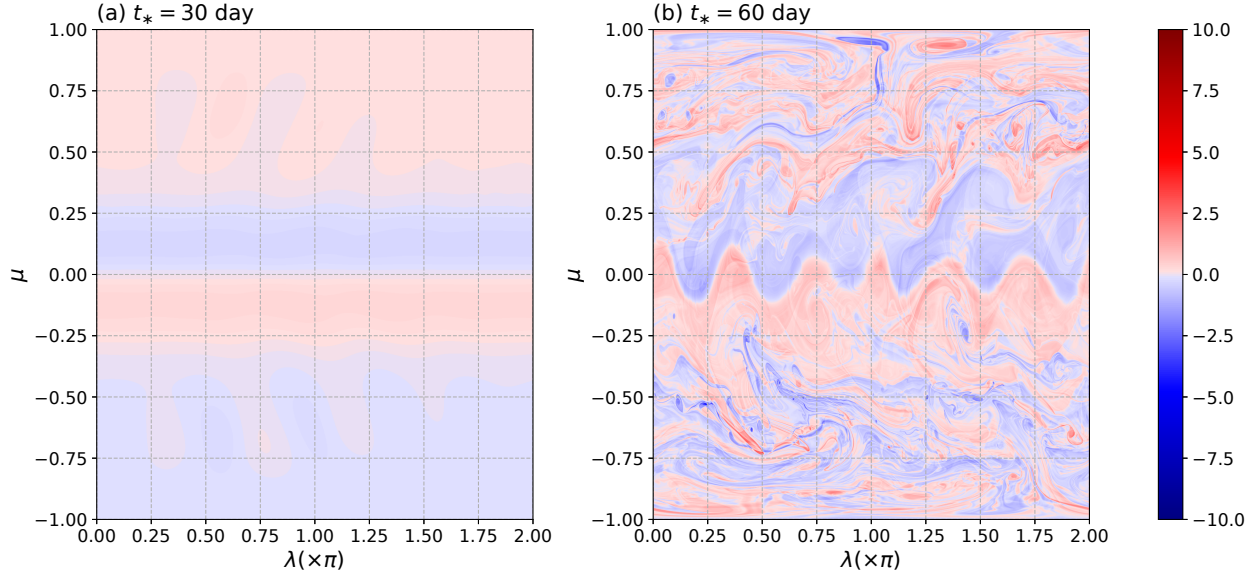


Figure 1: Snapshots of the non-dimensional baroclinic vorticity field ζ_c in RUN01 (T3413, harmonic viscosity) at two developing stages: (a) $t_* = 30$ day and (b) $t_* = 60$ day. Color bars indicate the non-dimensional vorticity values (warm colors represent positive/cyclonic vorticity and cool colors represent negative/anticyclonic vorticity). The horizontal axis represents longitude λ , and the vertical axis represents sine latitude $\mu = \sin \phi$.

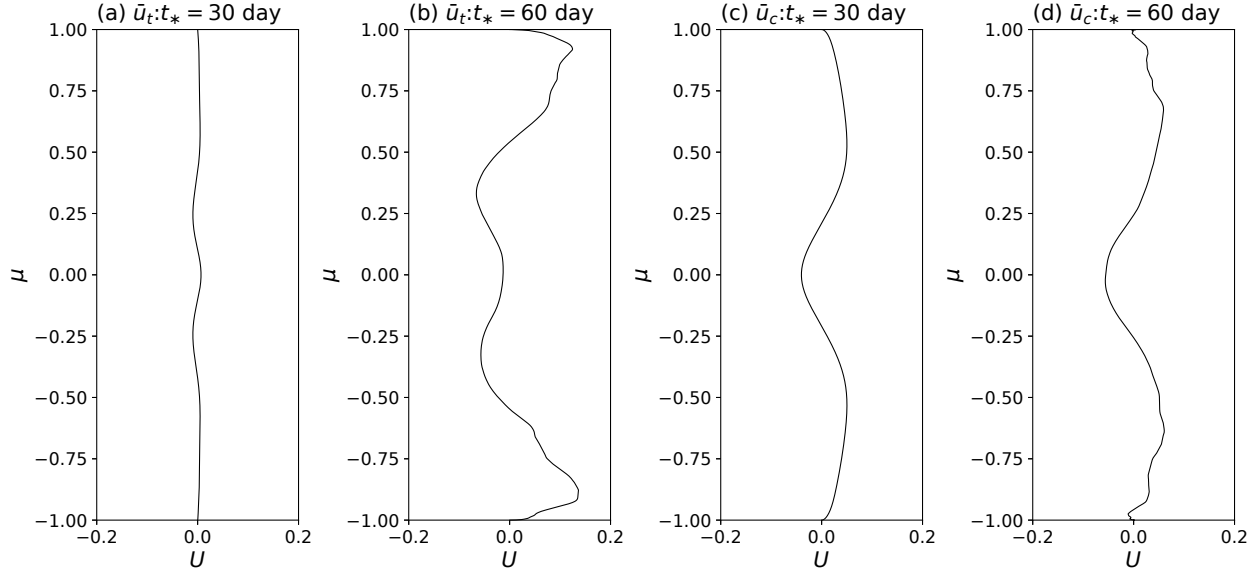


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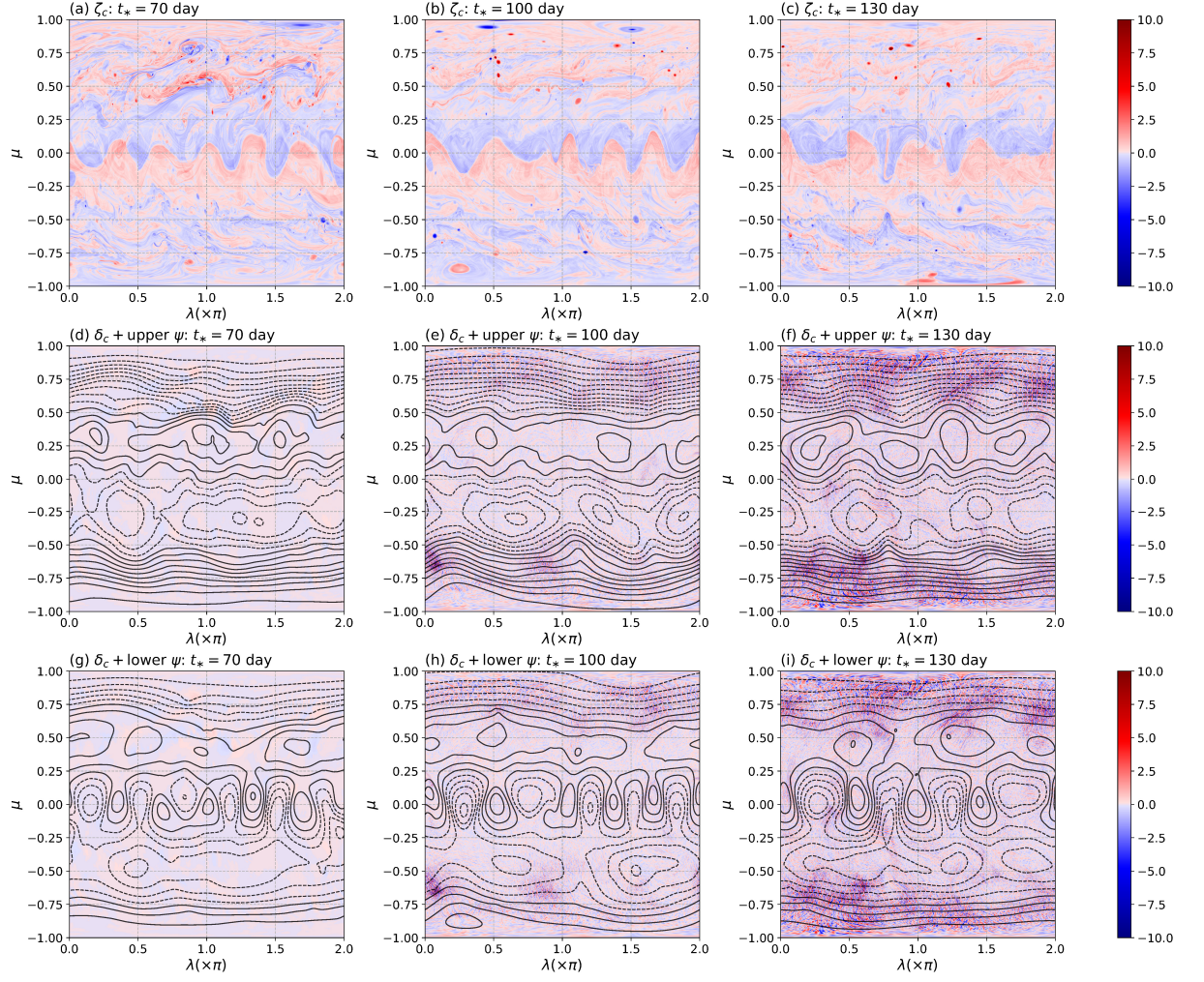


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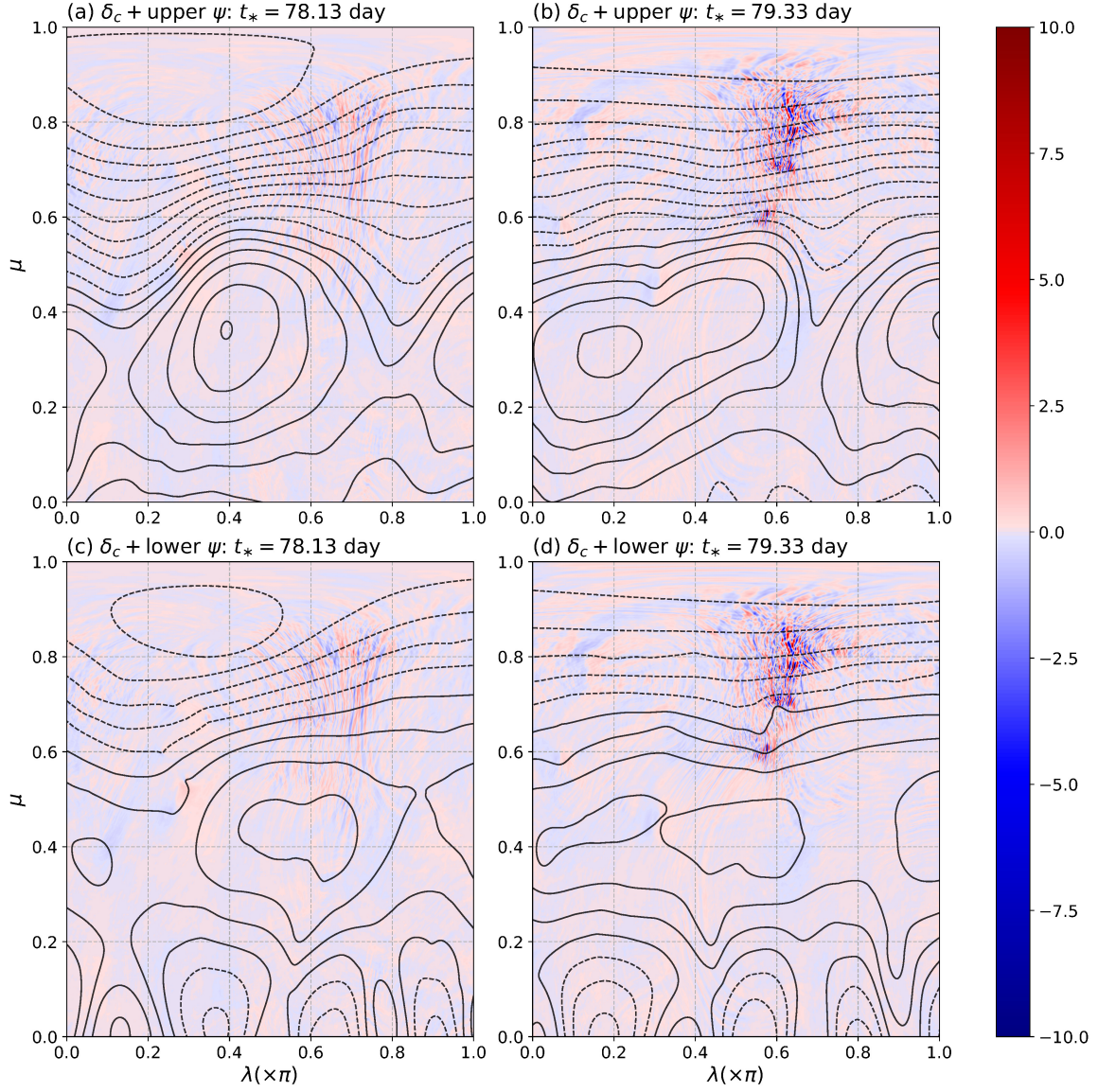


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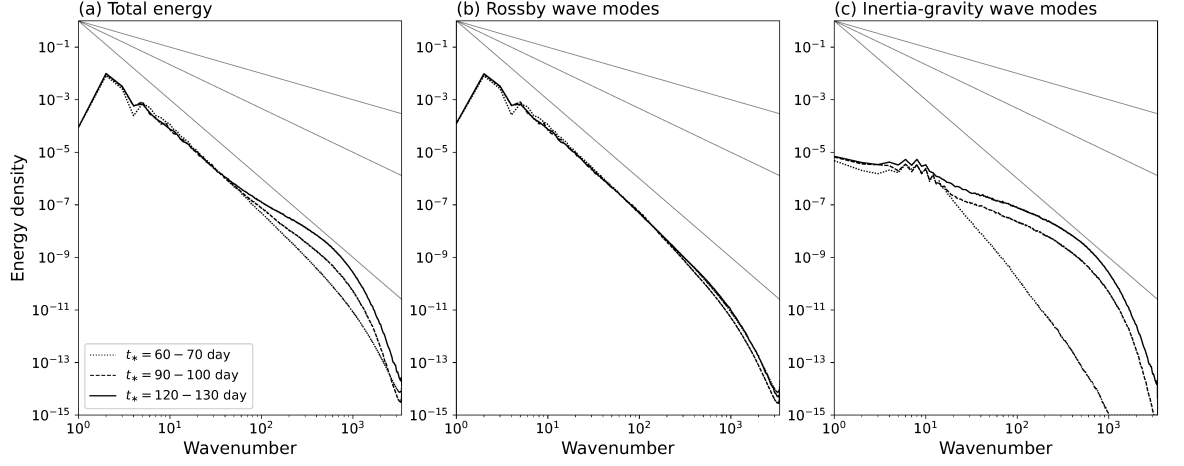


Figure 5: Energy spectra averaged over three distinct time intervals as functions of total wavenumber n in RUN01 (T3413): (a) total energy spectrum, (b) inertia-gravity wave modes (W), and (c) Rossby wave modes (V , barotropic plus baroclinic). Dotted, dashed, and solid curves represent time averages over $t_* = 60\text{--}70$ day, $t_* = 90\text{--}100$ day, and $t_* = 120\text{--}130$ day, respectively. Solid straight lines indicate reference power-law slopes of n^{-1} , $n^{-5/3}$, and n^{-3} .

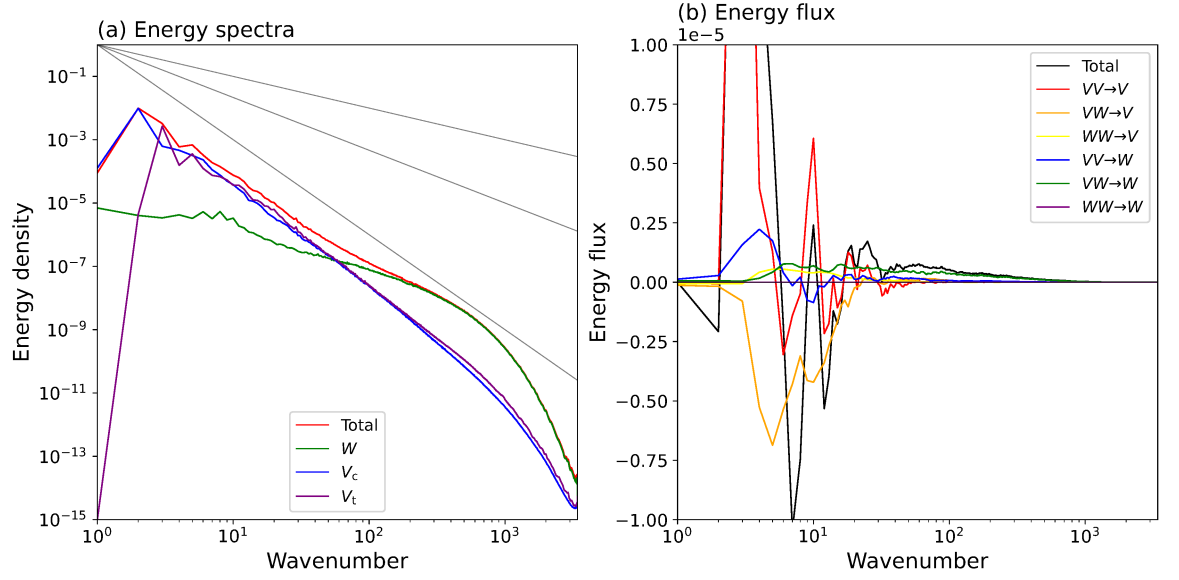


Figure 6: Time-averaged energy spectra and cumulative energy fluxes in statistical equilibrium ($t_* = 120\text{--}130$ day) for RUN01 (T3413): (a) decomposed energy spectra, where red, green, blue, and purple curves represent total energy, inertia-gravity wave modes, baroclinic Rossby wave modes, and barotropic Rossby wave modes, respectively; (b) cumulative nonlinear energy flux $\Pi(n)$ decomposed into specific triad interactions (e.g., $VV \rightarrow V$, $VV \rightarrow W$, and $VW \rightarrow W$).

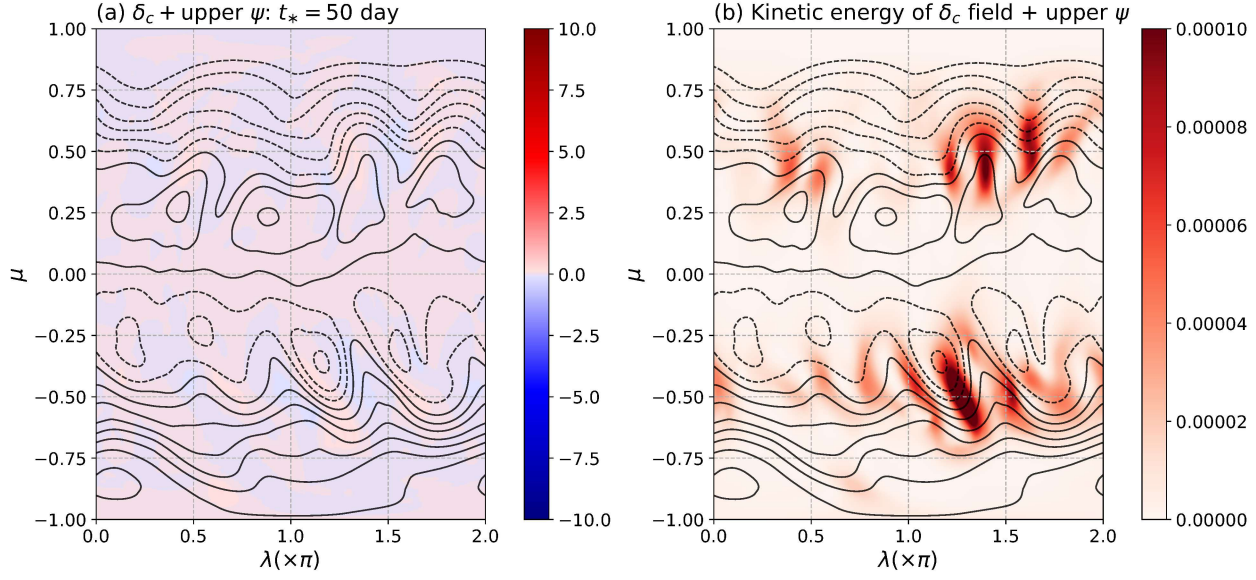


Figure 7: Spatial structure at $t_* = 50$ day in RUN01 (T3413): (a) divergence field δ_c , and (b) kinetic energy density distribution corresponding to the divergent velocity component. Color bars indicate (a) the magnitude of divergence δ_c and (b) the kinetic energy density of the divergent flow field. Black contours in (b) denote upper-layer streamfunction (contour interval: 0.01). The horizontal axis represents longitude λ , and the vertical axis represents sine latitude $\mu = \sin \phi$.

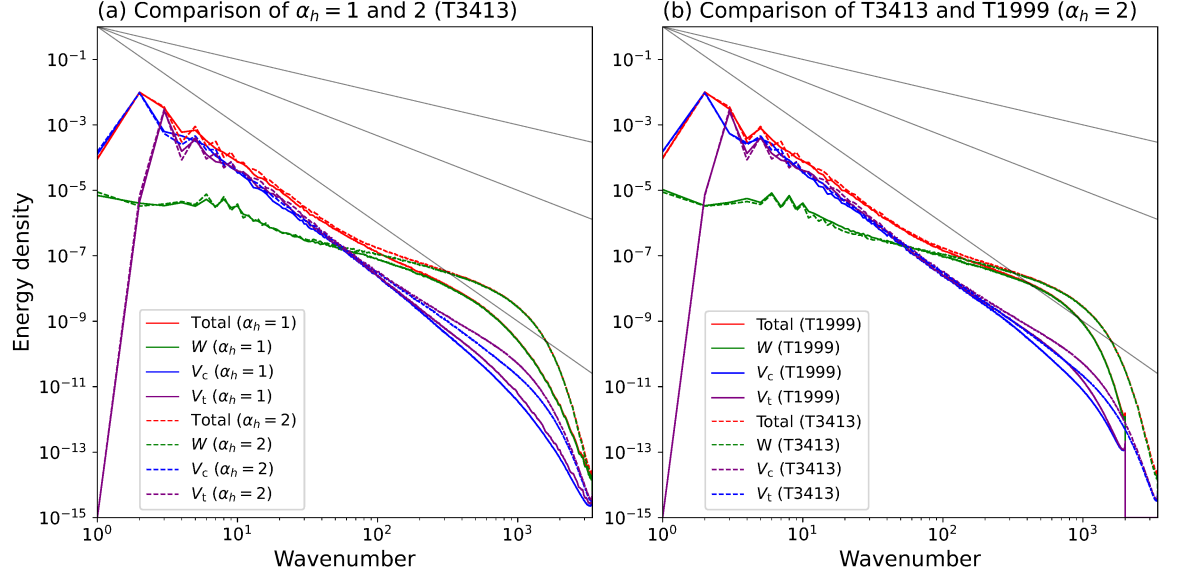


Figure 8: Comparison of time-averaged energy spectra over $t_* = 120$ – 130 day across experiments with different orders of high-wavenumber dissipation and truncation wavenumbers: (a) harmonic viscosity ($\alpha_h = 1$, solid curve, RUN01) versus hyperviscosity ($\alpha_h = 2$, dashed curve, RUN02) at resolution T3413; (b) resolution T1999 (solid curve, RUN03) versus T3413 (dashed curve, RUN02) using hyperviscosity ($\alpha_h = 2$).

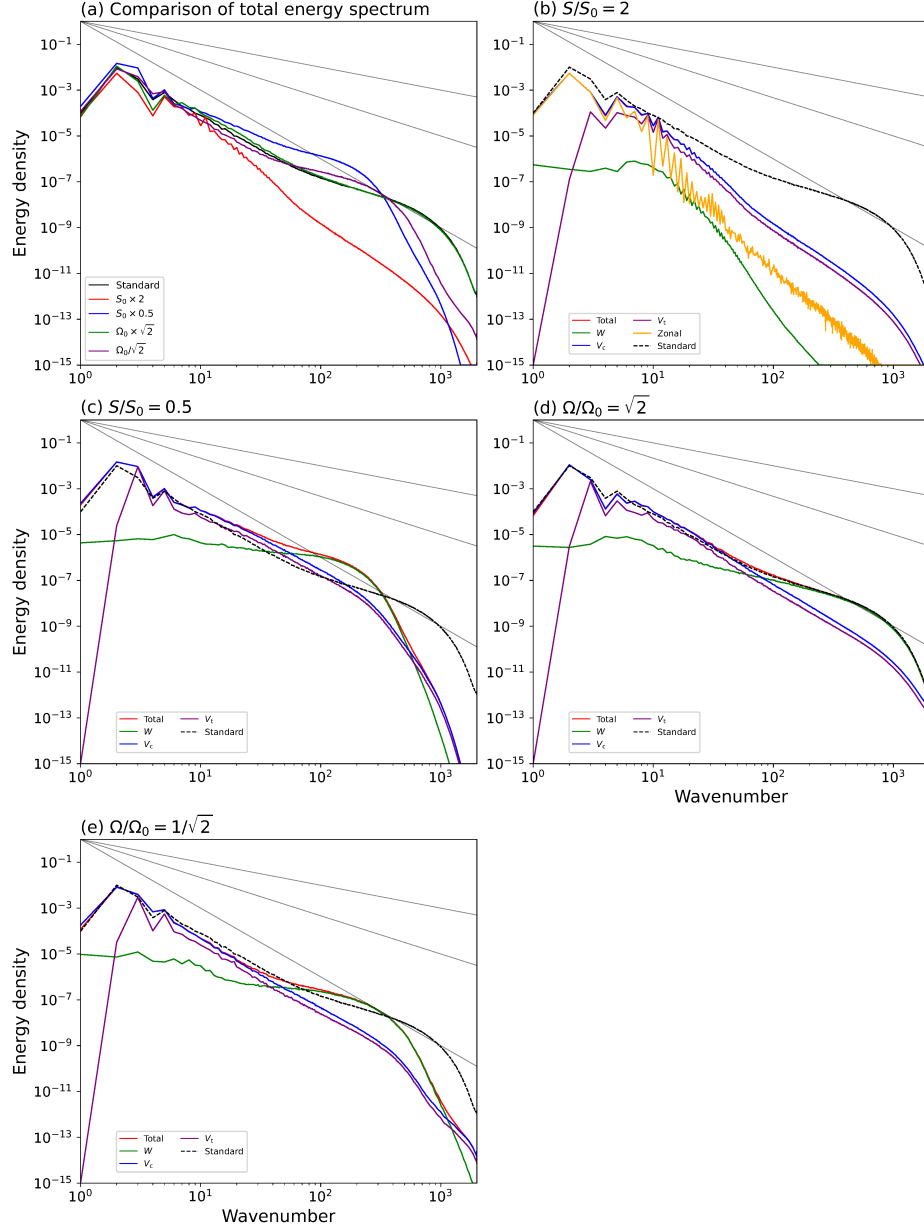


Figure 9: Sensitivity of time-averaged energy spectra over $t_* = 120\text{--}130$ day as functions of total wavenumber n to physical parameters at T1999 resolution: (a) superimposed total energy spectra across all cases; (b) doubled static stability ($S/S_0 = 2.0$, RUN04), where the orange curve specifically denotes the energy spectrum of the zonal vortical mode component ($V, m = 0$); (c) halved static stability ($S/S_0 = 0.5$, RUN05); (d) $\sqrt{2}$ times planetary rotation rate ($\Omega/\Omega_0 = \sqrt{2}$, RUN06); and (e) $1/\sqrt{2}$ times planetary rotation rate ($\Omega/\Omega_0 = 1/\sqrt{2}$, RUN07).

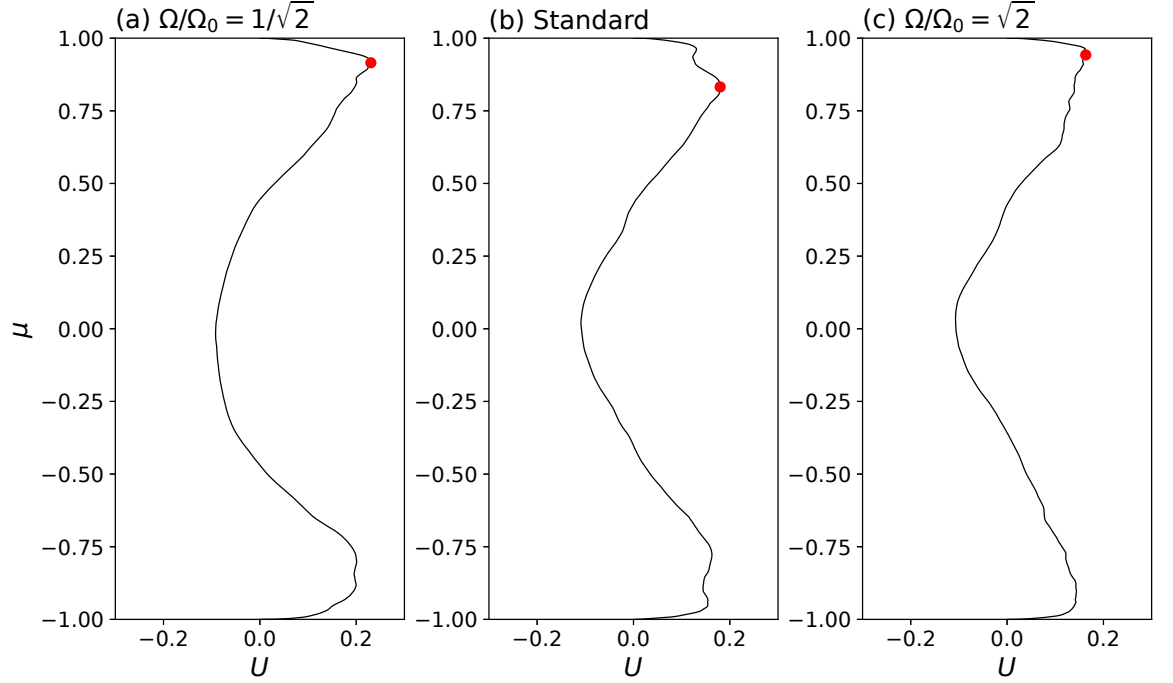


Figure 10: Upper-layer zonal mean zonal wind profiles at $t_* = 130$ day for sensitivity experiments with varying planetary rotation rates: (a) $\Omega/\Omega_0 = 1/\sqrt{2}$ (RUN07), (b) standard rotation rate $\Omega/\Omega_0 = 1.0$ (RUN03), and (c) $\Omega/\Omega_0 = \sqrt{2}$ (RUN06). The vertical axis represents sine latitude $\mu = \sin \phi$, and the horizontal axis denotes non-dimensional eastward wind velocity. Red dots indicate the maximum zonal wind speed $U_{\max}$.

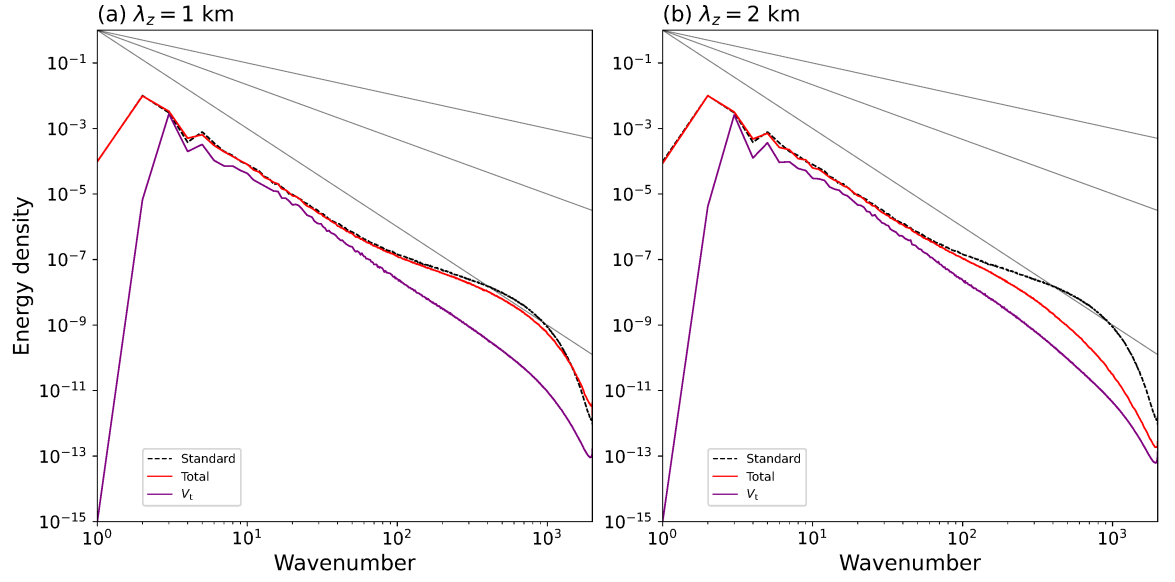


Figure 11: Time-averaged energy spectra over $t_* = 120\text{--}130$ day obtained from hypothesis-verification experiments incorporating k -dependent dissipation applied exclusively to the divergent component: (a) RUN08 ($\lambda_z = 1$ km) and (b) RUN09 ($\lambda_z = 2$ km). Red and purple curves denote total energy and barotropic Rossby wave modes, respectively.