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Research Article

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Steady-state nonlinear vibrations of a fractional cantilever beam with an eccentric tip mass element

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Abstract

The paper presents nonlinear steady-state vibrations of a cantilever beam with a eccentric mass element attached to its free end. The Euler-Bernoulli beam theory and the fractional Kelvin-Voigt model of the beam material are assumed. A fractional Caputo derivative of an order of $\mathbf{0} < \boldsymbol{\gamma} \leq \mathbf{0}$ is used to formulate a viscoelastic material law. The beam nonlinear equation of motion is derived using Hamilton's principle. The characteristic equation, modal frequencies, eigenfunction and orthogonality condition are obtained for the linear beam vibrations. The multiple scales method is used to solve the equation of motion of the beam in the vicinity of the first resonance. The first eigenfunction of linear vibrations is used as an approximate solution for the nonlinear vibrations. The impact of order of the fractional derivative on the beam vibrations is studied. The performed analysis shows that the order of the fractional derivative affects the nonlinear oscillation. An increase of the order of a fractional derivative decreases amplitude of vibrations.

Keywords: Fractional Viscoelasticity, Beam Vibration, Nonlinear Vibrations, Method of multiple scales

1 Introduction

In the engineering practice, a cantilever beam with a tip mass element is widely used to model various engineering structures, such as moving cranes, masts, tall buildings,

offshore structures, energy harvesters, accelerometers, military airplane wings, Stockbridge dampers, turbine blades [1–6]. The analysis of a cantilever beam with a mass element attached to its free end have been carried out in many studies [6–23]. In many practical applications, the aforementioned structures require a detailed dynamic analysis. Unfortunately, many studies have omitted (neglected) some important problems, such as the effects of material damping and eccentricity on system dynamics, especially in the analysis of nonlinear vibration problems. Its commonly known, that viscoelastic material properties may significantly change dynamics response oh the studied system. Therefore, an appropriate viscoelastic material model should be employed in a dynamic analysis. Many engineering materials show a low frequency dependence of their damping properties within a wide frequency range [25, 26]. The description of this property is complex and it is usually performed using integer order derivatives [25]. In recent decades, fractional derivatives have been more often used for modeling viscoelastic material properties. [26–29]. These derivatives allows more accurate modeling of viscoelastic material behavior in broad range of frequencies [25, 26]. Similarly, eccentricity of a tip mass element may affect on the dynamics behavior of the analyzed system, thus it is necessary to take the eccentricity into account in dynamic analysis of the analyzed beam. [6, 13, 15, 17–20].

Therefore, in this work steady-state nonlinear vibrations of a fractional cantilever beam with a rigid mass element attached at its free end, whose center of mass does not coincide with the point of its attachment. The study is a continuation and extension of the author's earlier works [22–24]. The first two papers mentioned focused on linear vibration of a fractional viscoelastic cantilever beam with end mass element, whose center of gravity coincide with the point of its attachment. In the first mentioned work, a fractional viscoelastic Kelvin-Voigt material model was used [22], whereas the second work adopted a fractional viscoelastic Zener material model [23]. The third paper deals with numerical analysis nonlinear transient vibrations of a fractional viscoelastic cantilever beam with an eccentric end mass element.

2 Problem formulation

In this paper, nonlinear vibrations of homogeneous cantilever beam with a heavy mass element attached to its free end are analyzed. The beam of length l with tip element of mass m_p and moment of inertia I_c is analyzed. The mass center of the mass element does not coincide with the free end of the beam and there is an eccentricity of distance r (Fig. 1). The analyzed beam has a uniform cross-section A and mass density ρ.

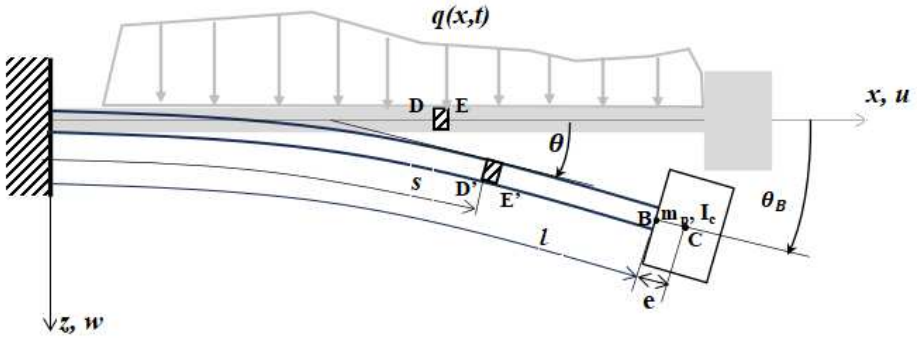


Fig. 1 Schematic of the analyzed beam.

It is assumed that the viscoelastic properties of the beam material can be described using a fractional Kelvin-Voigt model. This model is defined as below [26, 30, 31].

$$\sigma(t) = E\left(\epsilon(t) + \mu_\gamma \mathrm{D}^{(\gamma)}(\epsilon(t))\right) \tag{1}$$

where $\sigma(t)$ and $\epsilon(t)$ are the stress and strain functions of time, μ_γ is a time constant [31], E is the relaxed modulus, t is time and $D^{(\gamma)}(\cdot)$ is the Caputo fractional derivative of the order γ, formulated as [25, 32, 33]

$$\mathrm{D}^{(\gamma)}(f(t)) \equiv \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}(f(t)) \equiv \frac{1}{\Gamma(p-\gamma)} \int_0^t \frac{\mathrm{D}^{(p)}(f(\tau))}{(t-\tau)^{\gamma+1-p}} \mathrm{d}\tau \tag{2}$$

where $\Gamma(p-\gamma)$ is the Euler gamma function [33], $\mathrm{D}^{(p)}(f(\cdot)) = \frac{\partial^p}{\partial t^p}(\cdot)$ is the p-th derivative of a function $f(\cdot)$ with respect to time, $\mathfrak{p}$ is a positive integer number satisfying the inequality $p-1 < \gamma < p$, and $t > 0$.

Many real materials exhibit properties such that the fractional derivative is often in the range of $0 < \gamma \leq 1$ [25, 26], where $\gamma = 1.0$ is in the case of integer order derivative [32, 33].

A thin inextensible beam subjected to large deformation is analyzed and the Euler-Bernoulli theory is applied. Therefore, the rotary inertia and shear deformation of the beam are neglected. The motion of the beam is assumed to be only in the xz-plane and that the gravitational force is perpendicular to this plane, thus the gravitational force has no impact on the beam motion.

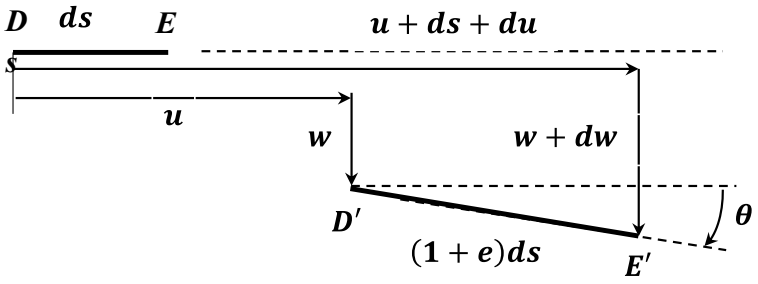


Fig. 2 Deformation of an elementary beam segment.

The analysis of deformation of an elementary beam segment is shown in Fig. 2. The analysis of deformation shows that

$$R \cdot \mathrm{d}\theta = \mathrm{d}s \rightarrow \frac{1}{R} = \kappa = \frac{\partial \theta}{\partial s} = \theta', \text{and } \sin\theta = \frac{\partial w}{\partial s} = w' \Rightarrow \theta = \arcsin(w') \tag{3}$$

where R is a radius of curvature, κ - is a curvature.

Because the beam is assumed to be inextensible, therefore

$$\cos\theta = \frac{u + \mathrm{d}s + \mathrm{d}u - u}{(1+e)\mathrm{d}s} = \frac{\mathrm{d}s + \mathrm{d}u}{(1+e)\mathrm{d}s} = \frac{1+u'}{(1+e)} \text{ , for } e = 0 \quad \cos\theta = 1 + u' \tag{4}$$

and

$$\kappa = \theta' = \frac{\partial}{\partial s}\left(\arcsin\left(w'\right)\right) = \left(1 - \left(w'\right)^2\right)^{-\frac{1}{2}} w'' \approx w''\left(1 + \frac{1}{2}\left(w'\right)^2\right) \tag{5}$$

Based on the beam theory, the strain is

$$\varepsilon\left(t\right) = -z\kappa = -zw''\left(1 + \frac{1}{2}\left(w'\right)^2\right) \tag{6}$$

The extended Hamilton's principle is used to derive the equation of motion [34]

$$\int_{t_1}^{t_2} (\delta T - \delta\Pi + \delta L)\mathrm{d}t = 0 \tag{7}$$

where: δT and $\delta\Pi$- are the variations of total kinetic and potential energy of the system, respectively, δL is the total virtual work done by non-conservative forces.

The total kinetic energy of the system is the sum of the kinetic energy of a beam and the end mass element. In the presented analysis, it is assumed that the effect of the longitudinal velocity of the beam $\dot{u}$ on the total kinetic energy of the system is

small and can be neglected, thus the total system kinetic energy is expressed as

$$T = \frac{1}{2}\int_0^l m\dot{w}^2 \mathrm{d}x + \frac{1}{2}m_p\dot{w}_C^2 + \frac{1}{2}I_C\left(\dot{\theta}_C\right)^2 \tag{8}$$

where over-dots $\dot{(\cdot)}$ and $\ddot{(\cdot)}$ indicate first and second derivatives respect time, m is a mass density per unit length of a beam, m_p is a mass of the end mass element, I_C is a moment of inertia of the end mass element respect its mass center, $\dot{w}$ is the velocity of neutral beam axis point in z direction, $\dot{w}_C$ is the velocity of the center of mass C of the end mass element, $\dot{\theta}_C$ is the angular velocity of the end mass element (Fig. 1).

Substituting the derivative of angle θ (Eq. (3)) with respect to time into equation (8), the kinetic energy can be expressed as

$$\begin{aligned} T = {} & \frac{1}{2}\int_0^l m\dot{w}^2 \mathrm{d}x + \frac{1}{2}m_p\left(-\frac{1}{\sqrt{\left(1-\left(w'_B\right)^2\right)}}\dot{w}'_B r\cdot\sin\theta_B\right)^2 \\ & + \frac{1}{2}m_p\left(\dot{w}_B + \frac{1}{\sqrt{\left(1-\left(w'_B\right)^2\right)}}\dot{w}'_B r\cdot\cos\theta_B\right)^2 + \frac{1}{2}I_C\left(\frac{1}{\sqrt{\left(1-\left(w'_B\right)^2\right)}}\dot{w}'_B\right)^2 \end{aligned} \tag{9}$$

Using the approximate relationship

$$\frac{1}{\sqrt{\left(1-\left(w'\right)^2\right)}} \approx 1 + \frac{1}{2}\left(w'\right)^2$$

The variation of the kinetic energy of the system is expressed as

$$\begin{aligned} \delta T = {} & \delta\left(\frac{1}{2}\int_0^l m\dot{w}^2 \mathrm{d}s + \frac{1}{2}m_p\left(\dot{w}_B^2 + 2\dot{w}_B\dot{w}'_B r\right)\right) + \\ & \delta\left(\frac{1}{2}\left(m_p r^2 + I_C\right)\left(\dot{w}'_B\right)^2\left(1+\left(w'_B\right)^2\right)\right) \end{aligned} \tag{10}$$

According to the assumptions made earlier, the total potential energy of the system is the strain energy of the beam. Using expressions (5) and (6), the variation of the

beam strain energy is expressed as

$$\Pi_b = \frac{1}{2}\int\limits_A \int_0^l E\varepsilon^2 \mathrm{d}A\mathrm{d}x = \frac{1}{2}\int\limits_A \int_0^l E\left(z\right)^2 \kappa^2 \mathrm{d}A\mathrm{d}x =$$
$$\frac{1}{2}\int_0^l EJ\kappa^2 \mathrm{d}x \Rightarrow \delta\Pi_b = \delta\left(\frac{1}{2}\int_0^l EJ\frac{\left(w''\right)^2}{\left(1-\left(w'\right)^2\right)}\mathrm{d}x\right) \approx \tag{11}$$
$$\delta\left(\frac{1}{2}\int_0^l EJ\left(\left(w'\right)^2 + \left(w''\right)^2\left(w'\right)^2\right)\mathrm{d}x\right)$$

where E is the Young's modulus of a beam material, A is a cross-section area of the beam, J is a beam cross-section moment of inertia respect the neutral beam axis.

Virtual work of non-conservative forces is a sum of work done by internal dissipation forces and external forces acting on the beam, namely

$$\delta\ L_{nc} = -\int_0^l \int\limits_A \sigma_{dis}\delta\varepsilon \mathrm{d}A + \int_0^l q\delta w \mathrm{d}s =$$
$$-E'_\gamma J\cdot\int_0^l \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w''\left(1+\frac{1}{2}\left(w'\right)^2\right)\right)\left(1+\frac{1}{2}\left(w'\right)^2\right)\delta\left(w''\right)\mathrm{d}s \tag{12}$$
$$-E'_\gamma J\int_0^l \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w''\left(1+\frac{1}{2}\left(w'\right)^2\right)\right)w''w'\delta\left(w'\right)\mathrm{d}s + \int_0^l q\delta w\mathrm{d}s$$

Substituting expanded and transformed expression for variations of the kinetic energy (Eq. (10)), the strain potential energy (Eq. (11)), and virtual work (Eq. (12)) into Hamilton's extended principle Eq. (7), the following equation of motion and boundary conditions of the analyzed system are obtained

$$m\ddot{w} + EJw'''' + EJ\left(w''''\left(w'\right)^2 + 6w''w'''w' + 3\left(w''\right)^3\right)$$
$$+E'_\gamma J\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w''''\left(1+\frac{1}{2}\left(w'\right)^2\right)+3w''w'''w'+\left(w''\right)^3\right)\cdot\left(1+\frac{1}{2}\left(w'\right)^2\right) \tag{13}$$
$$+E'_\gamma J\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w'''\left(1+\frac{1}{2}\left(w'\right)^2\right)+\left(w''\right)^2 w'\right)w'w'' = q$$

The boundary conditions are established using Hamilton's principle, and for $s = 0$, a beam deflection and slope equals 0, thus

$$w = w' = 0 \tag{14}$$

Whereas, boundary conditions for $s = l$ are as follows

$$
\begin{aligned}
&- m_p\ddot{w}_B - m_p\ddot{w}'_B r + EJ\left(w'''' + w'''\left(w'\right)^2 + 2w'\left(w''\right)^2 - \left(w''\right)^2 w'\right) \\
&+ E'_\gamma J \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w'''\left(1 + \frac{1}{2}\left(w'\right)^2\right) + \left(w''\right)^2 w'\right)\left(1 + \frac{1}{2}\left(w'\right)^2\right) = 0 \\
&- \left(m_p r^2 + I_C\right)\left(\ddot{w}'_B\left(1 + \left(w'_B\right)^2\right) + 2\left(\dot{w}'_B\right)^2 w'_B\right) - \; m_p\ddot{w}_B r - EJ\left(w'' + w''\left(w'\right)^2\right) \\
&- E'_\gamma J\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(w''\left(1 + \frac{1}{2}\left(w'\right)^2\right)\right)\left(1 + \frac{1}{2}\left(w'\right)^2\right)\right) = 0
\end{aligned}
\tag{15}
$$

Introducing dimensionless parameters

$$
\begin{aligned}
&\tau = \sqrt{\frac{EJ}{\rho A\, l^4}}t = c \cdot t, \quad \widetilde{x} = \frac{s}{l}, \quad \widetilde{w} = \frac{w}{l}, \quad \widetilde{\mu}_\gamma = \mu_\gamma\left(\frac{EJ}{\rho A\, l^4}\right)^{\frac{\gamma}{2}} = \mu_\gamma c^\gamma, \quad \widetilde{q} = \frac{ql^3}{EJ} \\
&\alpha = \frac{m_p}{\rho A l}, \quad \beta = \frac{I_C}{\rho A\, l^3}, \quad z = kl, \quad \eta = \frac{r}{l}
\end{aligned}
\tag{16}
$$

and next substituting them into equation (13), the following dimensionless equation of motion is derived

$$
\begin{aligned}
&\frac{\partial^2\widetilde{w}}{\partial\tau^2} + \frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4} + \left(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\left(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)^2 + 6\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}} + 3\left(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)^3\right) \\
&+ \widetilde{\mu}_\gamma\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\left(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\left(1 + \frac{1}{2}\left(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)^2\right) + 3\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}} + \left(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)^2\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)\left(1 + \frac{1}{2}\left(\frac{\partial\widetilde{w}'}{\partial\widetilde{x}}\right)^2\right)\right) \\
&+ \widetilde{\mu}_\gamma\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\left(\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\left(1 + \frac{1}{2}\left(\frac{\partial\widetilde{w}'}{\partial\widetilde{x}}\right)^2\right) + \left(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)^2\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right) = \widetilde{q}
\end{aligned}
\tag{17}
$$

with dimensionless boundary conditions for $\widetilde{x}=0,\ \widetilde{w}=\widetilde{w}'=0$ and for $\widetilde{x}=1$

$$
\begin{aligned}
&-\alpha\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}+\eta\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}\right)+\left(\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}+\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2+\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\right)^2\right)\\
&+\widetilde{\mu}_\gamma\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\left(1+\frac{1}{2}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)\right)\left(1+\frac{1}{2}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)\right)=0\\
&-\left(\alpha\eta^2+\beta\right)\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}\left(1+\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)+2\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau\partial\widetilde{x}}\right)^2\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)\\
&-\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}+\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)\\
&-\mu_\gamma\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\left(1+\frac{1}{2}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)\right)\left(1+\frac{1}{2}\left(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right)^2\right)\right)=0
\end{aligned}
\tag{18}
$$

In this work, the dynamic behavior of the system is investigated in the vicinity of the first resonance. Therefore, the approximate solution is assumed to be in the form of the first eigenfunction of the linearized nonlinear equation Eq. (17). Moreover, the equation of motion is simplified, namely, only terms up to the third order are considered. Thus, grouping linear and nonlinear terms, we obtain

$$
\frac{\partial^2\widetilde{w}}{\partial\tau^2}+\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}+\widetilde{\mu}_\gamma\frac{d^\gamma}{d\tau^\gamma}\left(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\right)+\mathrm{NL}=\widetilde{q}
\tag{19}
$$

Where

$$
\begin{aligned}
\mathrm{NL}=&\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\left(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)^2+6\frac{\partial^2\widetilde{w}\,(1,\tau)}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}}+3\left(\frac{\partial^2\widetilde{w}\,(1,\tau)}{\partial\widetilde{x}^2}\right)^3\\
&+\widetilde{\mu}_\gamma\left(\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\left(\frac{1}{2}\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\left(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)^2+3\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}}+\left(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)^3\right)+\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\left(\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\right)\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right)
\end{aligned}
\tag{20}
$$

The linearized equation (19) has a form

$$
\frac{\partial^2\widetilde{w}}{\partial\tau^2}+\left(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}+\mu_\gamma\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\left(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\right)\right)=\widetilde{q}
\tag{21}
$$

with the following linear boundary conditions, viz for the clamped beam end $\tilde{x}=0$

$$
\widetilde{w}(0,\tau)=\frac{\partial\widetilde{w}(0,\tau)}{\partial\widetilde{x}}=0
\tag{22}
$$

and for $\tilde{x} = 1$

$$\alpha\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}+\eta\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}\right)-\left(\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}+\widetilde{\mu}_\gamma\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\right)=0$$
$$\alpha\eta\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}+\left(\alpha\eta^2+\beta\right)\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}+\left(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}+\widetilde{\mu}_\gamma\frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma}\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\right)=0 \tag{23}$$

The solution to Eqs (21)-(23) is sought in a form of a convergent series of the dimensionless eigenfunctions of the beam

$$\widetilde{w}(\widetilde{x},\tau)=\sum_{n=1}^{\infty}\xi_n(\tau)\,W_n(\widetilde{x}) \tag{24}$$

where $W_n(\widetilde{x})$ is n-th eigenfunction of the beam , $\xi_n(\tau)$ is the n-th time depended generalized coordinate [34].

The functions $W_n(\widetilde{x})$ can be obtained utilizing a well-known procedure, i.e. by solving Eq. (21) with its right-hand side equal to zero (homogeneous equation). Namely, using of separation of variables method, the following equation can be derived

$$W''''(\widetilde{x})-\hat{k}^4W(\widetilde{x})=0 \tag{25}$$

The solution to the equation (25) is sought in a form

$$W(\widetilde{x})=\mathcal{A}\sin\left(\hat{k}\widetilde{x}\right)+\mathcal{B}\cos\left(\hat{k}\widetilde{x}\right)+\mathcal{C}\sinh\left(\hat{k}\widetilde{x}\right)+\mathcal{D}\cosh\left(\hat{k}\widetilde{x}\right) \tag{26}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are arbitrary unknown constants.

Using the first two boundary conditions Eq. (22), the following relations between constant may be found, namely, $\mathcal{A}=-\mathcal{C}$ and $\mathcal{B}=-\mathcal{D}$. Next, using these relationships and the second two boundary conditions Eq. (23), the following system of equations for constants $\mathcal{A}$ and $\mathcal{B}$ is obtained

$$\begin{aligned}&\mathcal{A}\left(\alpha\hat{k}\left(\sin\left(\hat{k}\right)-\sinh\left(\hat{k}\right)\right)+\alpha\hat{k}^2\eta\left(\cos\left(\hat{k}\right)-\cosh\left(\hat{k}\right)\right)-\left(\cos\left(\hat{k}\right)+\cosh\left(\hat{k}\right)\right)\right)\\&+\mathcal{B}\left(\alpha\hat{k}\left(\cos\left(\hat{k}\right)-\cosh\left(\hat{k}\right)\right)-\alpha\hat{k}^2\eta\left(\sin\left(\hat{k}\right)+\sinh\left(\hat{k}\right)\right)-\left(\sin\left(\hat{k}\right)-\sinh\left(\hat{k}\right)\right)\right)=0\end{aligned}$$

$$\begin{aligned}&\mathcal{A}\left(\alpha\eta\hat{k}^2\left(\sin\left(\hat{k}\right)-\sinh\left(\hat{k}\right)\right)+\left(\alpha\eta^2+\beta\right)\hat{k}^3\left(\cos\left(\hat{k}\right)-\cosh\left(\hat{k}\right)\right)+\left(\sin\left(\hat{k}\right)+\sinh\left(\hat{k}\right)\right)\right)\\&+\mathcal{B}\left(\alpha\eta\hat{k}^2\left(\cos\left(\hat{k}\right)-\cosh\left(\hat{k}\right)\right)-\left(\alpha\eta^2+\beta\right)\hat{k}^3\left(\sin\left(\hat{k}\right)+\sinh\left(\hat{k}\right)\right)+\left(\cos\left(\hat{k}\right)+\cosh\left(\hat{k}\right)\right)\right)=0\end{aligned} \tag{27}$$

The system of equations presented above (Eq. (27)) is satisfied if the determinant of the coefficients matrix of the system of equations equals zero. Then, equating the

determinant of the system of equation Eq. (27) to zero, after tedious and complicated mathematical transformations, the characteristic equation of the system can be obtained

$$\begin{aligned}
&-\hat{k}^4\alpha\cdot\beta\cdot\left(1-\cos\left(\hat{k}\right)\cosh\left(\hat{k}\right)\right)+\hat{k}^3\cdot\left(\beta+\alpha\cdot\eta^2\right)\cdot\left(\cos\left(\hat{k}\right)\sinh\left(\hat{k}\right)+\sin\left(\hat{k}\right)\cosh\left(\hat{k}\right)\right)\\
&+2\alpha\cdot\eta\cdot\hat{k}^2\sin\left(\hat{k}\right)\sinh\left(\hat{k}\right)-\alpha\cdot\left(\cos\left(\hat{k}\right)\cosh\left(\hat{k}\right)-\sin\left(\hat{k}\right)\cosh\left(\hat{k}\right)\right)\\
&-\cos\left(\hat{k}\right)\cosh\left(\hat{k}\right)-1=0
\end{aligned} \tag{28}$$

The characteristic equation (28) has of a countable infinitive set of roots $\hat{k}_n$ corresponding to the n-th natural undamped dimensionless frequency of the beam. Then, substituting calculated roots into equations (25) and (27), the expression for eigenfunctions can be formulated as

$$W_n\left(\widetilde{x}\right)=\mathcal{A}_n\left(\left(\sin\left(\hat{k}_n\widetilde{x}\right)-\sinh\left(\hat{k}_n\widetilde{x}\right)\right)-\lambda_n\left(\cos\left(\hat{k}_n\widetilde{x}\right)-\cosh\left(\hat{k}_n\widetilde{x}\right)\right)\right) \tag{29}$$

where

$$\lambda_n=\frac{\left(\alpha\hat{k}_n\left(\sin\left(\hat{k}_n\right)-\sinh\left(\hat{k}_n\right)\right)+\alpha\hat{k}_n^2\eta\left(\cos\left(\hat{k}_n\right)-\cosh\left(\hat{k}_n\right)\right)-\left(\cos\left(\hat{k}_n\right)+\cosh\left(\hat{k}_n\right)\right)\right)}{\left(\alpha\hat{k}_n\left(\cos\left(\hat{k}_n\right)-\cosh\left(\hat{k}_n\right)\right)-\alpha\hat{k}_n^2\eta\left(\sin\left(\hat{k}_n\right)+\sinh\left(\hat{k}_n\right)\right)-\left(\sin\left(\hat{k}_n\right)-\sinh\left(\hat{k}_n\right)\right)\right)}$$

The eigenfunctions (29) must satisfy the orthogonality condition. Using well-known standard procedure (see e.g. [34]), it can be shown that the orthogonality condition has a following form

$$\begin{aligned}
&\int_0^1 W_m\left(\widetilde{x}\right)W_n\left(\widetilde{x}\right)\mathrm{d}\widetilde{x}+\alpha\left(W_n\left(1\right)W_m\left(1\right)+\eta W_n'\left(1\right)W_m\left(1\right)\right)\\
&+\alpha\eta\,W_n\left(1\right)W_m'\left(1\right)+\left(\alpha\eta^2+\beta\right)W_n'\left(1\right)W_m'\left(1\right)=\delta_{mn}
\end{aligned} \tag{30}$$

Utilizing the orthogonality condition Eq. (29) and eigenfunction Eq. (29), coefficients A_n in Eq. (29) can be calculated as

$$\mathcal{A}_n=\frac{1}{\sqrt{\int_0^1\widetilde{W}_n^2\left(\widetilde{x}\right)d\widetilde{x}+\alpha\left(\widetilde{W}_n^2\left(1\right)+2\eta\widetilde{W}_n'\left(1\right)\widetilde{W}_n\left(1\right)\right)+\left(\alpha\eta^2+\beta\right)\widetilde{W}_n'^2\left(1\right)}} \tag{31}$$

Because, the eigenfunction W_n are determined, thus the approximate solution to the problem described by Eq. (19) can be obtained using linear mode approximation. Assuming the first mode of vibrations, namely $\widetilde{w}\left(\widetilde{x},\tau\right)=W_1\left(\widetilde{x}\right)\xi_1\left(\tau\right)$, the following equation is obtained

$$W_1\mathrm{D}^{(2)}\left(\xi_1\right)+W_1''''\xi_1+\widetilde{\mu}_\gamma W_1''''\mathrm{D}^{(\gamma)}\left(\xi_1\right)+W_1''''\xi_1\left(W_1'\xi_1\right)^2+6W_1''\xi_1W_1'''\xi_1W_1'\xi_1+3\left(W_1''\xi_1\right)^3$$
$$+\widetilde{\mu}_\gamma\left(\mathrm{D}^{(\gamma)}\left(\frac{1}{2}W_1''''\xi_1\left(W_1'\xi_1\right)^2+3W_1''\xi_1W_1'''\xi_1W_1'\xi_1+\left(W_1''\xi_1\right)^3\right)+\mathrm{D}^{(\gamma)}\left(W_1'''\xi_1\right)W_1'\xi_1W_1''\xi_1\right)=\widetilde{q} \tag{32}$$

where W_1 is the first eigenfunction of the linearized nonlinear equation Eq. (17).

Next, multiplying both sides of Eq. (32) by W_1, substituting relationship (Eq. (25)) into it

$$W''''\left(\widetilde{x}\right)=\hat{k}^4W\left(\widetilde{x}\right)$$

and integrating from 0 to 1, we obtain equation of the generalized coordinate

$$\mathrm{D}^{(2)}(\xi_1)+\hat{k}^4\xi_1+\widetilde{\mu}_\gamma\hat{k}^4\mathrm{D}^{(\gamma)}(\xi_1)+a_3\xi_1^3+b_3\widetilde{\mu}_\gamma\mathrm{D}^{(\gamma)}\left(\xi_1^3\right)+b_2\widetilde{\mu}_\gamma\mathrm{D}^{(\gamma)}\left(\xi_1\right)\xi_1^2=Q \tag{33}$$

where

$$\begin{aligned}
a_3&=\frac{\int_0^1\hat{k}^4W_1^2\left(W_1'\right)^2\mathrm{d}\widetilde{x}+6\int_0^1W_1''W_1'''W_1'W_1\mathrm{d}\widetilde{x}+3\int_0^1\left(W_1''\right)^3W_1\mathrm{d}\widetilde{x}}{\int_0^1W_1^2\mathrm{d}\widetilde{x}},\\
b_2&=\frac{\int_0^1W_1'''W_1'W_1''W_1\mathrm{d}\widetilde{x}}{\int_0^1W_1^2\mathrm{d}\widetilde{x}},\\
b_3&=\frac{\int_0^1\frac{1}{2}\hat{k}^4W_1^2\left(W_1'\right)^2\mathrm{d}\widetilde{x}+3\int_0^1W_1''W_1'''W_1'W_1\mathrm{d}\widetilde{x}+\int_0^1\left(W_1''\right)^3W_1\mathrm{d}\widetilde{x}}{\int_0^1W_1^2\mathrm{d}\widetilde{x}},\\
Q&=\frac{\int_0^1\widetilde{q}\left(\widetilde{x},\tau\right)W_1\mathrm{d}\widetilde{x}}{\int_0^1W_1^2\mathrm{d}\widetilde{x}}
\end{aligned} \tag{34}$$

An approximate solution to equation Eq. (33) can be obtained using the method of multiple scales [36, 37]. Rossikhin and Shitikova presented an approximated solution to a similar equation in [38]. First, we introduce a small nondimensional parameter ϵ as a formal small parameter, and rewrite equation (33) as

$$\ddot{\xi}+\hat{\omega}^2\xi+\epsilon\mu_\xi\hat{\omega}^2\dot{\xi}^{(\gamma)}+\epsilon\left(a_3\xi^3+b_3\epsilon\mu_\xi\left(\dot{\xi}^3\right)^{(\gamma)}+\epsilon\mu_\xi b_2\dot{\xi}^{(\gamma)}\xi^2\right)=\epsilon q_\xi \tag{35}$$

where it is assumed that $\widetilde{\mu}_\gamma=\epsilon\mu_\xi$, $Q=\epsilon q_\xi$, $q_\xi=q\cos\Omega\tau$, $\hat{\omega}=\hat{k}^2$, Ω is a dimensionless excitation frequency.

The approximate solution of equation (35) for small vibrations can be expressed by expansion with different time scales, as presented below

$$\xi\left(t,\epsilon\right)=\xi_0\left(T_0,T_1\right)+\epsilon\xi_1\left(T_0,T_1\right)+\cdots \tag{36}$$

where

$$T_n=\varepsilon^n\tau\quad(n=0,1,2,3\ldots) \tag{37}$$

are new independent variables, ε is a formal small parameter, $T_0 = \tau$ is the fast timescale and T_1, $T_2 \ldots T_n$ are slow timescale

Next, utilizing the chain rule, the integer order derivatives can be expanded in series of a small parameter ε

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \mathrm{D}_0 + \epsilon\ \mathrm{D}_1 + \epsilon^2 \mathrm{D}_2 + \ldots \tag{38a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = \mathrm{D}_0^2 + 2\epsilon \mathrm{D}_0 \mathrm{D}_1 + \epsilon^2 (\mathrm{D}_1^2 + 2\mathrm{D}_0 \mathrm{D}_2) + \ldots \tag{38b}$$

The fractional order derivative can be expanded into a series of a small parameter, as it has been proposed by Rossikhin and Shitikova [28, 39, 40]

$$\begin{aligned} &\mathrm{D}^\gamma = \frac{\mathrm{d}^\gamma}{\mathrm{d}\tau^\gamma} = \left(\mathrm{D}_0 + \epsilon \mathrm{D}_1 + \epsilon^2 \mathrm{D}_2 + \ldots\right)^\gamma = \\ &\mathrm{D}_0^\gamma + \epsilon\gamma\ \mathrm{D}_0^{\gamma-1} \mathrm{D}_1 + \frac{1}{2}\epsilon^2 \gamma \left((\gamma - 1)\, \mathrm{D}_0^{\gamma-2} \mathrm{D}_1^2 + 2\mathrm{D}_0^{\gamma-1} \mathrm{D}_2\right) \end{aligned} \tag{39}$$

where D_0^γ, $\mathrm{D}_0^{\gamma-1}$, $\mathrm{D}_0^{\gamma-2}$ are the Caputo fractional derivatives with respect to time T_0.

Substituting relationships (36)-(39) into (35) and comparing terms standing at the equal powers of the small parameter ε and limiting the approximate solution by terms of ε^1, a following system of recurrent equations is obtained

for ε^0

$$\mathrm{D}_0^2 \xi_0 + \omega^2 \xi_0 = 0 \tag{40}$$

for ε^1

$$\mathrm{D}_0^2 \xi_1 + \omega^2 \xi_1 = q \cos \Omega\tau - 2\mathrm{D}_0 \mathrm{D}_1 \xi_0 - \mu_\xi \hat{\omega}^2 \mathrm{D}_0^\gamma \xi_0 - a_3 \xi_0^3 \tag{41}$$

whrere, instead of $\hat{\omega}$ is used ω for convenience.

The solution to Eq. (40) is

$$\xi_0 = A\,(T_1)\, \mathrm{e}^{\mathrm{i}\omega T_0} + \bar{A}\,(T_1)\, \mathrm{e}^{-\mathrm{i}\omega T_0} \tag{42}$$

where $A\,(T_1)$ is an arbitrary complex function of time scale T_1 and overbar denotes a complex conjugate function. Since further in the current analysis the expansions for general coordinate ξ is limited by the expressions (41) of order ϵ^1, thus it is assumed that the amplitude ξ_1 is a function of time T_1 only.

Substituting solution of the first approximation Eq. (42) into second approximation Eq. (41), and using fractional derivatives of exponential function in a form presented by Caputo and afterwards by Rossikhin and Shitikova [39, 41]

$$\mathrm{D}_0^\gamma \mathrm{e}^{\mathrm{i}\omega\tau} = (\mathrm{i}\omega)^\gamma\, \mathrm{e}^{\mathrm{i}\omega\tau} + \mathrm{i}\omega \frac{\sin \pi\gamma}{\pi} \int_0^\infty \frac{u^{\gamma-1}}{u + i\omega} \mathrm{e}^{-u\tau} \mathrm{d}u \tag{43}$$

the following equation can be obtained

$$\begin{aligned}
&\mathrm{D}_0^2\xi_1+\omega^2\xi_1=\frac{1}{2}q\left(\mathrm{e}^{\mathrm{i}\Omega T_0}+\mathrm{e}^{-\mathrm{i}\Omega T_0}\right)-2\mathrm{D}_0\left(\mathrm{D}_1\left(A\left(T_1\right)\right)\mathrm{e}^{\mathrm{i}\omega T_0}+\mathrm{D}_1\left(\bar{A}\left(T_1\right)\right)\mathrm{e}^{-\mathrm{i}\omega T_0}\right)\\
&-\mu_\xi\omega^2\left(A\left(T_1\right)(\mathrm{i}\omega)^\gamma\,\mathrm{e}^{\mathrm{i}\omega T_0}+\bar{A}\left(T_1\right)(-\mathrm{i}\omega)^\gamma\,\mathrm{e}^{-\mathrm{i}\omega T_0}\right)\\
&-\mu_\xi\omega^2\left(A\left(T_1\right)\mathrm{i}\omega\frac{\sin\pi\gamma}{\pi}\int_0^\infty\frac{u^{\gamma-1}}{u+\mathrm{i}\omega}\mathrm{e}^{-ut}\mathrm{d}u-\bar{A}\left(T_1\right)\mathrm{i}\omega\frac{\sin\pi\gamma}{\pi}\int_0^\infty\frac{u^{\gamma-1}}{u-\mathrm{i}\omega}\mathrm{e}^{-ut}\mathrm{d}u\right)\\
&-a_3\left(A^3\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega 3T_0}+\bar{A}^3\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega 3T_0}+3A^2\left(T_1\right)\bar{A}\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega T_0}+3A\left(T_1\right)\bar{A}^2\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega T_0}\right)
\end{aligned}\tag{44}$$

The improper integrals in the Eq. (44) can be neglected in certain circumstances [41], e.g. for sufficient large values of τ. In further analysis, it is assumed that improper integrals can be omitted.

In the presented analysis, we assume that vibrations occurs in the vicinity of the first natural frequency ω, namely

$$\Omega=\omega+\epsilon\sigma\tag{45}$$

where σ is a detuning parameter.

The solvability condition of Eq. (44) requires the elimination of secular terms. The secular terms are produced by functions $e^{\pm i\omega T_0}$. Therefore, secular terms vanish in Eq. (44) if

$$\begin{aligned}
&\frac{1}{2}q\left(\mathrm{e}^{\mathrm{i}(\omega+\epsilon\sigma)T_0}+\mathrm{e}^{-\mathrm{i}(\omega+\epsilon\sigma)T_0}\right)-2\left(\mathrm{D}_1\left(A\left(T_1\right)\right)\mathrm{i}\omega\mathrm{e}^{\mathrm{i}\omega T_0}-\mathrm{D}_1\left(\bar{A}\left(T_1\right)\right)\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega T_0}\right)\\
&-\mu_\xi\omega^2\left(A\left(T_1\right)(\mathrm{i}\omega)^\gamma\,\mathrm{e}^{\mathrm{i}\omega T_0}+\bar{A}\left(T_1\right)(-\mathrm{i}\omega)^\gamma\,\mathrm{e}^{-\mathrm{i}\omega T_0}\right)\\
&-a_3\left(3A^2\left(T_1\right)\bar{A}\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega T_0}+3A\left(T_1\right)\bar{A}^2\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega T_0}\right)=0
\end{aligned}\tag{46}$$

Next, omitting secular terms and improper integrals, the equation (44) has the following form

$$\mathrm{D}_0^2\xi_1+\omega^2\xi_1=-a_3\left(A^3\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega 3T_0}+\bar{A}^3\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega 3T_0}\right)\tag{47}$$

The solution to Eq. (47) consists of homogeneous and particular parts, namely

$$\xi_{1h}=B\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega T_0}+\bar{B}\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega T_0}\tag{48a}$$

$$\xi_{1p}=C\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega 3T_0}+\bar{C}\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega 3T_0}\tag{48b}$$

Functions $B\left(T_1\right)$ and $\bar{B}\left(T_1\right)$ may be determined employing initial conditions to Eq. (47), whereas functions $C\left(T_1\right)$ and $\bar{C}\left(T_1\right)$ can be obtained using demand that the particular solution Eq. (48b) must fulfill Eq. (47).

Substituting the particular solution Eq. (48b) into Eq. (47), functions $C\left(T_1\right)$ and $\bar{C}\left(T_1\right)$ can be determined as

$$C\left(T_1\right)=\frac{a_3 A^3\left(T_1\right)}{\left(9\omega^2-1\right)},\ \bar{C}\left(T_1\right)=\frac{a_3 \bar{A}^3\left(T_1\right)}{\left(9\omega^2-1\right)} \tag{49}$$

Consequently, the solution to Eq. (47) may be expressed as

$$\begin{aligned}\xi_1=&B\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega T_0}+\bar{B}\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega T_0}\\&+\frac{a_3}{\left(9\omega^2-1\right)}\left(A^3\left(T_1\right)\mathrm{e}^{\mathrm{i}\omega 3T_0}+\bar{A}^3\left(T_1\right)\mathrm{e}^{-\mathrm{i}\omega 3T_0}\right)\end{aligned} \tag{50}$$

The function $A(T_1)$ can be found solving Eq. (46). We assume that functions $A(T_1)$ and $\bar{A}(T_1)$ are expressed as

$$A\left(T_1\right)=\frac{1}{2}a_r\mathrm{e}^{\mathrm{i}\psi},\ \bar{A}(T_1)=\frac{1}{2}a_r\mathrm{e}^{-\mathrm{i}\psi} \tag{51}$$

where the amplitude a_r is a function of time T_1.

Substituting expression (51) into Eq. (46) and noting that time $T_1=\epsilon T_0$, a following equation is derived

$$\frac{1}{2}q\mathrm{e}^{\mathrm{i}(\sigma T_1-\psi)}-\mathrm{i}\omega a'_r+\omega\ a_r\psi'-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}a_r\mathrm{e}^{\mathrm{i}\frac{\pi\gamma}{2}}-a_3\frac{3}{8}a_r^3=0 \tag{52}$$

where " $'$ " denotes derivative D_1.

In order to remove terms containing explicit time T_1, a new variable is introduced, namely

$$\Phi=\sigma\ T_1-\psi\Rightarrow\psi=\sigma\ T_1-\Phi\Rightarrow\psi'=\sigma-\Phi' \tag{53}$$

Substituting expression (53) into Eq. (52) and separating real and imaginary parts of Eq. (52), the following system of equation can be obtained

$$\begin{cases}\frac{1}{2}q\cos\left(\Phi\right)+\omega a_r(\sigma-\Phi')-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}a_r\cos\frac{\pi\gamma}{2}-a_3\frac{3}{8}a_r^3=0\\ \frac{1}{2}q\sin\left(\Phi\right)-\omega a'_r-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}a_r\sin\frac{\pi\gamma}{2}=0\end{cases} \tag{54}$$

For a steady-state vibrations, we can assume that $a'_r=0$ and $\Phi'=0$, then

$$\begin{cases}\frac{1}{2}q\cos\left(\Phi\right)+a_r\left(\omega\sigma-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}\cos\frac{\pi\gamma}{2}\right)-a_3\frac{3}{8}a_r^3=0\\ \frac{1}{2}q\sin\left(\Phi\right)-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}a_r\sin\frac{\pi\gamma}{2}=0\end{cases} \tag{55}$$

By squaring both sides of Eq. (55) and adding, the following equation for amplitude a_r can be obtained

$$\begin{aligned}&\frac{9}{16}a_3^2a_r^6+4a_r^2\left(\omega^2\sigma^2+\frac{1}{4}\mu_\xi^2\omega^{2(2+\gamma)}-\sigma\mu_\xi\omega^{(3+\gamma)}\cos\frac{\pi\gamma}{2}\right)\\&-3a_3a_r^4\left(\omega\sigma-\frac{1}{2}\mu_\xi\omega^{(2+\gamma)}\cos\frac{\pi\gamma}{2}\right)-q^2=0\end{aligned} \tag{56}$$

The equation above is a polynomial with respect to a_r^2 as shown below

$$A_p u_r^3 + B_p u_r^2 + C_p u_r + D_p = 0 \tag{57}$$

where

$$\begin{aligned} &u_r = a_r^2,\ A_p = \frac{9}{16}a_3^2,\ B_p = 3a_3\left(\frac{1}{2}\mu_\xi \omega^{(2+\gamma)}\cos\frac{\pi\gamma}{2} - \omega\sigma\right), \\ &C_p = 4\left(\omega^2\sigma^2 + \frac{1}{4}\mu_\xi^2\omega^{2(2+\gamma)} - \sigma\mu_\xi\omega^{(3+\gamma)}\cos\frac{\pi\gamma}{2}\right),\ D_p = -q^2 \end{aligned} \tag{58}$$

The equation above may have one, two or three real roots. The number of the roots depends on value of the equation discriminant. If the discriminant is greater than 0, the equation (57) has three distinct real roots, and if the discriminant is less than 0, equation (57) has one real root.

Therefore, solving Eq. (57) amplitudes a_r can be calculated if real roots of Eq. (57) are greater than zero, and then a phase angle Φ can be determined as

$$\Phi = \arctan\left(\frac{\mu_\xi\omega^{(2+\gamma)}\sin\frac{\pi\gamma}{2}}{\frac{3}{4}a_3a_r^2 - 2\left(\omega\sigma - \frac{1}{2}\mu_\xi\ \omega^{(2+\gamma)}\cos\frac{\pi\gamma}{2}\right)}\right) \tag{59}$$

Thus, if Eq. (57) has at least one positive real root, thus amplitudes a_r and phase angles Φ may be determined. Therefore, the approximated value of the generalized coordinate ξ (Eqs (35) and (36)) can be determined for each instant of dimensionless time τ.

3 Numerical calculations and discussion

The performed numerical calculation demonstrate impact of the order of a fractional derivative and an eccentricity parameter η on nonlinear vibrations of the analyzed beam. Sample calculations of dependence of amplitudes a_r (Eq. (51)) and the order of a fractional derivative γ are performed (Eq. (2)). The relationships obtained in the preceding section are used to study the impact of the fractional derivative order and other parameters of the system on its forced vibrations.

The relationship between the order of a fractional derivative γ and detuning parameter σ for varied values of the order of a fractional derivative γ and for two different parameters η are performed. The relationship is obtained solving numerically Eq. (57). The solution is calculated using procedure 'roots' of the 'Matlab' computer program. Because the detuning parameter σ changes the excitation frequency Ω (Eq. (45)), the relationship between the amplitude a_r and the detuning parameter σ describes the frequency response curve for the amplitude a_r on the excitation frequency Ω (e.g. [36]).

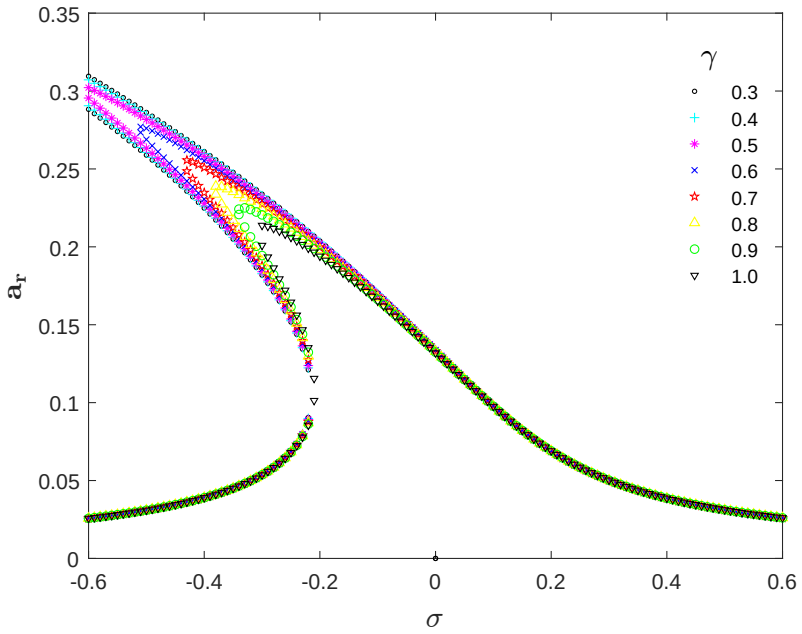


Fig. 3 Dependence of the amplitude a_r on the parameter σ for different orders of the fractional derivative γ, $\mu_\xi = 0.008$, $\eta = 0.05$

The calculations are performed for load $q_\xi = 0.0454$. The results obtained for different values of the order of a fractional derivative γ and eccentricity parameter $\eta = 0.05$ are presented in Figs 3 and 4, whereas results obtained for $\eta = 0.15$ are presented in Figs 5 and 6. It can be seen from figures mentioned earlier that the relationship between vibration amplitude and excitation frequency is such as for the nonlinear Duffing oscillator with soft characteristic of the nonlinear stiffness [36]. Therefore, more than one steady-state system response may occur for one excitation frequency. The response that occurs is determined by the initial conditions imposed on the system [36].

Curves shown in Figs 3- 6, indicate that the increase in the order of a fractional derivative decreases vibration amplitudes a_r. Comparing curves presented in Figs 3 and 4 or curves presented in Figs 5 and 6, it can be noticed that the decrease in amplitude a_r is grater for the larger damping coefficient μ_ξ. Thus, the impact of the order of a fractional derivative γ on amplitudes a_r is greater for higher value of damping coefficient μ_ξ.

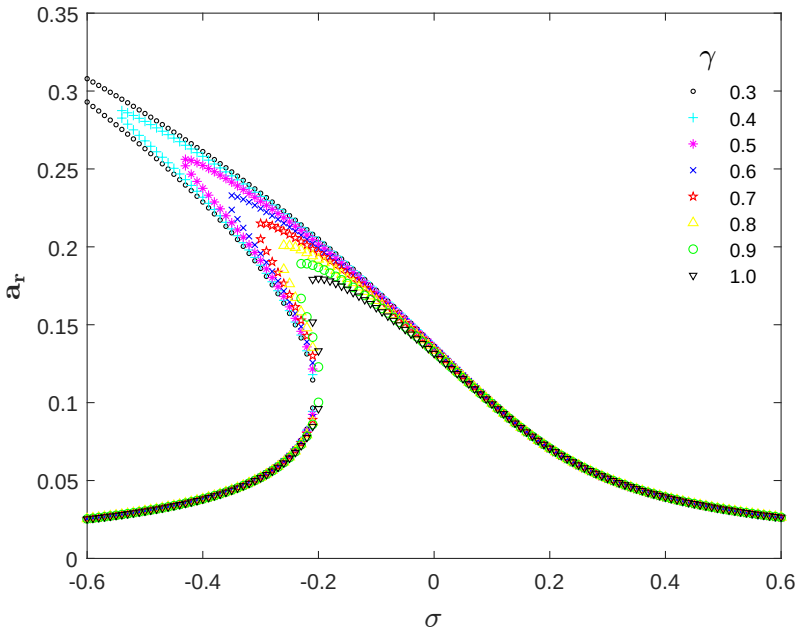


Fig. 4 Dependence of the amplitude a_r on the parameter σ for different orders of the fractional derivative γ, $\mu_\xi = 0.016$, $\eta = 0.05$.

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The obtained calculation results show that curves for the eccentricity coefficient $\eta = 0.15$ (Figs 5 and 6) are more bent then curves for the eccentricity coefficient $\eta = 0.05$ (Figs 3 and 4). Similarly, amplitudes a_r calculated for the given detuning parameter σ are grater for the eccentricity coefficient $\eta = 0.15$ then calculated for the eccentricity parameter $\eta = 0.05$. Furthermore, comparing curves calculated for the eccentricity parameter $\eta = 0.05$ (Figs 3 and 4) and eccentricity parameter $\eta = 0.15$ (Figs 5 and 6), we can notice that the point of inflection of the concave portion of the curves for eccentricity parameter $\eta = 0.15$ (Figs 5 and 6) occurs for smaller value of the detunig parameter σ then for it occurs for eccentricity parameter $\eta = 0.05$ (Figs 3 and 4).

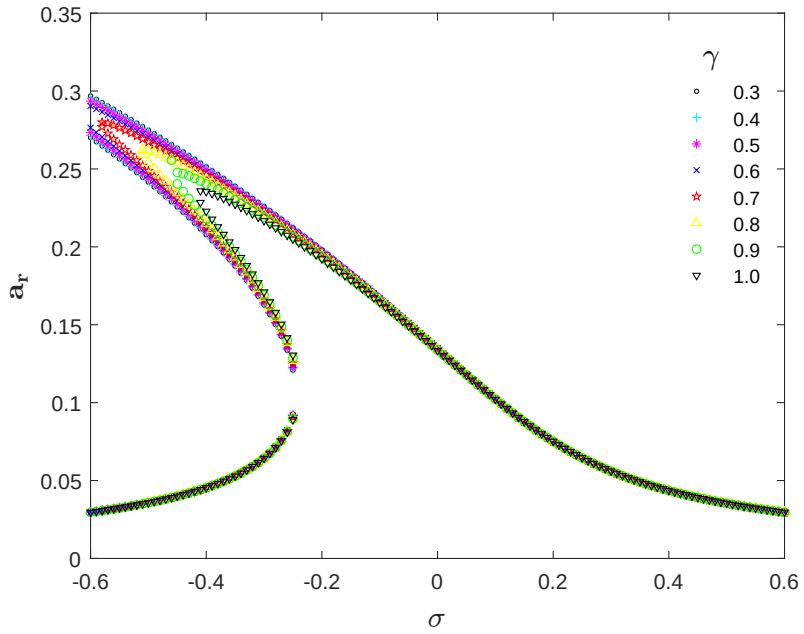


Fig. 5 Dependence of the amplitude a_r on the parameter σ for different orders of the fractional derivative γ, $\mu_\xi = 0.008$, $\eta = 0.15$.

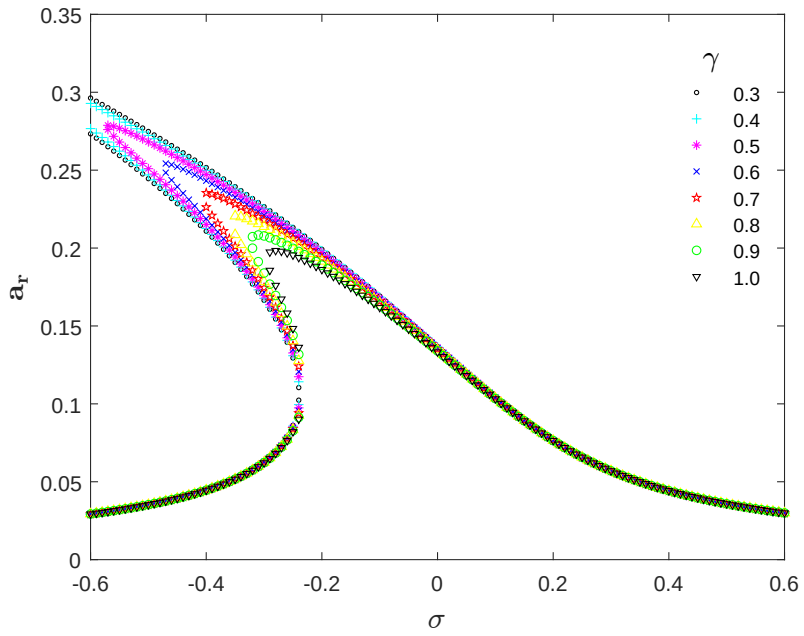


Fig. 6 Dependence of the amplitude a_r on the parameter σ for different orders of the fractional derivative γ, $\mu_\xi = 0.016$, $\eta = 0.15$.

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4 Conclusions

In this paper, the study nonlinear steady-state vibrations of a fractional cantilever beam with an eccentric mass element is presented. The fractional Kelvin-Voigt viscoelastic material model is assumed as a beam material. The equation of motion of a cantilever beam with an eccentric tip mass is derived using the Hamilton's principle. The characteristic equation, modal frequencies, eigenfunction and orthogonality condition are obtained for the linear beam vibrations.

The nonlinear equation for generalized coordinate of the sytem is solved with the help of method of multiple scales. From the first order approximation, expression for approximated values the generalized coordinate is obtained, which allows the calculation of the nonlinear forced vibrations of the analyzed beam in the steady-state.

The analysis shows that the order of the fractional derivative affects the nonlinear vibrations. An increase in the order of the fractional derivative decreases the amplitude of oscillation. The decrease is greater for the larger damping coefficient μ_ξ. Moreover, the value of the eccentricity parameter η affects nonlinear forced vibrations in the steady state. Namely, vibration amplitudes are greater for the larger eccentricity parameter η and the amplitude-frequency curves are more bent for larger eccentricity coefficient η.

Declaration of competing interest

The author declare that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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