

Supplementary Information

Quantum oblique eigenprojection

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S1 Standard block encoding of resolvent integral

In this appendix, we provide details on the standard resolvent-integral block encoding described in the main text.

S1.1 Resolvent integral and generalized eigensubspaces

Let A be a square matrix acting on a Hilbert space $\mathcal{H}$. By the primary decomposition theorem [3, Theorem 12.2.14][6, Theorem 7.6], the entire space splits as a direct sum of generalized eigensubspaces

$$\mathcal{H} = \bigoplus_{\lambda \in \mathbf{Spec}(A)} \mathcal{H}_\lambda, \quad \mathcal{H}_\lambda = \mathbf{Ker}((A - \lambda I)^{m_\lambda}), \quad (\text{S1})$$

for positive integers $m_\lambda \leq \dim(\mathcal{H})$. Each $\mathcal{H}_\lambda$ is invariant under A , and the restriction $N_\lambda := (A - \lambda I)|_{\mathcal{H}_\lambda}$ is nilpotent, with $N_\lambda^{m_\lambda} = 0$.

Let $\mathcal{C}$ be a closed curve that encloses part of the spectrum $\mathbf{Spec}_{\text{in}}(A)$ and excludes the rest $\mathbf{Spec}_{\text{out}}(A)$, so that $\mathbf{Spec}(A) = \mathbf{Spec}_{\text{in}}(A) \cup \mathbf{Spec}_{\text{out}}(A)$ with $\mathbf{Spec}_{\text{in}}(A)$ inside $\mathcal{C}$ and $\mathbf{Spec}_{\text{out}}(A)$ outside. The Riesz projector associated with $\mathbf{Spec}_{\text{in}}(A)$ is defined by

$$\Pi_{\text{in}} = \frac{1}{2\pi i} \int_{\mathcal{C}} (zI - A)^{-1} dz, \quad (\text{S2})$$

the curve traversed counterclockwise. The following lemma shows that Π_{in} is precisely the oblique projection onto the enclosed generalized eigensubspaces $\bigoplus_{\lambda \in \mathbf{Spec}_{\text{in}}(A)} \mathcal{H}_\lambda$ along the excluded ones $\bigoplus_{\lambda \in \mathbf{Spec}_{\text{out}}(A)} \mathcal{H}_\lambda$.

Lemma S1. *Let A be a square matrix on $\mathcal{H}$ with the generalized eigensubspace decomposition $\mathcal{H} = \bigoplus_{\lambda \in \mathbf{Spec}(A)} \mathcal{H}_\lambda$ as in (S1), and let Π_{in} be the Riesz projector (S2) for a curve $\mathcal{C}$ enclosing $\mathbf{Spec}_{\text{in}}(A) \subseteq \mathbf{Spec}(A)$ and excluding $\mathbf{Spec}_{\text{out}}(A) = \mathbf{Spec}(A) \setminus \mathbf{Spec}_{\text{in}}(A)$. Then*

$$\Pi_{\text{in}}^2 = \Pi_{\text{in}}, \quad \mathbf{Im}(\Pi_{\text{in}}) = \bigoplus_{\lambda \in \mathbf{Spec}_{\text{in}}(A)} \mathcal{H}_\lambda, \quad \mathbf{Ker}(\Pi_{\text{in}}) = \bigoplus_{\lambda \in \mathbf{Spec}_{\text{out}}(A)} \mathcal{H}_\lambda. \quad (\text{S3})$$

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Proof. The resolvent $(zI - A)^{-1}$ preserves the decomposition (S1), so it suffices to evaluate Π_{in} on each summand $\mathcal{H}_\lambda$.

Writing $A = \lambda I + N_\lambda$ on $\mathcal{H}_\lambda$, for $z \notin \mathbf{Spec}(A)$ the restricted resolvent is the finite Neumann series

$$(zI - A)^{-1}|_{\mathcal{H}_\lambda} = ((z - \lambda)I - N_\lambda)^{-1} = \sum_{k=0}^{m_\lambda-1} \frac{N_\lambda^k}{(z - \lambda)^{k+1}}, \quad (\text{S4})$$

terminating because $N_\lambda^{m_\lambda} = 0$. Integrating term by term over $\mathcal{C}$,

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{(z - \lambda)^{k+1}} = \begin{cases} 1 & k = 0 \text{ and } \lambda \in \mathbf{Spec}_{\text{in}}(A), \\ 0 & \text{otherwise,} \end{cases} \quad (\text{S5})$$

since the integrand has vanishing residue for $k \geq 1$, and for $k = 0$ the integral is the winding number of $\mathcal{C}$ about λ .

Thus Π_{in} restricts to the identity on $\mathcal{H}_\lambda$ for $\lambda \in \mathbf{Spec}_{\text{in}}(A)$ and to 0 for $\lambda \in \mathbf{Spec}_{\text{out}}(A)$. The first of these gives $\mathcal{H}_\lambda \subseteq \mathbf{Im}(\Pi_{\text{in}})$ and $\Pi_{\text{in}}^2 = \Pi_{\text{in}}$ on $\mathcal{H}_\lambda$; the second gives $\mathcal{H}_\lambda \subseteq \mathbf{Ker}(\Pi_{\text{in}})$ and again $\Pi_{\text{in}}^2 = \Pi_{\text{in}}$. Summing over the decomposition (S1) yields $\Pi_{\text{in}}^2 = \Pi_{\text{in}}$, $\mathbf{Im}(\Pi_{\text{in}}) = \bigoplus_{\lambda \in \mathbf{Spec}_{\text{in}}(A)} \mathcal{H}_\lambda$, and $\mathbf{Ker}(\Pi_{\text{in}}) = \bigoplus_{\lambda \in \mathbf{Spec}_{\text{out}}(A)} \mathcal{H}_\lambda$. $\square$

S1.2 Block encoding of resolvent integral

Appendix S1.1 treated a general curve $\mathcal{C}$. We now specialize to the unit circle. Parametrizing $z = e^{i\theta}$ in (S2), with $dz = ie^{i\theta} d\theta$ and $e^{i\theta}(e^{i\theta}I - A)^{-1} = (I - e^{-i\theta}A)^{-1}$, the Riesz projector (S2) becomes

$$\Pi_{\text{in}} = \frac{1}{2\pi i} \int_{|z|=1} (zI - A)^{-1} dz = \frac{1}{2\pi} \int_0^{2\pi} (I - e^{-i\theta}A)^{-1} d\theta. \quad (\text{S6})$$

This can be seen as the zeroth Fourier coefficient of the periodic function $\theta \mapsto (I - e^{-i\theta}A)^{-1}$.

We discretize the integral (S6) by the n -point trapezoidal rule at the roots of unity ω_n^j , where $\omega_n = e^{2\pi i/n}$ and $j = 0, \dots, n-1$,

$$\Pi_{\text{in}} \approx S_0 := \frac{1}{n} \sum_{j=0}^{n-1} (I - \omega_n^{-j}A)^{-1}. \quad (\text{S7})$$

Because the integrand is analytic on an annulus containing the unit circle, this approximation converges geometrically in n , with a rate we quantify in Appendix S2. The node count n enters only through the size $\lceil \log_2 n \rceil$ of the ancilla register indexing the roots of unity, and does not affect the query complexity of the construction below, which depends only on the normalization factor of the block encoding and the target precision.

The discretization (S7) expresses Π_{in} through resolvents of A , which can be realized on a quantum computer via block encodings. We recall standard block-encoding operations used throughout, referring to [2, 4] for details. In all complexity statements, the parameters are known bounds on the underlying quantities, such as upper bounds on normalization factors, norms, and spectral-set constants, and lower bounds on gaps, overlaps, and smallest nonzero singular values. The algorithms use these bounds to choose their internal parameters, and the stated costs hold with the bounds in place of the exact values.

Lemma S2 (Linear combination). *Given block encodings of matrices $A_1/\alpha_1, \dots, A_m/\alpha_m$ with normalization factors $\alpha_1, \dots, \alpha_m > 0$, and coefficients $c_1, \dots, c_m$, one can construct a block encoding of*

$$\frac{1}{\sum_{k=1}^m |c_k| \alpha_k} \sum_{k=1}^m c_k A_k \quad (\text{S8})$$

with normalization factor $\sum_{k=1}^m |c_k| \alpha_k$, using a single query to a controlled unitary that selects among the input block encodings, whose cost is the maximum of their individual query costs.

Lemma S3 (Product). *Given block encodings of matrices A_1/α_1 and A_2/α_2 with normalization factors $\alpha_1, \alpha_2 > 0$, one can construct a block encoding of*

$$\frac{1}{\alpha_1 \alpha_2} A_1 A_2 \quad (\text{S9})$$

with normalization factor $\alpha_1 \alpha_2$, using a single query to each of the two input block encodings.

Lemma S4 (Inversion). *Given a block encoding of A/α_A with normalization factor $\alpha_A > 0$, and an upper bound $\alpha_{A^{-1}} \geq \|A^{-1}\|$ on the inverse, for any $\varepsilon > 0$ one can construct a block encoding of*

$$\frac{1}{2\alpha_{A^{-1}}} A^{-1} \quad (\text{S10})$$

with normalization factor $2\alpha_{A^{-1}}$ and accuracy ε , using

$$\mathbf{O}\left(\alpha_A \alpha_{A^{-1}} \log \frac{1}{\varepsilon}\right) \quad (\text{S11})$$

queries to the block encoding of A/α_A .

Lemma S5 (Scaling). *Given a block encoding of A/α_A with normalization factor $\alpha_A > 0$, and a value $\alpha'_A \geq \|A\|$ upper bounding the norm:*

1. *if $\alpha'_A \geq \alpha_A$, one can construct a block encoding of*

$$\frac{1}{\alpha'_A} A \quad (\text{S12})$$

with normalization factor α'_A , using a single query to the block encoding of A/α_A ;

2. *if $\alpha'_A < \alpha_A$, then for any $\varepsilon > 0$ one can construct a block encoding of*

$$\frac{1}{2\alpha'_A} A \quad (\text{S13})$$

with normalization factor $2\alpha'_A$ and accuracy ε , using

$$\mathbf{O}\left(\frac{\alpha_A}{\alpha'_A} \log \frac{1}{\varepsilon}\right) \quad (\text{S14})$$

queries to the controlled block encoding of A/α_A and its inverse.

When applied to implementing the resolvent integral, the standard block encoding approach has a normalization factor governed by the maximum size of the resolvent on the unit circle

$$\alpha_{\text{res}} = \max_{\theta \in [0, 2\pi)} \left\| \left(I - e^{-i\theta} A \right)^{-1} \right\|, \quad (\text{S15})$$

which can be bounded under two standard assumptions on A . We denote the compressions onto the inner and outer invariant subspaces (S1) by $A_{\text{in}} = A\Pi_{\text{in}}$ and $A_{\text{out}} = A\Pi_{\text{out}}$.

The first assumption controls the resolvent through the numerical range. Recall that for a matrix B , $\mathcal{W}(B) = \{ \langle \psi | B | \psi \rangle : \|\psi\| = 1 \}$ is its numerical range. We assume the per-block numerical ranges are separated from the unit circle,

$$\mathcal{W}(A_{\text{in}}) \subseteq \{ z : |z| \leq 1 - \delta_{\text{num}} \}, \quad \mathcal{W}(A_{\text{out}}^{-1}) \subseteq \{ z : |z| \leq 1 - \delta_{\text{num}} \}, \quad (\text{S16})$$

for some $0 < \delta_{\text{num}} < 1$. The Crouzeix–Palencia bound [1] then applies to each block, giving $\|g(A_{\text{in}})\| \leq (1 + \sqrt{2}) \max_{z \in \mathcal{W}(A_{\text{in}})} |g(z)|$ for any g analytic on a neighborhood of $\mathcal{W}(A_{\text{in}})$, and likewise for A_{out}^{-1} .

The second assumption is diagonalizability. We assume the compressions $A_{\text{in}} = V_{\text{in}} \Lambda_{\text{in}} V_{\text{in}}^{-1}$ and $A_{\text{out}} = V_{\text{out}} \Lambda_{\text{out}} V_{\text{out}}^{-1}$ are respectively diagonalizable, so that $\|g(A_{\text{in}})\| \leq \kappa_V \max_{\lambda \in \mathbf{Spec}(A_{\text{in}})} |g(\lambda)|$ for any g analytic on a neighborhood of $\mathbf{Spec}(A_{\text{in}})$, and likewise for A_{out} , where

$$\kappa_V = \max(\|V_{\text{in}}\| \|V_{\text{in}}^{-1}\|, \|V_{\text{out}}\| \|V_{\text{out}}^{-1}\|), \quad \delta_{\text{eig}} = \min_{\lambda \in \mathbf{Spec}(A)} |\lambda| - 1 \quad (\text{S17})$$

is the per-block eigenbasis condition number replacing the Crouzeix–Palencia constant. A non-diagonalizable matrix instead admits bounds through its Jordan condition number and largest Jordan block size, outside the spectral-set framework below [8, Page 444].

Both assumptions on the input matrices are instances of a single condition, which we now formalize.

Definition S1 (Spectral set). *Let B be a square matrix and let $\mathcal{K} \subseteq \mathbb{C}$ be a compact set containing $\mathbf{Spec}(B)$. We call $\mathcal{K}$ a spectral set for B with constant $c \geq 1$ if*

$$\|g(B)\| \leq c \sup_{z \in \mathcal{K}} |g(z)| \quad (\text{S18})$$

for every function g analytic on a neighborhood of $\mathcal{K}$.

Lemma S6 (Instances of spectral sets). *Let B be a square matrix.*

1. *The numerical range $\mathcal{W}(B)$ is a spectral set for B with constant $c = 1 + \sqrt{2}$.*
2. *If $B = V \Lambda V^{-1}$ with Λ diagonal, then $\mathbf{Spec}(B)$ is a spectral set for B with constant $c = \|V\| \|V^{-1}\|$.*

Proof. Part (1) is the Crouzeix–Palencia bound [1]. For part (2), any g analytic on a neighborhood of $\mathbf{Spec}(B)$ satisfies $g(B) = V g(\Lambda) V^{-1}$. Since $g(\Lambda)$ is diagonal with entries $g(\lambda)$, $\lambda \in \mathbf{Spec}(B)$,

$$\|g(B)\| \leq \|V\| \|V^{-1}\| \max_{\lambda \in \mathbf{Spec}(B)} |g(\lambda)|. \quad (\text{S19})$$

□

We say that the compressions A_{in} and A_{out}^{-1} admit spectral sets $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ (S18) with constant c and gap δ if both sets have the common constant c and satisfy

$$\mathcal{K}_{\text{in}} \subseteq \{z: |z| \leq 1 - \delta\}, \quad \mathcal{K}_{\text{out}} \subseteq \{z: |z| \leq 1 - \delta\} \quad (\text{S20})$$

for some $0 < \delta < 1$. Here, for a compression such as $A_{\text{in}} = A\Pi_{\text{in}}$, spectra, numerical ranges, and spectral sets are taken for its restriction to the invariant subspace $\mathcal{H}_{\text{in}}$, and similarly for A_{out} and A_{out}^{-1} on $\mathcal{H}_{\text{out}}$. By Lemma S6, this condition holds in two cases. Under the numerical-range assumption (S16), take $\mathcal{K}_{\text{in}} = \mathcal{W}(A_{\text{in}})$ and $\mathcal{K}_{\text{out}} = \mathcal{W}(A_{\text{out}}^{-1})$, with $c = 1 + \sqrt{2}$ and $\delta = \delta_{\text{num}}$. Under the diagonalizability assumption, take $\mathcal{K}_{\text{in}} = \mathbf{Spec}(A_{\text{in}})$ and $\mathcal{K}_{\text{out}} = \mathbf{Spec}(A_{\text{out}}^{-1})$, with $c = \kappa_V$ (S17). Every eigenvalue of A_{out} has modulus at least $1 + \delta_{\text{eig}}$. So the eigenvalues of A_{out}^{-1} have modulus at most $1/(1 + \delta_{\text{eig}}) \leq 1 - \delta_{\text{eig}}/2$, and the condition holds with $\delta = \delta_{\text{eig}}/2$.

Proposition S7 (Resolvent norm bound). *Let A be a square matrix with Riesz projector Π_{in} (S6) and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$. If A_{in} and A_{out}^{-1} admit spectral sets (S18) with constant c and gap δ satisfying (S20), then the maximum resolvent norm satisfies*

$$\alpha_{\text{res}} = \max_{\theta \in [0, 2\pi)} \left\| (I - e^{-i\theta} A)^{-1} \right\| \leq \frac{2c \|\Pi_{\text{in}}\|}{\delta}. \quad (\text{S21})$$

Moreover:

1. under the numerical-range assumption (S16) with gap δ_{num} ,

$$\alpha_{\text{res}} \leq \frac{2(1 + \sqrt{2}) \|\Pi_{\text{in}}\|}{\delta_{\text{num}}}. \quad (\text{S22})$$

2. if A_{in} and A_{out} are diagonalizable with per-block eigenbasis condition number κ_V and spectrum gap δ_{eig} (S17),

$$\alpha_{\text{res}} \leq \frac{2\kappa_V \|\Pi_{\text{in}}\|}{\delta_{\text{eig}}}. \quad (\text{S23})$$

Proof. The resolvent splits along the invariant subspaces as

$$(I - e^{-i\theta} A)^{-1} = (I - e^{-i\theta} A_{\text{in}})^{-1} \Pi_{\text{in}} + (I - e^{-i\theta} A_{\text{out}})^{-1} \Pi_{\text{out}}, \quad (\text{S24})$$

so that

$$\left\| (I - e^{-i\theta} A)^{-1} \right\| \leq \left\| (I - e^{-i\theta} A_{\text{in}})^{-1} \right\| \|\Pi_{\text{in}}\| + \left\| (I - e^{-i\theta} A_{\text{out}})^{-1} \right\| \|\Pi_{\text{out}}\|. \quad (\text{S25})$$

For an oblique projector and its complement, we have the tight bound $\|\Pi_{\text{out}}\| = \|I - \Pi_{\text{in}}\| = \|\Pi_{\text{in}}\|$ by Lemma S8 below, so it remains to bound the two restricted resolvents. For the purpose of deriving asymptotic bounds, the looser estimate $\|\Pi_{\text{out}}\| \leq 1 + \|\Pi_{\text{in}}\|$ from triangle inequality would also suffice.

For the inner block, take $g(z) = (1 - e^{-i\theta} z)^{-1}$. Its only pole, $z = e^{i\theta}$, lies outside $\mathcal{K}_{\text{in}}$, and $|1 - e^{-i\theta} z| \geq \delta$ on $\mathcal{K}_{\text{in}}$. By (S18), $\left\| (I - e^{-i\theta} A_{\text{in}})^{-1} \right\| \leq c/\delta$.

For the outer block, write $(I - e^{-i\theta} A_{\text{out}})^{-1} = h(A_{\text{out}}^{-1})$ with $h(w) = w(w - e^{-i\theta})^{-1}$. Its only pole, $w = e^{-i\theta}$, lies outside $\mathcal{K}_{\text{out}}$, and $|h(w)| \leq (1 - \delta)/\delta \leq 1/\delta$ on $\mathcal{K}_{\text{out}}$. By (S18), $\left\| (I - e^{-i\theta} A_{\text{out}})^{-1} \right\| \leq c/\delta$.

Combining the two blocks gives (S21). By Lemma S6, (S20) holds with $c = 1 + \sqrt{2}$ and $\delta = \delta_{\text{num}}$ under the numerical-range assumption, which gives (S22). Under the diagonalizability assumption, it holds with $c = \kappa_V$ and $\delta = \delta_{\text{eig}}/2$, which gives (S23) with prefactor $4\kappa_V$.

In the diagonalizable case, the prefactor can be improved to $2\kappa_V$ by bounding the outer block directly rather than through A_{out}^{-1} . For every $\lambda \in \mathbf{Spec}(A)$,

$$\left|1 - e^{-i\theta}\lambda\right| = \left|e^{i\theta} - \lambda\right| \geq \left||\lambda| - 1\right| \geq \delta_{\text{eig}}, \quad (\text{S26})$$

whether λ lies inside or outside the unit circle. Applying the eigenbasis bound of [Lemma S6](#) to $g(z) = (1 - e^{-i\theta}z)^{-1}$ on each block gives $\|(I - e^{-i\theta}A_{\text{in}})^{-1}\|, \|(I - e^{-i\theta}A_{\text{out}})^{-1}\| \leq \kappa_V/\delta_{\text{eig}}$. Hence $\alpha_{\text{res}} \leq 2\kappa_V \|\Pi_{\text{in}}\|/\delta_{\text{eig}}$. $\square$

Lemma S8 (Norm of an oblique projection [\[7\]](#)). *Let $\Pi^2 = \Pi$ be an oblique projection, with $\Pi \neq 0$ and $\Pi \neq I$. Then $\|\Pi\| = \|I - \Pi\|$.*

Proof. The entire space admits the orthogonal decomposition $\mathcal{H} = \mathbf{Im}(\Pi) \oplus \mathbf{Im}(\Pi)^\perp$. Every vector in $\mathbf{Im}(\Pi)$ is fixed by Π , and Π maps $\mathbf{Im}(\Pi)^\perp$ into $\mathbf{Im}(\Pi)$, so with respect to this splitting

$$\Pi = \begin{bmatrix} I & P \\ 0 & 0 \end{bmatrix}, \quad I - \Pi = \begin{bmatrix} 0 & -P \\ 0 & I \end{bmatrix}, \quad (\text{S27})$$

for some operator $P : \mathbf{Im}(\Pi)^\perp \rightarrow \mathbf{Im}(\Pi)$. Then

$$\Pi\Pi^\dagger = \begin{bmatrix} I + PP^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (I - \Pi)^\dagger(I - \Pi) = \begin{bmatrix} 0 & 0 \\ 0 & I + P^\dagger P \end{bmatrix}, \quad (\text{S28})$$

so $\|\Pi\|^2 = \|\Pi\Pi^\dagger\| = 1 + \|PP^\dagger\| = 1 + \|P\|^2$ and $\|I - \Pi\|^2 = \|(I - \Pi)^\dagger(I - \Pi)\| = 1 + \|P^\dagger P\| = 1 + \|P\|^2$. Hence $\|\Pi\| = \|I - \Pi\|$. $\square$

We now assemble the standard block encoding. Introduce an ancilla register indexing the roots of unity and form the block-diagonal matrix

$$M = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes (I - \omega_n^{-j}A), \quad (\text{S29})$$

whose blocks are the shifted matrices appearing in the discretization [\(S7\)](#). Each block is the combination $I - \omega_n^{-j}A$ of I and A , so [Lemma S2](#) block encodes M with normalization factor $\alpha_A + 1 = \mathbf{O}(\alpha_A)$ from the block encoding of A/α_A . Applying the inversion of [Lemma S4](#) block encodes $M^{-1} = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes (I - \omega_n^{-j}A)^{-1}$, and preparing the ancilla in the uniform superposition $\mathbf{Had}|0\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |j\rangle$ extracts the average

$$(\langle 0| \mathbf{Had} \otimes I) M^{-1} (\mathbf{Had} |0\rangle \otimes I) = \frac{1}{n} \sum_{j=0}^{n-1} (I - \omega_n^{-j}A)^{-1} = S_0. \quad (\text{S30})$$

Proposition S9 (Standard block encoding of the oblique eigenprojection). *Let A/α_A be a block encoding of a square matrix A with Riesz projector $\Pi_{\text{in}} = \frac{1}{2\pi i} \int_{|z|=1} (zI - A)^{-1} dz$, and let $\alpha_{\text{res}} = \max_{\theta \in [0, 2\pi)} \|(I - e^{-i\theta}A)^{-1}\|$ be the maximum resolvent norm. For any $\varepsilon > 0$, Π_{in} can be block encoded with normalization factor $\mathbf{O}(\alpha_{\text{res}})$ and accuracy ε , using*

$$\mathbf{O}\left(\alpha_A \alpha_{\text{res}} \log \frac{1}{\varepsilon}\right) \quad (\text{S31})$$

queries to the block encoding of A/α_A .

If the compressions A_{in} and A_{out}^{-1} admit spectral sets [\(S18\)](#) with constant c and gap δ satisfying [\(S20\)](#), then $\alpha_{\text{res}} = \mathbf{O}(c \|\Pi_{\text{in}}\|/\delta)$ by [Proposition S7](#). In particular:

1. under the numerical-range assumption (S16), $\alpha_{\text{res}} = \mathbf{O}(\|\Pi_{\text{in}}\|/\delta_{\text{num}})$;
2. under the diagonalizability assumption (S17), $\alpha_{\text{res}} = \mathbf{O}(\kappa_V \|\Pi_{\text{in}}\|/\delta_{\text{eig}})$.

Proof. The block encoding of M (S29) follows from Lemma S2, each block being the combination $I - \omega_n^{-j}A$ of I and A , with normalization factor $\alpha_A + 1 = \mathbf{O}(\alpha_A)$. The nonzero singular values of M are bounded below by $1/\alpha_{\text{res}}$, since $\|M^{-1}\| = \max_j \|(I - \omega_n^{-j}A)^{-1}\| \leq \alpha_{\text{res}}$, so Lemma S4 block encodes M^{-1} with the query count (S31) and normalization factor $\mathbf{O}(\alpha_{\text{res}})$. The Hadamard sandwich (S30) is a linear combination of the diagonal blocks, which preserves the normalization factor and adds only the ancilla preparation. The resulting block encoding is that of S_0 , which is ε -close to Π_{in} by (S7) once n is chosen as in Appendix S2. The two specializations are the bounds of Proposition S7. $\square$

S2 Discrete Fourier transform of resolvent and preconditioning

In this appendix, we provide details on the discrete Fourier transform of the resolvent and the two-sided block preconditioning described in the main text.

S2.1 Block encoding matrix resolvent in the Fourier basis

Recall the block-diagonal matrix (S29) whose inverse collects the shifted resolvents,

$$M^{-1} = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes (I - \omega_n^{-j}A)^{-1}. \quad (\text{S32})$$

Conjugating by the quantum Fourier transform **QFT**, acting on the ancilla register as $\mathbf{QFT}|j\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \omega_n^{jk} |k\rangle$, its (k, l) block is

$$\begin{aligned} (\langle k| \otimes I) (\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I) (|l\rangle \otimes I) &= \sum_{j=0}^{n-1} \langle k| \mathbf{QFT} |j\rangle\langle j| \mathbf{QFT}^\dagger |l\rangle (I - \omega_n^{-j}A)^{-1} \\ &= \sum_{j=0}^{n-1} \frac{\omega_n^{jk}}{\sqrt{n}} \cdot \frac{\omega_n^{-jl}}{\sqrt{n}} (I - \omega_n^{-j}A)^{-1} \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{j(k-l)} (I - \omega_n^{-j}A)^{-1}. \end{aligned} \quad (\text{S33})$$

This depends only on $k - l \bmod n$, so defining the discrete Fourier transform of the resolvent

$$S_m = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{jm} (I - \omega_n^{-j}A)^{-1}, \quad m = 0, \dots, n-1, \quad (\text{S34})$$

the conjugated operator is block circulant,

$$(\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I) = \begin{bmatrix} S_0 & S_{n-1} & S_{n-2} & \cdots & S_1 \\ S_1 & S_0 & S_{n-1} & \cdots & S_2 \\ S_2 & S_1 & S_0 & \cdots & S_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & S_{n-2} & S_{n-3} & \cdots & S_0 \end{bmatrix}. \quad (\text{S35})$$

The zeroth mode S_0 is the trapezoidal discretization (S7) of the oblique eigenprojection Π_{in} . The standard approach of Appendix S1 extracts S_0 alone through the Hadamard conjugation (S30), leaving the remaining blocks undetermined. The Fourier transform (S35) instead characterizes the entire operator with Fourier modes $S_1, \dots, S_{n-1}$, which is what makes the preconditioning below possible.

S2.2 Discrete Fourier transform of matrix resolvent

The S_m of (S34) are related to the continuous Fourier coefficients of the resolvent,

$$C_m = \frac{1}{2\pi} \int_0^{2\pi} e^{im\theta} (I - e^{-i\theta} A)^{-1} d\theta, \quad m \in \mathbb{Z}, \quad (\text{S36})$$

by aliasing. This aliasing relation is a form of the Poisson summation formula; we give an elementary proof using only the orthogonality of roots of unity.

Lemma S10 (Aliasing identity). *Fixing n , the discrete Fourier transform (S34) of the resolvent and its continuous Fourier coefficients (S36) satisfy*

$$S_m = \sum_{q \in \mathbb{Z}} C_{m+qn}. \quad (\text{S37})$$

Proof. Since the resolvent $\theta \mapsto (I - e^{-i\theta} A)^{-1}$ is analytic on an annulus containing the unit circle, its Fourier series converges absolutely,

$$(I - e^{-i\theta} A)^{-1} = \sum_{p \in \mathbb{Z}} C_p e^{-ip\theta}, \quad (\text{S38})$$

with the coefficients C_p of (S36). Evaluating (S38) at the nodes $\theta_j = 2\pi j/n$, where $e^{-i\theta_j} = \omega_n^{-j}$, and substituting into the definition (S34),

$$S_m = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{jm} \sum_{p \in \mathbb{Z}} C_p \omega_n^{-jp} = \sum_{p \in \mathbb{Z}} C_p \left(\frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{j(m-p)} \right), \quad (\text{S39})$$

where the interchange of the two sums is justified by the absolute convergence of (S38). The inner sum is the orthogonality relation of the roots of unity,

$$\frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{j(m-p)} = \begin{cases} 1 & p \equiv m \pmod{n}, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{S40})$$

which retains exactly the terms $p = m + qn$, $q \in \mathbb{Z}$, giving $S_m = \sum_{q \in \mathbb{Z}} C_{m+qn}$. $\square$

By Lemma S10, the discrete Fourier transform is determined by the continuous coefficients C_k through aliasing, which we now compute explicitly on each invariant subspace.

Lemma S11 (Continuous Fourier transform of matrix resolvent). *Let A be a square matrix whose spectrum is disjoint from the unit circle, with Riesz projector Π_{in} (S6), complementary projector $\Pi_{\text{out}} = I - \Pi_{\text{in}}$, and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$. The continuous Fourier coefficients C_k (S36) of the resolvent are*

$$C_k = \begin{cases} A_{\text{in}}^k & k \geq 1, \\ \Pi_{\text{in}} & k = 0, \\ -A_{\text{out}}^k & k \leq -1, \end{cases} \quad (\text{S41})$$

where A_{out}^k for $k \leq -1$ denotes a power of the inverse of A_{out} on $\mathcal{H}_{\text{out}}$, extended by zero on $\mathcal{H}_{\text{in}}$.

Proof. On $\mathcal{H}_{\text{in}}$ all eigenvalues of A_{in} have modulus below 1, so the integrand of (S36) expands as the norm-convergent series

$$e^{i\theta}(e^{i\theta}I - A_{\text{in}})^{-1} = (I - e^{-i\theta}A_{\text{in}})^{-1} = \sum_{m \geq 0} A_{\text{in}}^m e^{-im\theta}. \quad (\text{S42})$$

Multiplying by $e^{ik\theta}$ and integrating over $[0, 2\pi]$ selects the term $m = k$, giving A_{in}^k for $k \geq 0$ and 0 otherwise. On $\mathcal{H}_{\text{out}}$ all eigenvalues of A_{out} have modulus above 1, so $\|A_{\text{out}}^{-1}\| < 1$ and

$$e^{i\theta}(e^{i\theta}I - A_{\text{out}})^{-1} = -A_{\text{out}}^{-1} \frac{e^{i\theta}}{I - e^{i\theta}A_{\text{out}}^{-1}} = -\sum_{m \geq 0} A_{\text{out}}^{-(m+1)} e^{i(m+1)\theta}. \quad (\text{S43})$$

Multiplying by $e^{ik\theta}$ and integrating selects $m + 1 = -k$, that is $m = -k - 1 \geq 0$, which requires $k \leq -1$; this yields $-A_{\text{out}}^k$ for $k \leq -1$ and 0 otherwise. $\square$

Theorem S1 (Discrete Fourier transform of matrix resolvent). *Let A be a square matrix whose spectrum is disjoint from the unit circle, with Riesz projector Π_{in} (S6), complementary projector $\Pi_{\text{out}} = I - \Pi_{\text{in}}$, and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$. For $0 \leq m \leq n - 1$, the discrete Fourier transform (S34) of the resolvent is given by*

$$S_m = \frac{1}{n} \sum_{j=0}^{n-1} \omega_n^{jm} (I - \omega_n^{-j}A)^{-1} = A_{\text{in}}^m (I - A_{\text{in}}^n)^{-1} \Pi_{\text{in}} - A_{\text{out}}^{m-n} (I - A_{\text{out}}^{-n})^{-1} \Pi_{\text{out}}, \quad (\text{S44})$$

where A_{out}^{-1} denotes the inverse of A_{out} on $\mathcal{H}_{\text{out}}$ and $\omega_n = e^{2\pi i/n}$.

Proof. By Lemma S10, $S_m = \sum_{q \in \mathbb{Z}} C_{m+qn}$, and by Lemma S11 the coefficients C_k are nonzero only when $k \geq 0$ on $\mathcal{H}_{\text{in}}$ and $k \leq -1$ on $\mathcal{H}_{\text{out}}$. On $\mathcal{H}_{\text{in}}$, for $0 \leq m \leq n - 1$ the surviving terms are $m, m + n, m + 2n, \dots$,

$$S_m|_{\mathcal{H}_{\text{in}}} = \sum_{q \geq 0} A_{\text{in}}^{m+qn} = A_{\text{in}}^m (I - A_{\text{in}}^n)^{-1}. \quad (\text{S45})$$

On $\mathcal{H}_{\text{out}}$ the surviving terms are $m - n, m - 2n, \dots$,

$$S_m|_{\mathcal{H}_{\text{out}}} = -\sum_{q \geq 1} A_{\text{out}}^{m-qn} = -A_{\text{out}}^{m-n} (I - A_{\text{out}}^{-n})^{-1}. \quad (\text{S46})$$

Summing the two contributions gives (S44). $\square$

The closed form of Theorem S1 reduces the bounds we need on the Fourier modes S_m to the growth of the powers A_{in}^j and the inverse powers A_{out}^{-j} , which decay geometrically under either of our assumptions on A .

Corollary S12 (Fourier mode bounds). *Let A be a square matrix with Riesz projector Π_{in} (S6) and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$, and let S_m (S34) be the discrete Fourier transform of the resolvent (S44).*

1. *If the compressions A_{in} and A_{out}^{-1} admit spectral sets (S18) with constant c and gap δ satisfying (S20), then their powers decay geometrically,*

$$\|A_{\text{in}}^j\| \leq c(1 - \delta)^j, \quad \|A_{\text{out}}^{-j}\| \leq c(1 - \delta)^j, \quad j \geq 0. \quad (\text{S47})$$

In particular, this holds with:

- (a) $c = 1 + \sqrt{2}$ and $\delta = \delta_{\text{num}}$ under the numerical-range assumption (S16);
(b) $c = \kappa_V$ and $\delta = \delta_{\text{eig}}/2$ under the diagonalizability assumption (S17).

2. Whenever (S47) holds, once $n \geq \frac{1}{\delta} \log(2c)$:

- (a) the zeroth mode approximates the eigenprojection,

$$\|S_0 - \Pi_{\text{in}}\| \leq 4c(1 - \delta)^n \|\Pi_{\text{in}}\|; \quad (\text{S48})$$

- (b) every Fourier mode is uniformly bounded,

$$\|S_m\| \leq 4c \|\Pi_{\text{in}}\|, \quad 0 \leq m \leq n - 1. \quad (\text{S49})$$

Proof. For part (1), apply (S18) to $g(z) = z^j$ on $\mathcal{K}_{\text{in}}$ for A_{in} , and on $\mathcal{K}_{\text{out}}$ for A_{out}^{-1} . Both sets lie in $\{z: |z| \leq 1 - \delta\}$, so $\|A_{\text{in}}^j\| \leq c(1 - \delta)^j$ and $\|A_{\text{out}}^{-j}\| \leq c(1 - \delta)^j$. The two special cases follow from Lemma S6. In case (b), every eigenvalue of A_{out} has modulus at least $1 + \delta_{\text{eig}}$, so every eigenvalue of A_{out}^{-1} has modulus at most $1/(1 + \delta_{\text{eig}}) \leq 1 - \delta_{\text{eig}}/2$. This gives $\delta = \delta_{\text{eig}}/2$.

For part (2), assume (S47). Since $n \geq \frac{1}{\delta} \log(2c)$ and $(1 - \delta)^n \leq e^{-\delta n}$,

$$\|A_{\text{in}}^n\|, \|A_{\text{out}}^{-n}\| \leq c(1 - \delta)^n \leq c e^{-\delta n} \leq \frac{1}{2}, \quad (\text{S50})$$

so $(I - A_{\text{in}}^n)^{-1}$ and $(I - A_{\text{out}}^{-n})^{-1}$ have norm at most 2.

For (2a), setting $m = 0$ in the closed form (S44) and subtracting $\Pi_{\text{in}} = (I - A_{\text{in}}^n)^{-1}(I - A_{\text{in}}^n)\Pi_{\text{in}}$,

$$S_0 - \Pi_{\text{in}} = A_{\text{in}}^n(I - A_{\text{in}}^n)^{-1}\Pi_{\text{in}} - A_{\text{out}}^{-n}(I - A_{\text{out}}^{-n})^{-1}\Pi_{\text{out}}. \quad (\text{S51})$$

Taking norms with $\|\Pi_{\text{out}}\| = \|\Pi_{\text{in}}\|$ (Lemma S8), the inverse factors at most 2, and $\|A_{\text{in}}^n\|, \|A_{\text{out}}^{-n}\| \leq c(1 - \delta)^n$ gives (S48).

For (2b), the closed form (S44) with $\|A_{\text{in}}^m\|, \|A_{\text{out}}^{m-n}\| \leq c$ for $0 \leq m \leq n - 1$, the inverse factors $(I - A_{\text{in}}^n)^{-1}$ and $(I - A_{\text{out}}^{-n})^{-1}$ at most 2 in spectral norm, and $\|\Pi_{\text{out}}\| = \|\Pi_{\text{in}}\|$ gives

$$\|S_m\| \leq \|A_{\text{in}}^m\| \|(I - A_{\text{in}}^n)^{-1}\| \|\Pi_{\text{in}}\| + \|A_{\text{out}}^{m-n}\| \|(I - A_{\text{out}}^{-n})^{-1}\| \|\Pi_{\text{out}}\| \leq 4c \|\Pi_{\text{in}}\|. \quad (\text{S52})$$

□

S2.3 Two-sided block preconditioning of resolvent integral

The standard block encoding of Appendix S1 inverts the block-diagonal matrix $M = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes (I - \omega_n^{-j} A)$ (S29) and extracts the single block $S_0 \approx \Pi_{\text{in}}$. However, this normalizes Π_{in} by the norm of the whole inverse matrix $\|M^{-1}\| = \alpha_{\text{res}} = \mathbf{O}(\|\Pi_{\text{in}}\|/\delta)$, even though the extracted block has norm only $\mathbf{O}(\|\Pi_{\text{in}}\|)$ by Corollary S12. The two-sided preconditioning rescales the ancilla registers so that the extracted block is normalized near $\|\Pi_{\text{in}}\|$, while the condition number of the inversion, and hence its query complexity of quantum implementation, is preserved up to a logarithmic factor.

Fix two scaling parameters $0 < t_1, t_2 \leq 1$ and define diagonal scaling operators on the ancilla register,

$$T_1 = t_1|0\rangle\langle 0| + \sum_{k=1}^{n-1} |k\rangle\langle k| = \begin{bmatrix} t_1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad T_2 = t_2|0\rangle\langle 0| + \sum_{k=1}^{n-1} |k\rangle\langle k| = \begin{bmatrix} t_2 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \quad (\text{S53})$$

A block encoding of the preconditioned matrix

$$(T_1 \otimes I)(\mathbf{QFT} \otimes I) M (\mathbf{QFT}^\dagger \otimes I)(T_2 \otimes I) \quad (\text{S54})$$

is then obtained from the block encoding of M using the product rule from [Lemma S3](#). Note that T_1 and T_2 are diagonal contractions and the quantum Fourier transform $\mathbf{QFT}$ is a unitary; hence the preconditioned matrix can be block encoded with no normalization overhead.

Applying the inversion rule of [Lemma S4](#) block encodes the inverse of the preconditioned matrix ([S54](#)):

$$\left[(T_1 \otimes I)(\mathbf{QFT} \otimes I) M (\mathbf{QFT}^\dagger \otimes I)(T_2 \otimes I) \right]^{-1} = (T_2^{-1} \otimes I)(\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I)(T_1^{-1} \otimes I). \quad (\text{S55})$$

Here, the middle factor $(\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I)$ is the block-circulant operator ([S35](#)), with (k, l) block S_{k-l} . Taking the scaling operators T_1 and T_2 into account, the (k, l) block is S_{k-l} scaled by $1/(t_1 t_2)$ when $k = l = 0$, by $1/t_2$ on the rest of row 0, by $1/t_1$ on the rest of column 0, and left unchanged otherwise:

$$(T_2^{-1} \otimes I) (\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I) (T_1^{-1} \otimes I) = \begin{bmatrix} \frac{1}{t_1 t_2} S_0 & \frac{1}{t_2} S_{n-1} & \frac{1}{t_2} S_{n-2} & \cdots & \frac{1}{t_2} S_1 \\ \frac{1}{t_1} S_1 & S_0 & S_{n-1} & \cdots & S_2 \\ \frac{1}{t_1} S_2 & S_1 & S_0 & \cdots & S_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t_1} S_{n-1} & S_{n-2} & S_{n-3} & \cdots & S_0 \end{bmatrix}. \quad (\text{S56})$$

Proposition S13 (Preconditioned norm bound). *Let $0 < t_1, t_2 \leq 1$, and let S_m ([S34](#)) be the discrete Fourier transform of the resolvent. The inverse ([S56](#)) of the preconditioned matrix has norm bounded by*

$$\begin{aligned} \alpha_{\text{precond}} &= \left\| (T_2^{-1} \otimes I)(\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I)(T_1^{-1} \otimes I) \right\| \\ &\leq \frac{\|S_0\|}{t_1 t_2} + \frac{\sqrt{n-1}}{t_1} \max_m \|S_m\| + \frac{\sqrt{n-1}}{t_2} \max_m \|S_m\| + \|M^{-1}\|. \end{aligned} \quad (\text{S57})$$

Proof. Applying the triangle inequality to the four blocks of ([S56](#))—the corner, the rest of the top row, the rest of the left column, and the remaining submatrix—gives

$$\alpha_{\text{precond}} \leq \frac{\|S_0\|}{t_1 t_2} + \frac{1}{t_2} \left\| \begin{bmatrix} S_{n-1} & \cdots & S_1 \end{bmatrix} \right\| + \frac{1}{t_1} \left\| \begin{bmatrix} S_1 \\ \vdots \\ S_{n-1} \end{bmatrix} \right\| + \left\| \begin{bmatrix} S_0 & \cdots & S_2 \\ \vdots & \ddots & \vdots \\ S_{n-2} & \cdots & S_0 \end{bmatrix} \right\|. \quad (\text{S58})$$

Here, the first term is $\frac{\|S_0\|}{t_1 t_2}$. The second is a single row of $n - 1$ blocks, with norm at most

$$\begin{aligned} \|[S_{n-1} \ \cdots \ S_1]\| &= \left\| [S_{n-1} \ \cdots \ S_1] \begin{bmatrix} S_{n-1}^\dagger \\ \vdots \\ S_1^\dagger \end{bmatrix} \right\|^{1/2} = \left\| \sum_{l=1}^{n-1} S_{n-l} S_{n-l}^\dagger \right\|^{1/2} \\ &\leq \left(\sum_{l=1}^{n-1} \|S_{n-l}\|^2 \right)^{1/2} \leq \sqrt{n-1} \max_m \|S_m\|. \end{aligned} \quad (\text{S59})$$

The third is a single column of $n - 1$ blocks and can be handled similarly. Finally, the fourth is a principal submatrix of the unscaled block-circulant operator (S35), so its norm is at most $\|(\mathbf{QFT} \otimes I) M^{-1} (\mathbf{QFT}^\dagger \otimes I)\| = \|M^{-1}\|$. Substituting these into (S58) gives (S57). $\square$

As t_1, t_2 decrease, the corner and the block row and column in (S57) grow, so α_{precond} increases; but the extracted block encoding is normalized by $t_1 t_2 \alpha_{\text{precond}}$, which decreases at a higher rate. To manage this tradeoff, we take

$$t_1 = t_2 = \frac{1}{\sqrt{n}}. \quad (\text{S60})$$

Substituting into (S57),

$$\alpha_{\text{precond}} \leq n \|S_0\| + 2\sqrt{n(n-1)} \max_m \|S_m\| + \alpha_{\text{res}} \leq 3n \max_m \|S_m\| + \alpha_{\text{res}}, \quad (\text{S61})$$

and the extracted normalization factor becomes

$$\alpha_\Pi = t_1 t_2 \alpha_{\text{precond}} \leq 3 \max_m \|S_m\| + \frac{\alpha_{\text{res}}}{n}. \quad (\text{S62})$$

Theorem S2 (Quantum oblique eigenprojection on the unit disk). *Let A be a square matrix accessed through a block encoding A/α_A with normalization factor α_A , whose spectrum is disjoint from the unit circle, with Riesz projector Π_{in} (S6) and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$.*

Assume the compressions A_{in} and A_{out}^{-1} admit spectral sets (S18) with constant c and gap δ satisfying (S20). For any $\varepsilon > 0$, the two-sided block preconditioning (S56) with $t_1 = t_2 = 1/\sqrt{n}$ and $n = \Theta(\frac{1}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon})$ block encodes Π_{in} with normalization factor

$$\alpha_\Pi = \mathbf{O}(c \|\Pi_{\text{in}}\|), \quad (\text{S63})$$

near its optimal value $\|\Pi_{\text{in}}\|$, and accuracy ε , using

$$\mathbf{O}\left(\frac{c \alpha_A \|\Pi_{\text{in}}\|}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon} \log \frac{1}{\varepsilon}\right) \quad (\text{S64})$$

queries to the block encoding A/α_A . In particular, this holds:

1. *with $c = 1 + \sqrt{2}$ and $\delta = \delta_{\text{num}}$ under the numerical-range assumption (S16);*
2. *with $c = \kappa_V$ and $\delta = \delta_{\text{eig}}/2$ under the diagonalizability assumption (S17).*

Proof. By Proposition S13 with $t_1 = t_2 = 1/\sqrt{n}$, the extracted normalization factor is $\alpha_\Pi \leq 3 \max_m \|S_m\| + \alpha_{\text{res}}/n$ (S62), and the norm of the inverse preconditioned matrix is $\alpha_{\text{precond}} \leq 3n \max_m \|S_m\| + \alpha_{\text{res}}$ (S61). By part (1) of Corollary S12, the spectral-set hypothesis gives the geometric decay (S47). Part (2) of Corollary S12 then gives $\max_m \|S_m\| = \mathbf{O}(c \|\Pi_{\text{in}}\|)$. Proposition S7 gives $\alpha_{\text{res}} = \mathbf{O}(c \|\Pi_{\text{in}}\|/\delta)$.

The choice of n balances two competing effects. It must be large enough that the discretization error (S48) falls below ε , which requires $n = \Omega(\frac{1}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon})$. It must also be no larger than this, since the condition number $\alpha_{\text{precond}} = \mathbf{O}(n c \|\Pi_{\text{in}}\| + c \|\Pi_{\text{in}}\| / \delta)$ grows linearly in n . The choice $n = \Theta(\frac{1}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon})$ meets both. For sufficiently small ε , it also meets the threshold $n \geq \frac{1}{\delta} \log(2c)$ of Corollary S12.

With this n , the discretization error is at most ε by Corollary S12, and $\alpha_{\text{res}}/n = \mathbf{O}(c \|\Pi_{\text{in}}\|)$. So $\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|)$, which is (S63). Also, $\alpha_{\text{precond}} = \mathbf{O}(n c \|\Pi_{\text{in}}\|) = \mathbf{O}(\frac{c \|\Pi_{\text{in}}\|}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon})$. By (S54), the preconditioned matrix is block encoded at normalization $\mathbf{O}(\alpha_A)$. Inverting it with Lemma S4 uses $\mathbf{O}(\alpha_A \alpha_{\text{precond}} \log \frac{1}{\varepsilon})$ queries, which is (S64). Extracting the top-left block gives the block encoding of Π_{in} at normalization α_{Π} . The two special cases follow from Lemma S6, with $\delta = \delta_{\text{eig}}/2$ in case (ii) as in Corollary S12. $\square$

S3 Extension to general regions

In this appendix, we provide details on the extension to general regions described in the main text.

S3.1 Reduction to the unit disk

Appendix S2 treats the unit disk. We now reduce a general region to that case through a separating map.

Let A be a square matrix with spectrum split into $\mathbf{Spec}_{\text{in}}(A)$ and $\mathbf{Spec}_{\text{out}}(A)$, and $\mathcal{C}$ be a closed curve enclosing $\mathbf{Spec}_{\text{in}}(A)$ but not $\mathbf{Spec}_{\text{out}}(A)$, disjoint from $\mathbf{Spec}(A)$. Then, the oblique eigenprojection onto $\mathbf{Spec}_{\text{in}}(A)$ is given by the Riesz projector $\Pi_{\text{in}} = \frac{1}{2\pi i} \int_{\mathcal{C}} (wI - A)^{-1} dw$.

Suppose that $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ are disjoint open sets covering $\mathbf{Spec}_{\text{in}}(A)$ and $\mathbf{Spec}_{\text{out}}(A)$. Let f be analytic on $\mathcal{S}_{\text{in}} \cup \mathcal{S}_{\text{out}}$ and separate the two sets in modulus,

$$|f(w)| < 1 \quad (w \in \mathcal{S}_{\text{in}}), \quad |f(w)| > 1 \quad (w \in \mathcal{S}_{\text{out}}). \quad (\text{S65})$$

The next lemma states that the eigenprojection of A for the target region coincides with the eigenprojection of $f(A)$ for the unit disk.

Lemma S14 (Change of variables). *Let A be a square matrix with spectrum split into $\mathbf{Spec}_{\text{in}}(A)$ and $\mathbf{Spec}_{\text{out}}(A)$, and $\mathcal{C}$ be a closed curve enclosing $\mathbf{Spec}_{\text{in}}(A)$ but not $\mathbf{Spec}_{\text{out}}(A)$, disjoint from $\mathbf{Spec}(A)$. Suppose that $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ are disjoint open sets covering $\mathbf{Spec}_{\text{in}}(A)$ and $\mathbf{Spec}_{\text{out}}(A)$. Let f be analytic on $\mathcal{S}_{\text{in}} \cup \mathcal{S}_{\text{out}}$ with $|f(w)| < 1$ ($w \in \mathcal{S}_{\text{in}}$) and $|f(w)| > 1$ ($w \in \mathcal{S}_{\text{out}}$). Then,*

$$\Pi_{\text{in}} = \frac{1}{2\pi i} \int_{\mathcal{C}} (wI - A)^{-1} dw = \frac{1}{2\pi i} \int_{|z|=1} (zI - f(A))^{-1} dz. \quad (\text{S66})$$

Proof. Both integrals are Riesz projectors. By Lemma S1, the first is the projection onto the invariant subspaces of A associated with $\mathbf{Spec}_{\text{in}}(A)$, and the second is the projection onto the invariant subspaces of $f(A)$ associated with the part of its spectrum inside the unit circle. By the spectral mapping theorem [3, Theorem 12.7.1][6, Theorem 8.3], $\mathbf{Spec}(f(A)) = f(\mathbf{Spec}(A))$. The modulus separation map f sends $\mathbf{Spec}_{\text{in}}(A)$ inside the unit circle and $\mathbf{Spec}_{\text{out}}(A)$ outside, so the two projections have matching spectral parts. The holomorphic functional calculus preserves the invariant subspaces of A , so $f(A)$ acts on each invariant subspace of A and shares its spectral projection onto $\mathbf{Spec}_{\text{in}}(A)$. The two integrals therefore coincide. $\square$

Theorem S3 (Quantum oblique eigenprojection for general regions). *Let A be a square matrix with Riesz projector Π_{in} (S66) for a region whose boundary is the separating curve $\mathcal{C}$, and compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$. Let f be an analytic separating map (S65), and suppose $f(A)$ is accessed through a block encoding $f(A)/\alpha_{f(A)}$ with normalization factor $\alpha_{f(A)}$.*

If the compressions A_{in} and A_{out} admit spectral sets $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ (S18) with constant c , and f is analytic on neighborhoods of $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ with gap δ_f satisfying

$$\max_{z \in \mathcal{K}_{\text{in}}} |f(z)| \leq 1 - \delta_f, \quad \min_{z \in \mathcal{K}_{\text{out}}} |f(z)| \geq 1 + \delta_f, \quad (\text{S67})$$

then for any $\varepsilon > 0$, the two-sided block preconditioning of Theorem S2 applied to $f(A)$ block encodes Π_{in} with normalization factor

$$\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|), \quad (\text{S68})$$

near its optimal value $\|\Pi_{\text{in}}\|$, and accuracy ε , using

$$\mathbf{O}\left(\frac{c \alpha_{f(A)} \|\Pi_{\text{in}}\|}{\delta_f} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon} \log \frac{1}{\varepsilon}\right) \quad (\text{S69})$$

queries to the block encoding $f(A)/\alpha_{f(A)}$. In particular, this holds:

1. *with $c = 1 + \sqrt{2}$, $\mathcal{K}_{\text{in}} = \mathcal{W}(A_{\text{in}})$, and $\mathcal{K}_{\text{out}} = \mathcal{W}(A_{\text{out}})$, under the numerical-range assumption;*
2. *with $c = \kappa_V$ (S17), $\mathcal{K}_{\text{in}} = \mathbf{Spec}(A_{\text{in}})$, and $\mathcal{K}_{\text{out}} = \mathbf{Spec}(A_{\text{out}})$, under the diagonalizability assumption.*

Proof. By Lemma S14, Π_{in} is the unit-disk Riesz projector of $f(A)$. We apply Theorem S2 to $f(A)$, with $f(A)/\alpha_{f(A)}$ as the input block encoding. It remains to show that the compressions $f(A)_{\text{in}}$ and $f(A)_{\text{out}}^{-1}$ admit spectral sets (S18) with constant c and gap $\delta_f/2$ satisfying (S20).

Since f preserves the invariant subspaces, $f(A)_{\text{in}}$ acts on $\mathcal{H}_{\text{in}}$ as $f(A_{\text{in}})$. Similarly, $f(A)_{\text{out}}^{-1}$ acts on $\mathcal{H}_{\text{out}}$ as $(1/f)(A_{\text{out}})$. The function $1/f$ is analytic on a neighborhood of $\mathcal{K}_{\text{out}}$, because $|f| \geq 1 + \delta_f > 0$ there.

We claim that $f(\mathcal{K}_{\text{in}})$ is a spectral set for $f(A_{\text{in}})$ with constant c . The set $f(\mathcal{K}_{\text{in}})$ is compact, and it contains $\mathbf{Spec}(f(A_{\text{in}})) = f(\mathbf{Spec}(A_{\text{in}}))$ by the spectral mapping theorem. Let g be analytic on a neighborhood of $f(\mathcal{K}_{\text{in}})$. Then $g \circ f$ is analytic on a neighborhood of $\mathcal{K}_{\text{in}}$, and $g(f(A_{\text{in}})) = (g \circ f)(A_{\text{in}})$. Applying (S18) to $g \circ f$ gives

$$\|g(f(A_{\text{in}}))\| \leq c \sup_{z \in \mathcal{K}_{\text{in}}} |g(f(z))| = c \sup_{w \in f(\mathcal{K}_{\text{in}})} |g(w)|. \quad (\text{S70})$$

The same argument with $1/f$ in place of f shows that $(1/f)(\mathcal{K}_{\text{out}})$ is a spectral set for $f(A)_{\text{out}}^{-1}$ with constant c .

By (S67), $f(\mathcal{K}_{\text{in}})$ lies in $\{|w| \leq 1 - \delta_f\}$. The out-block gap converts to a reciprocal bound: $|1/f| \leq \frac{1}{1+\delta_f} \leq 1 - \frac{\delta_f}{2}$ on $\mathcal{K}_{\text{out}}$. So $(1/f)(\mathcal{K}_{\text{out}})$ lies in $\{|w| \leq 1 - \delta_f/2\}$. Both sets lie in $\{|w| \leq 1 - \delta_f/2\}$, so (S20) holds with gap $\delta_f/2$. Theorem S2 with $\delta = \delta_f/2 = \Theta(\delta_f)$ then gives the normalization factor (S68) and the query count (S69). The two special cases follow from Lemma S6. $\square$

When $f(z) = z$, Theorem S3 reduces to the unit-disk setting, with the out-block hypothesis placed on A_{out} rather than A_{out}^{-1} . In the diagonalizable case, taking $\mathcal{K}_{\text{out}} = \mathbf{Spec}(A_{\text{out}})$ reproduces case (ii) of Theorem S2. In the numerical-range case, taking $\mathcal{K}_{\text{out}} = \mathcal{W}(A_{\text{out}})$ gives a different condition from (S16), and in general neither condition implies the other. Case (i) of Theorem S2

corresponds instead to the reciprocal set $\mathcal{K}_{\text{out}} = 1/\mathcal{W}(A_{\text{out}}^{-1})$. This set is not the numerical range of A_{out} . But it is still a spectral set for A_{out} with constant $1 + \sqrt{2}$, since the Crouzeix–Palencia bound applies to $g(1/z)$ over $\mathcal{W}(A_{\text{out}}^{-1})$.

More generally, the set $f(\mathcal{W}(A_{\text{in}}))$ is not the numerical range of $f(A_{\text{in}})$, and need not even be convex. It is nevertheless a spectral set for $f(A_{\text{in}})$ with constant $1 + \sqrt{2}$, since the Crouzeix–Palencia bound applies to $g \circ f$ over $\mathcal{W}(A_{\text{in}})$. The numerical range $\mathcal{W}(f(A_{\text{in}}))$ may be larger and may even meet the unit circle. This is why we state and apply [Theorem S2](#) for spectral sets rather than numerical ranges in the proof of [Theorem S3](#).

S3.2 Half-plane construction

We now focus on the case where the spectrum is split along the imaginary axis, which underlies the applications described in the main text. We instantiate [Theorem S3](#) with a degree-two rational map sending the right half-plane into the unit disk. Take

$$f(z) = \frac{(z-1)^2}{(z+1)^2}, \quad (\text{S71})$$

with a double zero at $z = 1$ and a double pole at $z = -1$. The map sends the right half-plane into the unit disk, the imaginary axis onto the unit circle, and the left half-plane outside. Moreover, the following lemma shows that a spectrum separation between half-planes transfers to the unit circle.

Lemma S15 (Gap preservation). *Let f be the map $f(z) = \frac{(z-1)^2}{(z+1)^2}$, and write $z = x + iy$. Then*

$$|f(z)| = \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} = 1 - \frac{4x}{(x+1)^2 + y^2}, \quad (\text{S72})$$

so $|f(z)| < 1$ for $x > 0$, $|f(z)| = 1$ for $x = 0$, and $|f(z)| > 1$ for $x < 0$. Moreover, for $|z| \leq 1/2$:

1. $|f(z)| \leq 1 - \frac{16}{9}\delta$ if $x \geq \delta$;
2. $|f(z)| \geq 1 + \frac{16}{9}\delta$ if $x \leq -\delta$.

Proof. Since $f(z) = \left(\frac{z-1}{z+1}\right)^2$, its modulus is the squared Möbius modulus $|f(z)| = \left|\frac{z-1}{z+1}\right|^2 = \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}$, which gives the explicit formula in [\(S72\)](#).

The sign of $4x$ determines whether $|f(z)|$ is below, equal to, or above 1. For $|z| \leq 1/2$ the denominator $(x+1)^2 + y^2$ is the squared distance from -1 to z , maximized at $z = 1/2$, so it is at most $(3/2)^2 = 9/4$. Hence for $x \geq \delta$,

$$|f(z)| = 1 - \frac{4x}{(x+1)^2 + y^2} \leq 1 - \frac{4\delta}{9/4} = 1 - \frac{16}{9}\delta. \quad (\text{S73})$$

The case $x \leq -\delta$ is symmetric, giving $|f(z)| \geq 1 + \frac{16}{9}\delta$. $\square$

The map $f(A)$ is applied without polynomial approximation. Under $\|A\| \leq 1/2$, the shifted matrix $A+I$ has singular values in $[1/2, 3/2]$, since $\|A+I\| \leq \frac{3}{2}$ and $\|(A+I)^{-1}\| \leq (1-\|A\|)^{-1} \leq 2$. Inverting it with [Lemma S4](#) block encodes $(A+I)^{-1}$ at normalization $\mathbf{O}(1)$, using $\mathbf{O}(\log(1/\varepsilon))$ queries to the block encoding of A . The product $f(A) = (A-I)^2(A+I)^{-2}$ then follows from [Lemma S3](#), with

$$\|f(A)\| \leq \|A-I\|^2 \|(A+I)^{-1}\|^2 \leq (3/2)^2 (2)^2 = 9. \quad (\text{S74})$$

So $f(A)$ is block encoded at normalization $\alpha_{f(A)} = \mathbf{O}(1)$ using $\mathbf{O}(\log(1/\varepsilon))$ queries.

For the half-plane, the spectral-set hypothesis of [Theorem S3](#) takes a concrete geometric form. Suppose the compressions A_{in} and A_{out} admit spectral sets $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ ([S18](#)) with constant c , separated from the imaginary axis into opposite half-planes,

$$\mathcal{K}_{\text{in}} \subseteq \{z: \Re(z) \geq \delta, |z| \leq \tfrac{1}{2}\}, \quad \mathcal{K}_{\text{out}} \subseteq \{z: \Re(z) \leq -\delta, |z| \leq \tfrac{1}{2}\}. \quad (\text{S75})$$

Since [Lemma S15](#) bounds $|f(z)|$ pointwise, it applies to every point of these sets. This gives $\max_{z \in \mathcal{K}_{\text{in}}} |f(z)| \leq 1 - \frac{16}{9}\delta$ and $\min_{z \in \mathcal{K}_{\text{out}}} |f(z)| \geq 1 + \frac{16}{9}\delta$. The only pole of f is at $z = -1$, which lies outside the disk $\{|z| \leq \frac{1}{2}\}$. So f is analytic on a neighborhood of each set. The hypothesis of [Theorem S3](#) therefore holds with $\delta_f = \frac{16}{9}\delta$.

In particular, ([S75](#)) holds with $c = 1 + \sqrt{2}$ when the per-block numerical ranges $\mathcal{W}(A_{\text{in}})$ and $\mathcal{W}(A_{\text{out}})$ satisfy it. It holds with $c = \kappa_V$ when A_{in} and A_{out} are diagonalizable and their spectra $\mathbf{Spec}(A_{\text{in}})$ and $\mathbf{Spec}(A_{\text{out}})$ satisfy it. In both cases, the condition $|z| \leq \frac{1}{2}$ is automatic under $\|A\| \leq \frac{1}{2}$.

Corollary S16 (Quantum oblique eigenprojection for the half-plane). *Let A be a square matrix accessed through a block encoding A/α_A with normalization factor α_A . Let Π_{in} be the Riesz projector onto the right-half-plane invariant subspace, with compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$.*

If the compressions A_{in} and A_{out} admit spectral sets $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ ([S18](#)) with constant c and gap δ satisfying

$$\mathcal{K}_{\text{in}} \subseteq \{z: \Re(z) \geq \delta, |z| \leq \alpha_A\}, \quad \mathcal{K}_{\text{out}} \subseteq \{z: \Re(z) \leq -\delta, |z| \leq \alpha_A\}, \quad (\text{S76})$$

then for any $\varepsilon > 0$, the eigenprojection Π_{in} can be block encoded with normalization factor

$$\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|), \quad (\text{S77})$$

near its optimal value $\|\Pi_{\text{in}}\|$, and accuracy ε , using

$$\mathbf{O}\left(\frac{c\alpha_A \|\Pi_{\text{in}}\|}{\delta} \text{polylog} \frac{c\alpha_A \|\Pi_{\text{in}}\|}{\delta\varepsilon}\right) \quad (\text{S78})$$

queries to the block encoding A/α_A . In particular, this holds:

1. *with $c = 1 + \sqrt{2}$, when the per-block numerical ranges $\mathcal{W}(A_{\text{in}})$ and $\mathcal{W}(A_{\text{out}})$ satisfy ([S76](#));*
2. *with $c = \kappa_V$ ([S17](#)), when A_{in} and A_{out} are diagonalizable and their spectra $\mathbf{Spec}(A_{\text{in}})$ and $\mathbf{Spec}(A_{\text{out}})$ satisfy ([S76](#)).*

In both cases, the condition $|z| \leq \alpha_A$ is automatic, since $\|A\| \leq \alpha_A$.

Proof. Scaling the block encoding A/α_A by $1/2$ block encodes $A_2 = A/(2\alpha_A)$ at normalization $\mathbf{O}(1)$, with $\|A_2\| \leq 1/2$. Scaling by the positive real $1/(2\alpha_A)$ fixes the imaginary axis. So A_2 has the same invariant subspaces as A , the same right- and left-half-plane split, and the same projector Π_{in} . Spectral sets rescale with the same constant: if $\mathcal{K}$ is a spectral set for B with constant c , then $\mathcal{K}/s$ is a spectral set for B/s with constant c , by applying ([S18](#)) to $g(z/s)$. So $\mathcal{K}_{\text{in}}/(2\alpha_A)$ and $\mathcal{K}_{\text{out}}/(2\alpha_A)$ are spectral sets for the compressions of A_2 with constant c . By ([S76](#)), they satisfy ([S75](#)) with gap $\delta' = \delta/(2\alpha_A)$.

Take f to be the map ([S71](#)). Under $\|A_2\| \leq 1/2$, the product $f(A_2) = (A_2 - I)^2(A_2 + I)^{-2}$ is block encoded at normalization $\mathbf{O}(1)$ using $\mathbf{O}(\log(1/\varepsilon))$ queries to the block encoding of A_2 . By the

discussion following (S75), the hypothesis of Theorem S3 holds for A_2 with $\delta_f = \frac{16}{9}\delta' = \Theta(\delta/\alpha_A)$. Applying Theorem S3 to $f(A_2)$ gives the normalization factor (S77), using

$$q_f = \mathbf{O}\left(\frac{c\alpha_A \|\Pi_{\text{in}}\|}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon} \log \frac{1}{\varepsilon}\right) \quad (\text{S79})$$

queries to the block encoding of $f(A_2)$.

Each of the q_f queries to $f(A_2)$ is realized by an ε' -accurate block encoding. Since block-encoding errors compose linearly, taking $\varepsilon' = \Theta(\varepsilon/q_f)$ gives final accuracy ε . This uses $\mathbf{O}(\log(q_f/\varepsilon))$ queries to A/α_A per query to $f(A_2)$. Since q_f is polynomial in the parameters, this adds only a polylogarithmic factor, which is included in (S78). $\square$

S4 Further analysis on applications

In this appendix, we provide details on the applications described in the main text.

S4.1 Eigenstate preparation

Suppose the target eigenvalue or cluster is enclosed by a disk $\{z : |z - z_0| < \rho\}$, with the target invariant subspace separated from the circle $|z - z_0| = \rho$ by an absolute gap δ . The separating map is the linear $f(z) = (z - z_0)/\rho$, so $f(A) = (A - z_0 I)/\rho$ is obtained from the block encoding of A by a shift and a rescaling.

Corollary S17 (Eigenstate preparation). *Let A be a square matrix accessed through a block encoding A/α_A , and let Π_{in} be the Riesz projector onto the invariant subspace whose eigenvalues lie in the disk $\{z : |z - z_0| < \rho\}$ of center z_0 and radius ρ , with compressions $A_{\text{in}} = A\Pi_{\text{in}}$, $A_{\text{out}} = A\Pi_{\text{out}}$. Let $|\psi\rangle$ be an initial state prepared by a unitary O_ψ , with overlap $\eta = \|\Pi_{\text{in}}|\psi\rangle\|$.*

If the compressions $A_{\text{in}} - z_0 I$ and $(A_{\text{out}} - z_0 I)^{-1}$ admit spectral sets $\mathcal{K}_{\text{in}}$ and $\mathcal{K}_{\text{out}}$ (S18) with constant c and gap $0 < \delta < \rho$ satisfying

$$\mathcal{K}_{\text{in}} \subseteq \{z : |z| \leq \rho - \delta\}, \quad \mathcal{K}_{\text{out}} \subseteq \left\{z : |z| \leq \frac{1}{\rho + \delta}\right\}, \quad (\text{S80})$$

then the projected state $\Pi_{\text{in}}|\psi\rangle/\eta$ can be prepared to accuracy ε using

$$\mathbf{O}\left(\frac{c^2(\alpha_A + |z_0|) \|\Pi_{\text{in}}\|^2}{\eta \delta} \text{polylog} \frac{c \|\Pi_{\text{in}}\|}{\eta \varepsilon}\right) \quad (\text{S81})$$

queries to the block encoding A/α_A . The initial state is prepared

$$\mathbf{O}\left(\frac{c \|\Pi_{\text{in}}\|}{\eta}\right) \quad (\text{S82})$$

times, using that many queries to O_ψ and its inverse. In particular,

1. *with $c = 1 + \sqrt{2}$ when the numerical ranges $\mathcal{W}(A_{\text{in}} - z_0 I)$ and $\mathcal{W}((A_{\text{out}} - z_0 I)^{-1})$ satisfy (S80);*
2. *with $c = \kappa_V$ (S17) when A_{in} and A_{out} are diagonalizable and the spectra $\mathbf{Spec}(A_{\text{in}} - z_0 I)$ and $\mathbf{Spec}((A_{\text{out}} - z_0 I)^{-1})$ satisfy (S80).*

Proof. Form a block encoding of $f(A) = (A - z_0 I)/\rho$ from the block encoding of A by [Lemma S2](#). Its normalization factor is $\alpha_{f(A)} = (\alpha_A + |z_0|)/\rho$. The block encoding is exact and uses one query to the block-encoding of A . Π_{in} is the unit-disk Riesz projector of $f(A)$. When rescaling by ρ , the spectral sets also rescale by ρ , so $\mathcal{K}_{\text{in}}/\rho$ and $\rho \mathcal{K}_{\text{out}}$ are spectral sets for $f(A)_{\text{in}}$ and $f(A)_{\text{out}}^{-1} = \rho (A_{\text{out}} - z_0 I)^{-1}$, respectively, both with constant c . By [\(S80\)](#), the first set lies in $\{z: |z| \leq 1 - \delta/\rho\}$. The second lies in $\{z: |z| \leq \rho/(\rho + \delta)\}$, and $\rho/(\rho + \delta) \leq 1 - \delta/(2\rho)$. So [\(S20\)](#) holds for $f(A)$ with gap $\delta/(2\rho)$.

[Theorem S2](#) applied to $f(A)$ block encodes Π_{in} with normalization factor $\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|)$. It uses

$$\mathbf{O}\left(\frac{c \alpha_{f(A)} \|\Pi_{\text{in}}\|}{\delta/(2\rho)} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon'} \log \frac{1}{\varepsilon'}\right) = \mathbf{O}\left(\frac{c(\alpha_A + |z_0|) \|\Pi_{\text{in}}\|}{\delta} \log \frac{c \|\Pi_{\text{in}}\|}{\varepsilon'} \log \frac{1}{\varepsilon'}\right) \quad (\text{S83})$$

queries per application to accuracy ε' . The factor ρ cancels between $\alpha_{f(A)}$ and the relative gap $\delta/(2\rho)$.

Applying this block encoding to $|\psi\rangle$ produces $\Pi_{\text{in}}|\psi\rangle$ with success amplitude η/α_{Π} . So fixed-point amplitude amplification prepares the normalized state to constant success amplitude with $\mathbf{O}(\alpha_{\Pi}/\eta) = \mathbf{O}(c \|\Pi_{\text{in}}\|/\eta)$ applications. Taking each application to accuracy $\varepsilon' = \Theta(\varepsilon\eta/\alpha_{\Pi})$ gives final accuracy ε . Then each logarithm is $\mathbf{O}(\log \frac{c \|\Pi_{\text{in}}\|}{\eta\varepsilon})$. Multiplying the number of applications by the queries per application gives [\(S81\)](#). The two special cases follow from [Lemma S6](#). $\square$

Alternatively, the resolvent-integral construction of [Appendix S1](#) (and [\[5\]](#)) applied to $f(A)$ block encodes the same projector with normalization factor $\alpha_{\text{res}} = \mathbf{O}(\rho \|\Pi_{\text{in}}\|/\delta)$ by [Proposition S7](#). Each application has the same query cost as before. Amplitude amplification then needs $\mathbf{O}(\alpha_{\text{res}}/\eta)$ rounds instead of $\mathbf{O}(\alpha_{\Pi}/\eta)$, a factor of order ρ/δ more. For $\rho = \Theta(1)$, the total gap dependence of queries to block encoding is $1/\delta^2$ for the resolvent integral versus $1/\delta$ with preconditioning. The latter matches the gap dependence of eigenstate preparation for nonnormal matrices with real spectra [\[4\]](#). Separately, the initial state is prepared once per round. So the resolvent integral uses $\mathbf{O}(\rho \|\Pi_{\text{in}}\|/(\eta\delta))$ queries to O_{ψ} and its inverse, compared with our gap-independent count [\(S82\)](#).

S4.2 Continuous-time algebraic Riccati equation

Consider the continuous-time algebraic Riccati equation $XQX - XP - P^{\dagger}X - R = 0$, where P, Q, R are square matrices of the same size with Q, R Hermitian. Its associated Hamiltonian matrix is $J = \begin{bmatrix} P & -Q \\ -R & -P^{\dagger} \end{bmatrix}$. The spectrum of J splits across the imaginary axis. Let Π_{in} be the projector onto the antistable (right-half-plane) invariant subspace, and write it in column blocks as $\Pi_{\text{in}} = [\Pi_1 \ \Pi_2]$. Then the stabilizing solution is $X = -\Pi_2^{\dagger}\Pi_1$ [\[5\]](#), where $(\cdot)^+$ is the pseudoinverse. Here, the smallest nonzero singular value of Π_2 satisfies $\mathbf{Sval}_{\text{min}}(\Pi_2) \geq 1/\sqrt{1 + \|X\|^2}$ [\[5\]](#). This bound does not depend on the gap.

Corollary S18 (Riccati equation). *Let P, Q, R be square matrices of the same size with Q, R Hermitian, and let $J = \begin{bmatrix} P & -Q \\ -R & -P^{\dagger} \end{bmatrix}$ be the associated Hamiltonian matrix, accessed through a block encoding J/α_J . Let $\Pi_{\text{in}} = [\Pi_1 \ \Pi_2]$ be the projector onto its antistable invariant subspace, with compressions $J_{\text{in}} = J\Pi_{\text{in}}$, $J_{\text{out}} = J\Pi_{\text{out}}$, and let $\sigma > 0$ be a lower bound on $\mathbf{Sval}_{\text{min}}(\Pi_2)$.*

Assume the compressions J_{in} and J_{out} admit spectral sets [\(S18\)](#) with constant c and gap δ satisfying [\(S76\)](#) with α_J in place of α_A . For any $\varepsilon > 0$, the stabilizing solution $X = -\Pi_2^{\dagger}\Pi_1$ of the continuous-time algebraic Riccati equation $XQX - XP - P^{\dagger}X - R = 0$ can be block encoded with normalization factor

$$\alpha_X = \mathbf{O}\left(\frac{c \|\Pi_{\text{in}}\|}{\sigma}\right) \quad (\text{S84})$$

and accuracy ε , using

$$\mathbf{O}\left(\frac{c^2 \alpha_J \|\Pi_{\text{in}}\|^2}{\sigma \delta} \text{polylog} \frac{c \alpha_J \|\Pi_{\text{in}}\|}{\sigma \delta \varepsilon}\right) \quad (\text{S85})$$

queries to the block encoding J/α_J .

Proof. By hypothesis, [Corollary S16](#) applies to J . It block encodes the projector Π_{in} with normalization factor $\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|)$ and accuracy ε' , using

$$\mathbf{O}\left(\frac{c \alpha_J \|\Pi_{\text{in}}\|}{\delta} \text{polylog} \frac{c \alpha_J \|\Pi_{\text{in}}\|}{\delta \varepsilon'}\right) \quad (\text{S86})$$

queries to J/α_J . Restricting the input register to the first or the second column block gives block encodings of Π_1/α_{Π} and Π_2/α_{Π} , with no additional queries.

The nonzero singular values of Π_2/α_{Π} are at least σ/α_{Π} . So the pseudoinverse version of quantum singular value transformation [\[2\]](#) block encodes Π_2^+ with normalization factor $\mathbf{O}(1/\sigma)$ and accuracy $\varepsilon/2$. This uses $\mathbf{O}(\frac{\alpha_{\Pi}}{\sigma} \log \frac{1}{\varepsilon})$ applications of the block encoding of Π_2/α_{Π} . [Lemma S3](#) then multiplies by Π_1/α_{Π} and absorbs the sign into a phase. The result is a block encoding of $X = -\Pi_2^+ \Pi_1$ with normalization factor $\mathbf{O}(\alpha_{\Pi}/\sigma)$, which is [\(S84\)](#).

In total, the projector block encoding is applied $\mathbf{O}(\frac{\alpha_{\Pi}}{\sigma} \log \frac{1}{\varepsilon})$ times. Errors in block encodings add up at most linearly. Taking $\varepsilon' = \Theta(\varepsilon \sigma / (\alpha_{\Pi} \log \frac{1}{\varepsilon}))$ therefore keeps the final accuracy at ε . The argument of every logarithm in the cost is then polynomial in c , α_J , $\|\Pi_{\text{in}}\|$, $1/\sigma$, $1/\delta$, and $1/\varepsilon$, so each logarithm is $\mathbf{O}(\log \frac{c \alpha_J \|\Pi_{\text{in}}\|}{\sigma \delta \varepsilon})$. Multiplying the number of applications by the cost of each application gives [\(S85\)](#). $\square$

The lower bound $\mathbf{Sval}_{\min}(\Pi_2) \geq 1/\sqrt{1 + \|X\|^2}$ [\[5\]](#) does not depend on the gap. Taking $\sigma = 1/\sqrt{1 + \|X\|^2}$, the normalization factor [\(S84\)](#) becomes $\alpha_X = \mathbf{O}(c \|\Pi_{\text{in}}\| \sqrt{1 + \|X\|^2})$, which does not depend on δ . In comparison, the resolvent-integral construction [\[5\]](#) block encodes the same projector with normalization factor $\alpha_{\text{res}} = \mathbf{O}(\|\Pi_{\text{in}}\|/\delta)$ in place of α_{Π} . This introduces a factor of $1/\delta$ to both the number of pseudoinverse applications and the solution normalization. Together with the $1/\delta$ cost of each projector application, the overall gap dependence is $1/\delta^3$. Preconditioning removes the two factors driven by the normalization, leaving only $1/\delta$.

S4.3 Sylvester equation

Consider the Sylvester equation $AX + XB = C$, where A, B, C are square matrices of the same size. Its augmented matrix is $K = \begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$, whose spectrum is the union of the spectra of A and $-B$. The projector onto the invariant subspace of the A branch, along that of the $-B$ branch, is $\Pi_{\text{in}} = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$. The solution X is therefore the top-right block of Π_{in} , and no pseudoinverse is needed to extract it.

Corollary S19 (Sylvester equation). *Let A, B, C be square matrices of the same size, and let $K = \begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$ be the associated augmented matrix, accessed through a block encoding K/α_K . Suppose the spectra of A and $-B$ lie in the right and left half-planes, respectively. Let $\Pi_{\text{in}} = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ be the projector onto the right-half-plane invariant subspace of K , with compressions $K_{\text{in}} = K\Pi_{\text{in}}$, $K_{\text{out}} = K\Pi_{\text{out}}$.*

Assume the compressions K_{in} and K_{out} admit spectral sets [\(S18\)](#) with constant c and gap δ satisfying [\(S76\)](#) with α_K in place of α_A . For any $\varepsilon > 0$, the solution X of the Sylvester equation $AX + XB = C$ can be block encoded with normalization factor

$$\alpha_X = \mathbf{O}(c \|\Pi_{\text{in}}\|) \quad (\text{S87})$$

and accuracy ε , using

$$\mathbf{O}\left(\frac{c \alpha_K \|\Pi_{\text{in}}\|}{\delta} \text{polylog} \frac{c \alpha_K \|\Pi_{\text{in}}\|}{\delta \varepsilon}\right) \quad (\text{S88})$$

queries to the block encoding K/α_K .

Proof. By hypothesis, [Corollary S16](#) applies to K . It block encodes the projector Π_{in} with normalization factor $\alpha_{\Pi} = \mathbf{O}(c \|\Pi_{\text{in}}\|)$ and accuracy ε , using the query count (S88). Restricting the output register to the first block and the input register to the second block extracts the top-right block X , with no additional queries. This gives a block encoding of X with the same normalization factor and accuracy, which is (S87). The special cases $c = 1 + \sqrt{2}$ and $c = \kappa_V$ carry over from [Corollary S16](#). $\square$

The normalization factor (S87) is close to optimal. From the block form of Π_{in} ,

$$\Pi_{\text{in}} \Pi_{\text{in}}^\dagger = \begin{bmatrix} I + XX^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (\text{S89})$$

so $\|\Pi_{\text{in}}\| = \sqrt{1 + \|X\|^2}$. Any block encoding of X has normalization factor at least $\|X\|$, so $\alpha_X = \mathbf{O}(c\sqrt{1 + \|X\|^2})$ is within a factor $\mathbf{O}(c)$ of $\max(1, \|X\|)$. In comparison, the direct augmented method [9, Remark 6.4] block encodes the same projector through the resolvent integral. Its normalization factor is $\alpha_{\text{res}} = \mathbf{O}(\|\Pi_{\text{in}}\|/\delta)$ by [Proposition S7](#). Preconditioning removes this factor of $1/\delta$ while keeping the query complexity essentially unchanged.

References

- [1] M. Crouzeix and C. Palencia, *The numerical range is a $(1 + \sqrt{2})$ -spectral set*, SIAM Journal on Matrix Analysis and Applications **38** (2017), no. 2, 649–655, [arXiv:1702.00668](#), DOI:10.1137/17M1116672.
- [2] András Gilyén, Yuan Su, Guang Hao Low, and Nathan Wiebe, *Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics*, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, pp. 193–204, 2019, [arXiv:1806.01838](#), DOI:10.1145/3313276.3316366.
- [3] Jeffrey Humpherys, Tyler J. Jarvis, and Emily J. Evans, *Foundations of applied mathematics, volume 1: Mathematical analysis*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2017, DOI:10.1137/1.9781611974904.
- [4] Guang Hao Low and Yuan Su, *Quantum eigenvalue processing*, SIAM Journal on Computing **55** (2026), no. 1, 135–215, [arXiv:2401.06240](#), DOI:10.1137/24M1689363.
- [5] Pablo Rodenas-Ruiz, Andrew Zhao, and Joonho Lee, *Quantum solvers for nonlinear matrix equations in quantum chemistry*, 2026, [arXiv:2605.16189](#), DOI:10.48550/arXiv.2605.16189.
- [6] Steven Roman, *Advanced linear algebra*, 3 ed., Graduate Texts in Mathematics, vol. 135, Springer, New York, 2008, DOI:10.1007/978-0-387-72831-5.
- [7] Daniel B. Szyld, *The many proofs of an identity on the norm of oblique projections*, Numerical Algorithms **42** (2006), no. 3–4, 309–323, DOI:10.1007/s11075-006-9046-2.

- [8] Lloyd N. Trefethen and Mark Embree, *Spectra and pseudospectra: The behavior of nonnormal matrices and operators*, Princeton university press, 2005, [DOI:10.1515/9780691213101](#).
- [9] Yanqiao Wang and Jin-Peng Liu, *Sign embedding quantum algorithms for matrix equations and matrix functions*, 2026, [arXiv:2604.25333](#), [DOI:10.48550/arXiv.2604.25333](#).