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## Research Article

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# Logarithmic Configurational Response of Braided Vortex Filaments: Local Helicity Projection, Low-Defect Asymptotics, and a Falsifiable Reduced Model

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We formulate a reduced statistical model for vortex-filament rearrangements that separates exact finite-state counting from physical closure assumptions. The model does not enlarge the standard braid group by assertion. Instead, a hard-feasible vortex state is assigned a measurable local effective degeneracy  $g_i = g_0 + n_i$ , where  $g_0 = N - 1$  is a reference mode count and  $n_i$  is an integer or measured local multiplicity. For independent truncated word positions, the configurational entropy is exactly logarithmic,  $S - S_0 = k_B \sum_i M_i \log[(g_0 + n_i)/g_0]$ . A continuum limit gives a logarithmic entropy density. If, and only if, the local multiplicity is empirically related to a phase-gradient field by  $n = \mathcal{C}|\nabla\theta|^2$ , the exact reduced free-energy variation is a nonlinear divergence, not a Laplacian at arbitrary defect strength. The quadratic gradient entropy and Laplacian response follow as controlled low-defect asymptotics. We connect the reduced coordinate to local helicity without identifying the two: local helicity is a physical fluid projection, while the multiplicity field is a statistical state coordinate. Numerical checks verify finite-word counting, the discrete variational derivative, and the breakdown of the quadratic approximation as the dimensionless defect grows. The construction yields a bounded, falsifiable model; it does not by itself derive reconnection thresholds, mechanical forces, or universal turbulent laws.

Keywords: vortex filaments, local helicity, braid statistics, configurational entropy, topology-state reduction

## I. INTRODUCTION

Fluid helicity and its spatial organization provide a natural bridge between vortex geometry and reduced topological descriptions. The global helicity integral compresses information that can remain visible in local-helicity fields. In controlled vortex-tube states, local-helicity distributions can distinguish configurations that have similar global measures, motivating local rather than purely integral diagnostics<sup>1</sup>. Complete measurements of helicity and its dynamics in vortex tubes further show that twist, writhe, and linking can be resolved experimentally<sup>2</sup>. Recent high-resolution work also demonstrates that twisting vortex lines can alter extreme-event statistics in Navier–Stokes turbulence<sup>3</sup>.

The present paper develops a statistical branch model for this continuity problem. The aim is deliberately limited. We seek an exact finite-state entropy relation, a controlled continuum reduction, and a measurable response coordinate. We do not claim that an effective mode count is a new generator set of the standard braid group, nor that a reduced entropy gradient is automatically a mechanical force.

The construction follows four ordered layers: hard feasibility, a computable topology-state coordinate, free-energy variation on that reduced state space, and projection to an observable. This order is important because a global linking class is invariant only during a

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non-reconnecting evolution, whereas local helicity and local geometric multiplicity can vary continuously. Reconnection is therefore treated as an event outside the fixed-class ensemble.

## II. HARD-FEASIBLE STATES AND LOCAL EFFECTIVE MULTIPLICITY

Let  $\Omega \subset \mathbb{T}^3$  be a periodic domain containing  $N$  closed vortex filaments. Let  $\mathcal{A}$  denote the set of states satisfying the prescribed core resolution, periodicity, nonintersection, and no-reconnection conditions during a time window. The global linking class is fixed on  $\mathcal{A}$ . We partition  $\Omega$  into cells  $V_i$  and assign a reference local mode count  $g_0 = N - 1$ .

The reduced state coordinate is a local effective multiplicity

$$n_i \in \mathbb{N}_0, \quad g_i = g_0 + n_i. \quad (1)$$

The integer  $n_i$  is a measured or algorithmically counted local multiplicity of admissible geometric continuations. Equation (1) is a definition of the reduced ensemble, not a claim that standard braid algebra has acquired new generators. If a reconstruction produces a noninteger effective degeneracy,  $g_i$  is treated as a measured statistical weight and its uncertainty must be reported.

## III. EXACT FINITE-STATE ENTROPY

Suppose cell  $i$  contains  $M_i$  truncated word positions and positions are conditionally independent after the hard-feasibility gate. Its number of effective words is

$$\Omega_i = g_i^{M_i}. \quad (2)$$

Consequently, the entropy increment relative to  $n_i = 0$  is exactly

$$S - S_0 = k_B \sum_i M_i \log \left( 1 + \frac{n_i}{g_0} \right). \quad (3)$$

No quadratic defect law has been used. The assumptions are finite truncation, a specified local multiplicity, and the stated conditional product measure. Correlations between cells can be included by replacing the product measure; Eq. (3) then becomes a controlled baseline rather than an identity.

For a spatially resolved continuum description, define  $m(\mathbf{x}) = M_i/|V_i|$  and  $n(\mathbf{x}) = n_i$  in the cell limit. The corresponding entropy density is

$$s[n](\mathbf{x}) = k_B m(\mathbf{x}) \log \left( 1 + \frac{n(\mathbf{x})}{g_0} \right), \quad S - S_0 = \int_{\Omega} s[n] dV. \quad (4)$$

The continuum statement requires mesh refinement and a finite local correlation length; it is not a consequence of braid algebra alone.

## IV. PHASE-GRADIENT CLOSURE AND EXACT REDUCED RESPONSE

To connect Eq. (4) to a smooth phase coordinate, one may test the constitutive relation

$$n(\mathbf{x}) = \mathcal{C} |\nabla \theta(\mathbf{x})|^2, \quad \mathcal{C} > 0. \quad (5)$$

This relation is an empirical closure to be measured, not an endogenous theorem. For constant  $m$  and  $\mathcal{C}$ , and with  $E_{\text{real}}$  independent of  $\theta$ , the reduced free energy is

$$F[\theta] = E_0 - T k_B m \int_{\Omega} \log \left( 1 + \frac{\mathcal{C} |\nabla \theta|^2}{g_0} \right) dV. \quad (6)$$

Periodic variation gives

$$\frac{\delta F}{\delta \theta} = 2Tk_B m \nabla \cdot \left[ \frac{\mathcal{C} \nabla \theta}{g_0 + \mathcal{C} |\nabla \theta|^2} \right]. \quad (7)$$

If  $E_{\text{real}} = E_{\text{real}}[\theta]$ , its functional derivative must be added to Eq. (7). The quantity  $-\delta F/\delta \theta$  is a reduced-state selection field. A mechanical force requires a separately measured constitutive projection.

### A. Embedding in the projected topology-state framework

The functional derivative above is the field-coordinate representation of the projected topology-state gradient used in the author's earlier NSDS formulation. Let a finite discretization or a feature map define

$$\boldsymbol{\xi} = \Phi[\theta, n, h, \text{geometry}], \quad \mathcal{F}_{\text{top}}(\boldsymbol{\xi}) = -E_0 \log P(\boldsymbol{\xi}) + \text{const.}, \quad (8)$$

where  $h = \mathbf{u} \cdot \boldsymbol{\omega}$  is the physical local-helicity projection and  $P$  is the empirical or ensemble density on the declared feasible branch. When the entropy model supplies the density,  $\mathcal{F}_{\text{top}} = E_{\text{real}} - TS$  up to an additive constant. The reduced response and its mobility-weighted evolution are For the constrained finite ensemble this identification follows from

$$P(\boldsymbol{\xi} | H_0, \mathcal{A}) = Z^{-1} \exp[-E_{\text{real}}(\boldsymbol{\xi})/(k_B T)] \Omega(\boldsymbol{\xi}; H_0, \mathcal{A}), \quad (9)$$

so that  $-k_B T \log P = E_{\text{real}} - TS + \text{const.}$  with  $S = k_B \log \Omega$ . The conditioning on the hard-feasible set  $\mathcal{A}$  and fixed helicity sector  $H_0$  is essential; an unconstrained empirical density would describe a different ensemble.

$$\mathbf{f}_{\boldsymbol{\xi}} = -\nabla_{\boldsymbol{\xi}} \mathcal{F}_{\text{top}}, \quad \dot{\boldsymbol{\xi}} = \mathbf{M}_{\boldsymbol{\xi}} \mathbf{f}_{\boldsymbol{\xi}} + \boldsymbol{\eta}. \quad (10)$$

For an observable  $Y = \mathcal{P}_{\text{obs}}(\boldsymbol{\xi})$ , the measurable response is the push-forward

$$\dot{Y} = D\mathcal{P}_{\text{obs}}(\boldsymbol{\xi}) \mathbf{M}_{\boldsymbol{\xi}} \mathbf{f}_{\boldsymbol{\xi}} + D\mathcal{P}_{\text{obs}}(\boldsymbol{\xi}) \boldsymbol{\eta}. \quad (11)$$

Thus Eq. (7) is not an additional Navier–Stokes body force. It is the continuum-coordinate form of  $\mathbf{f}_{\boldsymbol{\xi}}$  before mobility and observable projection. This identification preserves the earlier NSDS claim boundary: topology-state gradients are reduced response coordinates, whereas the local Navier–Stokes dynamics remain the physical evolution law.

### B. Low-defect limit

With  $u = \mathcal{C} |\nabla \theta|^2 / g_0 \ll 1$ ,

$$\log(1 + u) = u - \frac{u^2}{2} + O(u^3). \quad (12)$$

Thus

$$S - S_0 = k_B m \frac{\mathcal{C}}{g_0} \int |\nabla \theta|^2 dV - k_B m \frac{\mathcal{C}^2}{2g_0^2} \int |\nabla \theta|^4 dV + \dots, \quad (13)$$

$$\frac{\delta F}{\delta \theta} = 2Tk_B m \frac{\mathcal{C}}{g_0} \nabla^2 \theta + O(u \nabla^2 \theta, \nabla u \cdot \nabla \theta). \quad (14)$$

The commonly used quadratic entropy is therefore an asymptotic approximation with a stated small parameter.

## V. RELATION TO LOCAL HELICITY

For an incompressible velocity field  $\mathbf{u}$ , vorticity is  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  and local helicity is  $h(\mathbf{x}) = \mathbf{u} \cdot \boldsymbol{\omega}$ . The previous work of Pan established local helicity as a branch-sensitive physical diagnostic rather than merely a post hoc global integral<sup>1</sup>. In the present model,  $h$  is a projected observable extracted from the fluid state, while  $n$  or  $\theta$  is a reduced statistical coordinate. They must not be identified without data.

A practical test is to compare rearrangement rate against three candidate predictors:

$$R_{\text{exact}} = \nabla \cdot \left[ \frac{\mathcal{C} \nabla \theta}{g_0 + \mathcal{C} |\nabla \theta|^2} \right], \quad (15)$$

$$R_{\text{quad}} = \nabla^2 \theta, \quad (16)$$

$$R_W = \nabla^2 W, \quad W = \log \left( 1 + \frac{\mathcal{C} |\nabla \theta|^2}{g_0} \right). \quad (17)$$

The statement  $\dot{\gamma}_{\text{top}} = \kappa R_W$  is an independent dynamical hypothesis, not a consequence of the counting theorem. Vortex-centreline tracking, local-helicity reconstruction, time-resolved braid distance, and a time-shuffled negative control are required for a physical test.

## VI. NUMERICAL CHECKS

The accompanying scripts use periodic grids and a fixed random seed. They are algebraic checks and do not solve Navier–Stokes or model reconnection.

The finite-word experiment compares  $M \log(1+q)$  with  $Mq$  for  $N = 8$ ,  $\mathcal{C} = 0.37$ , and  $K = 1.9$ . The variational experiment checks the discrete derivative of  $F = -\alpha T \sum |\nabla \theta|^2$ . The prediction control plants  $\dot{\gamma} = 0.8 \nabla^2 W$  plus noise and recovers slope 0.7992; an independent random field gives correlation  $-0.0035$ . Finally, the nonlinear test increases the maximum  $u = \mathcal{C} |\nabla \theta|^2 / g_0$  from  $7.9 \times 10^{-4}$  to 0.564, while the relative error of the quadratic response increases from approximately 2.5% to 33.0%.

### A. Fixed-helicity-sector branch experiment

To test the central constrained-branch statement directly, we performed an exact finite enumeration separate from the synthetic response positive control. Eight cells were assigned  $g_0 = 3$  neutral baseline states. At a branch-opening step, each cell received  $b$  additional states with helicity charges  $+1$  and  $b$  states with charge  $-1$ . Dynamic programming counted the number  $\Omega_{H=0}$  of global configurations in the fixed total-helicity sector  $H = 0$ , as well as the unconstrained count  $\Omega_{\text{all}}$ . This construction implements the conservation constraint explicitly: newly available branches enter the counted ensemble only through combinations whose charges sum to zero.

For  $b = 0$ ,  $\Omega_{H=0} = 6,561$ . Increasing to  $b = 6$  gives  $\Omega_{H=0} = 398,285,505$  and  $S_{H=0}/k_B$  increases from 8.789 to 19.803. The fraction of all local words that remain in the  $H = 0$

TABLE I. Numerical checks in the accompanying reproducible directory.

| Test                        | Result                                    | Interpretation                    |
|-----------------------------|-------------------------------------------|-----------------------------------|
| Finite-word expansion       | $q = 0.10043$ , error 4.94%               | error is controlled by $q$        |
| Discrete variation          | min. relative error $8.6 \times 10^{-15}$ | variation verified                |
| Positive prediction control | $R^2 = 0.9412$                            | planted linear relation recovered |
| Independent null control    | correlation $-0.0035$                     | no automatic correlation          |
| Nonlinear limit             | error 2.5% $\rightarrow$ 33.0%            | quadratic form fails at large $u$ |

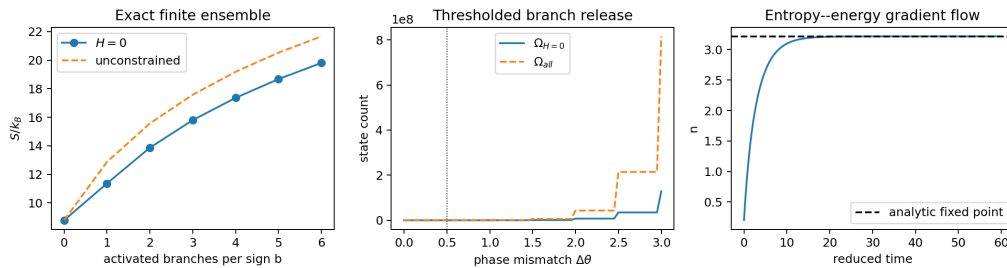


FIG. 1. **Constrained branch experiment.** Left: exact entropy in the fixed  $H = 0$  sector and in the unconstrained ensemble as compensated  $+/-$  branch pairs are added. Middle: a thresholded phase rule increases the number of fixed-sector states in discrete branch-opening steps. Right: the logarithmic entropy combined with a quadratic energetic term relaxes to the analytic NSDS fixed point.

sector decreases from 1 to 0.155, showing that conservation restricts the ensemble even while the absolute number of admissible compensated branches grows. A threshold rule that activates one additional  $+/-$  pair when  $\Delta\theta$  crosses  $\Delta\theta_c = 0.5$  produces the corresponding stepwise increase in  $\Omega_{H=0}$ ; the threshold is an imposed branch-opening rule, not inferred from the count.

The same exact logarithmic entropy was then used in a one-coordinate NSDS gradient flow,

$$F(n) = \frac{k}{2}n^2 - TM \log(1 + n/g_0), \quad \dot{n} = -\frac{\partial F}{\partial n}. \quad (18)$$

With  $(T, M, g_0, k) = (1, 4, 3, 0.2)$ , the analytic fixed point is  $n_* = 3.216990566$ . Explicit Euler integration with  $\Delta t = 0.002$  for 30,000 steps gives  $n = 3.216990536$ , an absolute error of  $3.0 \times 10^{-8}$ . Independent centered finite differences of  $F$  reproduce the analytic derivative with maximum absolute error  $2.6 \times 10^{-10}$ . These results validate the constrained finite-state entropy and its NSDS projected gradient internally. They do not establish the phase-opening threshold or the charge assignment for physical vortex filaments; those require centreline-resolved DNS or experiment.

## B. Physical projection bridge at fixed global helicity

The remaining bridge is from the discrete charge construction to a fluid field. We constructed periodic divergence-free velocity fields from three same-sign helical Fourier modes with  $|\mathbf{k}| = 1$ . Every mode has the same positive curl eigenvalue, so  $\boldsymbol{\omega} = \mathbf{u}$  and  $h = |\mathbf{u}|^2$ . Modal amplitudes were continuously redistributed while keeping  $\sum_j |A_j|^2$  fixed. The global helicity  $H = |\Omega|^{-1} \int h dV$  remained fixed to relative drift  $4.4 \times 10^{-16}$ , while the standard deviation of local helicity changed from 0.0998 to 1.045 and its 99th percentile changed from 2.193 to 3.938. Thresholded high- $h$  connected regions and phase-plaquette defects also changed with modal redistribution.

This experiment closes one physical projection link: local-helicity organization and phase-like branch proxies can change substantially inside one global-helicity sector. It does not identify a unique physical branch charge; connected-component and phase-defect counts must be calibrated against centreline braid distance. The result supports local helicity as a projected coordinate while leaving the phase-to-multiplicity constitutive map as an empirical closure.

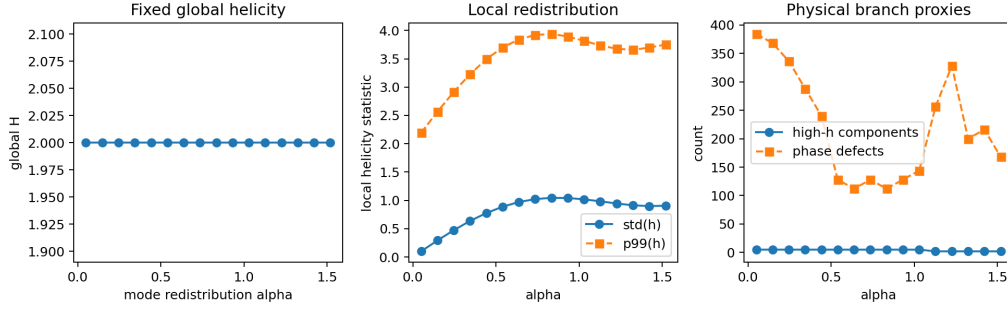


FIG. 2. **Physical projection bridge.** Same-sign helical Fourier fields preserve global helicity while modal redistribution changes local-helicity concentration and phase-like branch proxies. This demonstrates the separation between global conservation and local organization; it does not determine a unique filament branch number.

## VII. DISCUSSION AND CLAIM BOUNDARY

The strengthened chain has one exact combinatorial result, one explicitly defined continuum limit, and two measurable closures. The exact result is Eq. (3) under the stated product measure. Equation (5) is a physical hypothesis. Equation (7) is then an exact variation of the reduced functional. Equation (14) is an asymptotic expansion. This ordering prevents a fitted quadratic coefficient from being mistaken for a universal constant.

The model does not derive a reconnection threshold. Saturation of a reduced multiplicity can be used to define a candidate event coordinate, but reconnection requires a core-scale dynamical criterion and may change the global linking class. Nor does the reduced response constitute a body force. Its projection to momentum balance must be supplied by a system-specific constitutive law.

The connection to local helicity is nevertheless useful. Pan’s local-helicity work provides a physically measurable branch-sensitive observable, while the present model supplies a candidate statistical law for the accessible local state count. The two can be joined only by testing whether measured  $h(\mathbf{x}, t)$ , centreline rearrangements, and an independently reconstructed  $n(\mathbf{x}, t)$  obey the predicted response ordering. The correct outcome may be that the nonlinear predictor succeeds while the simple Laplacian law fails; that would refine rather than invalidate the finite-state entropy construction.

## VIII. CONCLUSION

We have written a reduced, bounded model for entropic response in braided vortex-filament states. Its exact content is a logarithmic finite-state entropy. A phase-gradient representation produces a nonlinear divergence response, with the quadratic gradient entropy and Laplacian term recovered only in the low-defect regime. Local helicity is retained as a measurable fluid projection, continuing the author’s earlier local-helicity program, but is not conflated with the statistical multiplicity coordinate. Numerical checks validate the algebraic and variational steps and quantify the range in which the quadratic approximation is reliable. A DNS or experiment with centreline tracking and local-helicity reconstruction is now the decisive next test.

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