

Supplementary Information

Realization of decoherence-induced averaged symmetry-protected topological phases on quantum processors

Ruizhe Shen¹, Xue-Jia Yu² and Ching Hua Lee¹

¹Department of Physics, National University of Singapore, Singapore 117551

²Eastern Institute of Technology, Ningbo 315200, China

Correspondence: e0554228@u.nus.edu; xuejiayu@eitech.edu.cn; phylch@nus.edu.sg

This Supplementary Information provides additional theoretical and experimental details supporting the main text. We present the cluster-state conventions and boundary terms, the ancilla-assisted realization of even-sublattice dephasing and the resulting strong-average symmetry structure, derivations of the single-copy and two-replica string diagnostics, and the analysis of the reduced-density-matrix spectrum and its characteristic twofold edge degeneracy. We also provide further details of the measurement-based comparison protocol, circuit implementations, and IBM Quantum hardware experiments.

Supplementary Note 1 Cluster-state conventions and boundary terms

We summarize the cluster-state conventions used in the main text and clarify the role of boundary terms. In the bulk of a one-dimensional chain, the cluster stabilizers are

$$K_j = Z_{j-1}X_jZ_{j+1}. \quad (\text{S1})$$

For an open chain, two closely related conventions are commonly used. In the graph-state convention adopted for state preparation in this work, the boundary stabilizers

$$K_1 = X_1Z_2, \quad K_L = Z_{L-1}X_L \quad (\text{S2})$$

are included together with Eq. (S1). The resulting cluster graph state is the unique simultaneous +1 eigenstate of the complete stabilizer set [1–3],

$$K_j|\psi_{\text{cl}}\rangle = |\psi_{\text{cl}}\rangle, \quad j = 1, \dots, L. \quad (\text{S3})$$

Alternatively, in the symmetry-preserving open-chain convention, only the bulk stabilizers K_j with $1 < j < L$ are included. The two unconstrained boundary degrees of freedom then generate the characteristic edge-state degeneracy of the cluster SPT phase. Thus, the first convention specifies the state prepared in our circuits.

The graph-state preparation circuit for the open-chain cluster state is

$$|\psi_{\text{cl}}\rangle = U_{\text{CZ}}^{\text{odd}}U_{\text{CZ}}^{\text{even}}|+\rangle^{\otimes L}, \quad (\text{S4})$$

with

$$U_{\text{CZ}}^{\text{even}} = \prod_{\substack{j \text{ even} \\ 1 \leq j < L}} \text{CZ}_{j,j+1}, \quad U_{\text{CZ}}^{\text{odd}} = \prod_{\substack{j \text{ odd} \\ 1 \leq j < L}} \text{CZ}_{j,j+1}. \quad (\text{S5})$$

Gates within the same sublayer act on disjoint pairs and can therefore be executed in parallel on a nearest-neighbour device.

Supplementary Note 2 Ancilla implementation of dephasing channels

We provide additional details on the implementation of dephasing channels in quantum circuits. For each even-sublattice system qubit q_j , an ancilla a_j is initialized in $|0\rangle_{a_j}$ and prepared as

$$R_y(2\theta)|0\rangle_{a_j} = \sqrt{1-p}|0\rangle_{a_j} + \sqrt{p}|1\rangle_{a_j}, \quad p = \sin^2 \theta. \quad (\text{S6})$$

The ancilla is then coupled to the system qubit through a controlled- Z operation,

$$U_{a_j q_j} = |0\rangle\langle 0|_{a_j} \otimes I_j + |1\rangle\langle 1|_{a_j} \otimes Z_j. \quad (\text{S7})$$

Measuring the ancilla in the computational basis gives the two Kraus operators [4–6]

$$M_{j,0} = \sqrt{1-p} I_j, \quad M_{j,1} = \sqrt{p} Z_j. \quad (\text{S8})$$

Discarding the ancilla outcome therefore produces the local dephasing channel

$$\mathcal{E}_j^Z(\rho) = \sum_{\eta_j=0,1} M_{j,\eta_j} \rho M_{j,\eta_j}^\dagger = (1-p)\rho + p Z_j \rho Z_j. \quad (\text{S9})$$

Applying the construction independently to all even-sublattice sites gives

$$\mathcal{E}_{\text{even}}^Z = \prod_{j \in \text{even}} \mathcal{E}_j^Z. \quad (\text{S10})$$

If the ancilla measurement record is retained, the same circuit admits a trajectory-resolved description [7–11]. We denote the outcome at even site j by $\eta_j \in \{0, 1\}$, corresponding respectively to the identity and Z_j branches. For the complete record $\boldsymbol{\eta} = \{\eta_j\}_{j \in \text{even}}$, the trajectory operator and its probability are

$$Z(\boldsymbol{\eta}) = \prod_{j \in \text{even}} Z_j^{\eta_j}, \quad P(\boldsymbol{\eta}) = \prod_{j \in \text{even}} p^{\eta_j} (1-p)^{1-\eta_j}. \quad (\text{S11})$$

The corresponding conditional system state is

$$|\psi_{\boldsymbol{\eta}}\rangle = Z(\boldsymbol{\eta})|\psi_{\text{cl}}\rangle, \quad (\text{S12})$$

and averaging over all measurement records yields

$$\rho_{\text{dec}} = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) |\psi_{\boldsymbol{\eta}}\rangle \langle \psi_{\boldsymbol{\eta}}|, \quad (\text{S13})$$

which is equivalent to applying $\mathcal{E}_{\text{even}}^Z$ to the initial cluster-state density matrix.

Supplementary Note 3 Detailed structure of the averaged cluster SPT

In this section, we justify the use of local Z dephasing in the decohered cluster-chain protocol and explain how the resulting mixed state realizes the strong-average symmetry structure of an ASPT phase. The decoherence introduced here is not treated as a generic uncontrolled error. Instead, it is an engineered local quantum channel with a definite symmetry action: the odd-sublattice symmetry remains strong, whereas the even-sublattice symmetry is reduced from a strong symmetry to a weak, or average, symmetry after ensemble averaging.

The local channel applied to each even-sublattice site is

$$\mathcal{E}_j^Z(\rho) = (1-p)\rho + p Z_j \rho Z_j, \quad (\text{S14})$$

where p is the probability of the local Z_j branch. This is the standard single-site Pauli- Z dephasing channel [4–6]. For a single-qubit density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \quad (\text{S15})$$

the channel gives

$$\mathcal{E}^Z(\rho) = \begin{pmatrix} \rho_{00} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & \rho_{11} \end{pmatrix}. \quad (\text{S16})$$

Thus, the computational-basis populations are unchanged, while the off-diagonal coherences are multiplied by $1 - 2p$. In the range $0 \leq p \leq 1/2$, p therefore provides a monotonic measure of the dephasing strength: $p = 0$ corresponds to the identity channel, whereas $p = 1/2$ gives complete dephasing in the Z basis.

Applying the channel independently to all even-sublattice sites gives

$$\rho_{\text{dec}} = \left(\prod_{j \in \text{even}} \mathcal{E}_j^Z \right) (\rho_{\text{cl}}), \quad (\text{S17})$$

where $\rho_{\text{cl}} = |\psi_{\text{cl}}\rangle\langle\psi_{\text{cl}}|$ is the initial cluster-state density matrix. Equivalently, the resulting state can be written as the trajectory ensemble

$$\rho_{\text{dec}} = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}), \quad (\text{S18})$$

with

$$Z(\boldsymbol{\eta}) = \prod_{j \in \text{even}} Z_j^{\eta_j}, \quad P(\boldsymbol{\eta}) = \prod_{j \in \text{even}} p^{\eta_j} (1-p)^{1-\eta_j}, \quad (\text{S19})$$

where $\eta_j \in \{0, 1\}$ labels the identity and Z_j branches, respectively.

This form makes the symmetry action transparent. Since the dephasing operators act only on the even sublattice, they commute with the odd-sublattice generator U_{odd} , so the odd symmetry remains strong. By contrast, the local Z_j operators anticommute with the corresponding X_j factors in U_{even} , so individual trajectories generally carry different even-sublattice symmetry charges. After averaging over the trajectories, however, the density matrix remains invariant under conjugation by U_{even} ,

$$U_{\text{even}} \rho_{\text{dec}} U_{\text{even}}^\dagger = \rho_{\text{dec}}, \quad (\text{S20})$$

while it no longer needs to satisfy the stronger condition $U_{\text{even}} \rho_{\text{dec}} = \rho_{\text{dec}}$. The engineered dephasing therefore converts the even-sublattice symmetry from strong to average while preserving the odd-sublattice symmetry as strong, yielding the hybrid symmetry structure

$$\mathbb{Z}_2^{\text{odd, strong}} \times \mathbb{Z}_2^{\text{even, average}}, \quad (\text{S21})$$

which underlies the decoherence-induced averaged cluster SPT construction used in this work [7–9, 12, 13].

We now explain why this channel produces the desired averaged cluster-chain SPT. The ideal cluster state is protected by the two $\mathbb{Z}_2$ symmetry generators

$$U_{\text{odd}} = \prod_{j \in \text{odd}} X_j, \quad U_{\text{even}} = \prod_{j \in \text{even}} X_j. \quad (\text{S22})$$

We assume a boundary convention for which the reference cluster state lies in the $+1$ sector of both symmetry generators, for example periodic boundary conditions or an open-chain fixed-point state with the boundary degrees of freedom fixed. Its density matrix

$$\rho_{\text{cl}} = |\psi_{\text{cl}}\rangle\langle\psi_{\text{cl}}| \quad (\text{S23})$$

then obeys

$$U_\alpha \rho_{\text{cl}} = \rho_{\text{cl}} U_\alpha = \rho_{\text{cl}}, \quad \alpha \in \{\text{odd, even}\}. \quad (\text{S24})$$

Here, a strong symmetry means that the density matrix is fixed by the symmetry operator when the latter acts from either the left or the right.

In the decohered protocol, Z -dephasing is applied only to the even sublattice:

$$\rho_{\text{dec}} = \prod_{j \in \text{even}} \mathcal{E}_j^Z (\rho_{\text{cl}}). \quad (\text{S25})$$

Equivalently, the mixed state can be written as an ensemble average

$$\rho_{\text{dec}} = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}), \quad (\text{S26})$$

where

$$Z(\boldsymbol{\eta}) = \prod_{j \in \text{even}} Z_j^{\eta_j}, \quad \eta_j \in \{0, 1\}, \quad (\text{S27})$$

and

$$P(\boldsymbol{\eta}) = \prod_{j \in \text{even}} p^{\eta_j} (1-p)^{1-\eta_j}. \quad (\text{S28})$$

The binary string $\boldsymbol{\eta}$ labels the Kraus trajectory of the engineered dephasing channel. In an ancilla-assisted implementation, the same label can be identified with the corresponding recorded measurement branch, and Eq. (S26) is obtained after averaging over the measurement record.

The selectivity of the protocol follows from the sublattice structure of the noise. For $j \in \text{even}$, the operator Z_j commutes with U_{odd} , because U_{odd} acts only on odd sites:

$$[U_{\text{odd}}, Z_j] = 0, \quad j \in \text{even}. \quad (\text{S29})$$

Therefore

$$U_{\text{odd}} Z(\boldsymbol{\eta}) = Z(\boldsymbol{\eta}) U_{\text{odd}}. \quad (\text{S30})$$

Using Eq. (S24), one obtains

$$\begin{aligned} U_{\text{odd}} \rho_{\text{dec}} &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) U_{\text{odd}} Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}) \\ &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) Z(\boldsymbol{\eta}) U_{\text{odd}} \rho_{\text{cl}} Z(\boldsymbol{\eta}) \\ &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}) = \rho_{\text{dec}}. \end{aligned} \quad (\text{S31})$$

The same argument for right multiplication gives

$$\rho_{\text{dec}} U_{\text{odd}} = \rho_{\text{dec}}. \quad (\text{S32})$$

Hence the odd symmetry remains strong after decoherence:

$$U_{\text{odd}} \rho_{\text{dec}} = \rho_{\text{dec}} U_{\text{odd}} = \rho_{\text{dec}}. \quad (\text{S33})$$

The even symmetry behaves differently. Since U_{even} contains X_j on every even site, while the dephasing channel inserts Z_j on even sites, one has

$$X_j Z_j = -Z_j X_j, \quad j \in \text{even}. \quad (\text{S34})$$

Therefore,

$$U_{\text{even}} Z(\boldsymbol{\eta}) = (-1)^{|\boldsymbol{\eta}|} Z(\boldsymbol{\eta}) U_{\text{even}}, \quad |\boldsymbol{\eta}| = \sum_{j \in \text{even}} \eta_j. \quad (\text{S35})$$

Acting from the left gives

$$\begin{aligned} U_{\text{even}} \rho_{\text{dec}} &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) U_{\text{even}} Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}) \\ &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) (-1)^{|\boldsymbol{\eta}|} Z(\boldsymbol{\eta}) U_{\text{even}} \rho_{\text{cl}} Z(\boldsymbol{\eta}) \\ &= \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta}) (-1)^{|\boldsymbol{\eta}|} Z(\boldsymbol{\eta}) \rho_{\text{cl}} Z(\boldsymbol{\eta}). \end{aligned} \quad (\text{S36})$$

For generic p , this is not equal to ρ_{dec} . Hence U_{even} is no longer a strong symmetry:

$$U_{\text{even}}\rho_{\text{dec}} \neq \rho_{\text{dec}}. \quad (\text{S37})$$

Equivalently, the individual trajectory state $Z(\boldsymbol{\eta})|\psi_{\text{cl}}\rangle$ carries a trajectory-dependent sign under the left action of U_{even} .

Nevertheless, the even symmetry is preserved in the weak, or average, sense. Under conjugation by U_{even} , the signs from the left and right actions cancel:

$$\begin{aligned} U_{\text{even}}Z(\boldsymbol{\eta})\rho_{\text{cl}}Z(\boldsymbol{\eta})U_{\text{even}}^\dagger \\ = (-1)^{|\boldsymbol{\eta}|}Z(\boldsymbol{\eta})U_{\text{even}}\rho_{\text{cl}}U_{\text{even}}^\dagger Z(\boldsymbol{\eta})(-1)^{|\boldsymbol{\eta}|} \\ = Z(\boldsymbol{\eta})\rho_{\text{cl}}Z(\boldsymbol{\eta}). \end{aligned} \quad (\text{S38})$$

After summing over all trajectories, this gives

$$U_{\text{even}}\rho_{\text{dec}}U_{\text{even}}^\dagger = \rho_{\text{dec}}. \quad (\text{S39})$$

Thus the even generator is not a strong symmetry of the decohered density matrix, but it remains a weak symmetry of the ensemble-averaged mixed state.

The resulting mixed state therefore has the symmetry structure

$$\mathbb{Z}_2^{\text{odd,strong}} \times \mathbb{Z}_2^{\text{even,weak}}. \quad (\text{S40})$$

This is the symmetry structure characteristic of the decohered averaged cluster-chain SPT.

This also clarifies why the choice of even-sublattice Z -dephasing is selective. If the noise commuted with both U_{odd} and U_{even} , both symmetries would remain strong and no strong-to-weak conversion would occur. Conversely, if the noise destroyed the conjugation invariance under U_{even} , the ASPT structure would be lost. Even-sublattice Z -dephasing realizes the intermediate case required here: it preserves one $\mathbb{Z}_2$ symmetry strongly while converting the other into an average symmetry.

Supplementary Note 4 Ordinary and doubled string diagnostics

We provide the details behind the ordinary and doubled string diagnostics used in the main text.

4.1 Single-copy string diagnostic

Let the endpoints i and k lie on the same sublattice, so that $k - i$ is even. The corresponding cluster string operator is

$$\mathcal{O}_{i,k} = Z_i \left(\prod_{r=i+1, i+3, \dots, k-1} X_r \right) Z_k. \quad (\text{S41})$$

We denote the set of interior sites carrying X support by $\mathcal{X}_{i,k} = \{i+1, i+3, \dots, k-1\}$.

The ideal cluster state used throughout this work is prepared from $|+\rangle^{\otimes L}$ by nearest-neighbor controlled- Z gates,

$$|\psi_{\text{cl}}\rangle = \left(\prod_{j=1}^{L-1} \text{CZ}_{j,j+1} \right) |+\rangle^{\otimes L}, \quad (\text{S42})$$

and satisfies the bulk stabilizer conditions $K_r|\psi_{\text{cl}}\rangle = |\psi_{\text{cl}}\rangle$, with $K_r = Z_{r-1}X_rZ_{r+1}$. The string operator is the product of the stabilizers on $\mathcal{X}_{i,k}$,

$$\mathcal{O}_{i,k} = \prod_{r \in \mathcal{X}_{i,k}} K_r. \quad (\text{S43})$$

It therefore follows directly that

$$\mathcal{O}_{i,k}|\psi_{\text{cl}}\rangle = |\psi_{\text{cl}}\rangle. \quad (\text{S44})$$

We first derive the single-copy decay under even-sublattice dephasing. The local dephasing channel on an even site j is

$$\mathcal{E}_j^Z(\rho) = (1-p)\rho + pZ_j\rho Z_j. \quad (\text{S45})$$

In the Heisenberg picture, we have for a Pauli operator P

$$(\mathcal{E}_j^Z)^\dagger(P) = (1-p)P + pZ_j P Z_j, \quad (\text{S46})$$

which simplifies to

$$(\mathcal{E}_j^Z)^\dagger(P) = \begin{cases} P, & [P, Z_j] = 0, \\ (1-2p)P, & \{P, Z_j\} = 0. \end{cases} \quad (\text{S47})$$

For the engineered dephasing channel acting on the even sublattice, let $\mathcal{D}_{\text{even}}$ denote the set of dephased sites. Its adjoint action on the string operator is

$$(\mathcal{E}_{\text{even}}^Z)^\dagger(\mathcal{O}_{i,k}) = (1-2p)^{N_{\text{dep}}(i,k)}\mathcal{O}_{i,k}, \quad (\text{S48})$$

where $N_{\text{dep}}(i,k) = |\mathcal{X}_{i,k} \cap \mathcal{D}_{\text{even}}|$ counts the dephased even-sublattice sites on which the string operator has X support. The string expectation value in the decohered state therefore becomes

$$\begin{aligned} \langle \mathcal{O}_{i,k} \rangle_{\text{dec}} &= \text{Tr}[\mathcal{O}_{i,k} \mathcal{E}_{\text{even}}^Z(\rho_{\text{cl}})] \\ &= \text{Tr}[(\mathcal{E}_{\text{even}}^Z)^\dagger(\mathcal{O}_{i,k}) \rho_{\text{cl}}] \\ &= (1-2p)^{N_{\text{dep}}(i,k)} \text{Tr}[\mathcal{O}_{i,k} \rho_{\text{cl}}] \\ &= (1-2p)^{N_{\text{dep}}(i,k)}. \end{aligned} \quad (\text{S49})$$

Thus, the conventional string correlator is suppressed by averaging over dephasing trajectories with different string signs.

4.2 Double-copy string diagnostic

We now turn to the doubled diagnostic. The essential point is that the averaged-SPT order is encoded in the density matrix rather than in a linear single-copy expectation value. We therefore use the Hilbert–Schmidt inner product

$$\langle\langle A|B \rangle\rangle = \text{Tr}(A^\dagger B) \quad (\text{S50})$$

and define the doubled string action by

$$\mathcal{O}_{i,k}^{(2)} : |A\rangle\rangle \mapsto |\mathcal{O}_{i,k} A \mathcal{O}_{i,k}\rangle\rangle. \quad (\text{S51})$$

For Hermitian Pauli strings with $\mathcal{O}_{i,k}^2 = I$, the normalized doubled-string expectation is

$$\begin{aligned} \langle \mathcal{O}_{i,k}^{(2)} \rangle_{\rho_{\text{dec}}} &= \frac{\langle\langle \rho_{\text{dec}} | \mathcal{O}_{i,k}^{(2)} | \rho_{\text{dec}} \rangle\rangle}{\langle\langle \rho_{\text{dec}} | \rho_{\text{dec}} \rangle\rangle} \\ &= \frac{\text{Tr}[\rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k}]}{\text{Tr}[\rho_{\text{dec}}^2]} \equiv C_{i,k}^{(2)}. \end{aligned} \quad (\text{S52})$$

This quantity is nonlinear in ρ_{dec} and is therefore sensitive to density-matrix-level order that may be invisible to ordinary single-copy observables.

Although Eq. (S52) is nonlinear in one copy of the density matrix, it is measured as a linear observable on two independently prepared replicas. Let $\rho_A = \rho_B = \rho_{\text{dec}}$ denote identical copies of ρ_{dec} in replicas A and B , and let

$$S_{AB} = \prod_{j=1}^L S_{j_A j_B} \quad (\text{S53})$$

be the global SWAP operator between replicas A and B . We use the identity

$$\text{Tr}(AB) = \sum_{j_A, j_B} \langle j_A, j_B | A \otimes B | j_B, j_A \rangle = \text{Tr}[(A \otimes B) S_{AB}] \quad (\text{S54})$$

for arbitrary operators A and B on the same Hilbert space, where the trace on the LHS is performed over a single copy, while the RHS trace is over both copies.

The denominator of Eq. (S52) is therefore

$$\begin{aligned} D &= \text{Tr}(\rho_{\text{dec}}^2) \\ &= \text{Tr}[(\rho_A \otimes \rho_B) S_{AB}]. \end{aligned} \quad (\text{S55})$$

The numerator of Eq. (S52), denoted by $N_{i,k}$, can be obtained in two equivalent ways. First, using Eq. (S54), we have

$$\begin{aligned} N_{i,k} &= \text{Tr}[\rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k}] \\ &= \text{Tr}[(\rho_A \mathcal{O}_{i,k}) \otimes (\rho_B \mathcal{O}_{i,k}) S_{AB}] \\ &= \text{Tr}[(\rho_A \otimes \rho_B)(\mathcal{O}_{i,k}^A \otimes \mathcal{O}_{i,k}^B) S_{AB}]. \end{aligned} \quad (\text{S56})$$

Second, in the circuit implementation, we can equivalently insert the Pauli strings on only one replica, i.e. A , immediately before the SWAP readout. Since $\mathcal{O}_{i,k}$ is Hermitian this gives

$$\begin{aligned} \langle S_{AB} \rangle_{\mathcal{O}^A \text{ inserted}} &= \text{Tr}[(\mathcal{O}_{i,k}^A \rho_A \mathcal{O}_{i,k}^A \otimes \rho_B) S_{AB}] \\ &= \text{Tr}[\mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}}] \\ &= \text{Tr}[\rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k}] = N_{i,k}. \end{aligned} \quad (\text{S57})$$

Thus one does not need to insert Pauli strings into both replicas in the actual circuit implementation of $N_{i,k}$. Applying $\mathcal{O}_{i,k}$ to one copy before the destructive SWAP measurement is sufficient.

We next give the details of the destructive SWAP readout used to evaluate the two-replica diagnostic. For two replicas A and B , we express the global SWAP operator in terms of local SWAPs,

$$S_{AB} = \prod_{j=1}^L S_{j_A j_B}. \quad (\text{S58})$$

This operator satisfies

$$\text{Tr}[(\rho_A \otimes \rho_B) S_{AB}] = \text{Tr}(\rho_A \rho_B). \quad (\text{S59})$$

Thus, when the two replicas are prepared in the same mixed state, $\rho_A = \rho_B = \rho_{\text{dec}}$, the expectation value of S_{AB} directly gives the purity,

$$\langle S_{AB} \rangle = \text{Tr}(\rho_{\text{dec}}^2). \quad (\text{S60})$$

To measure S_{AB} destructively, each corresponding pair (j_A, j_B) is rotated from the Bell basis to the computational basis by

$$\text{CNOT}_{j_A \rightarrow j_B} \quad \text{followed by} \quad H_{j_A}, \quad (\text{S61})$$

after which both qubits are measured. In our implementation, the CNOT is compiled using the available CZ interaction as

$$\text{CNOT}_{j_A \rightarrow j_B} = H_{j_B} \text{CZ}_{j_A, j_B} H_{j_B}. \quad (\text{S62})$$

The Bell states are eigenstates of the local SWAP operator $S_{j_A j_B}$: the three symmetric Bell states have eigenvalue $+1$, whereas the antisymmetric singlet has eigenvalue -1 .

Under the Bell-basis rotation in Eq. (S61), the singlet is mapped to the computational-basis outcome $(b_{j,A}, b_{j,B}) = (1, 1)$, while the other three outcomes correspond to SWAP eigenvalue $+1$. Therefore, for each measured pair,

$$s_j = (-1)^{b_{j,A} b_{j,B}}, \quad b_{j,A}, b_{j,B} \in \{0, 1\}, \quad (\text{S63})$$

is a single-shot estimator of the local SWAP eigenvalue. Since the global SWAP is the product of the local SWAPs, the corresponding estimator for one experimental shot is

$$s_{AB} = \prod_{j=1}^L s_j = (-1)^{\sum_{j=1}^L b_{j,A} b_{j,B}}. \quad (\text{S64})$$

Averaging s_{AB} over repeated shots therefore gives the expectation value of S_{AB} .

For the denominator of the Rényi-2 string correlator, both replicas are measured directly with this destructive SWAP circuit, giving

$$D = \text{Tr}(\rho_{\text{dec}}^2) = \langle s_{AB} \rangle_{\text{no string}}. \quad (\text{S65})$$

For the numerator, the string operator $\mathcal{O}_{i,k}$ is first applied to replica A , after which the same destructive SWAP measurement is performed. Using $S_{AB} \mathcal{O}_{i,k}^A = \mathcal{O}_{i,k}^B S_{AB}$, we have

$$N_{i,k} = \text{Tr}[\rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k}] = \langle s_{AB} \rangle_{\text{string}}. \quad (\text{S66})$$

The measured two-replica diagnostic is therefore

$$C_{i,k}^{(2)} = \frac{N_{i,k}}{D} = \frac{\langle s_{AB} \rangle_{\text{string}}}{\langle s_{AB} \rangle_{\text{no string}}}. \quad (\text{S67})$$

We then show why $C_{i,k}^{(2)} = 1$ for the ideal decohered cluster construction. Using the trajectory decomposition, we have

$$\begin{aligned} D &= \text{Tr}(\rho_{\text{dec}}^2) \\ &= \sum_{\boldsymbol{\eta}, \boldsymbol{\eta}'} P(\boldsymbol{\eta}) P(\boldsymbol{\eta}') |\langle \psi_{\boldsymbol{\eta}} | \psi_{\boldsymbol{\eta}'} \rangle|^2, \end{aligned} \quad (\text{S68})$$

whereas

$$\begin{aligned} N_{i,k} &= \text{Tr}[\rho_{\text{dec}} \mathcal{O}_{i,k} \rho_{\text{dec}} \mathcal{O}_{i,k}] \\ &= \sum_{\boldsymbol{\eta}, \boldsymbol{\eta}'} P(\boldsymbol{\eta}) P(\boldsymbol{\eta}') s_{\boldsymbol{\eta}} s_{\boldsymbol{\eta}'} |\langle \psi_{\boldsymbol{\eta}} | \psi_{\boldsymbol{\eta}'} \rangle|^2. \end{aligned} \quad (\text{S69})$$

For the cluster graph state, distinct Z -dephasing trajectories are orthogonal. To see this, recall that $K_j = Z_{j-1} X_j Z_{j+1}$ and $K_j |\psi_{\text{cl}}\rangle = |\psi_{\text{cl}}\rangle$. A local Z_j anticommutes with K_j , while commuting with all $K_{\ell \neq j}$. Hence,

$$K_j Z_j |\psi_{\text{cl}}\rangle = -Z_j |\psi_{\text{cl}}\rangle, \quad (\text{S70})$$

so applying Z_j flips only the eigenvalue of the corresponding stabilizer from $+1$ to -1 . More generally, for a dephasing trajectory $|\psi_{\boldsymbol{\eta}}\rangle = Z(\boldsymbol{\eta}) |\psi_{\text{cl}}\rangle$, we have

$$K_j |\psi_{\boldsymbol{\eta}}\rangle = (-1)^{\eta_j} |\psi_{\boldsymbol{\eta}}\rangle, \quad j \in \mathcal{D}_{\text{even}}. \quad (\text{S71})$$

Thus, each trajectory $\boldsymbol{\eta}$ is labeled by a definite pattern of stabilizer eigenvalues. If $\boldsymbol{\eta} \neq \boldsymbol{\eta}'$, there is at least one dephased site j for which $\eta_j \neq \eta'_j$, so the two trajectory states have opposite eigenvalues of the same Hermitian stabilizer K_j . They must therefore be orthogonal,

$$\langle \psi_{\boldsymbol{\eta}} | \psi_{\boldsymbol{\eta}'} \rangle = 0, \quad \boldsymbol{\eta} \neq \boldsymbol{\eta}'. \quad (\text{S72})$$

Following these, we get

$$D = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta})^2, \quad (\text{S73})$$

$$N_{i,k} = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta})^2 s_{\boldsymbol{\eta}}^2 = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta})^2, \quad (\text{S74})$$

because $s_{\boldsymbol{\eta}} = \pm 1$ and hence $s_{\boldsymbol{\eta}}^2 = 1$. Therefore,

$$C_{i,k}^{(2)} = \frac{N_{i,k}}{D} = 1. \quad (\text{S75})$$

The distinction from the conventional single-copy string correlator is thus transparent. The latter directly averages the signed trajectory values $s_{\boldsymbol{\eta}}$, which can cancel in the mixed-state ensemble, whereas the two-replica diagnostic depends on $s_{\boldsymbol{\eta}}^2$ and is therefore insensitive to this trajectory-sign cancellation.

Supplementary Note 5 Half-chain entanglement spectrum under dephasing

We provide details of the half-chain entanglement-spectrum analysis discussed in the main text. For a density matrix ρ , the reduced density matrix of subsystem A is

$$\rho_A = \text{Tr}_{\bar{A}} \rho, \quad (\text{S76})$$

where $\bar{A}$ denotes the complement of A . If λ_{α} are the eigenvalues of ρ_A , the corresponding entanglement levels are defined as

$$\xi_{\alpha} = -\log \lambda_{\alpha}. \quad (\text{S77})$$

We first consider the ideal cluster graph state. Let $\mathcal{S}$ denote its stabilizer group, generated by L independent mutually commuting Pauli operators K_j . The subgroup containing only stabilizers fully supported inside A is denoted by $\mathcal{S}_A$,

$$\mathcal{S}_A = \langle G_1, \dots, G_{r_A} \rangle, \quad (\text{S78})$$

where G_{ℓ} are independent commuting generators and r_A is their number. Generally, any pure stabilizer state specified by L independent commuting Pauli stabilizers can be written as

$$\rho_{\text{cl}} = |\psi_{\text{cl}}\rangle\langle\psi_{\text{cl}}| = \frac{1}{2^L} \sum_{g \in \mathcal{S}} g. \quad (\text{S79})$$

For the cluster state, $\mathcal{S}$ is generated by the cluster stabilizers defined above.

Thus, only stabilizers supported entirely within A contribute to the reduced density matrix. We denote this subgroup by

$$\mathcal{S}_A = \{g \in \mathcal{S} \mid \text{supp}(g) \subseteq A\}. \quad (\text{S80})$$

For $g \in \mathcal{S}_A$, one can write $g = g_A \otimes I_{\bar{A}}$, such that $\text{Tr}_{\bar{A}}(g) = 2^{|\bar{A}|} g_A$. Using $L = |A| + |\bar{A}|$, we therefore obtain

$$\rho_A^{\text{cl}} = \frac{1}{2^{|A|}} \sum_{g \in \mathcal{S}_A} g_A. \quad (\text{S81})$$

Let $\mathcal{S}_A = \langle G_1, \dots, G_{r_A} \rangle$, where G_ℓ are r_A independent commuting stabilizer generators. The projector onto their simultaneous +1 eigenspace is

$$P_A^{(0)} = \prod_{\ell=1}^{r_A} \frac{I + G_\ell}{2}. \quad (\text{S82})$$

Expanding this projector gives

$$P_A^{(0)} = \frac{1}{2^{r_A}} \sum_{g \in \mathcal{S}_A} g_A, \quad (\text{S83})$$

and Eq. (S81) can consequently be written as

$$\rho_A^{\text{cl}} = \frac{1}{2^{|A|-r_A}} P_A^{(0)}. \quad (\text{S84})$$

This form has a simple interpretation. The r_A independent commuting stabilizers fully supported within A impose r_A independent constraints on the $|A|$ qubits of the subsystem. The simultaneous +1 eigenspace therefore has dimension

$$\text{rank } P_A^{(0)} = 2^{|A|-r_A}. \quad (\text{S85})$$

Since ρ_A^{cl} is proportional to the projector $P_A^{(0)}$, it is maximally mixed within this subspace. Its $2^{|A|-r_A}$ nonzero eigenvalues are therefore all equal,

$$\lambda_\alpha = 2^{-(|A|-r_A)}, \quad \alpha = 1, \dots, 2^{|A|-r_A}, \quad (\text{S86})$$

with all remaining eigenvalues equal to zero.

As a warm-up, we first consider the half-chain cut of the one-dimensional cluster state used here. There is one entanglement cut and hence one unresolved virtual boundary degree of freedom. In stabilizer terms, this means that one qubit inside A is not fixed by the stabilizers fully supported in A :

$$|A| - r_A = 1. \quad (\text{S87})$$

Thus

$$\rho_A^{\text{cl}} = \frac{1}{2} P_A^{(0)}, \quad (\text{S88})$$

where $P_A^{(0)}$ has rank two. Consequently, the reduced density matrix corresponding to the half-chain cut has two equal nonzero eigenvalues,

$$\lambda_1 = \lambda_2 = \frac{1}{2}, \quad (\text{S89})$$

up to the normalization convention for the chosen subsystem.

More generally, we now consider the effect of the engineered even-sublattice dephasing channel on the half-chain entanglement spectrum. For a fixed dephasing trajectory, the output state is a locally Z -dressed cluster state,

$$|\psi_\eta\rangle = Z(\eta) |\psi_{\text{cl}}\rangle, \quad Z(\eta) = \prod_{j \in \text{even}} Z_j^{\eta_j}, \quad \eta_j \in \{0, 1\}. \quad (\text{S90})$$

With respect to the bipartition $A \cup \bar{A}$, the trajectory operator factorizes as

$$Z(\eta) = Z_A(\eta_A) \otimes Z_{\bar{A}}(\eta_{\bar{A}}). \quad (\text{S91})$$

Therefore, after tracing out $\bar{A}$, the components acting only on $\bar{A}$ drop out by unitary invariance of the partial trace. The reduced density matrix for a single trajectory is

$$\rho_A(\eta) = Z_A(\eta_A) \rho_A^{\text{cl}} Z_A(\eta_A). \quad (\text{S92})$$

Thus each dephasing trajectory is related to the fixed-point reduced density matrix by a unitary conjugation acting entirely within subsystem A . This conjugation can rotate the eigenvectors of ρ_A^{cl} , but it cannot change its eigenvalues. Consequently, every individual trajectory has the same nonzero half-chain entanglement spectrum as the original cluster state.

The remaining question is whether the incoherent average over trajectories can split the twofold edge degeneracy. Given $\rho_A^{\text{cl}} = \frac{1}{2}P_A^{(0)}$, $P_A^{(0)} = \prod_{\ell=1}^{r_A} \frac{1+G_\ell}{2}$, and the Pauli operator $Z_A(\boldsymbol{\eta}_A)$, which either commutes or anticommutes with each reduced stabilizer generator G_ℓ , we define the corresponding syndrome signs by

$$Z_A(\boldsymbol{\eta}_A)G_\ell Z_A(\boldsymbol{\eta}_A) = \sigma_\ell(\boldsymbol{\eta}_A)G_\ell, \quad \sigma_\ell(\boldsymbol{\eta}_A) = \pm 1, \quad (\text{S93})$$

The corresponding syndrome vector is

$$\boldsymbol{\sigma}(\boldsymbol{\eta}_A) = \{\sigma_1(\boldsymbol{\eta}_A), \dots, \sigma_{r_A}(\boldsymbol{\eta}_A)\} \quad (\text{S94})$$

which labels the reduced stabilizer syndrome produced by the trajectory. We then obtain

$$\begin{aligned} \rho_A(\boldsymbol{\eta}) &= Z_A(\boldsymbol{\eta}_A) \left[\frac{1}{2}P_A^{(0)} \right] Z_A(\boldsymbol{\eta}_A) \\ &= \frac{1}{2} \prod_{\ell=1}^{r_A} \frac{1 + \sigma_\ell(\boldsymbol{\eta}_A)G_\ell}{2}. \end{aligned} \quad (\text{S95})$$

Defining the syndrome projector

$$P_A^{(\boldsymbol{\sigma})} = \prod_{\ell=1}^{r_A} \frac{1 + \sigma_\ell G_\ell}{2}, \quad (\text{S96})$$

the trajectory-reduced density matrix becomes

$$\rho_A(\boldsymbol{\eta}) = \frac{1}{2}P_A^{(\boldsymbol{\sigma}(\boldsymbol{\eta}_A))}. \quad (\text{S97})$$

This equation is the key point. A dephasing trajectory changes the reduced stabilizer syndrome sector, but it does not reduce the rank of the sector. Each projector $P_A^{(\boldsymbol{\sigma})}$ has the same rank as $P_A^{(0)}$, namely rank two for the half-chain cut. Thus the trajectory changes the bulk stabilizer label while leaving the two-dimensional virtual-edge subspace unresolved.

After the trajectory record is discarded, the reduced density matrix is a classical mixture over these reduced stabilizer-syndrome sectors:

$$\rho_A^{\text{dec}} = \sum_{\boldsymbol{\sigma}} W_{\boldsymbol{\sigma}} \frac{P_A^{(\boldsymbol{\sigma})}}{2}. \quad (\text{S98})$$

Here

$$W_{\boldsymbol{\sigma}} = \sum_{\boldsymbol{\eta}_A: \boldsymbol{\sigma}(\boldsymbol{\eta}_A) = \boldsymbol{\sigma}} P_A(\boldsymbol{\eta}_A) \quad (\text{S99})$$

is the total probability of all dephasing trajectories that produce the same reduced syndrome $\boldsymbol{\sigma}$ inside subsystem A .

Since each block $P_A^{(\boldsymbol{\sigma})}/2$ has rank two, the nonzero eigenvalues are

$$\lambda_{\boldsymbol{\sigma},1} = \lambda_{\boldsymbol{\sigma},2} = \frac{W_{\boldsymbol{\sigma}}}{2} \quad (\text{S100})$$

for every populated syndrome sector $\boldsymbol{\sigma}$.

For the $L = 8$ half-chain bipartition used here, we take $A = \{1, 2, 3, 4\}$. The reduced cluster-state density matrix is

$$\rho_A^{\text{cl}} = \frac{1}{16}(I + K_1)(I + K_2)(I + K_3). \quad (\text{S101})$$

Although the dephasing channel acts on the two even sites $j = 2$ and $j = 4$ within A , only Z_2 changes a stabilizer syndrome visible in the reduced density matrix. Specifically, Z_2 anticommutes with K_2 , whereas Z_4 commutes with all stabilizers fully supported within A . The latter therefore leaves ρ_A^{cl} invariant. Dephasing operations acting entirely in $\bar{A}$ likewise disappear under the partial trace. The reduced density matrix after dephasing consequently becomes

$$\rho_A^{\text{dec}} = \frac{1}{16}(I + K_1)[I + (1 - 2p)K_2](I + K_3). \quad (\text{S102})$$

Equivalently, it consists of two reduced-stabilizer sectors with weights $1 - p$ and p . Since each sector has rank two, the four nonzero eigenvalues are

$$\lambda_1 = \lambda_2 = \frac{1 - p}{2}, \quad \lambda_3 = \lambda_4 = \frac{p}{2}. \quad (\text{S103})$$

The corresponding entanglement levels are therefore

$$\xi_{1,2} = -\log\left(\frac{1 - p}{2}\right), \quad \xi_{3,4} = -\log\left(\frac{p}{2}\right). \quad (\text{S104})$$

For the dephasing strength $p = 0.1$ used in the main-text comparison, this gives

$$\lambda_{1,2} = 0.45, \quad \lambda_{3,4} = 0.05, \quad (\text{S105})$$

or, equivalently [see Fig. 7 in the main text],

$$\xi_{1,2} = -\ln(0.45) \simeq 0.799, \quad \xi_{3,4} = -\ln(0.05) \simeq 2.996. \quad (\text{S106})$$

Thus, the engineered dephasing generates an additional reduced-stabilizer sector and shifts the corresponding entanglement levels, while preserving the twofold pairing within each populated sector.

Supplementary Note 6 Measurement-and-feedback realization of mixed-state SPT order

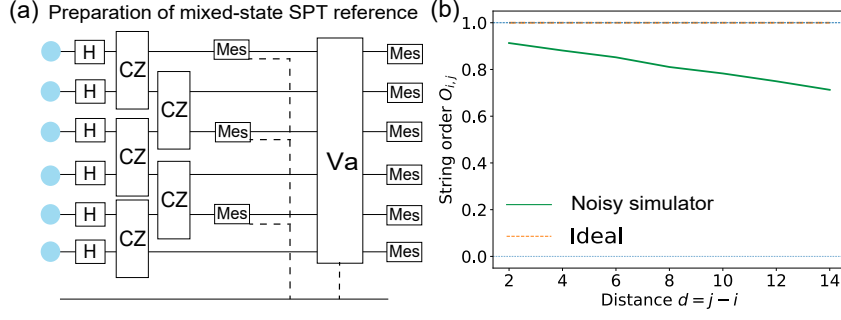
It is important to distinguish the decoherence-induced ASPT state studied in the main text from more general mixed-state SPT states, which need not exhibit the same strong-average symmetry structure. As a complementary reference, we consider the measurement-and-feedback construction of Ref. [14], shown in Supplementary Fig. 1 [15–17]. This protocol starts from the same one-dimensional cluster SPT state but treats the measurement record differently from the ASPT protocol. In the latter, the trajectory record associated with the engineered dephasing is averaged over, whereas here the measurement outcomes are actively used to apply a conditional feedback operation. The purpose of this comparison is therefore to illustrate a distinct route to mixed-state SPT order in which hidden cluster string order is converted into an ordinary long-range correlation.

Concretely, starting from $\rho_{\text{cl}} = |\psi_{\text{cl}}\rangle\langle\psi_{\text{cl}}|$, every even-sublattice qubit is measured in the X basis. We denote the measurement record by $\alpha = \{\alpha_i\}$, where $\alpha_i = \pm 1$ is the outcome of measuring X_{2i} . The projector associated with a given record is

$$P_\alpha = \prod_i \frac{1 + \alpha_i X_{2i}}{2}. \quad (\text{S107})$$

Without feedback, different measurement records generate trajectory-dependent signs in the remaining cluster correlations. To remove these signs, the measurement record is processed classically and used to determine the conditional unitary

$$V_\alpha = \prod_i X_{2i+1}^{\frac{1 - \prod_{m \leq i} \alpha_m}{2}}. \quad (\text{S108})$$



Supplementary Fig. 1: Measurement-and-feedback realization of mixed-state SPT order. **a**, Circuit for preparing the feedback-assisted mixed-state SPT. Starting from the cluster state, the even-sublattice qubits are measured in the X basis, and the resulting measurement record determines the conditional feedback unitary V_α in Eq. (S108). **b**, Long-range order obtained from the protocol in **a**. The plotted quantity is the feedback-mapped string correlator, equivalently the odd-sublattice two-point correlator $\langle Z_{2i+1} Z_{2j+1} \rangle$ in Eq. (S111), shown as a function of the endpoint separation $d = j - i$ for an $L = 16$ chain using a local noisy simulator.

Thus, an X operation is applied to the odd site $2i + 1$ whenever the cumulative measurement parity $\prod_{m \leq i} \alpha_m$ is -1 . The feedback therefore compensates the measurement-dependent sign accumulated along the corresponding cluster string.

For a fixed measurement record α , the conditional system state is

$$\rho_\alpha^{\text{fb}} = V_\alpha P_\alpha \rho_{\text{cl}} P_\alpha V_\alpha^\dagger. \quad (\text{S109})$$

Averaging over all measurement outcomes gives the feedback-prepared mixed state

$$\rho_{\text{fb}} = \sum_\alpha V_\alpha P_\alpha \rho_{\text{cl}} P_\alpha V_\alpha^\dagger. \quad (\text{S110})$$

Here the probabilities of the measurement outcomes are already contained in the unnormalized projected states $P_\alpha \rho_{\text{cl}} P_\alpha$, so no additional probability factor is required in Eq. (S110).

The effect of the feedback is particularly transparent in the cluster string order. For the non-local string operator $\mathcal{O}_{i,j}$ defined in the cluster string defined in the main text, the measurement outcomes of the intervening even-sublattice X operators determine the trajectory-dependent sign of the string. The feedback unitary V_α removes this sign and maps the hidden string order onto an ordinary two-point correlation between the remaining odd sites. Consequently,

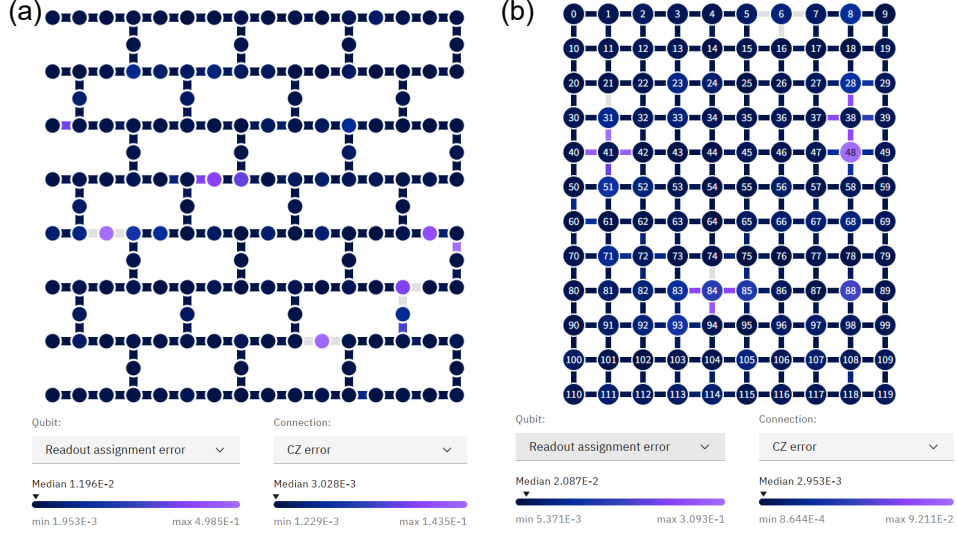
$$\begin{aligned} C_{i,j}^{\text{fb}} &= \text{Tr}[\rho_{\text{fb}} Z_{2i+1} Z_{2j+1}] \\ &= \text{Tr}[\rho_{\text{cl}} \mathcal{O}_{i,j}]. \end{aligned} \quad (\text{S111})$$

The nonlocal string order of the input cluster SPT can therefore be detected after feedback through a conventional single-copy long-range ZZ correlator [14–17].

Figure 1(b) shows the resulting correlator for an $L = 16$ chain. The feedback-mapped ZZ correlation remains finite as the endpoint separation increases, demonstrating that the measurement record can be used to recover long-range order from the underlying cluster string structure even after averaging over measurement outcomes. This provides a useful comparison with the decoherence-induced ASPT protocol. In the feedback construction, the classical record is retained and actively used to remove the trajectory-dependent signs, allowing the mixed-state order to be detected by a single-copy correlator.

Supplementary Note 7 IBM Quantum processors

Supplementary Fig. 2 shows the connectivity and calibration maps of the IBM processors considered for the experiment. The color of each qubit indicates its readout assignment error, while



Supplementary Fig. 2: Connectivity and calibration properties of the IBM quantum processors used in this work. Hardware coupling maps showing the readout assignment error of each qubit and the CZ-gate error of each connected qubit pair, as reported by the IBM Quantum calibration data.

the color of each connection indicates the corresponding CZ-gate error. For the processor in Supplementary Fig. 2(a), the median readout assignment error is 1.196×10^{-2} , with values ranging from 1.953×10^{-3} to 4.985×10^{-1} , while the median CZ error is 3.028×10^{-3} , with a range from 1.229×10^{-3} to 1.435×10^{-1} . For the processor in Supplementary Fig. 2(b), the median readout assignment error is 2.087×10^{-2} , with values ranging from 5.371×10^{-3} to 3.093×10^{-1} , and the median CZ error is 2.953×10^{-3} , with values ranging from 8.644×10^{-4} to 9.211×10^{-2} .

References

- [1] Robert Raussendorf and Hans J. Briegel. A one-way quantum computer. *Physical Review Letters*, 86(22):5188–5191, 2001. doi: 10.1103/PhysRevLett.86.5188.
- [2] Marc Hein, Jens Eisert, and Hans J. Briegel. Multi-party entanglement in graph states. *Physical Review A*, 69(6):062311, 2004. doi: 10.1103/PhysRevA.69.062311.
- [3] Marc Hein, Wolfgang Dür, Jens Eisert, Robert Raussendorf, Maarten Van den Nest, and Hans J. Briegel. Entanglement in graph states and its applications. *arXiv preprint*, 2006.
- [4] Karl Kraus. *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, volume 190 of *Lecture Notes in Physics*. Springer, Berlin, Heidelberg, 1983. ISBN 978-3-540-12732-1. doi: 10.1007/3-540-12732-1.
- [5] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, Cambridge, 2010.
- [6] Mark M. Wilde. *Quantum Information Theory*. Cambridge University Press, Cambridge, 2 edition, 2017. doi: 10.1017/9781316809976.
- [7] Ruochen Ma and Chong Wang. Average symmetry-protected topological phases. *Physical Review X*, 13(3):031016, 2023. doi: 10.1103/PhysRevX.13.031016.
- [8] Ruochen Ma, Jian-Hao Zhang, Zhen Bi, Meng Cheng, and Chong Wang. Topological phases with average symmetries: The decohered, the disordered, and the intrinsic. *Physical Review X*, 15(2):021062, 2025. doi: 10.1103/PhysRevX.15.021062.

- [9] Ruochen Ma and Alex Turzillo. Symmetry-protected topological phases of mixed states in the doubled space. *PRX Quantum*, 6(1):010348, 2025. doi: 10.1103/PRXQuantum.6.010348.
- [10] Zhehao Zhang, Utkarsh Agrawal, and Sagar Vijay. Quantum communication and mixed-state order in decohered symmetry-protected topological states. *Physical Review B*, 111(11):115141, 2025. doi: 10.1103/PhysRevB.111.115141.
- [11] Shuangyuan Lu, Lucas Q. Silveira, and Yizhi You. Nonequilibrium topological response under charge dephasing. *arXiv preprint arXiv:2602.23484*, 2026.
- [12] Hanyu Xue, Jong Yeon Lee, and Yimu Bao. Tensor network formulation of symmetry protected topological phases in mixed states. *arXiv preprint arXiv:2403.17069*, 2024.
- [13] Yuchen Guo, Jian-Hao Zhang, Hao-Ran Zhang, Shuo Yang, and Zhen Bi. Locally purified density operators for symmetry-protected topological phases in mixed states. *Physical Review X*, 15(2):021060, 2025. doi: 10.1103/PhysRevX.15.021060.
- [14] Tsung-Cheng Lu, Zhehao Zhang, Sagar Vijay, and Timothy H. Hsieh. Mixed-state long-range order and criticality from measurement and feedback. *PRX Quantum*, 4(3):030318, 2023. doi: 10.1103/PRXQuantum.4.030318.
- [15] Lorenzo Piroli, Georgios Styliaris, and J. Ignacio Cirac. Quantum circuits assisted by local operations and classical communication: Transformations and phases of matter. *Physical Review Letters*, 127(22):220503, 2021. doi: 10.1103/PhysRevLett.127.220503.
- [16] Tsung-Cheng Lu, Leonardo A. Lessa, Isaac H. Kim, and Timothy H. Hsieh. Measurement as a shortcut to long-range entangled quantum matter. *PRX Quantum*, 3(4):040337, 2022. doi: 10.1103/PRXQuantum.3.040337.
- [17] Nathanan Tantivasadakarn, Ryan Thorngren, Ashvin Vishwanath, and Ruben Verresen. Long-range entanglement from measuring symmetry-protected topological phases. *Physical Review X*, 14(2):021040, 2024. doi: 10.1103/PhysRevX.14.021040.