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Numerical homogenization of a heat transfer problem with phase transitions in heterogeneous frozen soils

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Abstract

Numerical homogenization of the two-phase Stefan problem in a heterogeneous frozen soil is considered. Effective coefficients are computed from cell problems, but on the coarse grid the phase-transition front passes inside macro-cells, and the homogenized problem has to be closed, i.e., the effective conductivity and the latent heat of a partially thawed cell have to be specified. Three closures are compared: a table of effective coefficients versus the cell-average temperature, a nodal scheme with frozen and thawed effective tensors, and h-smoothing. The comparison is made for the problem of thawing of soil beneath a house with a random conductivity field in the two- and three-dimensional formulations. It is shown that the accuracy of the homogenized model measured in temperature and in the front position is ranked differently: h-smoothing is more accurate in temperature, but, being non-conservative, it overestimates thawing. For a constant phase-transition temperature the table reduces to two tensors, while for a variable one it is 1.5–2.6 times more accurate than the two tensors in the front position.

Keywords: Stefan problem, permafrost, numerical homogenization, effective coefficients, phase transition

1 Introduction

Permafrost occupies a large part of the territory of Russia, and its thermal state determines the bearing capacity of the foundations of buildings, pipelines and roads.

Long-term monitoring shows that the permafrost temperature is rising, the active-layer thickness is increasing, and at the southern boundary of the permafrost zone the permafrost table is deepening [1, 2]. Thawing of a frozen foundation beneath a heated building leads to differential settlement and deformations of structures [3]; therefore, design in permafrost regions involves a thermal forecast: for a given climate and thermal action of the structure, the soil temperature and the depth of thawing beneath the building are determined over the whole service life [4, 5].

Heat transfer in soil with a phase transition of pore moisture is described by the Stefan problem, that is, by the heat conduction equation with an unknown moving interface between the frozen and thawed zones, at which latent heat is released or absorbed [6, 7]. In the multidimensional case it is more convenient not to single out the front explicitly, but to write the problem in enthalpy form and solve it by the whole-domain (apparent heat capacity) method: the enthalpy jump at the front is smoothed over a narrow temperature interval, and the problem reduces to a quasilinear parabolic equation with discontinuous coefficients in the whole domain [8–10]. The accuracy of such schemes depends on the width of the smoothing interval and on how it is matched with the grid; in [11] smoothing of the coefficients at the scale of the grid element is proposed, and in [12] a scheme that conserves energy at large time steps is proposed.

Real soils are heterogeneous: the conductivity varies at the scale of layers and lenses (tens of centimeters), while the thawed zone beneath a building has to be predicted at the scale of the building itself (meters and tens of meters). A computation on a grid resolving all layers, especially in the three-dimensional formulation and for many realizations of the soil properties, requires large computational costs. To build reduced-order models, homogenization [13, 14] and multiscale methods [15–17] are used, in which the problem is solved on a coarse grid with effective coefficients or multiscale basis functions computed from local problems. For the Stefan problem in frozen soils such approaches are developed, in particular, on the basis of the generalized multiscale finite element method (GMsFEM) and its discontinuous and meshfree variants: for the heat transfer problem with phase change [18], for artificial ground freezing [19–21], for the soil-thawing effect of piles [22], for the coupled filtration and heat transfer problem in frozen soils [23, 24].

Unlike the linear heat conduction problem, homogenization of the Stefan problem requires a closure: on the coarse grid the front passes inside macro-cells, and one has to decide which phase, which conductivity and which latent heat to assign to a cell that is partly frozen and partly thawed. The effective conductivity of such a cell depends on how the thawed zone is arranged with respect to the layers, and the latent heat depends on what volume fraction of the cell has already thawed. In practice three closures are used. In the *table approach*, effective coefficients are precomputed for a set of cell states and interpolated by the cell-average temperature, as in numerical homogenization methods for nonlinear problems [16, 25, 26]. In the *nodal two-tensor scheme* the thawed water fraction is taken out of the integral of the cell problem, so that only the frozen and thawed effective tensors are needed, and the phase is determined at the nodes, as in whole-domain enthalpy schemes [7, 8, 10]. In *h-smoothing* [11] the fraction of the thawed phase in the element is computed exactly from the linear temperature, i.e., the transition is smoothed at the scale of the grid.

These closures rely on different assumptions about the state of a cell containing the front, but, as far as the author knows, they have not been compared with each other under identical conditions, and usually only the temperature error is estimated, not the position of the phase boundary, which is exactly what is needed for the settlement forecast.

In this work the three closures are compared in a single implementation with one macrogrid, one time step and common effective tensors for the problem of thawing of soil beneath a heated building with a random conductivity field in the two- and three-dimensional formulations. Macrogrid solutions are compared with reference solutions on a fine grid; in addition to the temperature, the position of the phase boundary, the depth of thawing beneath the building and the energy balance are estimated. It is shown that the ranking of the methods by temperature and by front position differs, and the reason for this discrepancy is identified.

The paper is organized as follows. Section 2 states the mathematical model (the two-phase Stefan problem in a heterogeneous soil) and the formulation of the problem of thawing beneath a house. In Section 3 the cell problems and the three closures of the homogenized problem are described. Numerical results for the two- and three-dimensional problems are presented in Section 4, and the conclusions are formulated in Section 5.

2 Problem statement

Consider heat transfer in a water-saturated soil with a phase transition of pore moisture in the domain $\Omega = [0, L]^d$, $L = 12$ m, $d = 2, 3$. At time t the domain is split into thawed and frozen zones

$$\Omega^+(t) = \{\mathbf{x} \in \Omega : T(\mathbf{x}, t) > T^*\}, \quad \Omega^-(t) = \{\mathbf{x} \in \Omega : T(\mathbf{x}, t) < T^*\},$$

separated by the moving phase-transition boundary $S(t) = \{\mathbf{x} \in \Omega : T(\mathbf{x}, t) = T^*\}$, where T^* is the phase-transition temperature. In each zone the heat conduction equation holds

$$C^\pm \frac{\partial T}{\partial t} = \nabla \cdot (\lambda^\pm(\mathbf{x}) \nabla T), \quad \mathbf{x} \in \Omega^\pm(t),$$

and on the front $S(t)$ the temperature is continuous, $T = T^*$, and the Stefan condition holds

$$\left[\lambda \frac{\partial T}{\partial n} \right]_S = \lambda^- \frac{\partial T^-}{\partial n} - \lambda^+ \frac{\partial T^+}{\partial n} = \rho L v_n,$$

where $\mathbf{n}$ is the normal to S directed into the frozen zone, v_n is the normal velocity of the front, and ρL is the latent heat of the phase transition per unit volume of soil: the jump of the heat flux at the front is spent on melting (freezing) of the pore moisture [6, 7].

To avoid tracking the front explicitly, the problem is solved in the whole domain Ω by the whole-domain (apparent heat capacity) method [8, 9]. We introduce the fraction of thawed water $\varphi(T)$, equal to 0 for $T < T^*$ and 1 for $T > T^*$, and smooth

it over the interval $[T^* - \Delta, T^* + \Delta]$ (the phase transition zone) [5, 8, 11]:

$$\varphi_\Delta(T) = \min\left(1, \max\left(0, \frac{T - T^* + \Delta}{2\Delta}\right)\right).$$

Then the temperature in the whole domain satisfies the quasilinear parabolic equation

$$(C(\varphi_\Delta) + \rho L \varphi'_\Delta(T)) \frac{\partial T}{\partial t} = \nabla \cdot (\lambda(\varphi_\Delta, \mathbf{x}) \nabla T), \quad \mathbf{x} \in \Omega, \quad 0 < t \leq t_{\max}, \quad (1)$$

$$C(\varphi) = C^- + \varphi(C^+ - C^-), \quad \lambda(\varphi, \mathbf{x}) = \lambda^-(\mathbf{x}) + \varphi(\lambda^+(\mathbf{x}) - \lambda^-(\mathbf{x})),$$

in which the latent heat enters the effective heat capacity over the interval 2Δ . The width Δ determines the accuracy: for large Δ the front is smeared, and for small Δ a fine grid and a small time step are required, otherwise the front “jumps over” the nodes and energy is not conserved [11, 12].

The coefficients of equation (1) are determined by the soil composition, which is treated as a water-saturated porous medium with porosity $m = 0.2$, whose pores are filled with ice in the frozen zone and with water in the thawed zone. The volumetric heat capacities and conductivities of the zones are given by the mixing rule [18, 19]:

$$\begin{aligned} C^- &= (1 - m) c_{sc} \rho_{sc} + m c_{ice} \rho_{ice}, & C^+ &= (1 - m) c_{sc} \rho_{sc} + m c_w \rho_w, \\ \lambda^- &= (1 - m) \lambda_{sc} + m \lambda_{ice}, & \lambda^+ &= (1 - m) \lambda_{sc} + m \lambda_w, \end{aligned}$$

where the subscripts *sc*, *ice*, *w* refer to the soil skeleton, ice and water. The soil heterogeneity is specified by a random conductivity field of the skeleton $\lambda_{sc}(\mathbf{x})$; the skeleton heat capacity is constant. The field λ_{sc} is constructed by the Karhunen–Loève expansion [27] of a Gaussian field with covariance $\sigma^2 \exp(-\sum_i (\Delta x_i / \ell_i)^2)$, $\sigma^2 = 2$, $\ell = 0.6L$ horizontally and $0.05L = 0.6$ m vertically, using the 200 largest modes; the realization is linearly mapped onto $[1, 3]$ W/(m·K). Horizontal layers with a vertical scale of about 0.6 m are obtained (Fig. 1). It is assumed that $\lambda_{ice} = 2.21$, $\lambda_w = 0.56$ W/(m·K), $C^- = 1.842$, $C^+ = 2.302$ MJ/(m³·K), $\rho L = 125.5$ MJ/m³, $T^* = 0$ °C (the case of a variable $T^*(\mathbf{x})$ across layers is considered separately in Sec. 4). The conductivity jump at the phase transition $\lambda^- - \lambda^+ = m(\lambda_{ice} - \lambda_w)$ is the same over the whole domain.

Equation (1) is supplemented with boundary and initial conditions modeling the soil beneath a heated building. Let $\Gamma_h = \{x_d = L, 3 \leq x_i \leq 9, i < d\}$ denote the surface under the house, $\Gamma_a = \{x_d = L\} \setminus \Gamma_h$ the open surface, and Γ_0 the rest of the boundary. Then

$$-\lambda \frac{\partial T}{\partial n} = \begin{cases} h_h (T - T_h), & \mathbf{x} \in \Gamma_h, \\ h_a (T - T_a(t)), & \mathbf{x} \in \Gamma_a, \\ 0, & \mathbf{x} \in \Gamma_0, \end{cases} \quad T|_{t=0} = -3 \text{ °C}, \quad (2)$$

$h_h = 8.7$, $h_a = 14$ W/(m²·K), $T_h = 10$ °C, $T_a(t) = -10 + 30 \cos(2\pi(t - 287)/365)$ °C (t in days). The computation is performed for one year with a time step $\tau = 1$ day.

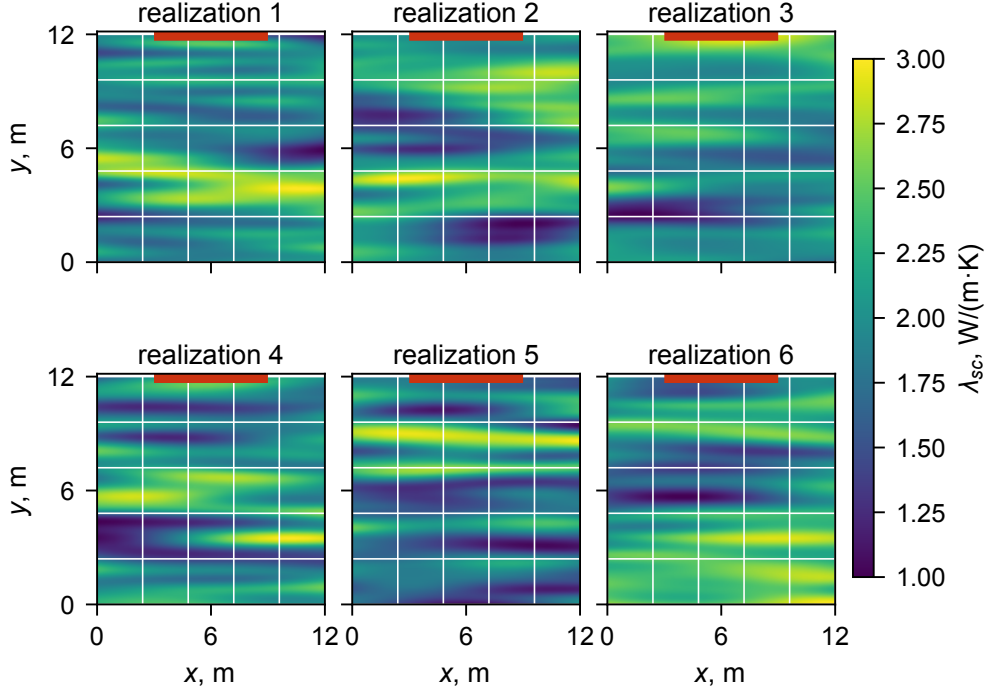


Fig. 1 Realizations of the field λ_{sc} (2D); white lines show the macrogrid 5×5 , the house is shown in red

The solution of problem (1)–(2) on a fine grid resolving the layers is used below as the reference solution. The homogenized problems constructed in Sec. 3 are solved on macrogrids of N^d squares (cubes) of size $H = L/N$ split into simplices, $N = 5$ –40 in 2D and 5–20 in 3D (Fig. 2, 3).

3 Numerical implementation

The homogenized problem is solved on a macrogrid whose cells are much larger than the soil layers. Away from the phase-transition front each macro-cell is either frozen or thawed, and the effective coefficients are those of the corresponding phase. Near the front, however, a macro-cell is partly frozen and partly thawed, and the macroscopic unknowns do not tell which part of the cell has thawed. Therefore, the homogenized problem has to be closed: one has to specify which fraction of the thawed phase, and hence which conductivity and latent heat, is assigned to a cell containing the front. The three closures compared in this paper differ exactly in this choice (Fig. 4). The table approach uses one value of the thawed fraction per element, computed from the element-average temperature; the nodal scheme determines the phase at the nodes; h-smoothing computes the exact thawed fraction of the element for a linear temperature. On coarse cells, where the temperature varies within a cell much more than over the phase-transition interval, the three closures assign noticeably different

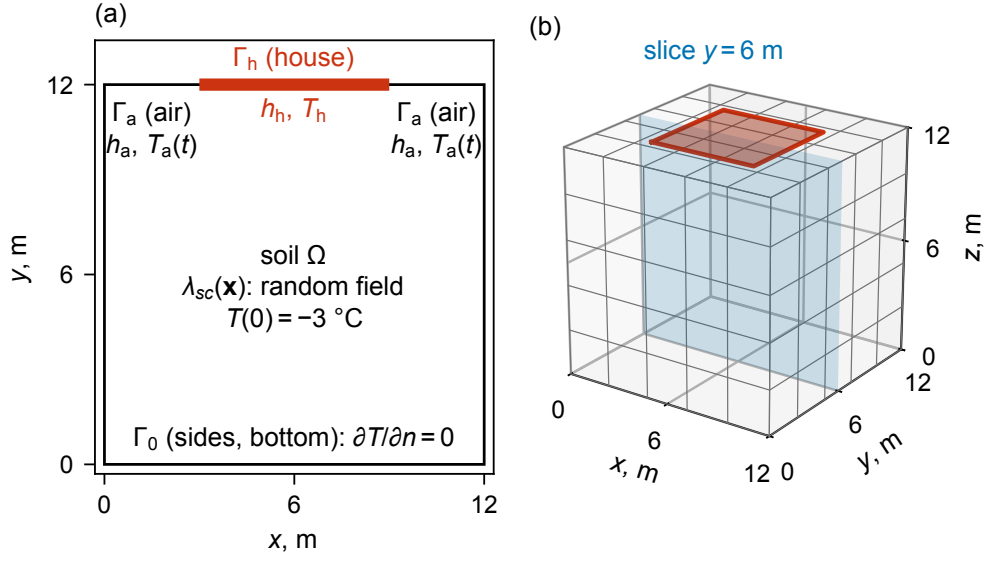


Fig. 2 Computational domain: (a) 2D: domain 12 × 12 m and boundary conditions; (b) 3D: cube 12³ m, house 6 × 6 m (red), macrogrid 5³ and slice plane $y = 6$ m (blue)

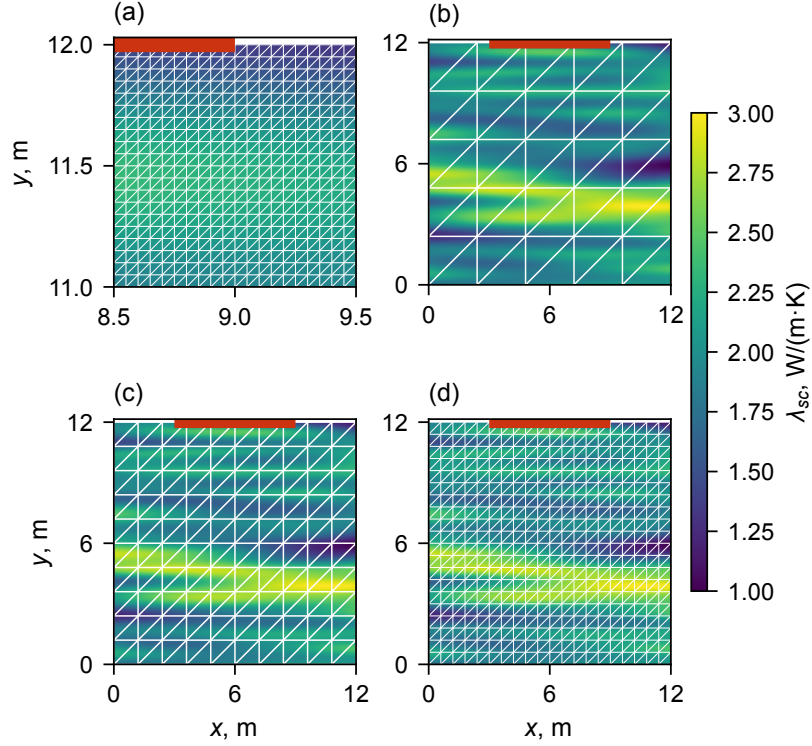


Fig. 3 Grids (2D, realization 1): (a) fragment 1 × 1 m of the fine reference grid ($h = 5$ cm) at the edge of the house; (b)–(d) macrogrids 5², 10², 20² ($H = 2.4, 1.2, 0.6$ m) overlaid on the field λ_{sc}

thawed fractions; on fine cells they almost coincide. Below, the cell problems for the effective coefficients are described first, and then the three closures (Cases 1–3).

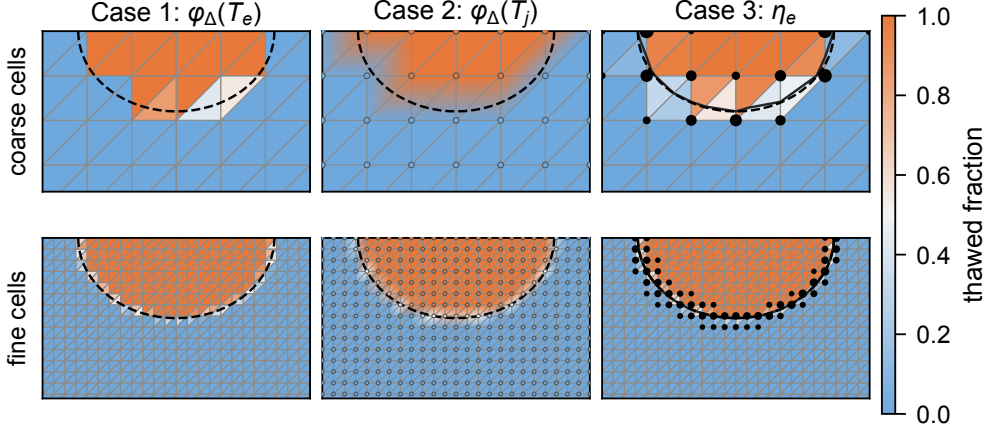


Fig. 4 Fraction of the thawed phase that each method uses in cells containing the front: Case 1 (table), $\varphi_\Delta(\bar{T}_e)$ per element; Case 2 (nodal scheme), $\varphi_\Delta(T_j)$ at the nodes; Case 3 (h-smoothing), exact thawed fraction η_e per element (dots: nodal latent-heat capacity, solid line: zero level of the linear temperature). Notation as in Cases 1–3 below; dashed line: the true front. Top: coarse cells (temperature variation in a cell ≈ 2.3 °C), bottom: fine cells (≈ 0.6 °C)

Computing problem (1) on a grid resolving layers of thickness of order 0.6 m over the whole domain requires a large number of unknowns, especially in 3D. In numerical homogenization [13–16] the small-scale heterogeneity is accounted for not by the grid but by effective coefficients, which are computed from local cell problems, and the problem is solved on a coarse macrogrid. For the Stefan problem the homogenized problem reads

$$\frac{\partial \bar{H}(\bar{T})}{\partial t} = \nabla \cdot (K_{\text{eff}} \nabla \bar{T}), \quad \mathbf{x} \in \Omega, \quad (3)$$

with the boundary and initial conditions (2), where $\bar{T}$ is the macroscopic temperature, $\bar{H}$ is the averaged enthalpy, and K_{eff} is the effective conductivity tensor.

For the discretization of problem (3) the domain is split into N^d cubes (macro-cells Y) of size $H = L/N$; each cube is split into $d!$ simplices e . The temperature is sought in the space of continuous piecewise linear functions with basis $\{\psi_j\}$, with the mass lumped at the nodes: $m_j = \sum_{e \ni j} |e|/(d+1)$. The stiffness matrix and the right-hand side are

$$A(K)_{ij} = \sum_e \int_e K_e \nabla \psi_j \cdot \nabla \psi_i \, d\mathbf{x} + \int_{\Gamma_h \cup \Gamma_a} h \psi_j \psi_i \, ds, \quad F_i = \int_{\Gamma_h \cup \Gamma_a} h T_{\text{out}} \psi_i \, ds,$$

where h and T_{out} equal h_h , T_h on Γ_h and h_a , $T_a(t)$ on Γ_a , and K_e is the effective tensor in element e . In time an implicit scheme with step τ is used.

It remains to determine the effective coefficients in each element. For this purpose, in each macro-cell, for the field $\lambda_\varphi = \lambda^- + \varphi(\lambda^+ - \lambda^-)$ with a given thawed water

fraction φ , the problems

$$\nabla \cdot (\lambda_\varphi \nabla u_k) = 0, \quad k = 1, \dots, d, \quad K_{\text{eff}}(\varphi) = \langle \lambda_\varphi \nabla u \rangle_Y \langle \nabla u \rangle_Y^{-1}$$

are solved by the finite volume method on voxels (5×5 cm in 2D; $20 \times 20 \times 5$ cm in 3D). The tensors $K^- = K_{\text{eff}}(0)$ and $K^+ = K_{\text{eff}}(1)$ are called the frozen and thawed tensors.

If a macro-cell is entirely frozen or entirely thawed, then K_{eff} equals K^- or K^+ , and $\bar{H}(\bar{T}) = H(\bar{T})$, since $C^\pm$ and ρL are constant. If, however, the front passes through the cell, then its effective conductivity and its latent heat content depend on which part of the cell has thawed and on how the thawed zone is arranged with respect to the layers. At the macro level this information is unavailable, and the homogenized problem (3) has to be closed, that is, K_{eff} and $\bar{H}$ in a cell containing the front have to be expressed through the macroscopic unknowns. Three closures are considered below. In all of them the same macrogrid, the same time step and the same tensors $K^\pm$ are used; the cases differ only in how the conductivity and the latent heat are specified in a macro-cell with a phase transition.

Case 1: table of effective coefficients. For each macro-cell a table $K_{\text{eff}}(\varphi_k)$, $\varphi_k \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, is precomputed, and the averaged enthalpy $H_{\text{eff}}(\bar{T}) = \langle H(\bar{T}) \rangle_Y$ is computed for a uniform cell temperature $\bar{T}$ (here it coincides with $H(\bar{T})$, since $C^\pm$ and ρL are constant). In element e the coefficients are taken at the average temperature $\bar{T}_e$ by linear interpolation in the table:

$$K_e = K_{\text{eff}}(\varphi_\Delta(\bar{T}_e)), \quad \sum_{e \ni j} \frac{|e|}{d+1} \frac{H_\Delta(\bar{T}_e^{n+1}) - H_\Delta(\bar{T}_e^n)}{\tau} + (A(K)T^{n+1})_j = F_j^{n+1}, \quad (4)$$

where $\varphi_\Delta(T) = \min(1, \max(0, (T + \Delta)/2\Delta))$ is the transition smoothed over $\pm\Delta$, $H_\Delta = e + \rho L \varphi_\Delta$, and $e(T) = C^- \min(T, 0) + C^+ \max(T, 0)$ is the sensible heat. The method assumes that the state of the cell is described by a single number, the average temperature $\bar{T}_e$, which determines the phase.

Case 2: nodal scheme with two tensors. If $C(T)$ and $K(T)$ vary smoothly and the cell is small, then the thawed water fraction in the cell is almost constant, and φ is taken out of the integral of the cell problem: $K_{\text{eff}}(\varphi) \approx K^- + \varphi(K^+ - K^-)$. Then only two cell problems are needed, and the enthalpy and the phase fraction are associated with the nodes:

$$m_j \frac{H_\Delta(T_j^{n+1}) - H_\Delta(T_j^n)}{\tau} + (A(K)T^{n+1})_j = F_j^{n+1}, \quad (5)$$

$$K_e = K^- + \bar{\varphi}_e(K^+ - K^-), \quad \bar{\varphi}_e = \frac{1}{d+1} \sum_{j \in e} \varphi_\Delta(T_j).$$

Case 3: h-smoothing. Following [11], the temperature in the element is assumed linear, and instead of smoothing in T with parameter Δ the exact fraction of the thawed volume of the element η_e as a function of its nodal temperatures is used, i.e., smoothing at the scale of the grid h . For a simplex with nodal values $t_0, \dots, t_d$ of the

same sign, except the vertex i ,

$$\eta_e = \prod_{j \neq i} \frac{t_i}{t_i - t_j} \quad (t_i > 0), \quad \eta_e = 1 - \prod_{j \neq i} \frac{t_i}{t_i - t_j} \quad (t_i < 0),$$

and in 3D, with two thawed vertices, the thawed part is a prism whose volume is computed by splitting it into three tetrahedra. Further,

$$K_e = K^- + \eta_e(K^+ - K^-), \quad c_j = \sum_{e \ni j} \left(\frac{|e|}{d+1} (C^- + \eta_e(C^+ - C^-)) + \rho L |e| \frac{\partial \eta_e}{\partial T_j} \right),$$

$$\left(\frac{1}{\tau} \text{diag } c + A(K) \right) T^{n+1} = \frac{1}{\tau} \text{diag } c T^n + F^{n+1}, \quad (6)$$

the coefficients are evaluated at T^n , so that one linear system is solved per step. Since the latent heat is linearized over the previous time level, this scheme does not conserve energy exactly.

To separate the effect of the exact phase fraction from that of the linearization, a conservative variant of h-smoothing, referred to as Case 3c, is also considered. It uses the same η_e and K_e , but the latent heat enters as the increment of the thawed volume over the time step:

$$m_j \frac{e(T_j^{n+1}) - e(T_j^n)}{\tau} + \rho L \sum_{e \ni j} \frac{|e|}{d+1} \frac{\eta_e(T^{n+1}) - \eta_e(T^n)}{\tau} + (A(K^{n+1})T^{n+1})_j = F_j^{n+1}, \quad (7)$$

where $K_e^{n+1} = K^- + \eta_e(T^{n+1})(K^+ - K^-)$. The nonlinear system (7) is solved by Newton's method, as in Cases 1 and 2, and the energy balance holds up to the solver tolerance.

All three cases use the tensors $K^\pm$; in Case 1 the intermediate $K_{\text{eff}}(\varphi_k)$ are used in addition. The difference lies in (i) the point where the phase is determined (element center, nodes, or exact volume fraction), (ii) the width of the smoothing (Δ in temperature or the element size), and (iii) the linearization (Newton or coefficients from the previous step).

4 Numerical results

The computations are performed in 2D on macrogrids with $N = 5, 10, 20, 40$ for 20 realizations of the field λ_{sc} , and in 3D on macrogrids with $N = 5, 10, 20$ for 6 realizations. The smoothing width in Cases 1–2 is $\Delta = 0.5$ °C ($\Delta = 0.1, 0.25$ and 1 were also tested); the table has 5 values of φ (2, 3 and 9 were also tested). Newton iterations stop when $\max |\delta T| < 10^{-6}$ °C or when the residual decreases by a factor of 10^7 .

To compare with the reference, the macro solution is interpolated onto the nodes of the fine grid, after which the following quantities are computed: the relative temperature error in the $L^2(0, T; L^2(\Omega))$ norm, the fraction of the volume with a mismatched phase (relative to the volume of the thawed zone of the reference, averaged over

slices), the mean error of the thawing depth under the center of the house, the relative energy imbalance and the run time. A pairwise comparison of the methods on the same realizations is also given.

Consider first the temperature error (Tables 1, 2, Fig. 5). On the coarse grid h-smoothing is the most accurate; under refinement in 2D the nodal method catches up with it and at $H = 0.3$ m overtakes it, while in 3D h-smoothing remains the best; the table is not better than the nodal method on any grid. The ranking of the methods is preserved in all realizations. The dependence $K_{\text{eff}}(\varphi)$ is linear to within 0.02 %, so the number of table nodes and the smoothing width Δ hardly affect the result. The cost of the methods is given in Table 3: h-smoothing is several times cheaper.

Table 1 2D, 20 realizations: temperature error $L^2(0, T; L^2)$ (mean $\pm$ std. dev.), phase mismatch, error of the thawing depth under the house; Cases 1–3: table, nodal scheme, h-smoothing; the best value is in bold

grid	H , m	error in T , % $\downarrow$			phase mismatch, % $\downarrow$			depth error, m $\downarrow$		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
5^2	2.4	28.73 ± 1.32	26.13 ± 1.71	22.63 ± 1.59	74.0	96.7	84.6	0.49	0.85	0.74
10^2	1.2	11.08 ± 0.62	9.06 ± 0.70	8.45 ± 0.75	29.3	30.8	37.5	0.26	0.25	0.32
20^2	0.6	5.17 ± 0.25	3.99 ± 0.27	3.85 ± 0.28	12.1	12.7	19.3	0.09	0.09	0.15
40^2	0.3	2.26 ± 0.09	1.51 ± 0.10	1.71 ± 0.09	4.8	5.1	11.0	0.04	0.03	0.09

Table 2 3D, 6 realizations: the same as in Table 1

grid	H , m	error in T , % $\downarrow$			phase mismatch, % $\downarrow$			depth error, m $\downarrow$		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
5^3	2.4	19.54 ± 1.25	19.91 ± 1.60	16.27 ± 1.56	93.7	105.2	99.9	0.55	0.83	0.71
10^3	1.2	7.50 ± 0.59	7.48 ± 0.78	6.02 ± 0.75	31.4	47.2	42.7	0.20	0.24	0.24
20^3	0.6	3.39 ± 0.22	3.29 ± 0.29	2.66 ± 0.29	12.4	16.4	18.8	0.06	0.09	0.12

However, in terms of the position of the phase boundary (Tables 1, 2, Fig. 6) the ranking is different: on grids with $H \leq 1.2$ m h-smoothing has a larger phase mismatch than the table (in 2D also than the nodal method) and systematically overestimates the thawing depth. The reason is the energy balance: the table and the nodal method are conservative, whereas h-smoothing, linearizing the latent heat over the previous time level, gains an extra 3–9 % of heat. This is confirmed by Tables 4 and 5: the conservative variant of the same scheme removes the excess phase error, losing its advantage in temperature, and for a step $\tau = 0.25$ day the imbalance and the phase error decrease together, whereas the nodal method does not change. On the coarse grid each method errs in its own way (Fig. 7, 8): the table produces a spurious cold region below the front, in the nodal method the front jumps over several nodes, and for h-smoothing the front is deeper than in the reference.

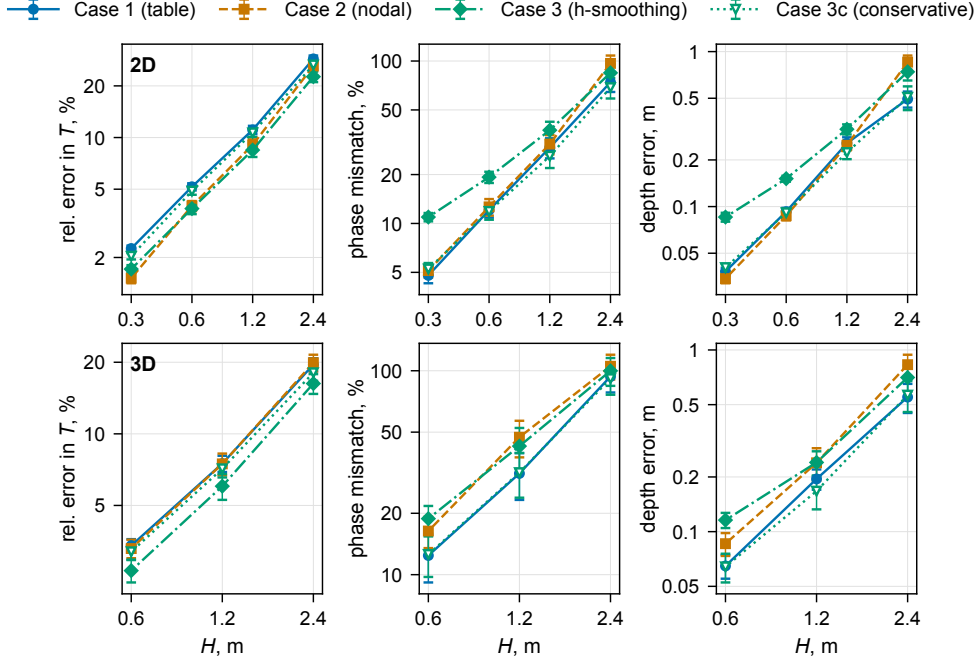


Fig. 5 Errors versus the macro-cell size H : relative temperature error in $L^2(0, T; L^2)$, phase mismatch and error of the thawing depth; 2D, 20 realizations (top) and 3D, 6 realizations (bottom); mean values, error bars: standard deviation

So far the phase-transition temperature has been constant, and the table was in essence reduced to two tensors. The situation is different when T^* varies across layers (Table 6): for $T^* = -2$ °C in layers with $\lambda_{sc} < 2$ W/(m·K), the table built from the cell-average temperature is 1.5–2.6 times more accurate than the two tensors in the front position, while remaining the worst in temperature.

5 Conclusions

Three closures of the homogenized Stefan problem (a table versus the average temperature, a nodal scheme with two tensors, and h-smoothing) are compared in a single implementation, on the same grids and with the same effective tensors, in 2D and 3D against reference solutions on a fine grid.

1. In terms of the temperature error on the coarse grid, h-smoothing is the most accurate; under grid refinement in 2D the nodal method catches up with it, and in 3D it remains the best.
2. In terms of the position of the phase boundary and the thawing depth, h-smoothing is worse than the conservative schemes and systematically overestimates thawing because it does not conserve energy; the conservative variant of the same scheme removes this error at the price of temperature accuracy.

Table 3 Cost on one core: cell problems (for Case 1), run time of one year, Newton iterations per step, energy balance of Case 3

grid	cells, s	year, s ↓			iterations		balance, %
	Case 1	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
5 ²	2.7	1.0	0.9	0.6	2.3	1.9	-0.1
10 ²	4.0	1.9	1.7	0.8	3.1	2.9	+8.8
20 ²	8.2	5.3	4.3	1.6	3.5	3.1	+7.9
40 ²	19.9	20.4	16.0	4.9	4.0	3.5	+6.0
5 ³	481.6	4.7	3.0	2.2	3.2	1.8	-0.9
10 ³	255.4	48.0	34.2	12.6	3.7	3.1	+3.9
20 ³	347.2	1378.0	1007.2	306.2	4.4	3.3	+3.2

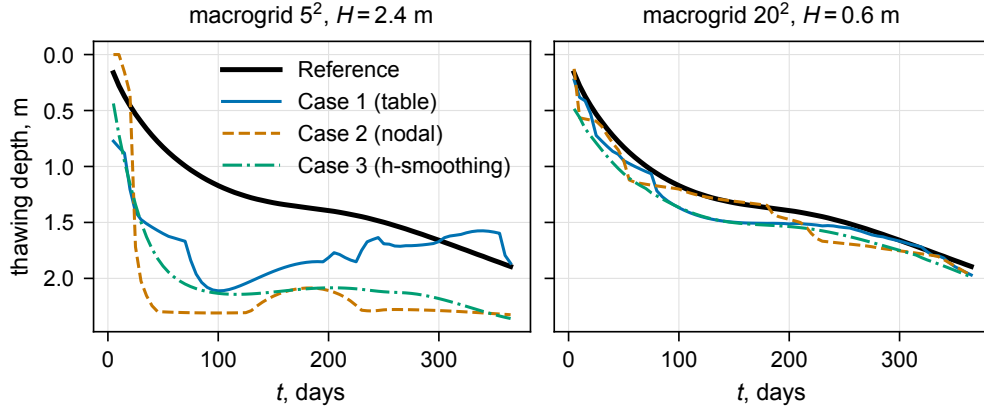


Fig. 6 Thawing depth under the center of the house versus time, 2D, realization 1: macrogrids 5^2 (left) and 20^2 (right)

Table 4 Conservative variant of h-smoothing (Case 3c) versus the original one (Case 3) and the nodal method (Case 2); the energy balance is given on the right. Top: 2D, bottom: 3D

grid	error in T , % ↓			phase mismatch, % ↓			depth error, m ↓			balance, %	
	Case 2	Case 3	Case 3c	Case 2	Case 3	Case 3c	Case 2	Case 3	Case 3c	Case 3	Case 3c
5^2	26.13	22.63	26.50	96.7	84.6	68.7	0.855	0.741	0.508	-0.1	$< 10^{-6}$
10^2	9.06	8.45	10.65	30.8	37.5	26.0	0.248	0.316	0.223	+8.8	$< 10^{-6}$
20^2	3.99	3.85	4.88	12.7	19.3	11.9	0.087	0.151	0.091	+7.9	$< 10^{-6}$
40^2	1.51	1.71	2.03	5.1	11.0	5.3	0.034	0.086	0.040	+6.0	$< 10^{-6}$
5^3	19.91	16.27	18.08	105.2	99.9	91.5	0.830	0.706	0.565	-0.9	$< 10^{-6}$
10^3	7.48	6.02	7.13	47.2	42.7	31.7	0.239	0.240	0.167	+3.9	$< 10^{-6}$
20^3	3.29	2.66	3.19	16.4	18.8	12.6	0.086	0.116	0.064	+3.2	$< 10^{-6}$

Table 5 Time step $\tau = 1$ and 0.25 days: h-smoothing (Case 3) and the nodal method (Case 2), 2D, 20 realizations

grid	τ , days	Case 3				Case 2		
		T , %	phase, %	depth, m	bal., %	T , %	phase, %	depth, m
5^2	1	22.63	84.6	0.741	-0.1	26.13	96.7	0.855
	0.25	22.70	86.5	0.752	+3.8	26.15	96.8	0.856
10^2	1	8.45	37.5	0.316	+8.8	9.06	30.8	0.248
	0.25	8.27	32.2	0.282	+1.8	9.08	30.9	0.249
20^2	1	3.85	19.3	0.151	+7.9	3.99	12.7	0.087
	0.25	3.66	14.6	0.117	+1.4	4.02	12.7	0.088
40^2	1	1.71	11.0	0.086	+6.0	1.51	5.1	0.034
	0.25	1.44	7.5	0.059	+2.0	1.55	5.3	0.036

Table 6 Variable phase-transition temperature (2D, 5 realizations): Case 1: table versus $\bar{T}$; Cases 2, 3, 3c: as in Table 4

grid	H , m	error in T , % $\downarrow$				phase mismatch, % $\downarrow$				depth error, m $\downarrow$			
		Case 1	Case 2	Case 3	Case 3c	Case 1	Case 2	Case 3	Case 3c	Case 1	Case 2	Case 3	Case 3c
5^2	2.4	28.85	25.87	23.02	25.74	60.5	89.5	87.8	59.7	0.521	0.777	0.798	0.547
10^2	1.2	11.17	9.38	9.03	10.29	25.3	38.5	48.1	25.2	0.275	0.294	0.442	0.277
20^2	0.6	5.08	4.09	4.49	4.61	11.1	16.6	29.2	12.2	0.115	0.097	0.202	0.102

3. For a spatially constant conductivity jump the table reduces to two tensors and is inferior to the nodal scheme in temperature; for a layerwise variable phase-transition temperature it is more accurate than the two tensors in the front position.
4. h-smoothing is several times cheaper than the methods with Newton iterations. In practice: if the temperature on a coarse grid is needed, h-smoothing should be used; if the boundary of the thawed zone is needed, a conservative scheme should be used.

One geometry, one family of fields and a constant porosity were considered; a variable phase-transition temperature was studied only in 2D. Possible continuations are soils with variable porosity and an unfrozen water curve, multi-year computations, and a scheme that combines the exact phase fraction of h-smoothing with a conservative treatment of the latent heat.

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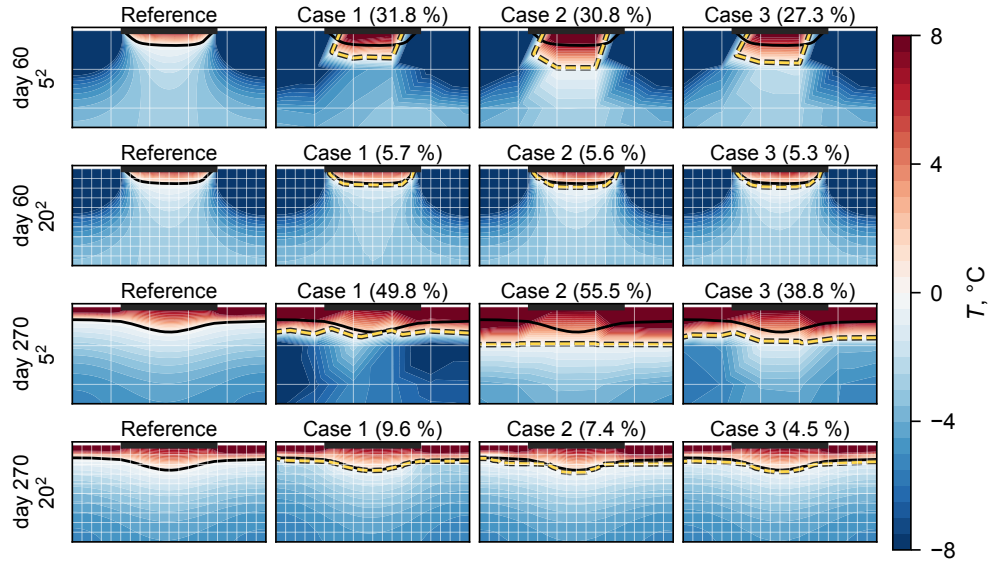


Fig. 7 Temperature on days 60 and 270, 2D, realization 1, macrogrids 5^2 and 20^2 (rows); white lines: the macrogrid, dark bar: the house. Solid black line: the reference front, yellow dashed line: the front of the method; the relative temperature error is given in parentheses

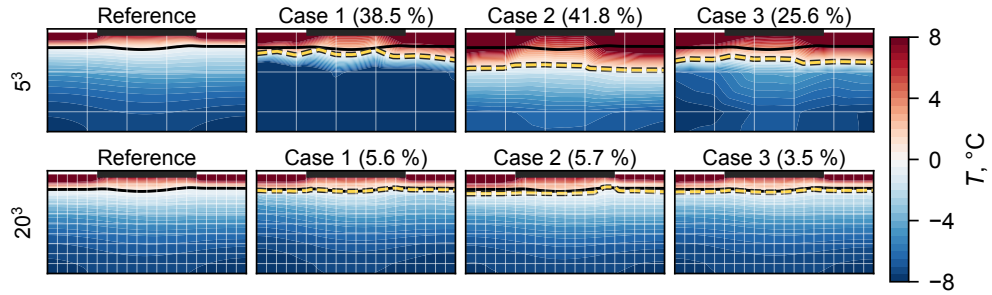


Fig. 8 3D, realization 1: temperature on the slice $y = 6$ m through the center of the house, day 270, macrogrids 5^3 and 20^3 . Notations as in Fig. 7

Statements and Declarations

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Data availability. The code and the data supporting the findings of this study are available from the author upon reasonable request.

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