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Research Article

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Posted Date: November 23rd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-11353057/v1

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Additional Declarations: No competing interests reported.

A wage proposal for the worker-machine assignment problem

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Abstract

For the worker-machine assignment problem, we modify the Hungarian method to simultaneously obtain an efficient assignment solution and the orthogonality conditions of the dual problem. We use this result to propose fair wages for the workers. Finally, we axiomatically characterize the proposed wages based on three axioms: marginal contribution, equity, and weak nullity.

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1 Introduction

Consider the typical assignment problem in operations research, where n workers must be efficiently assigned to n jobs. We assume that a matrix A is given, where i is the worker, j is the job, and a_{ij} is the efficiency of worker i on job j. We wish to develop algorithms to assign fair wages to workers, considering their relative efficiency. We

assume that workers are assigned to jobs using the typical OR assignment problem:

$$\begin{aligned} \max\ & z = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij} \\ & \sum_{i=1}^{n} x_{ij} = 1 \\ & \sum_{j=1}^{n} x_{ij} = 1 \\ & x_{ij} \geq 0 \end{aligned} \tag{1}$$

where $x_{ij} = 1$ if worker i is assigned to job j, and $x_{ij} = 0$ otherwise. Under this assumption, we propose and justify the workers' wages based on certain assumptions. Our goal is to compensate each worker based on their contribution to the total profit generated.

In this study, the assignment problem is solved using the Hungarian method (Kuhn 1955 or 2005), and a wage proposal for workers is generated. We have three main assumptions. First, we assume that the maximum efficiency of the system (the optimal value of the objective function) is equal to the amount to be distributed as wages among the workers. Alternatively, the total amount to be distributed as wages could be given exogenously, and most of our results would remain valid. Second, we consider only problems in which the number of workers is equal to the number of jobs. Finally, we assume that each worker has an equal right to be assigned to any job; therefore, all workers must agree to the final assignment.

Note that, as a general rule, it is not fair to pay workers based on their individual contributions to the optimal solution because, in that case, some workers are often sacrificed for the sake of overall efficiency. For example, in the optimal solution to the problem with the following matrix,

$$A = \begin{bmatrix} 8 & 15 \\ 0 & 10 \end{bmatrix} \tag{2}$$

The first worker is assigned to Job 1 and the second to Job 2; however, it is not fair to pay the first worker eight units and the second ten units, since, on a job-by-job basis, the first worker is more productive than the second. Therefore, to determine a wage for workers, it is necessary to consider their relative abilities to perform different jobs.

The Hungarian method is based on the fact that if the same value is added to or subtracted from a row or column of an efficiency matrix, the solution to the problem (1) remains the same. Furthermore, the objective function increases or decreases by this amount. If, after applying these operations to an initial matrix, we obtain a matrix with non-positive entries and a zero assignment, then that assignment must be optimal for the matrix. We prove that this assignment is also optimal for the original problem.

In Samet et al. (1984), prices are determined axiomatically for classes of cost functions that are continuously differentiable in the intervals. One such class consists of all the cost functions derived from transportation problems. Tauman, Y. (1988) presented a survey of this type of characterization. Sattinger, M. (1995) considers a model in which different types of workers are matched with different types of employers, and derives a method for calculating the opportunity costs of workers and jobs. Shi,S. (2001) describes efficient allocation and demonstrates that it can be decentralized through a market mechanism. Stovall, J. E. (2021) analyzed workers' compensation from an axiomatic perspective. He analyzes the egalitarian, individual contribution, and Shapley Value rules. Cousy et al. (2022) showed with examples that it is possible to strike a reasonable balance between efficiency and fairness in task assignment using the proposed models. In the context of digital platform applications,Kuhn, K. M., & Maleki, A. (2017) analyze how work platforms determine payments to workers through allocation mechanisms. Sanchez, F. (2004) addressed this problem, but from a different perspective.

The literature on the wage-based allocation problem—from Shapley and Shubik's (1972) Assignment Game to modern extensions in matching with contracts Hatfield

and Melohar Milgrom 2005 and digital platforms—demonstrates how wages can ensure stability and efficiency in labor markets. However, much of this research focuses on equilibrium stability and bilateral bargaining, relegating the dimensions of marginal contributions and equity in the distribution of benefits. This is where our research comes in: designing cooperation mechanisms that ensure efficient and stable allocations and promote the principles of distributive justice. This approach seeks to integrate the tradition of cooperative game theory with the principles of fair allocation, offering models applicable to both labor markets and institutional contexts, where cooperation and equity are essential.

The remainder of this paper is organized as follows. The second section begins with the notation and definitions used later. We present the Hungarian method and introduce additional variables that we have added for its application in this study, and then analyze the properties that emerge from the results obtained. This section concludes with a definition of wages and an explanation of how they are obtained. The section ends with a couple of examples. In Section 3, wages are characterized by three axioms, which follow naturally from the Hungarian method. In the fourth section, several properties of the proposed wages are established. The final section demonstrates the independence of these axioms.

2 The Hungarian Method and the Wage Proposal

Let $\mathcal{M} = \{A \,|A \text{ matrix } n\times n, \text{ with non-negative real entries}\}$ be the set of matrices that measure worker efficiency, $\mathcal{P} = \{P \subset \mathcal{M} | \sum_{i=1}^{n} p_{ij} = 1, \sum_{j=1}^{n} p_{ij} = 1 \text{ and } p_{ij} \geq 0\}$ the set of permutation matrices, and $z : \mathcal{M}\times\mathcal{P} \to \mathbb{R}$, given by $z(A, P) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}p_{ij}$ which we will call the objective function. We will denote matrices by uppercase letters and their entries by the corresponding lowercase letters. With this, we can rewrite the problem (1) as

$$\max\{z(A, P)|P \in \mathcal{P}\}$$

We wish to assign a wage vector to each matrix in $\mathcal{M}$; that is, we wish to determine a function

$$\varphi : \mathcal{M} \to \mathbb{R}^n$$

where $\varphi(A)$ is a "fair" wage vector for workers with efficiency matrix A.

Definition 1 $\overline{P}$ is said to be optimal for A if and only if $z(A, \overline{P}) \geq z(A, P)$ for all $P \in \mathcal{P}$.

Definition 2 We say that $\overline{P}$ is a zero assignment for A if and only if $z(A, \overline{P}) = 0$.

We want to assign workers in a way that maximizes efficiency; therefore, we need to solve problem (1). One of the most elegant methods for solving assignment problems is the Hungarian method. It involves adding or subtracting the same number from a row or column until the problem becomes trivial. This does not change the solution to the problem, and the objective function changes by the amount added or subtracted. These operations continue until a matrix with a zero assignment and all its entries are non-positive is reached. At this point, it is impossible to obtain an assignment with a positive objective function, and because we already have one with a zero objective function, it must be optimal. The entries marked by this zero assignment must be the solution to the original problem, since the optimal solution was maintained at every step.

Let us review the Hungarian method by adding a few variables to it.

Hungarian method:

1. Subtract the largest entry in each row from all the entries in that row. (There are no longer any positive entries.)
2. Add up the largest entry in each column from all the entries in that column.
3. Repeat until you get a zero assignment.

(a) The zeros are covered using the minimum number of lines. If the minimum number of covering lines is equal to n, there exists a zero assignment, which is optimal, and the process ends. Otherwise, let a^* be the largest uncovered number.

(b) a^* is subtracted from each uncovered row.

(c) Add a^* to each covered column.

We denote by R^k the $n \times n$ matrix with ones in row k and zeros in the other entries, by C^l the $n \times n$ matrix with ones in column l and zeros in the other entries, by $N = \{1, \ldots, n\}$ the set over which the subscripts run, and by ι the vector of ones in $\mathbb{R}^n$.

To determine the wages, we maintain running totals for the operations performed in each row and column of the process. Let s and t be two vectors in $\mathbb{R}^n$, where s_i is the result of all the additions and subtractions of the quantities applied to row i, and t_j is the corresponding result of the operations applied to column j.

Thus, at the end of the Hungarian method, we obtain: (a) the vectors s and t, (b) the final matrix of the Hungarian method

$$B = A + \sum_{i \in N} s_i R^i + \sum_{j \in N} t_j C^j \tag{3}$$

and (c) the zero-setting, which we will denote by P^* hereafter; that is, if $p^*_{ij} = 1$, then $b_{ij} = 0$. In addition,

$$z(B, P^*) = 0 \tag{4}$$

because it is the optimal solution of the Hungarian method.

The following statement highlights how the solution changes when the same amount is added to or subtracted from the elements of a row or a column.

Proposition 1 *For every $A \in \mathcal{M}$, $P \in \mathcal{P}$, $k, l \in N$ and real numbers α and β it must be*

- $z(A + \alpha R^k, P) = z(A, P) + \alpha.$
- $z(A + \beta C^l, P) = z(A, P) + \beta.$

Proof Let $A \in \mathcal{M}$, $P \in \mathcal{P}$, $\alpha \in \mathbb{R}$ and $k \in N$ be, then

$$z(A + \alpha R^k, P) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} + \alpha r_{ij}^k)p_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}p_{ij} + \alpha = z(A, P) + \alpha$$

since r_{ij}^k eliminates the entries outside row k and p_{ij} leaves only one of the entries in row k. The second part of the proof is similar to the first. □

Corollary 1 $z(A, P^*) = -\iota's + -\iota't.$

Proof In the following expression, the first equality holds because B is the final matrix of the Hungarian method, the second because of (3), and the third because of the previous statement.

$$0 = z(B, P^*) = z(A + \sum_{i \in N} s_i R^i + \sum_{j \in N} t_j C^j, P^*) = z(A, P^*) + \sum_{i \in N} s_i + \sum_{j \in N} t_j$$
$$= z(A, P^*) + \iota's + \iota't.$$

from where $z(A, P^*) = -\iota's + -\iota't.$ □

Proposition 2 *The permutation P^* is optimal for A.*

Proof The assignment of zeros P^* is optimal for B, as a positive assignment cannot be obtained with non-positive inputs, and a solution with the objective function equal to zero already exists. Furthermore, by Proposition 1, for any $P \in \mathcal{P}$, $z(B, P) = z(A, P) + \sum s_i + \sum t_j$. Thus,

$$z(A, P) + \sum s_i + \sum t_j = z(B, P) \leq z(B, P^*) = z(A, P^*) + \sum s_i + \sum t_j$$

from where $z(A, P^*) \geq z(A, P)$ for every $P \in \mathcal{P}$. □

The following proposition contains one of the main ideas of this study: Using the vectors $-s$ and $-t$ obtained by the Hungarian method, the orthogonality conditions for the dual of assignment problem (1) are derived. (See Lawler E.L. (1976).

Proposition 3 *If* $p^*_{ij} = 1$ *then* $a_{ij} = -s_i - t_j$.

Proof Let us assume $p^*_{kl} = 1$; that is, the entry kl in B is part of the zero assignment in the final matrix of the Hungarian method, and therefore $b_{kl} = 0$. Now, from (3), we have that

$$0 = a_{kl} + \sum_{i \in N} s_i R^i_{kl} + \sum_{j \in N} t_j C^j_{kl}$$

However, from the first sum, only s_k remains, since $R^i_{kl} = 0$ if $i \neq k$ and $R^k_{kl} = 1$. Using the same reasoning, we find that only t_l remains from the second sum. Thus,

$$0 = a_{kl} + s_k + t_l$$

and therefore $a_{kl} = -s_k - t_l$. □

Definition 3 Let $\phi : \mathcal{M} \to \mathbb{R}^n$ be given by,

$$\phi(A) = -s - \widehat{t}\,\iota$$

where s and t are the cumulative vectors obtained by applying the Hungarian method to A and $\widehat{t} = \frac{1}{n}\sum_{j=1}^{n} t_j$.

Suppose that we use the Hungarian method to find the optimal allocation. Wages are determined by the same quantities used to modify the matrix but with the signs reversed. Furthermore, whenever a column is modified, the wages are distributed equally, whereas changes to the rows affect only the corresponding workers. The algorithm can be summarized as follows: We start with a wage of zero for all workers. We apply the Hungarian method, and each time an amount is added to or subtracted

from row i, that same amount is subtracted from or added to the wage of Worker i. The remainder is distributed equally.

Remark 1 If there are multiple optimal solutions, the wages from the assignment are applied.

Example 1 Let us apply the Hungarian method to the problem we mentioned in the introduction:

$$A = \begin{bmatrix} 8 & 15 \\ 0 & 10 \end{bmatrix}.$$

First, subtract 15 and 10 from Rows 1 and 2, respectively, and then add 7 to Column 1. The result is

$$B = \begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix} \quad -s = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \quad t = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad \text{and} \quad P^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so, $\phi(A)' = (15 - \frac{7}{2}, 10 - \frac{7}{2}) = (11.5, 6.5)$. The proposed wages compensate Worker 1; he earns just under twice as much as Worker 2.

Example 2 The entries in the following example were generated using a uniform distribution of integers from 1 to 20. The numbers added to each row or column when applying the Hungarian method are bolded. Let

$$A = \begin{bmatrix} 15 & 13 & 6 & 2 \\ 12 & 4 & 6 & 5 \\ 17 & 8 & 5 & 2 \\ 18 & 20 & 11 & 19 \end{bmatrix}.$$

Step 1.

$$\left.\begin{array}{cccc|c} 15 & 13 & 6 & 2 & \mathbf{-15} \\ 12 & 4 & 6 & 5 & \mathbf{-12} \\ 17 & 8 & 5 & 2 & \mathbf{-17} \\ 18 & 20 & 11 & 19 & \mathbf{-20} \end{array}\right. \rightarrow \begin{bmatrix} 0 & -2 & -9 & -13 \\ 0 & -8 & -6 & -7 \\ 0 & -9 & -12 & -15 \\ -2 & 0 & -9 & -1 \end{bmatrix}$$

Step 2

$$\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{6} & \mathbf{1} \\ \hline 0 & -2 & -9 & -13 \\ 0 & -8 & -6 & -7 \\ 0 & -9 & -12 & -15 \\ -2 & 0 & -9 & -1 \end{array} \rightarrow \begin{bmatrix} 0 & -2 & -3 & -12 \\ 0 & -8 & 0 & -6 \\ 0 & -9 & -6 & -14 \\ -2 & 0 & -3 & 0 \end{bmatrix}$$

Step 3 (a) The zeros in the matrix are covered with lines in rows 2 and 4 and column 1. The largest uncovered number is $a^* = -2$.

Step 3 (b) and (c),

$$\begin{array}{cccc|c} \mathbf{-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\ \hline 0 & -2 & -3 & -12 & \mathbf{2} \\ 0 & -8 & 0 & -6 & \mathbf{0} \\ 0 & -9 & -6 & -14 & \mathbf{2} \\ -2 & 0 & -3 & 0 & \mathbf{0} \end{array} \rightarrow \begin{bmatrix} 0 & 0 & -1 & -10 \\ -2 & -8 & 0 & -6 \\ 0 & -7 & -4 & -12 \\ -4 & 0 & -3 & 0 \end{bmatrix}$$

The last matrix already has a zero-based assignment with non-positive entries; therefore, we already have an optimal solution.

In summary, we obtain,

$$B = \begin{bmatrix} 0 & 0 & -1 & -10 \\ -2 & -8 & 0 & -6 \\ 0 & -7 & -4 & -12 \\ -4 & 0 & -3 & 0 \end{bmatrix} \quad -s = \begin{bmatrix} 13 \\ 12 \\ 15 \\ 20 \end{bmatrix} \quad t = \begin{bmatrix} -2 \\ 0 \\ 6 \\ 1 \end{bmatrix} \quad P^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and therefore the proposed wages are,

$$\phi(A) = -\begin{bmatrix} -15 \\ -12 \\ -17 \\ -20 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{(7-2)}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11.75 \\ 10.75 \\ 13.75 \\ 18.75 \end{bmatrix}$$

which adds up to $11.75+10.75+13.75+18.75 = 55.0$. Below, we highlight the solution in the initial matrix in bold,

$$\begin{bmatrix} 15 & \mathbf{13} & 6 & 2 \\ 12 & 4 & \mathbf{6} & 5 \\ \mathbf{17} & 8 & 5 & 2 \\ 18 & 20 & 11 & \mathbf{19} \end{bmatrix}$$

with $z(A, P^*) = 17 + 13 + 6 + 19 = 55$. All wages are higher than the average of the workers' self-assessments. In particular, the second worker receives significantly higher pay than his productivity warrants, because he is assigned to a task that is unattractive to the others.

3 Axiomatic characterization of ϕ

The Hungarian method generates a sequence of matrices, and Marginal Contributions and Equity Axioms propose a natural way to link wages between any two consecutive matrices in this sequence. Ultimately, the sequence of matrices converges to an optimal solution, and the links converge to the proposed wages. Marginal Contributions Axiom states that if the change in efficiency is due to a particular agent, the agent's wage changes accordingly. Equity Axiom states that if all agents are equally responsible for a change in efficiency, everyone's wage must change by the same amount. Weak Nullity Axiom states that when the maximum achievable efficiency is zero, everyone is wage must be zero. Furthermore, if Weak Nullity Axiom is not required, φ is determined only up to an additive constant.

Axiom 1 (Marginal Contributions) $\varphi(A + \alpha R^k) = \varphi(A) + \alpha e_k$

Axiom 2 (Equity) $\varphi(A + \beta C^k) = \varphi(A) + \frac{\beta}{n}\iota$

Definition 4 We will say that φ satisfies Weak Nullity if $z(A, P^*) = 0$ then $\varphi(A) = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$

Axiom 3 φ satisfies Weak Nullity.

Proposition 4 *ϕ satisfies the above axioms.*

Proof To prove that ϕ satisfies marginal contributions axiom, note that there is only one difference when applying the Hungarian method to A and to $A + \alpha R^k$: in Step 1, for the first case, we subtract the maximum value of row k from that row, and in the second case, we subtract that maximum plus α. Thus,

$$\phi(A) + \alpha e_k = \phi(A + \alpha R^k)$$

The same applies to the proof of the second axiom, except now with column k. Finally, let $z(B, P^*) = 0$. When applying the Hungarian method to matrix B, the method terminates immediately: in step 1, the maximum in each row is zero, and the same is true in step 2 for the columns. Thus, $s = 0$ and $t = 0$, and therefore ϕ satisfies the Weak Nullity Axiom. □

Proposition 5 *If φ satisfies the Marginal Contributions, Equity and Weak Nullity Axioms then $\varphi = \phi$.*

Proof Suppose that φ satisfies the axioms, let A be arbitrary and applying the Hungarian method to A. By (4), $Z(B, P^*) = 0$, and by Weak Nullity Axiom, $\varphi(B) = 0$. Since $z(B, P^*) = 0$,

$$\mathbf{0} = \varphi(B) = \varphi(A + \sum_{i=1}^{n} s_i R^i + \sum_{j=1}^{n} t_j C^j) = \varphi(A) + \sum_{i=1}^{n} s_i e_i + \frac{1}{n}\sum_{i=1}^{n} t_j\, \iota$$

where the second equality follows from (3) and the third from Marginal Contributions and Equity Axioms. Finally, using the definition of ϕ,

$$\mathbf{0} = \varphi(A) + s + \widehat{t}\,\iota = \varphi(A) - \phi(A)$$

from where $\varphi(A) = \phi(A)$. □

4 Properties of ϕ

In this section, we establish some properties of the proposed wages. We assume that efficiency is maximized and wages are distributed with that total amount.

Definition 5 We say that the function $\varphi : \mathcal{M} \to R^n$ is efficient if

$$\iota'\varphi(A) = z(A, \overline{P})$$

where $\overline{P}$ is an optimal solution for A.

Proposition 6 *The function ϕ is efficient.*

Proof We just need to add up the terms involving ϕ.

$$\iota'\varphi(A) = -\iota's - \widehat{t}\,\iota'\iota = -\iota's - \iota't = z(A, P^*)$$

where the last equality follows from the Corollary 1. □

Note that for any square matrix A and permutation matrix P, PA is a permutation of the rows of A, and AP is a permutation of the columns of A.

The following statement asserts that if workers exchange efficiencies among themselves, the solution changes accordingly, and if efficiencies are exchanged among jobs, the solution remains unchanged.

Definition 6 We say that φ is symmetric if:

(a) $\varphi(PA) = P\varphi(A)$ and

(b) $\varphi(AP) = \varphi(A)$.

Proposition 7 *The function ϕ is symmetric.*

Proof It must be
(a) If $p_{ij} = 1$, then row i of PA is equal to row j of A; thus, $\phi_i(PA) = -s_j - \widehat{t}$. On the other hand, the entry i of $P\phi(A)$ is given by $\sum_k p_{ik}\phi_k(A) = \phi_j(A)$. Thus, $\varphi_i(PA) = [P\varphi(A)]_i$.
(b) The product AP swaps the columns of A, but $\phi(AP)$ does not depend on the order of the columns in A. □

Assuming that workers' preferences align with their productivity, if all workers prefer different jobs, they are assigned accordingly, and each earns what they produce. The proof is straightforward and has been omitted. In the first stage of the Hungarian method, the maximum value of each row is subtracted, yielding a feasible zero-sum allocation.

A special case occurs when the productivity matrix is a diagonal matrix; that is, if $A = diag(d)$ then $\varphi(A) = d$.

Proposition 8 *For all $A \in \mathcal{M}$ and $\lambda > 0$ it follows that $\phi(\lambda A) = \lambda\phi(A)$.*

Proof If s and t result from applying the Hungarian method to A, then λs and λt result from applying it to λA. □

For example, if we have an efficiency matrix whose values are in euros and we want to convert them to Mexican pesos, the wages before and after the conversion remain the same. If we assume that λ pesos equals 1 euro, then λA is the efficiency matrix in pesos, $\phi(\lambda A)$ represents wages in pesos, $\phi(A)$ represents wages in euros, and $\lambda\phi(A)$ represents wages converted to pesos.

5 Axiomatic independence

In this section, we demonstrate the independence of these axioms. For each axiom, we provide a function that satisfies the other two axioms but not this one.

An example of a function that satisfies Equity and Weak Nullity Axioms but not the Marginal Contributions Axiom. Let $\psi_i^1(A) = \frac{1}{n} z(A, P^*)$ be, then

(a) $\psi^1(A)$ does not satisfy the Marginal Contribution Axiom, since

$$\psi_i^1(A + \alpha R^k) = \frac{1}{n} z(A + \alpha R^k, P^*)$$

for Proposition 1,

$$= \frac{1}{n} \left[z(A, P^*) + \alpha \right] = \frac{1}{n} z(A, P^*) + \frac{\alpha}{n}$$

from where

$$= \psi_i^1(A) + \frac{\alpha}{n} \neq \psi_k^1(A) + \alpha.$$

(b) $\psi^1(A)$ satisfies Equity Axiom, since

$$\psi_i^1(A + \beta C^k) = \frac{1}{n} z(A + \beta C^k, P^*) = \frac{1}{n} \left[z(A, P^*) + \beta \right] = \psi_i^1(A) + \frac{\beta}{n}.$$

(c) $\psi^1(A)$ satisfies Weak Nullity Axiom, since if $z(A, \overline{P}) = 0$ with $\overline{P}$ optimal for A then $\psi_i^1(A) = 0$.

An example of a function that satisfies Marginal Contributions and Weak Nullity Axioms but not the Equity Axiom. Let $w \in R^n$ be such that $\iota' w = n$ and $\psi^2(A) = -s - \widehat{t} w$ then

(a) $\psi^2(A)$ satisfies Marginal Contributions Axiom, since

$$\psi^2(A + \alpha R^k) = -s - \alpha e_k - \widehat{t} w = \psi^2(A) + \alpha e_k$$

(b) $\psi^2(A)$ does not satisfy Equity Axiom

$$\psi^2(A + \beta C^k) = -s - \widehat{t}w - \beta \neq \psi^2(A) + \frac{\beta}{n}\iota$$

(c) $\psi^2(A)$ satisfies Weak Nullity Axiom, since if $z(A, \overline{P}) = 0$ with $\overline{P}$ optimal for A then s and t are vectors of zeros and $\psi^2(A) = 0$.

An example of a function that satisfies Marginal Contributions and Equity Axioms but not Weak Nullity Axiom. Let $\mathbf{0} \neq w \in \mathbb{R}^n$ and $\psi^3(A) = \phi(A) + w$ be, then

(a) $\psi^3(A)$ satisfies Marginal Contributions, since

$$\psi^3(A + \alpha R^k) = \phi(A + \alpha R^k) + w = \phi(A) + \alpha e_k + w = \psi^3(A) + \alpha e_k.$$

(b) $\psi^3(A)$ satisfies Equity Axiom, since

$$\psi^3(A + \beta C^k) = \phi(A + \beta C^k) + w = \phi(A) + \frac{\beta}{n}\iota + w = \psi^3(A) + \frac{\beta}{n}\iota.$$

(c) $\psi^3(A)$ does not satisfy Weak Nullity Axiom, since if P^* is optimal for A then

$$\psi^3(A) = \phi(A) + w = w \neq \mathbf{0}.$$

Thus, the axioms are independent.

References

[1] Cousy K., Lujak, M., Salvatore A., Fernández A., & Giordani, S. (2022). On Balancing Fairness and Efficiency of Task Assignment in Agent Societies. 95-107. https://doi.org/10.1007/978-3-031-18697-4_8.

[2] Kuhn H. W. (1955). The Hungarian method for the assignment problem, Naval Research Logistics Quarterly, 2:83–97.

[3] Kuhn H. W. (2005). The Hungarian method for the assignment problem, Naval Research Logistics, 52(1):7–21. doi 10.1002/nav.20053. (Republished).

[4] Kuhn K. M., & Maleki, A. (2017). Micro-Entrepreneurs, Dependent Contractors, and Instaserfs: Understanding Online Labor Platform Workforces. Academy of Management Perspectives, 31, 183-200. https://doi.org/10.5465/amp.2015.0111.

[5] Lawler E.L. (1976). Combinatorial Optimization: Networks and Matroids, Holt Rinehart and Winston.

[6] Hatfield J.W. and Milgrom P. R.(2005). Matching with Contracts, American Economic Review, vol. 95, no. 4, pp. 913–935.

[7] Samet D., Tauman Y. and Zang I. (1984). An application of the Aumann-Shapley prices for cost allocation in transportation problems. Mathematics of Operations Research, Vol. 9, No. 1, pp. 25-42.

[8] Sanchez Sanchez Francisco (2004). A Salary System for the Assignment Problem, "Economics Bulletin", Vol. 3, No. 5, pp-1-9.

[9] Sattinger M. (1995). Search and the Efficient Assignment of Workers to Jobs. International Economic Review, 36, 283-302. https://doi.org/10.2307/2527197.

[10] Shapley L., and Shubik M. (1972) "The Assignment Game I: The Core," International Journal of Game Theory, 111-130.

[11] Shi S. (2001). Frictional Assignment. I. Efficiency. J. Econ. Theory, 98, 232-260. https://doi.org/10.1006/jeth.2000.2713.

[12] Stovall J. E. (2021). Fair Compensation. arXiv:2109.04583 [econ.TH]. https://doi.org/10.4337/9781788119702.00013

[13] Tauman Y. (1988). The Aumann-Shapley prices: a survey, in: A. Roth, ed., The Shapley Value (Cambridge University Press, Cambridge) pp. 279-304.