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Analytical and numerical methods to solve the Cauchy and Inverse problems

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Abstract

In this paper second-order quasi-linear equations are considered in which the coefficients of the second-order derivatives depend on the first derivatives of the unknown function and the equations admit parabolic degeneracy. The general integrals of the equations are applied to solve nonlinear Cauchy problems. A finite-difference scheme on the characteristic grid is developed for the formulated initial-value problem; its convergence is proved and second-order accuracy with respect to the grid step is established. Numerical experiments confirm the stability and efficiency of the algorithm and agree with the theoretical convergence rate. Finally, an inverse Cauchy problem is investigated for one of the equations, and the existence of its solution is proved under regularity conditions.

Keywords: Quasi-linear Hyperbolic equation, parabolic degeneracy, Cauchy problem, Difference Scheme, Inverse problem.

MSC Classification: 35L72 , 35L80 , 65M25

1 Introduction

Quasi-linear second-order partial differential equations with two independent variables play a significant role in mathematical physics and the theory of nonlinear waves. Of particular interest is the class of equations in which the coefficients of the second-order derivatives depend on the first-order derivatives of the unknown solution. This feature implies that both families of characteristics depend on the solution itself; consequently, the geometry of the characteristics is not known a priori but must be determined simultaneously with the solution.

Research into equations of mixed type dates back to the classical works of Francesco Tricomi, who obtained the first fundamental results in this field in the twenties of the 20th century [1]. Since then, the theory of mixed-type equations has evolved into a major branch of mathematical physics with applications ranging from transonic gas dynamics to plasticity theory and problems involving fluid flow in gravitational fields. Also noteworthy are the contributions of A. Bitsadze, who proposed methods that have become important in the theory of mixed-type equations [2]. His work and the research school he established continue to influence the development of this field.

This paper examines a class of quasi-linear second-order equations of the type described above. Such equations admit parabolic degeneracy: on a certain set, two characteristic directions may coincide, causing the equation to change type within that set. For some equations in this class, general integrals have been constructed in the form of a sum of two arbitrary functions [3, 4, 5]. These general integrals facilitate the study of the nonlinear Cauchy problem, the description of the domain of the global solution, and the analysis of the structure of characteristic curves in specific cases. Alongside the analytical study, a finite-difference scheme is constructed for the numerical solution of the formulated initial-value problems. The scheme is based on a characteristic grid associated with two families of characteristics and employs the Riemann invariants of the equation in conjunction with differential relations. The convergence of the proposed scheme is proved, and its second-order accuracy with respect to the grid step is established under natural assumptions regarding smoothness of functions and the absence of degeneracy. The theoretical results are illustrated by numerical experiments that confirm the stability and efficiency of the proposed algorithm and demonstrate that the observed convergence rate is fully consistent with the theoretical prediction. The paper also investigates a variant of the inverse Cauchy problem and presents an example of a solution to the formulated inverse problem.

First, we consider the following equation:

$$L(u) \equiv (u_y^2 - u_y)u_{xx} - (2u_x u_y + u_y - u_x - 1)u_{xy} + (u_x^2 + u_x)u_{yy} = \Phi(u_x, u_y, y). \quad (1)$$

The main part of (1) is a non-strictly hyperbolic second-order operator L. For an equation from this class, in article [3] we construct a general solution and study the characteristic problem.

Characteristic roots defined by operator L

$$\lambda_1 = -\frac{p+1}{q}, \quad \lambda_2 = -\frac{p}{q-1}, \quad (2)$$

where $p = u_x, q = u_y$ are Monge designations, behave differently with different functions $u \in C^2(R^2)$: with some $u(x, y)$ they may coincide at all points. Along such functions operator (1) ceases to be hyperbolic and parabolically degenerates. This class of the function is defined by means of the condition

$$p - q + 1 = 0. \tag{3}$$

If the solution of the given equation belongs to this class it will be a parabolic solution. It follows from the structure of the roots (2) that when having parabolic solutions their values not only coincide but they both equal to $\lambda_1 = \lambda_2 = -1$. Accordingly, in such case characteristic directions coincide with the direction of the family of lines $x+y = c$. If the conditions (3) are not fulfilled at all points but only at the determined number of points then the solution is related to the parabolically degenerated hyperbolic class ([2-12]). Generally, some of the equations of this class admit order degeneration either. In particular case when

$$L(u) = -\frac{1}{y}p(p + 1)(p - q + 1) \tag{4}$$

the equation can be fully integrated and its general integral has the following form ([3, 5])

$$f(u + x) + g(u - y) = y^2, \tag{5}$$

with the arbitrary functions $f, g \in C^2(R^1)$.

Since no general theory of nonlinear equations is available, studying some particular classes and particular equations is valuable, as this widens the class of quasi-linear equations for which the general integrals are constructed and the various problems are posed correctly.

First, we will concentrate on the case when $L(u) = 0$, so the equation has the form

$$(u_y^2 - u_y)u_{xx} - (2u_xu_y + u_y - u_x - 1)u_{xy} + (u_x^2 + u_x)u_{yy} = 0. \tag{6}$$

And the general integral is

$$f(u + x) + g(u - y) = x, \tag{7}$$

However, we can easily prove that equivalent expression of the general integral is

$$f(u + x) + g(u - y) = y, \tag{8}$$

It should be noted that the general integral (8) of the equation (6) as well as the integral (5) of the concrete equation (4) are not connected to any kind of correct problems. Equivalence of general integrals and corresponding to them equations is proved in [3, 5].

We discuss also an initial and inverse problems for the following second order non-strictly hyperbolic equation

$$(u_y^2 - 1)u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0 \tag{9}$$

This equation is of interest for physical applications, as it is related to the process of two-dimensional flow of an ideal incompressible fluid in a gravitational field ([4]).

Equation (9) is hyperbolic everywhere, except for the points at which the derivative u_x of the sought solution $u(x, y)$ has zero values. At these points, it parabolically degenerates. Hence equation (9) has mixed type. As we know, unlike the linear equations the characteristic invariants depend on an unknown solution and its first derivatives in case of quasi-linear equations. In Aerodynamics they are called the Riemann invariants.

Solving the equations $\frac{dy}{dx} = -\frac{u_x}{u_y+1}$, $\frac{dy}{dx} = -\frac{u_x}{u_y-1}$ we obtain that the characteristic invariants for equation (9) are $u + y = const$ and $u - y = const$.

We formulate a variant of the inverse problem of the Cauchy problem for equation (9) (see Problem 3). We prove that the considered inverse problem has a solution under certain regularity condition (Theorem 3). We illustrate the Cauchy and the inverse problems in Examples. We also constructed a numerical scheme for solving the posed initial problem. The convergence of the proposed scheme is proved. Numerical experiments confirm the stability and efficiency of the proposed algorithm.

2 Initial value problem

The Cauchy problem is considered when the initial data support is entirely the segment of the straight-line $y = 0$. The statement of the problem is the following:

Cauchy problem 1. Find a regular solution of equation (6) together with its domain of definition, satisfying the conditions

$$u|_{y=0} = \varphi(x), u_y|_{y=0} = \psi(x), x \in [a, b], \tag{10}$$

when $\varphi \in C^2[a, b]$, $\psi \in C^1[a, b]$ are the given functions.

Theorem 1 *Implicit solution to the problem (6), (10) is written in the following form:*

$$\int_{\nu(u-y)}^{\tau(u+x)} \frac{\varphi(t)(\varphi\prime(t) + 1)}{\varphi\prime(t) - \psi(t) + 1}\, dt = y, \tag{11}$$

were $\tau(\xi) = x$, $\nu(\eta) = x$ are inverse functions of $\varphi(x) + x$ and $\varphi(x)$ consequently. Here we assume, that functional equations $\varphi(x) + x = \xi$, $\varphi(x) = \eta$ are uniquely solvable on the segment $[a, b]$ and $\xi \in [\varphi(a), \varphi(b)]$, $\eta \in [\varphi(a) + a, \varphi(b) + b]$.

Proof see in [8].

Remark. The integral (8) can be written in equivalent form:

$$\int_{\nu(u-y)}^{\tau(u+x)} -\frac{\varphi\prime(\varphi\prime + 1)}{\varphi\prime - \psi + 1}\, dt + \tau(u + x) = x \tag{12}$$

It is easy to show, because if we take equivalent form (7) of the general integral and apply the same the process of constructing the integral for the problem, we obtain exactly (12). However, it is also easy to prove that from (11) we can obtain (12) directly.

The integral (11) (as well as (12)), allows us to describe structure of both characteristic families. For the family of characteristics on which invariant $u+x$ preserves is constant value, we can write the equation describing the family of curves:

$$\int_{\nu(\varphi(c)+c-x-y)}^{\tau(\varphi(c)+c)} \frac{\varphi\prime(\varphi\prime+1)}{\varphi\prime-\psi+1}\,dt = y. \tag{13}$$

This is equation of the family of characteristics which depends on one parameter c.

For the second family of characteristics we have the following expression:

$$\int_{\nu(\varphi(c))}^{\tau(\varphi(c)+x+y)} \frac{\varphi\prime(\varphi\prime+1)}{\varphi\prime-\psi+1}\,dt = y. \tag{14}$$

This equation also describes family of characteristic curves with one parameter. This second family corresponds to invariant $u-y$.

The obtained results allow us to describe the structure of areas of definition of solution in each specific case. For example, let us take the following functions:

$$\varphi = kx + l,\ \ \psi = p, \tag{15}$$

where k, l and p are constants. In other wards let φ function be linear and ψ function constant from (13) we obtain that the characteristic family of curve corresponding to invariant $u+x$ will be

$$\int_{(kc+c-x-y)/k}^{c} \frac{k(k+1)}{k-p+1}\,dt = y.$$

After simple calculations we have the equation of straight line:

$$y = -\frac{k+1}{p}(x-c),$$

where c is a parameter. Similarly, from (14) for the family of characteristics corresponding to invariant $u-y$ we can write

$$\int_{c}^{(kc+x+y)/(k+1)} \frac{k(k+1)}{k-p+1}\,dt = y.$$

Again, after simple calculations we obtain the equation of straight line:

$$y = -\frac{k(x-c)}{(p-1)}.$$

It means, that in case of initial conditions of form (15), both families of characteristics are the families of straight lines. If we take more specific example $\varphi = x + 1,\ \psi = 1$, it is easy to see that the characteristic families are $y = -2x + 2c,\ x = c$ for $u + x$ and $u - y$ invariants consequently. The area of definition of solution in this case will be a parallelogram which consists of intersection points of these two families.

So, after solving the Cauchy problem (6),(10) – in case of quasi-linear equation, we obtained, that domain of definition of solution is parallelogram bounded by straight lines

$$y = -2x + 2a,\ x = a,\ y = -2x + 2b,\ x = b.$$

It appears, that we obtained the similar situation to well-know equation $u_{xx} - u_{yy} = 0$, where the domain of definition of solution is a square, bounded by straight lineas from families $x + y = const$ and $x - y = const$.

Situation may appear different when initial conditions are both linear functions. For example, if $\varphi = x + 5,\ \psi = x + 2$, from (11) we have $y = log\frac{4(u-y-5)^2}{(u+x-5)^2}$ and the regular solution does not exist on the straight line $y = -x$. From (13) and (14) we can see that characteristic curves of both families intersect at the point $(-ln4, ln4)$. So, in this case we have node type singular point.

Let us discuss an initial problem on the data support $[a, b] \in R$ for the second order non-strictly hyperbolic equation (9).

Cauchy problem 2. *Let $\tau(x)$, respectively $\nu(x)$ be real functions, which are twice, respectively once continuously differentiable. Find a function $u(x, y)$, which satisfies equation (9) and the conditions*

$$u(x, 0) = \tau(x), \quad u_y(x, 0) = \nu(x), \quad a \le x \le b. \tag{16}$$

It is also required to find a domain where the solution can be completely defined.

Theorem 2 *If $\tau'(x) \neq 0$ for all $x \in [a, b]$, then the integral of the problem (9), (16) can be written in the form*

$$x = a + \frac{1}{2}\int_a^{T(u+y)} (1 - \nu(t))dt + \frac{1}{2}\int_a^{T(u-y)} (1 + \nu(t))dt, \tag{17}$$

where $x = T(z)$ denotes the inverse function of $z = \tau(x)$.

Proof is in [4].

Taking (17) into account, one can obtain the equations of the characteristics in implicit form. All characteristic curves of the family $u + y = const$ which pass through the points of the data support have the representation

$$x = a + \int_a^c F(t)dt + \int_a^{T(\tau(c)-2y)} G(t)dt; \tag{18}$$

for all characteristic curves of the family $u - y = const$ we obtain

$$x = a + \int_a^{T(\tau(c)+2y)} F(t)dt + \int_a^c G(t)dt, \tag{19}$$

where the functions $F(t)$ and $G(t)$ are defined by relations

$$F(t) = \frac{1-\nu(t)}{2},\ G(t) = \frac{1+\nu(t)}{2}.$$

This representation is obtained if condition $\tau'(x) \neq 0$ is fulfilled for all $x \in [a, b]$ and the expressions under the integral sign are integrable.

In (18) and (19) the parameter c takes its values from the interval $[a, b]$. For both families, there is a one-to-one correspondence between the equation for a characteristic curve and the parameter value. This fact is stipulated by the condition $\tau'(x) \neq 0$ for all $x \in [a, b]$.

3 Inverse problem

Now we consider the inverse Cauchy problem for equation (9). Below we will investigate a variant of inverse problem which requires to find initial data under the condition that the characteristic curves of both families are known a priori.

As has been mentioned above, equation (9) is of hyperbolic type and the set of parabolic degeneration is not defined a priori because it depends on the behavior of an arbitrary solution $u(x, y)$ and its derivative with respect to the variable x. The characteristic invariants (see [4]) corresponding to equation (9) are given as follows:

$$u + y = const, \quad \text{for} \quad \lambda_1 = -\frac{u_x}{u_y + 1}, \tag{20}$$

$$u - y = const, \quad \text{for} \quad \lambda_2 = -\frac{u_x}{u_y - 1}. \tag{21}$$

In the case of equation (9), we can admit in the role of characteristics the families of characteristic curves along which relations (20) and (21) are fulfilled. Unlike linear equations, here characteristic families can be therefore given a priori in an arbitrary manner. The statement of the inverse problem considered here rests on this fact.

Let us consider two one-parameter families of plane curves which are given explicitly by the equations

$$y = \varphi_1(x, c), \tag{22}$$

$$y = \varphi_2(x, c), \tag{23}$$

where φ_1, φ_2 are given, twice differentiable functions with respect to the variable x. Let φ_1, φ_2 be defined for any value of the real parameter c. Assume that any curve of these families necessarily intersects once the straight line $y = 0$. We denote by D_1 the domain of the plane (x, y) which is completely covered by the family of characteristic curves given by equation (22) if the parameter c runs continuously through all real values. Analogously, we denote by D_2 the domain of the plane (x, y) which is completely covered by the characteristic lines given by equation (23). Also, we introduce the notation $D = D_1 \cap D_2$, $I = D \cap \{y = 0\}$.

Problem 3 (Inverse problem). *Find the initial data of a regular solution $u(x, y)$ of equation (9) and its derivative with respect to the normal direction on the interval I of the straight line $y = 0$ if the families of plane curves given by equations (22)*

are the characteristics corresponding to invariants (20), while (23) is the family of characteristics corresponding to invariants (21).

Theorem 3 *If the condition* $\varphi_1'(x,c') \neq \varphi_2'(x,c'')$*, is fulfilled for all* c'*,* $c'' \in R$*, then there exists a solution of Problem 3.*

Proof To investigate the posed problem, we need to analyse a structure of the characteristic curves of equation (2). As we know, at every point the characteristic roots, corresponding to equation (2), define two characteristic directions:

$$\frac{dy}{dx} = -\frac{u_x}{u_y + 1} \equiv \lambda_1(x,y), \tag{24}$$

$$\frac{dy}{dx} = -\frac{u_x}{u_y - 1} \equiv \lambda_2(x,y). \tag{25}$$

It follows from the posed problem that the family of characteristic curves defined by equation (22) corresponds to the root λ_1 given by (24), while that given by equation (23) corresponds to the root λ_2 given by (25). Therefore we can calculate the values of first derivatives u_x, u_y of the unknown solution $u(x,y)$ along any characteristic curve. Note that the parameters contained in families (22) and (23) can be defined by the abscissa of the intersection point of the concrete curve and the axis $y = 0$. Indeed, let the curve corresponding to the parameter c^*, intersecting the straight line $y = 0$ at a point x_0 be

$$\varphi_1(x_0, c^*) = 0.$$

Solving the equation for the parameter $c^*(x_0) = c$, family (22) takes the form

$$y = \varphi_1(x, x_0),$$

φ_1 becomes zero for $x = x_0$. If family (22) corresponds to the characteristic root λ_1, then the relation

$$\frac{d\varphi_1(x,x_0)}{dx} = -\frac{u_x(x,\varphi_1(x,x_0))}{u_y(x,\varphi_1(x,x_0)) + 1}. \tag{26}$$

is fulfilled. Analogously, along the second family of the characteristic curves the following equality is fulfilled

$$\frac{d\varphi_2(x,x_0)}{dx} = -\frac{u_x(x,\varphi_2(x,x_0))}{u_y(x,\varphi_2(x,x_0)) - 1}. \tag{27}$$

From equalities (26), (27) at the points of the interval I we have:

$$\frac{d\varphi_1(x,x_0)}{dx}\,|_{x=x_0} = -\frac{\tau'(x_0)}{\nu(x_0) + 1}, \quad x_0 \in I, \tag{28}$$

$$\frac{d\varphi_2(x,x_0)}{dx}\,|_{x=x_0} = -\frac{\tau'(x_0)}{\nu(x_0) - 1}, \quad x_0 \in I. \tag{29}$$

Since the point $(x_0, 0)$ is arbitrarily chosen from the interval I, x_0 in equations (28), (29) can be replaced with x. Hence we can easily define the unknown functions τ' and ν. Integrating the functions τ', we finally obtain the solution $\tau(x)$, $\nu(x)$ of the problem. Naturally, the function τ is defined up to an integration constant which will be defined uniquely if we give the value of the function $u(x,y)$ at one arbitrary point of the interval I. □

From the above general argumentation we can draw a conclusion: If families (22) and (23) do not have common characteristic directions at anyone of the points, i.e. if equation (9) is of strictly hyperbolic type, it is possible to find the solution of the inverse problem (see Example).

Example. We solve the Cauchy and inverse problems in the following concrete example.
Cauchy problem: Let $u(x,0) = \tau(x)$ be the function x and $u_y(x,0) = \nu(x)$ be the function $1-e^x$. According to (17) the solution of the Cauchy problem in implicit form is given by equation

$$\frac{1}{2}e^{u+y} + u - y - \frac{1}{2}e^{u-y} - x = 0.$$

The characteristic curves (18) of the family $u+y = const$, as parametric curve $f(x,y,c)$ with respect to the parameter c are given by

$$0 = \frac{1}{2}e^{c} + c - 2y - \frac{1}{2}e^{c-2y} - x = f_1(x,y,c). \tag{30}$$

The characteristic curves (19) of the family $u-y = const$, as parametric curve $f(x,y,c)$ with respect to the parameter c are given by

$$0 = \frac{1}{2}e^{c+2y} + c - \frac{1}{2}e^{c} - x = f_2(x,y,c). \tag{31}$$

Differentiating the function $f_1(x,y,c)$ in (30) with respect to the parameter c we get

$$0 = \frac{1}{2}e^{c}(1-e^{-2y}) + 1 = \frac{\partial f_1}{\partial c}(x,y,c). \tag{32}$$

Expressing from (32) the parameter c we have

$$c = \ln\left(\frac{2e^{2y}}{1-e^{2y}}\right). \tag{33}$$

Putting expression (33) into equation (30) we obtain $e^{1+x} = \frac{2}{1-e^{2y}}$.

Inverse problem: Assume that the families of characteristic curves are given by:

$$x = \varphi_1(y,c) \equiv \frac{1}{2}e^{c} - \frac{1}{2}e^{c-2y} + c - 2y, \quad \text{for} \quad \lambda_1, \tag{34}$$

$$x = \varphi_2(y,c) \equiv -\frac{1}{2}e^{c} + \frac{1}{2}e^{c+2y} + c, \quad \text{for} \quad \lambda_2. \tag{35}$$

From (28), (29) we get

$$-\frac{u_x(x_0,0)}{u_y(x_0,0)+1} = \frac{1}{e^{x_0}-2}, \tag{36}$$

for family (34) and

$$-\frac{u_x(x_0,0)}{u_y(x_0,0)-1} = \frac{1}{e^{x_0}}, \tag{37}$$

for family (35).

The right-hand sides of the equalities (36), (37) depend only on x_0. Since the point $(x_0, 0)$ is arbitrarily chosen from the interval I, x_0 can be replaced by x and from the relations (36) and (37) we find the values of the functions τ' and ν:

$$\tau'(x) = 1, \tag{38}$$

$$\nu(x) = 1 - e^x.$$

From equality (38) we obtain by integration the value of the solution $u(x, y)$ on the axis $y = 0$:

$$\tau(x) = x + c, \quad c = const. \tag{39}$$

The constant c in (39) can be obtained if we know the value of $\tau(x)$ in one arbitrary point.

4 Finite-difference scheme for the Cauchy Problem

For the equation (9) we have the following systems of differential relations:

$$p\,dx + (q+1)\,dy = 0,\ dp - \frac{p}{q-1}\,dq = 0,\ \text{for } \gamma \text{ characteristic, and}$$

$$p\,dx + (q-1)\,dy = 0,\ dp - \frac{p}{q+1}\,dq = 0,\ \text{for } \delta \text{ characteristic.}$$

The two characteristic directions coincide precisely when $p = 0$. We therefore assume that there exists a constant $c > 0$ such that

$$|p(x, y)| \geq c > 0, \qquad |q(x, y)| > 1, \tag{40}$$

everywhere within $\overline{\Omega}-$ the domain of the solution.

Analogues of Rimann invariants for the equation (9) have the form:

$$u + y = \text{const}, \quad \frac{p}{q-1} = \text{const}\ , \ \text{for } \gamma \text{ characteristic,} \tag{41}$$

$$u - y = \text{const}, \ \frac{p}{q+1} = \text{const}, \ \text{for } \delta \text{ characteristic.} \tag{42}$$

On the initial line $y = 0$ we introduce the uniform grid $x_i = a + ih, \qquad h = \frac{b-a}{N}$, with initial data

$$u_{i,0} = \phi(x_i), \qquad p_{i,0} = \phi'(x_i), \qquad q_{i,0} = \psi(x_i).$$

A node $C = (i, j+1)$ of the characteristic grid is obtained from two known nodes: $A = (i+1, j)$, lying on the γ-characteristic; $B = (i, j-1)$, lying on the δ-characteristic.

From (41) and (42), we have: $u_C + y_C = u_A + y_A, \qquad u_C - y_C = u_B - y_B$.

The exact solution of this linear system is

$$u_C = \frac{(u_A + y_A) + (u_B - y_B)}{2}, \tag{43}$$

$$y_C = \frac{(u_A + y_A) - (u_B - y_B)}{2}. \tag{44}$$

From (41) on γ we get $p_C(q_A - 1) - p_A q_C = -p_A$, and from and (42) on δ we have $p_C(q_B + 1) - p_B q_C = p_B$. This is a linear algebraic system for p_C and q_C, with determinant

$$\Delta = p_A(q_B + 1) - p_B(q_A - 1). \tag{45}$$

Solving it yields

$$p_C = \frac{2p_A p_B}{\Delta}, \tag{46}$$

$$q_C = \frac{p_A(q_B + 1) + p_B(q_A - 1)}{\Delta}. \tag{47}$$

Equations (46) and (47) are exact and require no trapezoidal approximation.

From the characteristic relations

$$\frac{dx}{dy} = -\frac{q+1}{p}\ , \qquad \frac{dx}{dy} = -\frac{q-1}{p}\ ,$$

the trapezoidal rule gives

$$x_C - x_A = -c_\alpha\,(y_C - y_A), \qquad x_C - x_B = -c_\beta\,(y_C - y_B), \tag{48}$$

where

$$c_\alpha = \frac{1}{2}\left(\frac{q_A + 1}{p_A} + \frac{q_C + 1}{p_C}\right), \qquad c_\beta = \frac{1}{2}\left(\frac{q_B - 1}{p_B} + \frac{q_C - 1}{p_C}\right).$$

From (48) we obtain x_C explicitly:

$$x_C = \frac{x_A - c_\alpha(y_C - y_A) + x_B - c_\beta(y_C - y_B)}{2}. \tag{49}$$

Theorem 4. Assume that:

$u \in C^3(\overline{\Omega})$ is the classical solution of (9); $\phi \in C^2[a,b]$, $\psi \in C^1[a,b]$; there exists $c > 0$ such that $|p(x,y)| \geq c > 0$ and $|q(x,y)| > 1$ on $\overline{\Omega}$;

Then, for a sufficiently small grid step size $h \leq h_0$, the discrete characteristic scheme is well-posed, the determinant (45) is non-zero:

$$|\Delta(x,y)| \geq c_0 > 0 \quad \text{on } \overline{\Omega};$$

also the scheme (43), (44), (46), (47), (49) is uniquely solvable for sufficiently small h and converges with second order:

$$\max_{i,j}\Big(|u_{i,j} - u(x_i,y_j)| + |p_{i,j} - p(x_i,y_j)| + |q_{i,j} - q(x_i,y_j)| + |x_{i,j} - x_i|\Big) \leq Kh^2,$$

where $K > 0$ is independent of h.

Proof Under the assumption $u \in C^3$,

$$R_C^u = 0, \quad R_C^y = 0, \quad R_C^p = 0, \quad R_C^q = 0, \quad |R_C^x| \le M_1 h^3.$$

Equation (49) uses the trapezoidal rule for a C^2 function, so the error is $O(h^3)$.

Define

$$\varepsilon_{i,j}^u = u_{i,j} - u(x_i, y_j), \quad \varepsilon_{i,j}^p = p_{i,j} - p(x_i, y_j), \quad \varepsilon_{i,j}^q = q_{i,j} - q(x_i, y_j), \quad \varepsilon_{i,j}^x = x_{i,j} - x_i.$$

From (46), (47) we have:

$$A_{i,j} \begin{pmatrix} \varepsilon_C^p \\ \varepsilon_C^q \end{pmatrix} = B_{i,j} \begin{pmatrix} \varepsilon_A^p \\ \varepsilon_A^q \end{pmatrix} + C_{i,j} \begin{pmatrix} \varepsilon_B^p \\ \varepsilon_B^q \end{pmatrix} + \boldsymbol{\tau}_{i,j}, \tag{50}$$

where

$$A_{i,j} = \begin{pmatrix} q_A - 1 & -p_A \\ q_B + 1 & -p_B \end{pmatrix}, \qquad \det A_{i,j} = \Delta,$$

and $\|\boldsymbol{\tau}_{i,j}\| = 0$ because the local approximation error vanishes.

It can easily be shown that the inequality $|\Delta| \ge c_0$, $\|A_{i,j}^{-1}\| \le C_1$, where C_1 is independent of h, holds under the conditions of the theorem.

Thus

$$\|\boldsymbol{\varepsilon}_C^{(p,q)}\| \le C_2 \big(\|\boldsymbol{\varepsilon}_A^{(p,q)}\| + \|\boldsymbol{\varepsilon}_B^{(p,q)}\|\big). \tag{51}$$

From (49) we have:

$$\varepsilon_C^x = \tfrac{1}{2}(\varepsilon_A^x + \varepsilon_B^x) - \tfrac{1}{2}\left[c_\alpha(\varepsilon_C^y - \varepsilon_A^y) + c_\beta(\varepsilon_C^y - \varepsilon_B^y)\right] + \tfrac{1}{2}(\delta c_\alpha)(y_C - y_A) - \tfrac{1}{2}(\delta c_\beta)(y_C - y_B) + R_C^x,$$

where $\delta c_\alpha, \delta c_\beta$ are the errors of c_α, c_β induced by p, q. Since $(y_C - y_A) \sim h$ and $(y_C - y_B) \sim h$,

$$|\varepsilon_C^x| \le \tfrac{1}{2}(|\varepsilon_A^x| + |\varepsilon_B^x|) + L_2 h \big(\|\boldsymbol{\varepsilon}_A^{(p,q)}\| + \|\boldsymbol{\varepsilon}_B^{(p,q)}\|\big) + M_1 h^3. \tag{52}$$

From (43) we obtain:

$$\varepsilon_C^u = \tfrac{1}{2}(\varepsilon_A^u + \varepsilon_B^u) + \tfrac{1}{2}(\varepsilon_A^y - \varepsilon_B^y). \tag{53}$$

Let

$$E_j = \max_i \Big(|\varepsilon_{i,j}^u| + |\varepsilon_{i,j}^y| + |\varepsilon_{i,j}^p| + |\varepsilon_{i,j}^q| + |\varepsilon_{i,j}^x|\Big).$$

Combining (51), (52), (53), we obtain

$$E_j \le C_3 h\, E_j + C_4 E_{j-1} + C_5 h^3.$$

For sufficiently small h (so that $C_3 h \le 1/2$), we can write that

$$E_j \le (1 + C_6 h) E_{j-1} + C_7 h^3. \tag{54}$$

For error on the first layer we can easily see that the following holds true

$$E_1 \le C_8 h^2. \tag{55}$$

Iterating (54) from $j = 1$ to j gives

$$E_j \le (1 + C_6 h)^{j-1} E_1 + C_7 h^3 \sum_{m=0}^{j-2} (1 + C_6 h)^m.$$

By (55), the sum equals

$$\sum_{m=0}^{j-2} (1 + C_6 h)^m = \frac{(1 + C_6 h)^{j-1} - 1}{C_6 h}.$$

Since $(1 + C_6 h)^{j-1} \le e^{C_6 (j-1)h} \le e^{C_6 Y_{\max}}$ (because $jh \le Y_{\max}$), we have

$$E_j \le e^{C_6 Y_{\max}} C_8 h^2 + C_7 h^3 \cdot \frac{e^{C_6 Y_{\max}} - 1}{C_6 h} \le C_8 e^{C_6 Y_{\max}} h^2 + \frac{C_7 \big(e^{C_6 Y_{\max}} - 1\big)}{C_6} h^2.$$

We finally obtain

$$\max_{i,j} E_{i,j} \le K h^2,$$

when

$$K = C_8 e^{C_6 Y_{\max}} + \frac{C_7 \big(e^{C_6 Y_{\max}} - 1\big)}{C_6}.$$

The theorem is proved. □

4.1 Numerical Example

To verify the efficiency and the convergence rate of the proposed finite-difference scheme, we consider a test problem with an exact nonlinear solution

$$u(x, y) = y + \ln(x), \qquad x \in [1, 2], \quad y \ge 0.$$

The partial derivatives are $p = u_x = \frac{1}{x} > 0, \qquad q = u_y = 1$. On the domain $x \in [1, 2]$ the hyperbolicity and non-degeneracy conditions are satisfied, namely $|p| \ge 0.5 > 0$.

Numerical experiments were carried out on several grids with step sizes (see table 1) $h \in \{0.1,\ 0.05,\ 0.025,\ 0.0125,\ 0.00625\}$. For $h = 0.025$ the maximum absolute error is $\max |u_h - u| = 1.697 \times 10^{-5}$. When the step size is halved, the error decreases by approximately a factor of 4, $\frac{E(h)}{E(h/2)} \approx 4.00$, and the convergence rate is $p = \log_2\Big(\frac{E(h)}{E(h/2)}\Big) \approx 2.00$.

Table 1 Maximum error and numerical convergence order for $u(x, y) = y + \ln(x)$.

N	h	$E(h) = \|u - u_h\|_\infty$	Ratio	Order (p)
10	0.1000	2.5213e-04	–	–
20	0.0500	6.6297e-05	3.803	1.927
40	0.0250	1.6971e-05	3.906	1.966
80	0.0125	4.2918e-06	3.954	1.983
160	0.0063	1.0790e-06	3.977	1.992

The numerical test confirms the theoretically proved second-order ($\mathcal{O}(h^2)$) convergence and stability of the scheme.

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