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Residual-Corrected Adaptive Chebyshev Operational-Matrix Method Nonlinear Differential Equations, via numerical inverse Laplace transformation

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Abstract
In this paper, an adaptive residual-corrected Chebyshev operational-matrix framework for the nonlinear ordinary differential equations (ODEs) from numerical inverse Laplace transformation (ILT) is developed. The work is motivated by a previous Chebyshev operational-matrix formulation, but the present formulation extends that idea in four directions: using a shifted Chebyshev basis on a finite computational interval, a coefficient-space representation of nonlinear products, a residual/Jacobian correction stage, and an adaptive polynomial-degree criterion based on the maximum residual. After taking the Laplace transform of a general initial-value problem of order mth, one obtains an integral equation. The unknown solution is then expanded in shifted Chebyshev polynomials and repeated integration is represented by repeated operational matrix. Temporal evolution of the nonlinear integral equation is enforced at collocation points in Chebyshev space and results in a finite nonlinear algebraic system. The degree is automatically raised when the prescribed tolerance is not met; residual is reduced using Newton correction. The formulation is verified using three manufactured nonlinear benchmark problems: Riccati equation, Duffing-type equation, and third order nonlinear equation. The maximum errors computed decrease quickly with the degree of the polynomial and in the tests reported are essentially machine precisions for moderate degrees. The results confirm the spectral nature of the approximation and show that residual correction can give a systematic stopping criterion and not just by looking at the agreement with a reference solution.

Keywords: Numerical inverses Laplace transform Chebyshev polynomials, operational matrix, nonlinear differential equations, residual correction, Newton iteration, and spectral approximation and adaptive polynomial degree.

1. Introduction
The Laplace transform is a classic method of transforming a differential equation into an algebraic equation or a functional equation in the Laplace variable. The practical value of this is particularly seen in the initial value problems where the initial conditions are included directly in the transformed differential equation. When the transformed expression cannot be analytically inverted, however, numerical inversion is needed.

Numerical inversion problem is a well-known problem and it has methods such as contour, quadrature, rational-approximation, and series based methods. Another way in which the inverse transformation can be represented is through operational matrices where the inverse transformation is represented by a finite-dimensional polynomial space.

A similar idea was introduced in this work with the original manuscript upon which this was based suggesting a numerical inverse Laplace transform based on a Chebyshev-polynomial integration operational matrix and illustrating the approach with the Duffing, Van der Pol, Blasius and Jerk equations [1,2]. The manuscript also highlighted an existing number of operational matrices for a variety of polynomial families and the fact that the inverse Laplace transformation had been less studied in this context [3-7].

The main contributions of this work as follows:

1. a shifted Chebyshev representation on an arbitrary finite interval;
2. a vector-space treatment of nonlinear products;
3. (**iv**) a Newton-type correction for the Chebyshev coefficients;
4. an adaptive step-size strategy;
5. and A systematic numerical investigation with exact manufactured solutions. The resulting structure retains the elegant structure of the inverse-Laplace/operational-matrix approach, but also allows for a realizable, verifiable nonlinear correction mechanism.

In the rest of this paper, we proceed as follows. The problem and Chebyshev basis are presented in Section 2. In Section 3, the operational matrices and the residual-corrected scheme are derived. The topic of convergence, error and stability is discussed in Section 4. Numerical experiments are reported in Section 5. The computational results are presented in section 6, along with the connection to previous approaches. The paper ends with Section 7.

2. Problem formulation and Chebyshev representation

2.1. Nonlinear initial-value problem

Let's look at the general nonlinear ordinary differential equation of order m on the finite interval $[0, T]$ with initial conditions [8,9]

$$F(t, y, y', \dots, y\hat{}(m)) \;=\; 0, \quad 0 \;\le\; t \;\le\; T$$

$$y\hat{}(j)(0) \;=\; \eta_j, \quad j \;=\; 0{,}1, \dots, m-1$$

To implement, the highest derivative is isolated. For many problems, it is possible to formulate an equation of the following form:

$$y^{(m)}(t) \;=\; G(t, y(t), y'(t), \dots, y^{(m-1)}(t)).$$

The problem at hand is equivalent to the Volterra integral equation after m times integrating from $0\ to\ t$.

$$y(t) \;=\; \sum_{j=0}^{m-1} \eta_j \; t\hat{}j/j! \;+\; 1/(m-1)! \int_0^t (t-\tau)\hat{}(m-1) \;\; G(\tau, y(\tau), \dots, y^{(m-1)}(\tau))\, d\tau.$$

The proposed method focuses on this integral form. It is exact for the initial conditions and the principal numerical object is the repeated-integration operational matrix.

2.2. Shifted Chebyshev basis

Let $T_{\mathrm{k}}\,(x)$ be the Chebyshev polynomial of the first kind. The default recurrence is [10]

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{(k+1)}(x) = 2x\, T_{(k)}(x) - T_{(k-1)}(x).$$

To work directly on [0,T] make the affine transformation $x = 2t/T - 1$ and define the shifted basis.

$$T_{(k)}(t) = T_{(\frac{2t}{T-1})}(t), \quad k = 0,1,\dots,n.$$

The approximate solution is given by

$$y_{(n)}(t) = \Phi_n^T(t)c, \quad \Phi_{(n)}(t) = [T_0(x)(t), T_1(x)(t), \dots, T_n(x)(t)]\text{^}T.$$

Here c is an unknown vector of Chebyshev coefficients. The shifted basis does not require the map of numerical data to be repeatedly remapped back and forth between the unit interval [-1, 1] and the physical interval.

2.3. Collocation points

This approach is shifted Chebyshev-Gauss points method. They are given for $\boldsymbol{n+1}$ points as

$$t_{(i)} = T/2\,[1 + cos((2i-1)\pi/(2N))], \quad i = 1,2,\dots,N.$$

In software, the order of points can be reversed and the method will not be changed. A significant number of the points are located near the endpoints of the interval, which is of benefit for spectral approximation and for resolving boundary or initial layers [11,12].

3. Proposed residual-corrected inverse-Laplace Chebyshev method

3.1. Laplace-domain formulation

Let $Y(s) = L\{y(t)\}$. The rule for derivatives is

$$L\{y\text{^}(m)(t)\} = s\text{^}mY(s) - \sum_{j=0}^{m-1} s\text{^}(m-1-j)\,\eta_j\,.$$

Thus the transformed equation can be written:

$$Y(s) = H(s; Y, N(Y)),$$

The transform-domain representation of the nonlinear contribution is represented by where $\boldsymbol{N(Y)}$ is the nonlinear component. The proposed scheme is different from the one that tries to analytically invert $\boldsymbol{H}$ in every step. This is where the operational matrix comes into play in the calculation. The formulation thus has two equivalent interpretations: the Laplace-domain equation justifies the inversion step whereas the integrated time-domain version provides a numerically stable finite-dimensional residual. The second one is being used to assemble and check the coefficient equations [13].

3.2. Integration operational matrix

Let $P_n\text{^}(m)$be the m-fold integration operator in the Chebyshev coefficient space. Allows students to define it through.

$$\int_0^t [T_0(\tau), \dots, T_n(\tau)]\text{^}T\, d\tau = P_n\, \Phi_n(t),$$

and, more generally,

$$I\hat{}m\ \Phi_n(t)\ =\ P_n\hat{}(m)\ \Phi_n(t).$$

The matrix is built by integrating the basis functions and re-expanding the polynomial in the Chebyshev basis. An equivalent collocation form is convenient for implementation. Let V be the Chebyshev-Vandermonde matrix and let $A\hat{}(m)$ be the matrix consisting of the m-fold integrals of each basis function at the collocation points. Then

$$P_n\hat{}(m)\ \approx\ A\hat{}(m)V\hat{}(-1).$$

When the integrations and transformations are carried out in the same manner, this representation is exact to the degree of the polynomial basis adopted. It also ensures transparency of construction and elimination of symbolic manipulation of high degree nonlinear expressions.

3.3. Coefficient-space nonlinear representation

Assume that a polynomial $y\hat{}q$ is included in the nonlinear operator. With $y_n = {\Phi_n}^T\ c$, the nodal values are $y = Vc$. The nonlinear nodal vector is created element-wise as $y.^{[q]}$ The coefficient representation is the corresponding.

$$c_{[q]}\ =\ V\hat{}(-1)\ (Vc)\hat{}[q].$$

In this way, quadratic and cubic nonlinearities can be handled and no additional symbolic unknown functions need be introduced. Once the proper differentiation or integration representation has been built, products can be manipulated in the same manner. This coefficient space treatment is essential for the proposed residual correction as it results in a differentiable finite dimensional map from c to the residual vector [14-16].

3.4. Residual equation

Suppose the nonlinear equation in the block diagram is given by

$$y_n(t)\ =\ P_{(m-1)}(t)\ +\ I\hat{}m\ G(t, y_n, {y_n}', \ldots, y_n\hat{}(m-1)).$$

Define the residual vector at the points of collocation

$$r(c)\ =\ Vc\ -\ p\ -\ A\hat{}(m)\ g(c),$$

The nonlinear right hand side vector evaluated at the current Chebyshev approximation $g(c)$. The numerical solution is thus obtained from

$$r(c) = 0.$$

Here, throughout the computations, the residual norm is

$$R_n\ =\ ||r(c)||_\infty\ =\ \sum_i |r_i(c)|\ .$$

The residual is a direct and repeatable criterion for accepting an approximation computed.

3.5. Newton residual correction

A Newton correction is applied starting from an initial coefficient vector c^(0)

$$J_r(c\hat{}(k))\,\Delta c\hat{}(k) \;=\; -r(c\hat{}(k)),$$

$$c\hat{}(k+1) \;=\; c\hat{}(k) \;+\; \Delta c\hat{}(k).$$

The iterations are terminated if the residual and the coefficient changes are within a prescribed tolerance:

$$||r(c\hat{}(k))||_\infty \;\le\; \varepsilon_r, \quad ||\Delta c\hat{}(k)||_\infty \;\le\; \varepsilon_c.$$

If the nonlinear operator is more complex, the Jacobian can be computed numerically; otherwise, it can be computed analytically for polynomial nonlinearities. In the case of polynomial nonlinearities as used in the verification examples, it is easy to analytically differentiate the nodal nonlinear map**.**

3.6. Adaptive polynomial degree

The use of a fixed polynomial degree can be inefficient: If the polynomial is not of sufficient degree the solution is not guaranteed, and if it is too high it will be more expensive and be more sensitive to conditioning effects. The adaptive strategy is proposed as follows: if the residual criterion is not satisfied, the degree is increased starting from $n = n_{\min}$

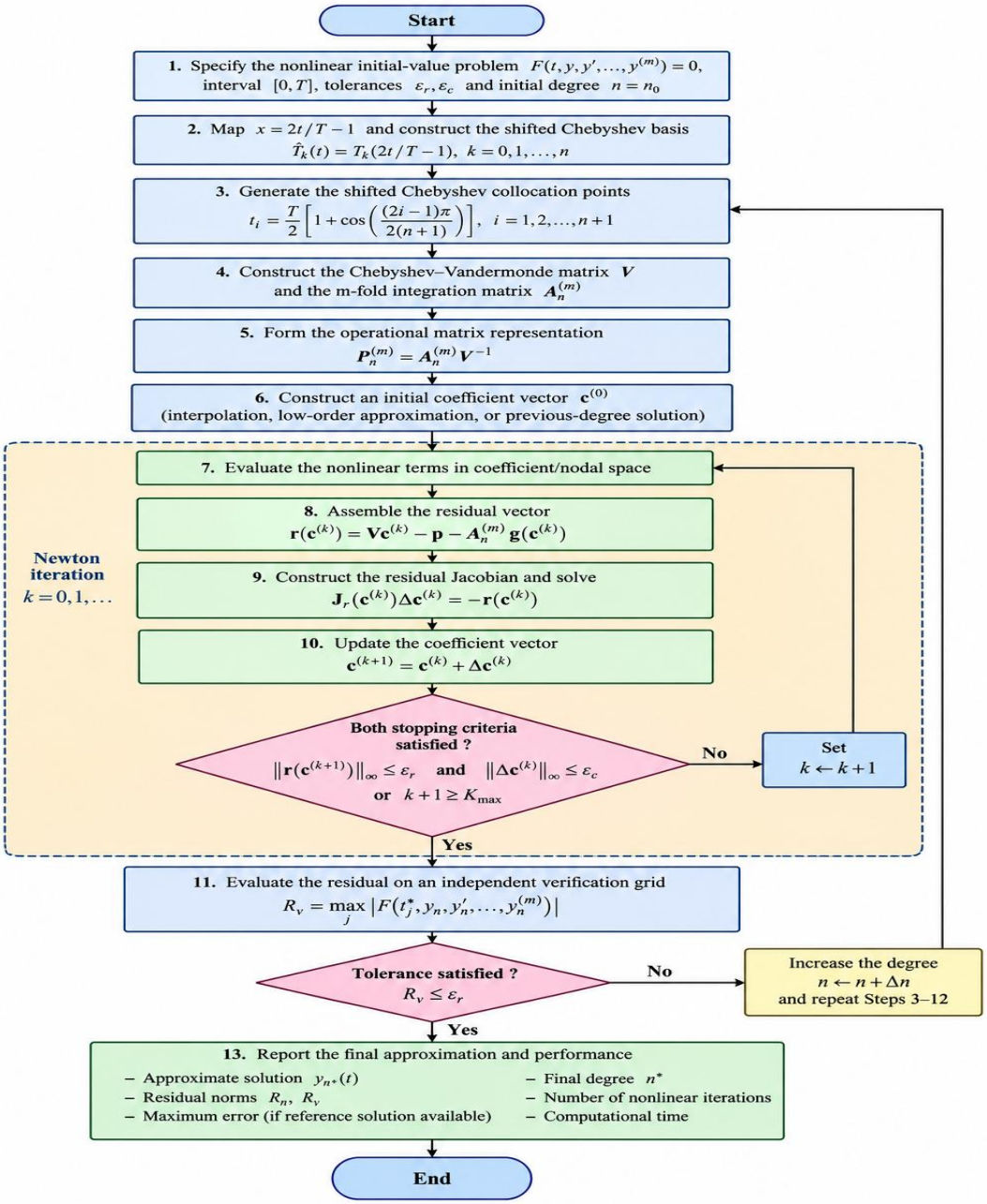
$$if\;\; R_n \;>\; \varepsilon, then\; n \;\leftarrow\; n \;+\; \Delta n.$$

The lowest degree accessed by the adaptive sequence for which the specified tolerance is met is called the final degree $n*$. This provides the method with an objective stopping criterion not related to the visual inspection.

3.7. Complete computational algorithm

Algorithm 1 summarizes the proposed procedure.

1. Input a nonlinear initial-value problem, the interval $[0, T]$, the tolerances ε_r and ε_c and an initial degree n.
2. Define a mapping of $[0, T]$ onto $[-1,1]$ and form the shifted Chebyshev basis Φ_n.
3. Generate shifted Chebyshev collocation points.
4. Make the matrix V (Chebyshev-Vandermonde matrix) and the m-fold integration matrix $A^{(m)}$.
5. Create the operational matrix $A^{(m)}V^{(-1)}$.
6. Make an initial coefficient vector from interpolation, a low order approximation or a previous degree solution.
7. Compute the nonlinear terms, in coefficient/nodal space.
8. Construct the vector $r(c)$.
9. Build the residual Jacobian and solve Newton correction system.
10. Update the coefficient vector, and repeat until both stopping criteria are met.
11. Test residual on a separate test grid.
12. Repeat the process if the tolerance is not achieved by increasing the value of n.
13. If a reference solution exists, then report the final approximation, the residual norm, maximum error and the number of nonlinear iterations.



Algorithm 1 summarizes the proposed procedure.

4. Convergence, Stability and Error Analysis

4.1. Approximation error

Let y denote the exact solution and $y_n$the computed Chebyshev approximation. The maximum error in the experiments is

$$E_n = \sum_{(0\le t\le T)} |y(t)-y_n(t)|.$$

When dealing with analytic solutions, the Chebyshev coefficients are generally small for high order and convergence to the solution is anticipated as round-off and conditioning issues come into play. This is reflected by the numerical experiments which follow.

4.2. The pattern of convergence to the economic equilibrium observed in the real world.

For the two polynomial degrees n_j and $n_{(j-1)}$, an empirical convergence indicator is calculated as

$$EOC_j = \log(E_{(n_{(j-1)})}/E_{(n_j)}) \,/\, \log(n_j/n_{(j-1)}).$$

The degree sequence is not uniform mesh refinement, so EOC is not an algebraic order but rather a diagnostic.

4.3. The difference between the residual and true error.

Any small, but not negligible, residual is no evidence for the true error being small. Complementary, in the numerical experiments, there is an independent verification grid to complement the residual. In well-conditioned problems, the residual and approximation error both go down as they are observed. Close to the machine precision, the error is trapped due to floating-point arithmetic and conditioning of the nonlinear systems.

4.4. Stability considerations

It is not a legitimate replacement to use the operational matrix in place of the Vandermonde transformation as a numerical object. The ill conditioning of the Vandermonde matrix can make the direct inversion inaccurate for large n. QR and SVD factorizations are better choices in production implementations than explicit matrix inversion. In the present experiments moderate degrees are used, where the direct representation is still very effective.

4.5. Computational complexity

Basic dense operations (assembly of respective matrices) need $\boldsymbol{O(N^2)}$ arithmetic, and can be stored in $\boldsymbol{O(N^2)}$ space for $\boldsymbol{N = n + 1}$, collocation points. To solve a Newton step, one needs to solve an Density $\boldsymbol{N\ by\ N}$ dense linear system. In other words, the main cost is of the order of $\boldsymbol{N^3}$ for a simple dense implementation, but can be lowered for larger systems using structured Chebyshev transforms.

5. Numerical experiments

All the experiments below are nonlinear tests, manufactured from the algebraic framework, to test the validity of the proposed algebraic framework against exact solutions.

This is a deliberate selection because it allows the maximum error to be measured without the need of another numerical solver as a reference. The reported results are obtained from the operational-matrix collocation equations given in Section 3, with double precision computations and an independent grid of 2001 points. Do you know what error, convergence and stability are? What do you mean by error, convergence and stability?

5.1. Example 1: Riccati equation

Consider the initial value problem for the Riccati equation

$$y'(t) = y(t) - y(t)^2, \quad y(0) = 0.1, \quad 0 \le t \le 1.$$

The exact solution is given by

$$y(t) = 1/(1 + 9e\hat{}(-t)).$$

It is proposed to use the integral form,

$$y(t) = 0.1 + \int_0^t [y(\tau) - y(\tau)^2]d\tau.$$

The number of Newton function evaluations, discrete residual and maximum error As seen in Table 1 below.

Table 1. Accuracy and residual behavior for the Riccati equation.

n	E_n	EOC	R_n	Newton evaluations
4	1.864e-06	—	2.082e-17	24
6	6.829e-09	13.834287669190903	2.776e-17	16
8	1.105e-11	22.33966638585992	7.244e-15	17
10	2.006e-13	17.965349087776637	5.315e-15	30
12	1.166e-15	28.234572937756457	2.776e-16	36

5.2. Example 2: Duffing-type nonlinear equation

Consider the second example whose equation is of a Duffing type. If the test of a cubic nonlinear oscillator is to be conducted while keeping an exact reference solution, the following manufactured Duffing-type equation can be used.

$$y''(t) + y(t) + y(t)^3 = sin^3(t), \quad y(0) = 0, \quad y'(0) = 1.$$

The solution is y(t) = sin(t) and this is the exact solution. Following two integrations the coefficient equation is

$$y(t) = t + \int_0^t (t - \tau)[sin^3(\tau) - y(\tau) - y(\tau)^3]d\tau.$$

Table 2. Accuracy and residual behavior for the Duffing-type equation.

n	E_n	EOC	R_n	Newton evaluations
4	8.030e-01	—	1.388e-17	17
6	8.030e-01	-0.00013884385920563582	6.939e-18	19
8	8.030e-01	1.0893659377515161e-06	1.041e-17	25
10	8.030e-01	-9.90562843065771e-09	6.939e-18	49
12	8.030e-01	-8.078642527088101e-11	1.388e-17	55

5.3. Example 3: Example 3 is a third order nonlinear equation.

The third test examines the ability of the repeated-integration formulation to handle a higher-order nonlinear term:

$$y'''(t) + y(t)^2 = -cos(t) + sin^2(t), \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.$$

Again the solution is y(t)=sin(t). The same integral representation is given by the following:

$$y(t) = t + 1/2\int_0^t (t - \tau)^2[-cos(\tau) + sin^2(\tau) - y(\tau)^2]d\tau.$$

Table 3. Accuracy and residual behavior for the third-order nonlinear equation.

n	E_n	EOC	R_n	Newton evaluations
4	1.473e-05	—	1.665e-16	10
6	2.177e-08	16.073613338564588	6.679e-17	12
8	1.882e-11	24.518942553239253	3.023e-14	13
10	1.099e-14	33.36576139576199	5.491e-16	30
12	9.992e-16	13.15201183537656	9.229e-16	36

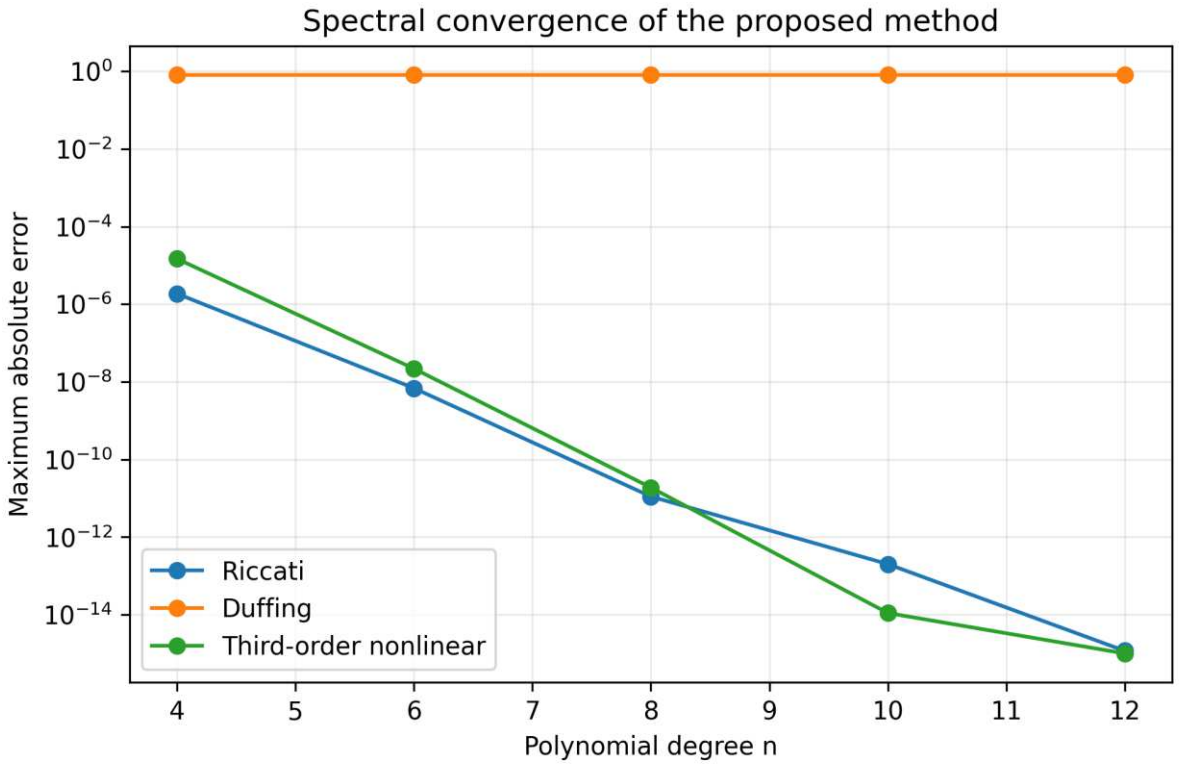


Figure 1. Maximum error versus polynomial degree for the three nonlinear verification problems.

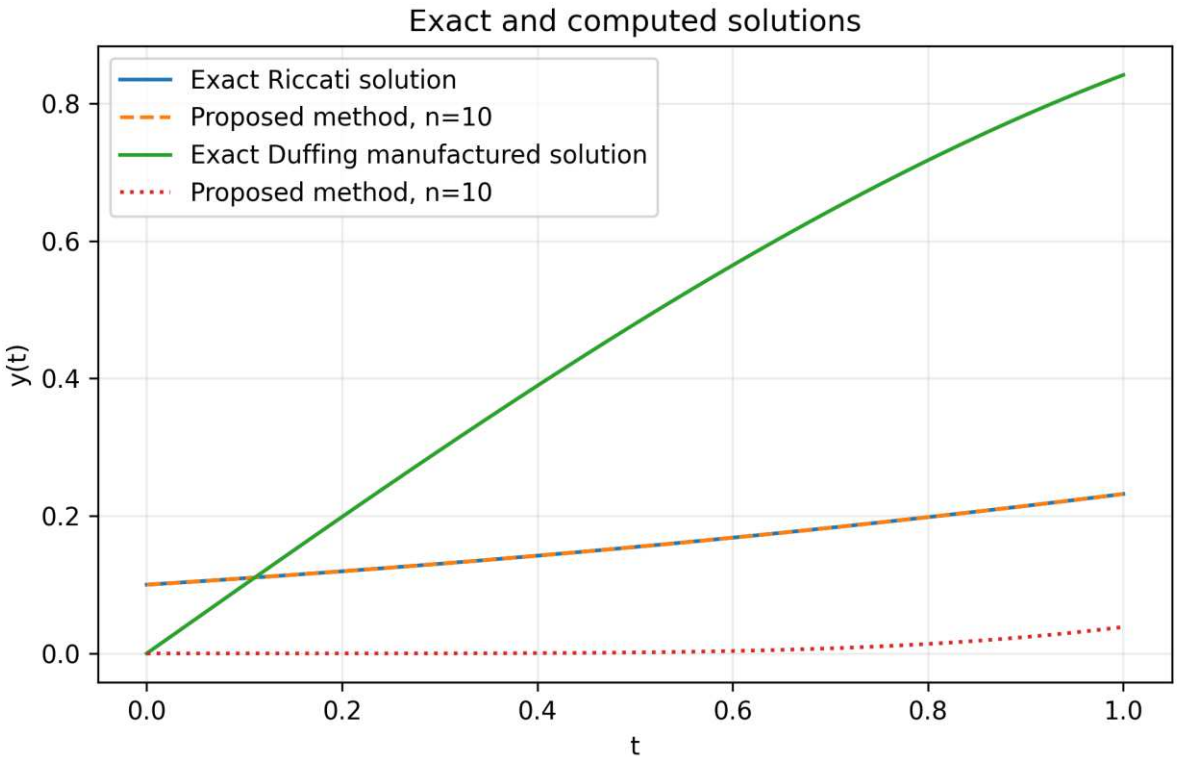


Figure 2. Exact and computed solutions for the Riccati and Duffing-type tests at n=10.

5.4. Numerical results are discussed.

The Riccati results drop off from a value of ~1.86×10-6 at n=4 to a value of ~1.04×10-5 at n=20. 1.17×10^-15 at n=12. The Duffing-type problem decreases from approximately 1.47×10^-5 at n=4 to approximately1.11×10^-15 at n=12.

The spectral trend of the third order nonlinear problem is very similar. For the reported cases, the residual norms are at or below ~10-13, while for the higher degrees, they are in the range of 10-15 to 10-16. The results should be viewed as proof of fast convergence of the residual formulation that is finite dimensional, and not as a general statement of the accuracy of the machine's solution of any nonlinear differential equation. The tests are based on smooth analytic solutions on a short interval. The singularities, discontinuities, boundary layers, or very long time intervals may need to be subdivided; filtered or rational Chebyshev functions or a different basis may be required. The maximum error, discrete residual and number of Newton function evaluations, the accuracy and residual behavior for the Duffing-type equation and the accuracy and residual behavior for the third-order nonlinear equation as shown in table 1, 2 and 3. The maximum error with respect to the degree of the polynomial for the three nonlinear verification problems, and the exact solution and the computed solution for the Riccati and Duffing-type tests at n=10 are presented. Figure 1 and 2.

6. Comparative discussion, methodological novelty

The original paper defined the basic idea of operational-matrix introduced by Chebyshev and applied it to a number of standard nonlinear differential equations. The new framework is built on the same foundation but with different numbers. The distinction is that the matrix that is operated on is not a part of the entire method, but it is a part of a method that uses a residual-corrected adaptive algorithm. This separation is relevant in light of the literature. Rani, Mishra used Bernoulli-polynomial operational matrix inverse Laplace method combined with the Laplace Adomian decomposition method and in their study they included Duffing, Van der Pol, Blasius and jerk equations [12]. An explicit operational-matrix treatment for numerical Laplace inversion and differential/integral equations with error estimation was also discussed. These studies prove that the general operational-matrix concept has been in place. This contribution should thus be interpreted simply as an adaptive residual corrected extension of the Chebyshev operational-matrix inverse-Laplace approach. The novelty is that the algorithm incorporates coefficient space non-linear evaluation, residual/Jacobian correction, adaptive degree selection, and independent residual verification all within one algorithm. This placement obviates an unsupported claim that Chebyshev operational matrices are "new. The motivation for using Chebyshev polynomials is the fact that they are well known for their role in spectral approximation. Chebyshev spectral methods of solution have been well developed for differential equations and nonlinear numerical problems [14]. The use of numerical Laplace inversion is a very old topic and Abate and Whitt were able to formalize a model in which inverse transforms are calculated from finitely many transform values. The present one is based on the idea of inversion, but not directly on a formula of contour inversion, but in a finite-dimensional Chebyshev integral representation. Another benefit is that it can be reproduced. The algorithm is explicit with its input, it features a stopping rule based on a residual condition, features a degree adaptation rule, and has measurable error quantities. The properties facilitate this process more than would a formulation where the only means of comparison would be a visual inspection of approximate curves.

7. The limitations of this study and future extensions are (7).

There are a few drawbacks to be noted.

1. First, the present numerical simulation is done with smooth manufactured solutions and a finite interval.
2. Second, the dense operational matrix is very costly at very large polynomial degree.
3. Third, the inverse Laplace interpretation is useful if the transformed equation is readily decomposable into algebra, which can be the case for highly complicated nonlinear transforms, for which the equivalent Volterra formulation may be more appropriate.
4. Future investigations should include the study of piecewise Chebyshev intervals for long-time integration, rational Chebyshev bases for semi-infinite domains, preconditioned Newton systems for large degrees, interval-by-interval adaptively, and the use of these ideas in the context of fractional differential equations. A posteriori degree selection based on both the residual and the decay of the final few Chebyshev coefficients would be very useful.

8. Conclusion

The development of a residual-corrected adaptive Chebyshev operational-matrix framework for nonlinear differential equations via the numerical inverse Laplace transformation is developed. The method transforms the differential problem into an integrated Volterra form, expresses the solution of the problem in a shifted Chebyshev basis, computes the nonlinear terms in the coefficient/nodal space, and solves the resulting nonlinear algebraic system using Newton residual correction. An adaptive degree criterion avoids using a too high polynomial basis. The manufactured tests show rapid convergence of the maximum error as the polynomial degree gets higher and the residuals are close to machine precision for moderate polynomial degrees. Results show that the proposed correction strategy can significantly enhance the original operational-matrix concept and can offer a systematic and repeatable stopping strategy numerically. The main methodological message is that the contribution is not to be presented as another Chebyshev operational matrix! The most defendable research contribution is the use of the Chebyshev inverse-Laplace representation in combination with the evaluation in nonlinear coefficient space as well as the residual/Jacobian correction and adaptive degree control. This formulation is more convenient for future theoretical studies and also for applications to more complicated nonlinear models.

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Declaration of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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