

Development of Turbulence: I. Stability and Smoothness of Navier-Stokes equations

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Short Report

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Posted Date: October 14th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11355091/v1>

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Additional Declarations: No competing interests reported.

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Abstract

Numerical analysis and Direct Numerical Simulations (DNS) are applied to investigate stability of Navier-Stokes equations (NS) to random perturbations of velocity at different scales defined by the grid resolution, $h \in [1.5 \times 10^{-7}, 1.5 \text{ cm}]$ and Reynolds numbers $Re \in [10^{-4}, 10^3]$. These phenomena are considered in conjunction with possible existence of infinite velocity singularities as recently proposed (OpenAI, 2026) by using a vortex model counterexample to smooth NS solutions. The present study involves a physically similar case of singular jets with or without external power. The overall assessment indicates that NS are stable below the local critical Reynolds number $Re_c = 1.5$, with singularities leading to infinite pressure gradients and external power inputs, while generally contradicting the NS energy balance. The current work also suggests that smooth, steady physical solutions to NS always exist at $h \rightarrow 0$.

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1 Introduction

Singularity concept is directly linked to the existence and smoothness of NS solutions and their physical nature. Any singularity in continuous flow constitutes an unphysical result which is contained in the phenomenon of finite time velocity blowup [1]. Infinite spatial energy density at singularity is also unphysical, albeit formally satisfying the overall energy constraints formulated in the description of existence and smoothness hypothesis [2]. This is a principal problem for the entire area of fluid mechanics which is based on NS. Moreover, from the first principles, NS directly follow from the fundamental laws of classical physics. If an unphysical situation manifests itself through NS, this inevitably means a contradiction to Newtonian dynamics under the general conditions for which these laws have been formulated. Any counterarguments that

discrete molecular structure or statistical energy considerations may prevent physical solutions for continuous fluids, are missing the point: NS computations involve nothing but the second Newton's law with inclusion of the viscosity terms and conservation of mass balance [3]. Thus, a detailed examination of this issue is of great scientific methodological significance and highly warranted.

In what follows below, the singularity problem is included in analysis of NS stability and smoothness at decreasing scale. A hypothetical (but physically similar to the vortex) singularity of growing velocity jet is examined using DNS, confirming that such systems are stabilized at maximum velocity defined by Re and external power input. Phenomena of energy dissipation, kinetic energy spatial distribution and velocity oscillations are briefly discussed. The next section presents a numerical analysis of NS leading to determination of critical local Reynolds number, Re_c , which defines the transformation between viscous regime at small scales, to turbulent flows at $Re \rightarrow \infty$. Finally, it is shown that decreasing Re is also responsible for velocity and pressure profiles being smooth at the scale limit $h \rightarrow 0$.

Following publications will discuss some important aspects of turbulent theory, based entirely on NS analysis, including: DNS methodology for $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ applications; NS averaging; energy dissipation, conversion and transfer; spectral characteristics and Kolmogorov hypothesis; origin of oscillations and intermittency; impact of boundary conditions and initial pressure-velocity profiles; and, finally, propagation of errors, criteria for energy blowup and control.

2 Methods

The present methodology is centered on conversion of NS into a discrete form suitable for analysis of discontinuous functions as described in the next section. The detailed DNS framework will be presented in later publications. Briefly, it utilizes a finite-difference method applied for a cubic grid with elemental unit h . Following computational stability criteria developed, the dimensionless time increment, Δt , (also known as Courant-Friedrichs-Lewy number) was $\sim 10^{-5}$. The computational steps involved a modified central difference algorithm, with accuracy similar to the fourth-order Runge-Kutta (RK4), but significantly faster. Options for boundary conditions included periodic boundary, fixed velocity/pressure, fixed gradients, and variable velocity/pressure. Flexibility for energy input allowed for velocity/pressure pulses with specific amplitude and frequency, application of bulk forces and/or variable boundary conditions. Singular jets were generated by velocity pulses perpendicular to the (x,y) plane at a central point of 40^3 grid, designed for short computations. Fixed velocity was enforced at the boundary. Dynamic measurements comprised velocity/pressure profiles, kinetic and potential energies, rate of energy dissipation, momentum/energy transfer rates, spatial velocity profile roughness and fractal dimensions, and, in addition, spectral energies characteristics and running error parameters. Air flow at normal pressure and temperature was used as a model medium. The method has been validated for its adherence to the criteria of energy and momentum conservation.

3 Discrete NS form

For compressible fluids the continuity equation reads:

$$\frac{\partial \rho}{\partial t} = -\nabla \bullet (\rho \mathbf{u}) \quad (3.1)$$

In the case of isotropic Newtonian fluids and adiabatic process, if density fluctuations are relatively small, pressure change is defined by its first derivative:

$$\Delta P = K_s \Delta \rho; \quad K_s = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (3.2)$$

For incompressible fluids, the modulus $K_s \rightarrow \infty$, and in the case of ideal gases, $K_s = \gamma P / \rho$, but in the present case, this coefficient can simply be considered a large constant. Introducing criteria for “nearly incompressible” fluids, the following applies:

$$|\mathbf{u} \bullet \nabla \rho| \ll |\rho \nabla \bullet \mathbf{u}|; \quad \left| \frac{\Delta \rho}{\rho} \right| \ll 1; \quad (\nabla \bullet \mathbf{u}) \rightarrow 0; \quad (\nabla \bullet \mathbf{u}) \neq 0 \quad (3.3)$$

so that equation (1) can be rewritten as:

$$\frac{\partial P}{\partial t} = -K_s \rho (\nabla \bullet \mathbf{u}) \quad (3.4)$$

thus, pressure and velocity fluctuations are directly proportional to each other.

Conditions above correspond to a borderline between more general NS for compressible fluids [3], and incompressible NS [1-3]. In the present work, incompressible fluids are treated as limit case. The advantage of using such an approach is that, whilst fluid density can be considered practically constant, the continuity mass-balance equation still contains a general pressure-velocity relationship required for computations. This also means no loss of generality, because all the principal conclusions related to the definition of scale, Re and energy functions can also be extended to compressible NS. As will be discussed later, conditions (3.3) satisfy most practical applications such as air or liquid flows at normal pressure and temperature. Applying the vector form of incompressible NS [3]:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \bullet \nabla) \mathbf{u} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad (3.5)$$

Energy form of this equation can be obtained by scalar multiplication of all parts with $\mathbf{u}$:

$$\frac{\partial E}{\partial t} = -(\mathbf{u} \bullet \nabla) (E + U) + \nu \nabla^2 E + \varepsilon + w \quad (3.6)$$

$$\text{where } E = \frac{1}{2} (\mathbf{u})^2; \quad U = \frac{P}{\rho}; \quad \varepsilon = -\nu (\nabla \mathbf{u})^2 \text{ and } w = (\mathbf{u} \bullet \mathbf{f})$$

Equation (3.6) contains some hidden parameters. Firstly, heat exchange is not considered, as motion is adiabatic and entropy of each fluid element remains constant, but it does affect the value of K_s . Secondly, there is a periodic conversion of kinetic energy, E , into elastic potential energy of fluid element deformation. This may seem to be unimportant for incompressible fluids, but it is not so. Whatever the magnitude of K_s in equation (3.2), the kinetic energy must convert into potential energy at least partially. DNS of the present work indicate that such conversion plays an important role in the total energy balance, intermittency and propagation of sonic waves in turbulent fluids. Thus, one should consider underlying potential energy terms in equation (3.6). This can be achieved by separating pressure into two parts:

$$P = \bar{P} + \tilde{P} \quad (3.7)$$

where the fluctuating part, $\tilde{P}$, describes relative deviation of pressure from its base value, $\bar{P}$, and caused by the velocity fluctuations in equation (3.4). Calculating work of deformation, $d\tilde{U} = \tilde{P}dV$, for a fluid element with volume per unit mass, $V = 1/\rho$, after integration:

$$\tilde{U} = -\frac{1}{K_s} \left(\frac{\tilde{P}}{\rho} \right)^2 \quad (3.8)$$

where $\tilde{U}$ is the potential energy of fluid deformation, similar to the energy of elastic strain of solids. By applying equation (3.4), the rate of elastic energy change is:

$$\frac{\partial \tilde{U}}{\partial t} = \frac{\tilde{P}}{\rho} (\nabla \bullet \mathbf{u}) \quad (3.9)$$

The next step is to convert equations (3.4-3.6) into a discrete and dimensionless form suitable for computations. Because NS equations represent the universal conservation laws of momentum (3.5) and energy (3.6), they are directly valid for any discrete values of these parameters, provided that non-linear terms $(\mathbf{u} \bullet \Delta) \mathbf{u}$ and $(\mathbf{u} \bullet \Delta) E$ are treated as momentum and energy transfer rates from one spatial point to another, and that spatial averages are taken for all variables, which can also be discontinuous functions. This theorem can be proven for a general case of finite variables. However, in the present work, limit $\Delta x \rightarrow 0$ implies differentials. The index form of converted equations is:

$$\frac{\Delta_t u'_i}{\Delta t'} = -u'_j \Delta_j u'_i - \Delta_i P' + \frac{1}{Re} \Delta_j^2 u'_i + f'_i \quad (3.10)$$

$$\frac{\Delta_t E'_i}{\Delta t'} = -u'_j \Delta_j (E'_i + U') + \frac{1}{Re} (\Delta_j^2 E'_i + \varepsilon'_i) + w'_i \quad (3.11)$$

$$\frac{\Delta_t P'}{\Delta t'} = -K'_s \Delta_i u'_i \quad (3.12)$$

Where the following applies:

$$Re = u_0 h / \nu; \quad K'_s = K_s / u_0^2; \quad \Delta t' = u_0 \Delta t / h; \quad \Delta x_i' = \Delta x_i / h; \quad u_i' = u_i / u_0; \quad P' = P / \rho u_0^2; \quad f'_i = f_i h / u_0^2; \quad E_i' = u_i'^2 / 2 = E_i / u_0^2; \quad U' = U / u_0^2; \quad \varepsilon_i' = -(\Delta_i u'_i)^2 = \varepsilon_i h / u_0^3 / \nu; \quad w_i' = w_i h / u_0^3; \quad \Delta x_i = h.$$

The last assignment leads to $\Delta x_i' = 1$ which can further be omitted from equations (3.10) and (3.12). Importantly, parameter h is now both the smallest scale and minimal unit of computational grid. u_0 is the absolute value of characteristic reference velocity, which is typically taken as the steady, non-fluctuating part of velocity field. Symbols Δ_t and Δ_j denote discrete variations of dimensionless time, t' , or coordinates, $\Delta x_i'$, whereas quantities E'_i , U' , ε'_i and w'_i correspond to dimensionless kinetic energy, potential energy, energy dissipation rate and external power. The prime symbol ($'$) is dropped in the following text.

The greatest advantage of using discrete form of equations (3.10-3.12) is that NS system of non-linear partial differential equations is modified into a less complex set of second order polynomials, which can be solved using algebraic analysis. It also enables discontinuous functions to be analyzed.

Let us consider development of singularity, $u \rightarrow \infty$, when $h \rightarrow 0$ and, correspondingly, $Re \rightarrow 0$. The last two conditions can be associated with viscous regime below the Kolmogorov microscale [3]. The reference velocity, u_0 , must be the same for each coordinate and at each scale, otherwise normalization of equations (3.10-3.12) is not applicable. As a consequence, u_0 cannot be the velocity at singularity. Futility of constructing blowup cases is already clear from the form of equations (3.10) and (3.11): with $Re \rightarrow 0$, any singularity leads to infinite derivatives of velocity, meaning infinite energy dissipation rate and infinite pressure gradients, not achievable in real physical systems or during simulations. In particular, any momentum transfer towards developing singularity would require unbounded pressure gradients and/or external forces and still negated by infinitely increasing friction at $Re \rightarrow 0$ as seen in equation (3.10). The physical mechanism of this phenomenon is contained within NS: because in all

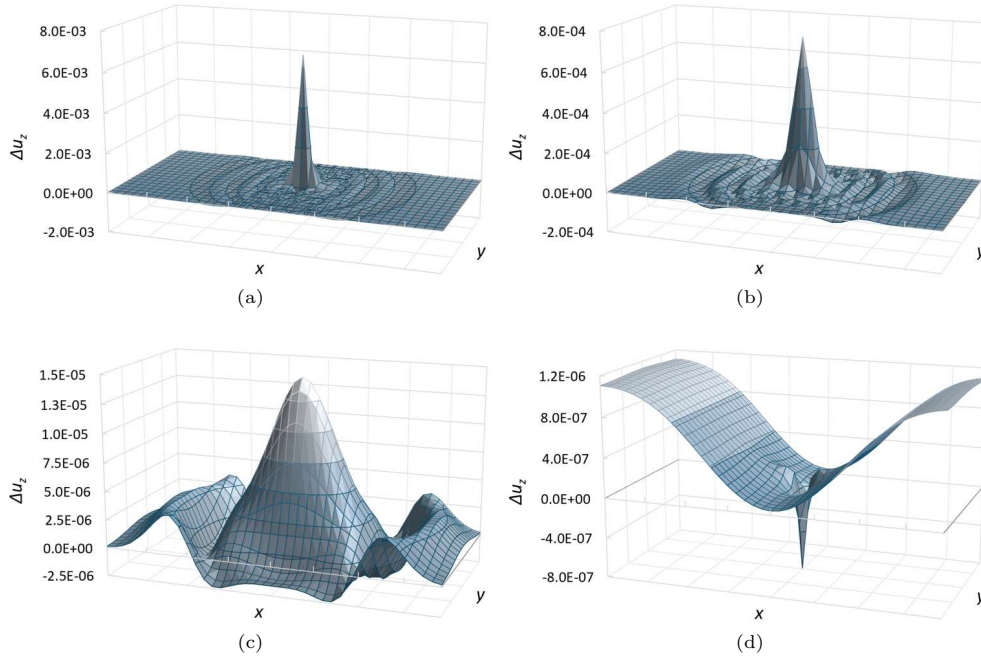


Fig. 1: Jet velocity profiles $\Delta u_z(x, y)$ originated with the same value and propagated at different Reynolds numbers: (a) $Re = 2$, (b) $Re = 0.1$, (c) $Re = 0.01$, and (d) $Re = 0.001$. These profiles were captured at the end of time interval in Fig.2

Newtonian fluids, the molecular transfer of momentum and energy depends on linear combination of derivatives, $\partial u_i / \partial x_k$, or in the discrete case, on magnitude of differentials $(\Delta_j u_i) / h$, such transfer rate per unit mass becomes infinite near singularity and therefore preventing such singularity to occur. The results above also contradict a major assumption that $Re \rightarrow \infty$ at singularity (see p.4, reference [1]).

4 A case against singularity

In the singularity model proposed [1], a vortex rotates with continuously decreasing radius and length, sustained by centripetal pressure and external forces. By retaining its angular momentum per unit mass, it is projected by the authors (at least in theory if not in practice) to reach infinite radial and axial velocities at finite time, suggesting a physical breakup of NS at the axial singularity point [1]. The motion of inner core can be approximated by a fluid cylinder rotating around its central axis passing through singularity and with radius, r , decreasing to zero with time. The momentum per unit mass, $L = \frac{1}{2} u_\theta r$, is constant, and therefore $u_\theta \sim r^{-1} \rightarrow \infty$. Also, the kinetic energy of rotation per unit mass $E = u_\theta^2 / 4 \sim r^{-2} \rightarrow \infty$, requiring an infinite power from outside the core. In order to balance this rotation speed, the centripetal pressure gradient can be calculated as: $grad P = \rho u_\theta^2 r^{-1} \sim r^{-3}$, directed from the axis radially and fast approaching infinity. Correspondingly, pressure must also rise to infinity near

singularity as $P \sim r^{-2}$. The same effect applies to the ideal fluid of Euler equation, obtained from equation (3.10) with $Re \rightarrow \infty$.

Considering the physics constraints above, one may suggest replacing vortex with a singular jet of increasing velocity but decreasing scale. Fig.1 shows velocity profiles of such jets originated with a constant initial velocity and no external energy input. These profiles vary dramatically as both scale and Re decrease. The evolution starts with a sharp and intense peak at higher $Re > Re_c$ (Fig.1a) passing through a transitional scale where this peak becomes smaller and wider (Fig.1b) ending up with very shallow morphologies (Figs.1c and d) which, at certain time points, can be reversed. The transformations observed illustrate three principal phenomena occurring with introduction of any velocity gradients: increasing dissipation of energy as scale decreases, spread of kinetic energy in fluids starting with sonic waves visible in Figs.1a and b, and periodic conversion between kinetic and elastic energies responsible for peak fluctuations and reversals at lower Re . Energy diagram of this process is shown in Fig.2. The energy dissipation and periodic energy conversions are clearly seen as functions of Re . At high Re however, another phenomenon emerges: a gradual increase in total energy due to computational errors associated with the finite values of dimensionless time increment, Δt . The later effect, although not physical, manifests itself as an additional energy source inherent to any computation. Consequently, it has to be addressed in order to prevent a real (for computer simulations) blowup of velocity and energy. The supplemental recordings (S1-S4) illustrate jet dynamics at different Re .

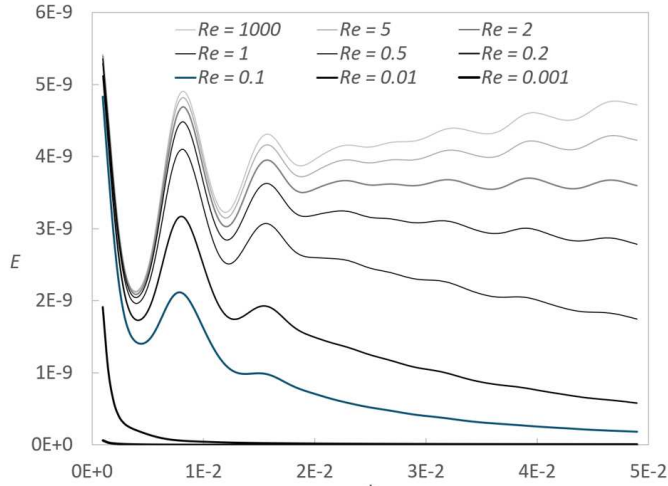


Fig. 2: Time-dependences of jet kinetic energy, E , on Reynolds number, Re

When energy per unit mass is supplied at constant rate, one can write kinetic energy balance from equation (3.11), neglecting the conservative energy transition terms:

$$\frac{\partial E}{\partial t} = \frac{\varepsilon}{Re} + w \quad (4.1)$$

In the present case of singular jet:

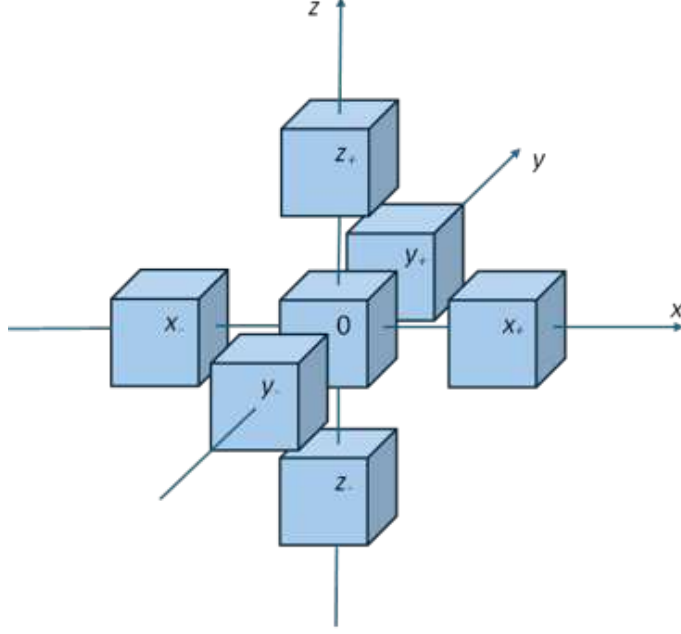


Fig. 3: The elemental unit of computation for discrete NS consisting of octahedral von Neumann neighborhood where the indexes (-) and (+) correspond to positions of cells relative to the central cell (0)

$$\varepsilon \sim -2E \quad (4.2)$$

By integrating equations (4.3) and (4.4):

$$E(t) = (E_o - \frac{w \, Re}{2})e^{-\frac{2t}{Re}} + \frac{w \, Re}{2} \quad (4.3)$$

This function plateaus at $E_{t \rightarrow \infty} = w \, Re/2$ and requires an infinite rate of energy input to sustain at $Re \rightarrow 0$. The evolution of such jet at constant external power, w , but decreasing scale is presented in supplemental video recording S5.

5 NS stability

Within the Eulerian framework of NS, time-dependent and spatial accelerations are mathematically separated. As a consequence for any simulation, any fluid system needs time to recover its balance after a disturbance occurred. This creates the following effect: if a small (theoretically infinitesimal) perturbation is made at any spatial point, and considering reaction of all forces: the pressure and friction responses are always negative (*i.e.* resisting change), but the inertial term (first on the right in equation (3.10)) is positive, formally leading to more instability. This phenomenon is exploited below. The elemental unit for computation here is characterized by octahedral von Neumann neighborhood [5,6] as shown in Fig.3. Let us consider a case whereby small perturbations of velocity, $\partial_{ij}u_i$, may appear randomly in any cell of this neighborhood. Here, index i is assigned to a specific velocity component and index j corresponds to the direction of this component change.

Let differential for any function dependent on velocity, φ , be:

$$\partial_{ij}\varphi(u_i) = \varphi(u_i + \partial_{ij}u_i) - \varphi(u_i) \quad (5.1)$$

and calculate the following terms:

$$\partial_{ij}(\Delta_j u_i) = \partial_{ij}u_{i+} - \partial_{ij}u_{i0} \quad (5.2)$$

$$\partial_{ij}(u_j \Delta_j u_i) = u_{j0}(\partial_{ij}u_{i+} - \partial_{ij}u_{i0}) \quad (5.3)$$

$$\partial_{ij}(\Delta_j^2 u_i) = \partial_{ij}u_{i+} + \partial_{ij}u_{i-} - 2\partial_{ij}u_{i0} \quad (5.4)$$

Equation (5.3) ignores differentials of the higher order. For the pressure term:

$$\partial_{ij}(\Delta_i P \delta_{ij}) = \partial_{ij}(\Delta_t P \delta_{ij}) = -K_s(\partial_{ij}u_{i0} - \partial_{ij}u_{i-})\Delta t \quad (5.5)$$

and, therefore, this term is of the second order in respect to time.

Moreover, $\delta_{ij}f_i = 0$, because the external force is considered independent of velocity fluctuations. The stability criterion is:

$$\partial_{ij}(\Delta_t u_i) = 0 \quad (5.6)$$

leading to the following relationships:

$$\chi_2 = Re_c \chi_1 \quad (5.7)$$

$$\chi_1 = u_{i0}(\partial_{ij}u_{i+} - \partial_{ij}u_{i0}); \quad \chi_2 = \partial_{ij}u_{i+} + \partial_{ij}u_{i-} - 2\partial_{ij}u_{i0} \quad (5.8)$$

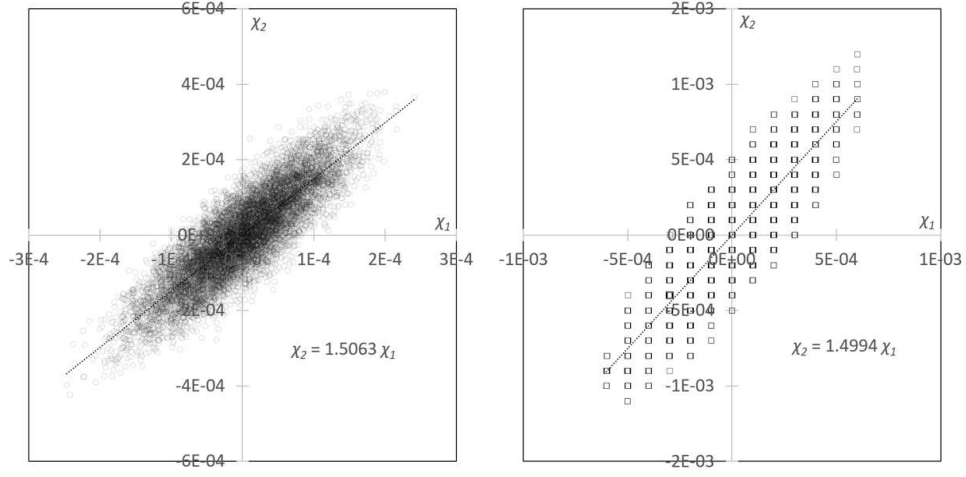


Fig. 4: Diagrams of NS stability response according to equation (5.7) with (a) random continuous and (b) random discrete perturbations of velocity in $\mathbb{R}^3$ space

Functions χ_1 and χ_2 correlate with each other defining the critical Reynolds number, Re_c , below which the friction term, χ_2 , dominates the inertial term, χ_1 . At $Re < Re_c$, NS are stable to small, random perturbations of velocity. Equation (5.7) is insensitive to the base velocity profile and amplitude or frequency of fluctuations, $\partial_{ij}u_i$, because these parameters affect both sides of this equation proportionally. It can also be noted that pressure fluctuations transform into velocity fluctuations according to equation (3.10), resulting in the same effect with some time delay. Fig.4 exhibits distributions χ_2 (χ_1) obtained with $\partial_{ij}u_i$ pulses randomly generated from continuous interval $[-\frac{a}{2}, \frac{a}{2}]$ (Fig.4a) or from discrete number set $(-a, 0, a)$ (Fig.4b), where the

amplitude $a = 10^{-4}$. In all cases, data points are distributed around straight lines with inclination $Re_c = 1.5$. The distribution graphically resembles an elongated hexagon or oblong ellipse. Although density of points changes with frequency of numbers generated, this inclination remains practically constant and robust, confirming that NS are statistically stable below Re_c .

6 The existence and smoothness of NS solutions

The results of previous sections establish that NS solutions have to be continuous functions without singularities at the scale limit $\Delta_i x = h \rightarrow 0$. The same methodology can be applied to examine existence and smoothness of these functions. According to equation (3.10), there are 24 boundary conditions of velocity and pressure in the six neighborhood cells (Fig.3). Three quadratic equations are to be solved in respect to three velocity components and pressure in the central cell with external force components, f_i , and in conjunction with the linear equation (3.12). Not all boundary conditions have to be defined, as some combinations can be too restrictive and incompatible. Let us consider when a steady physical solution can exist. For simplicity, the cell coordinate system is selected whereby the axis x is aligned with velocity vector, $\mathbf{u}$, in the central cell. This means that axis x is tangential to the velocity streamline and equations (3.10) and (3.12) convert into following:

$$u_{10}^2 - (u_{1+} + \frac{6}{Re})u_{10} + (\frac{\Sigma_1}{Re} - \Delta_1 P + f_1) = 0 \quad (6.1)$$

$$u_{2+}u_{10} - (\frac{\Sigma_2}{Re} - \Delta_2 P + f_2) = 0 \quad (6.2)$$

$$u_{3+}u_{10} - (\frac{\Sigma_3}{Re} - \Delta_3 P + f_3) = 0 \quad (6.3)$$

$$u_{1-} + u_{2-} + u_{3-} - u_{10} = 0 \quad (6.4)$$

Here, $\Delta_i P = (P_{i+} - P_{i0})$ and $\Sigma_i = \sum_{j=1}^3 (u_{ij+} + u_{ij-})$.

Equations (6.2) and (6.3) are now both linear, and equation (6.4) presents the incompressible continuity $\text{div } \mathbf{u} = 0$. This implies no loss of generality because such coordinates can always be assigned to a specific velocity on streamline. For a physical solution to exist, the discriminant of quadratic equation (6.1) should be non-negative:

$$D = (u_{1+} + \frac{6}{Re})^2 - 4(\frac{\Sigma_1}{Re} - \Delta_1 P + f_1) \geq 0 \quad (6.5)$$

It is easy to see that this relationship is always valid at $Re \rightarrow 0$ (or $h \rightarrow 0$) where it is dominated by the positive quantity $(\frac{6}{Re})^2$ and equation (6.1) becomes linear. Its solution, u_{10} , can be found as:

$$u_{10} = \frac{\Sigma_1}{6} + \mathcal{O}(Re) \quad (6.6)$$

Thus, on microscale, velocity in any central fluid cell can be calculated as average of corresponding velocity values in the boundary cells. Since pressure and velocity are related linearly on microscale, the same statement is also valid for pressure. Moreover, equation (6.6) is true for any fluid cell. It is therefore possible to build a manifold in which velocity and pressure for internal cells are completely defined by the boundary conditions on their surface, plus a linear combination of their residuals $\mathcal{O}(Re)$. Such manifold can be expanded globally until the boundary conditions of the macroscopic fluid system are met, and steady global solution to NS found after iterations.

This problem can also be considered from a different viewpoint. The local solution at $Re \rightarrow 0$, depends strongest on the neighborhood values of velocity, u_{ij} (see equations (6.1-6.4)). Solution of these equations is analytically related to Re , parameters u_{1+}

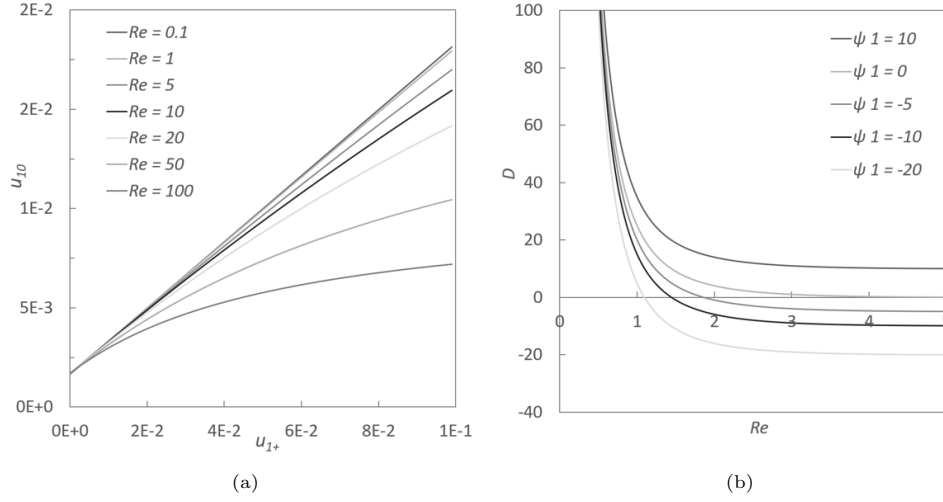


Fig. 5: (a) Local solution of equation (6.1) in coordinates $u_{10}(u_{1+})$ and for different Reynolds numbers, Re ; (b) Discriminant of equation (6.5) as function of Re , and the force parameter, ψ_1

and Σ_1 , and residuals, $\mathcal{O}(Re)$, which depend on the force function, $\psi_1 = (f_1 - \Delta_1 P)$. All these functions, by their structure, are smooth with $Re \rightarrow 0$ (with exception of the obvious point $Re = 0$). For example, Fig.5a shows dependence $u_{10}(u_{1+})$ at different Re and fixed boundary condition Σ_1 and force ψ_1 . For small velocity fluctuations, $u_{1+} \approx 1$; $\Sigma_1 \approx 6$, discriminant, D , converts into:

$$D = \left(1 - \frac{6}{Re}\right)^2 + \psi_1 \quad (6.7)$$

If $\psi_1 \geq 0$, a solution to equations (6.1-6.4) does exist in general. However, at higher Re (*i.e.* larger scale), this solution is not guaranteed to be smooth or even continuous, representing turbulence fluctuations on larger scales. When $\psi_1 < 0$, there is a maximum value of Re below which a steady solution can still be found as illustrated in Fig.5b. Not finding a steady solution with specific boundary conditions means that solution is not steady, *e.g.* as often observed in the important case of periodic boundary conditions at high Re . Non-steady solutions also appear when velocity or pressure undergo fluctuations on the boundary surface.

7 Conclusions

It is shown that motion of any fluid is stable to random perturbations of velocity and pressure below the critical Reynolds number $Re_c = 1.5$ as directly follows from the NS analysis. With $Re \rightarrow 0$, these equations result in smooth, steady solutions $(\mathbf{u}, P)$, defined by the boundary conditions and force parameters. Investigation into potential singularities has determined that blowups are incompatible with the definitions of NS scale, Re , and energy balance equations. Further results will demonstrate that NS are sufficient to address the major questions of turbulence theory.

Supplementary Recordings

Supplementary materials and recordings are available online. The supplemental recordings associated with this manuscript are listed below:

- S1:** Development of singular jet at $Re = 1000$
- S2:** Development of singular jet at $Re = 2$
- S3:** Development of singular jet at $Re = 0.1$
- S4:** Development of singular jet at $Re = 0.01$
- S5:** Kinetics of jet propagation at constant power input and $Re \rightarrow 0$

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Supplementary Files

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