

Online Resource 1 — Baseline-Aware Hysteretic Shape Modeling and Multilayer Design of P(VDF-TrFE-CTFE) Actuators for Continuum Robots

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Abstract

This Online Resource accompanies the main article. Section S1 verifies the planar CLT–Cosserat implementation used in the main text against closed-form laminate results, exact beam solutions, mesh refinement, and two independent constrained-shooting formulations. Section S2 extends the same sectional interface to tapered tubular robots with circumferential or helical strips and serial Cosserat sections, and verifies the new spatial operations numerically. Section S3 reports an image-space comparison for Sample 1, test 1 (SV1-a in the main article). Section S4 documents the sample identifiers, calibration settings, optical preprocessing, error metrics, and spatial-estimator sensitivity. Equation, figure, and table numbers without the prefix S refer to the main article.

S1 Numerical verification of the planar CLT–Cosserat implementation

This section verifies the implementation of Sections 3.3–3.4 against closed-form laminate results, exact beam solutions, mesh refinement, and two solution routes for the same constrained problem.

S1.1 Synthetic test case

All tests use a deterministic synthetic actuator with length $L = 20$ mm and width $b = 5$ mm, and a uniform active-layer eigenstrain $\varepsilon^* = 1.5 \times 10^{-3}$ is imposed directly, without the electrical law of Section 3.1. For the synthetic stack, the laminate model gives an intrinsic membrane strain of 1.4778×10^{-4} and a laminate curvature $\kappa_\ell = +17.024 \text{ m}^{-1}$, that is, a geometrical curvature $\kappa_g = -17.024 \text{ m}^{-1}$. With the right-handed rotation convention of the rod model, the laminate curvature enters the second Cosserat component directly, $\kappa_2 = \kappa_\ell = -\kappa_g$.

The rod is discretized with N_{mesh} nodes along its length. The refinement study uses $N_{\text{mesh}} = 21, 41, 81, 161$, and 321, so that the node spacing Δs is halved at each step. The analytical-limit and blocking calculations use $N_{\text{mesh}} = 1601$.

S1.2 Test families and acceptance limits

The implementation is exercised by 25 individual checks, grouped into the nine families of Table S1. Each family isolates one operation of the model: the transfer from the laminate to the rod, the integration of the rod, its response to loads, or the solution of the constrained problem.

Table S1: Planar verification checks and reference solutions.

Family (checks)	Check and independent reference
Reference-state recovery (1–3)	Zero-input centerline and tangent angle against the prescribed reference.
CLT interface (4–5)	Extension and curvature against the laminate closed form, Eq. (20).
Layer assembly (6–8)	Upper, lower and paired activation: symmetry, superposition and coupling $\mathbf{B}$.
Circular-arc limit (9–11)	Tip pose against the exact uniform-curvature arc.
Loaded limits (12–13)	Small-force Euler–Bernoulli deflection and exact pure-moment arc.
Reference-plane shift (14–15)	Curvature and shape invariance under a reference-plane shift.
Spatial refinement (16–18)	Tip pose on successively halved grids.
Blocking consistency (19–20)	Blocking force from nested scalar search and simultaneous shooting.
Boundary and rotation (21–25)	Exact terminal conditions, free-tip resultants, $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$, and $\det \mathbf{R} = 1$.

Acceptance limits are set by the type of reference. Where the reference is an exact algebraic identity (laminate transfer, reference-plane shift, layer assembly, rotation matrices, boundary residuals), the limit is close to machine precision, between 10^{-8} and 10^{-11} . Where the reference is a continuous solution approximated by the first-order discretization, the limit is a fraction of the rod length ($5 \times 10^{-4}L$ to $7 \times 10^{-4}L$)

or a relative error of 5×10^{-3} . Changes between the two finest grids must stay below 4×10^{-3} , and the observed convergence order must be at least 0.75. The two blocking formulations must agree within 2×10^{-6} .

S1.3 Results

All 25 checks pass. Table S2 gives the worst case of each family. Limits are upper bounds unless marked otherwise; the following paragraphs interpret the graphical checks.

Table S2: Planar verification results.

Test	Metric	Obtained	Limit
CLT interface	Intrinsic extension/curvature: max. relative error	1.72×10^{-16}	10^{-11}
Reference-state recovery	Centerline RMSE/ L	4.28×10^{-5}	5×10^{-4}
	Max. tangent-angle error (rad)	2.36×10^{-4}	3×10^{-4}
Circular-arc limit	Max. tip-position error/ L	4.48×10^{-4}	7×10^{-4}
	Tip-angle error (rad)	1.78×10^{-15}	10^{-10}
Loaded analytical limits	Small-force displacement error	4.61×10^{-5}	5×10^{-3}
	Pure-moment tip error/ L	1.24×10^{-4}	7×10^{-4}
Reference-plane shift	Field/shape: max. relative error	6.46×10^{-16}	10^{-10}
Spatial refinement	Loaded endpoint change/ L	5.35×10^{-4}	4×10^{-3}
	Angle change (rad)	2.40×10^{-4}	4×10^{-3}
	Observed order	0.996	≥ 0.75
Blocking consistency	Finest reaction change	3.11×10^{-3}	4×10^{-3}
	Nested/simultaneous difference	4.24×10^{-13}	2×10^{-6}
Boundary and rotation	Max. scaled boundary residual	1.77×10^{-13}	10^{-8}
	Max. SO(3) defect	2.01×10^{-14}	10^{-10}
Layer assembly	Symmetry/superposition error	3.47×10^{-12}	10^{-10}
	Curvature error without $\mathbf{B}$	37.4%	diagnostic

Reference-state recovery.

Figure S1 shows the first test. A nonuniform reference centerline is prescribed and the model is solved with zero electrical input; the computed nodes must fall on the prescribed curve. The error is reported in Table S2. This test isolates the initialization of the reference field from the electromechanical transfer, and it confirms that the measured baseline shape is not counted twice.

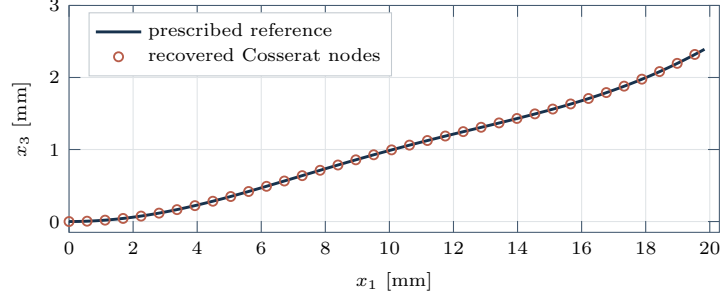


Fig. S1: Reference recovery. Line: prescribed centerline; markers: numerical nodes.

Analytical limits.

Figure S2 imposes five uniform curvatures, from 0 to 80 m^{-1} . Each numerical centerline must end on the tip of the corresponding exact arc. The largest endpoint discrepancy is $8.97 \mu\text{m}$ over the 20 mm rod, and the exponential rotation update recovers every terminal angle to roundoff.

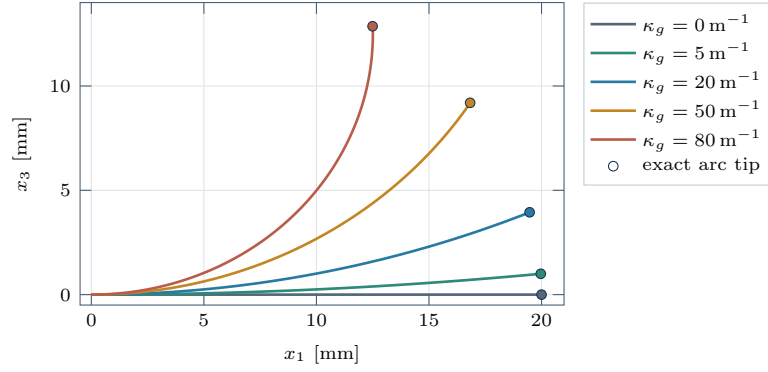


Fig. S2: Uniform-curvature benchmark. Lines: model; markers: exact arc tips.

Figure S3 applies external loads instead of an active strain. Under a small transverse tip force (a), the computed tip displacement is 0.133327 mm , against 0.133333 mm for the Euler–Bernoulli cantilever. Under a pure tip moment (b), the rod bends into an exact circular arc and the computed endpoint lies within $2.48 \mu\text{m}$ of it.

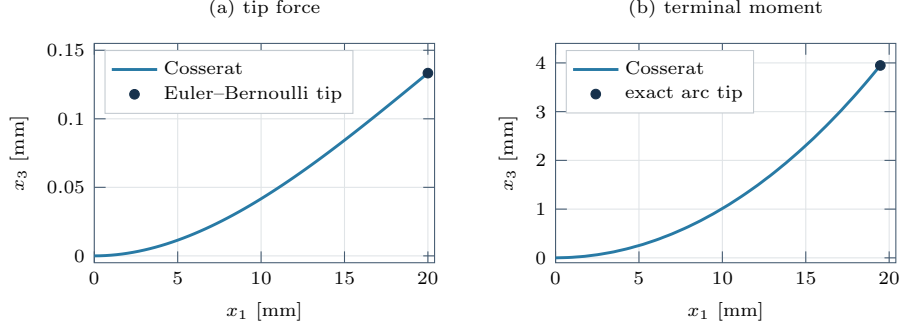


Fig. S3: Loaded limits: (a) tip force; (b) terminal moment.

Spatial refinement.

For a grid of N_{mesh} nodes, the change in tip position and tip rotation when the node spacing is halved is measured by

$$\begin{aligned} e_r^{(N_{\text{mesh}})} &= \frac{\|\mathbf{r}_{2N_{\text{mesh}}-1}(L) - \mathbf{r}_{N_{\text{mesh}}}(L)\|_2}{L}, \\ e_R^{(N_{\text{mesh}})} &= \|\text{vex}(\log[\mathbf{R}_{N_{\text{mesh}}}^\top(L) \mathbf{R}_{2N_{\text{mesh}}-1}(L)])\|_2, \end{aligned} \quad (\text{S1})$$

where the subscripts identify solutions computed with N_{mesh} and $2N_{\text{mesh}} - 1$ nodes, and vex is the inverse of the skew map $[\cdot]_\times$. For a first-order scheme, each halving of Δs halves these changes. The observed order p_{conv} is the slope of e_r against Δs on logarithmic axes.

Figure S4(a) shows this behavior for the free and loaded rods, and panel (b) for the blocking reaction: the points lie on approximately unit-slope lines, consistent with the first-order centerline update. The finest refinement satisfies all limits in Table S2; the blocking reaction reported below is computed on the finer $N_{\text{mesh}} = 1601$ grid.

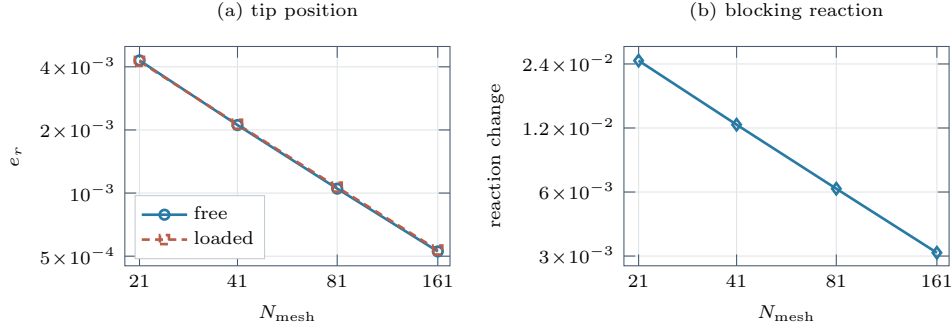


Fig. S4: Mesh refinement: (a) tip position; (b) blocking reaction.

Constrained and layer-assembly checks.

The blocking force is the transverse tip force that brings the tip back to zero displacement. It is computed by two independent formulations: a nested scalar root search on the force, and a simultaneous shooting on the force and the initial moment. Both give

$$F_{3,\text{block}} = 4.0165 \times 10^{-5} \text{ N} = 0.04016 \text{ mN}, \quad (\text{S2})$$

with agreement and constraint residuals near machine precision (Table S2). The small-deflection estimate is 0.03977 mN, about 1% lower, because it neglects the geometric nonlinearity. Figure S5 shows the tip coordinate against the applied tip force: the blocking force is the point where this curve crosses zero, and both formulations locate the same point.

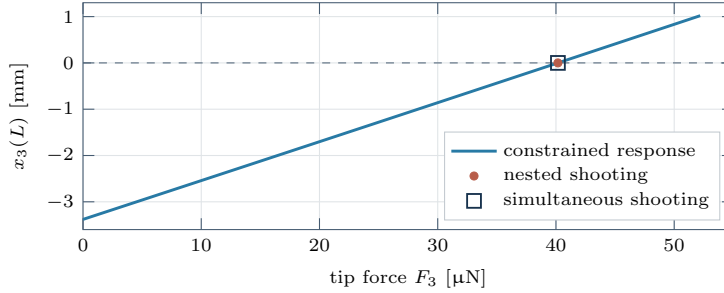


Fig. S5: Tip-coordinate response under transverse loading.

The layer checks apply the same active strain to the lower or to the upper layer of a symmetric test stack. The two cases give equal and opposite curvatures of $\pm 11.65 \text{ m}^{-1}$, and activating both layers cancels the curvature to roundoff. Replacing the coupled compliance by separate axial and bending divisions changes the curvature of the asymmetric test case by 37.4%, which shows that the extension–bending coupling $\mathbf{B}$ must be retained.

S2 Spatial, tapered, and multisection extension

S2.1 Reference configuration and sectional law

The experimentally tested actuator is planar, whereas a continuum robot can carry several EAP strips distributed around a tubular carrier. The general spatial kinematics and equilibrium equations are already given by Eqs. (25)–(28); the extension changes the sectional fields, the actuation map, and the serial domain, without introducing a second rod theory. The formulation remains quasi-static.

At actuator scale, the reference configuration is baseline-aware: the measured fabricated morphology $(\mathbf{r}^0, \mathbf{R}^0)$ is retained as an intrinsic field. The prospective robot design instead starts from a straight nominal reference, $\boldsymbol{\nu}_0 = [1, 0, 0]^\top$ and $\boldsymbol{\kappa}_0 = \mathbf{0}$,

because no tubular prototype morphology has yet been measured. The formulation retains s -dependent intrinsic fields, so a measured robot baseline can later be substituted without changing the model.

S2.2 Tubular cross-section

The tubular carrier reserves internal area for imaging, illumination, and fluid management while keeping the EAP strips near the outer surface, where their moment arm is largest (Fig. S6). The distal view shows the camera, illumination LEDs, and insufflation channels. Its cross-sectional axes are x_2 and x_3 , with x_1 pointing out of the page; the circumferential angle θ_i is measured from $+x_2$ toward $+x_3$. Mechanically, this replaces the planar laminate by a composite tubular section: the lumens are subtracted from the carrier properties and the exterior strips are added to them.

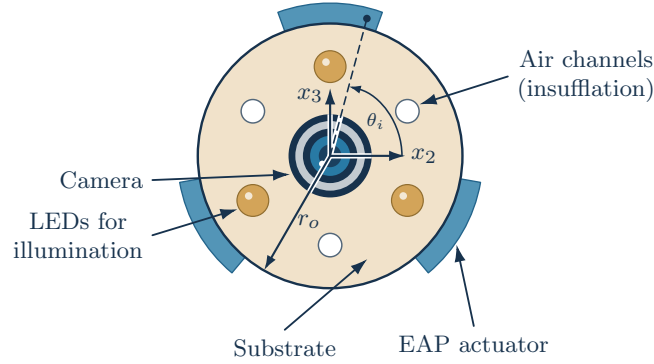


Fig. S6: Tubular cross-section viewed from the distal tip. Gold: substrate; blue: active polymer.

For an outer carrier radius r_o , consider lumen h of radius r_h , whose center is located at $(x_{2,h}, x_{3,h})$ in the cross-sectional frame. Its area is $A_h = \pi r_h^2$, and the effective carrier area, second moments of area about the two transverse axes, and polar moment are

$$\begin{aligned}
 A_s &= \pi r_o^2 - \sum_h A_h, \\
 I_{s,2} &= \frac{\pi r_o^4}{4} - \sum_h \left(\frac{\pi r_h^4}{4} + A_h x_{3,h}^2 \right), \\
 I_{s,3} &= \frac{\pi r_o^4}{4} - \sum_h \left(\frac{\pi r_h^4}{4} + A_h x_{2,h}^2 \right), \\
 J_s &= \frac{\pi r_o^4}{2} - \sum_h \left[\frac{\pi r_h^4}{2} + A_h (x_{2,h}^2 + x_{3,h}^2) \right].
 \end{aligned} \tag{S3}$$

The EAP actuators are bonded to the outer surface of the carrier as N_a identical active strips, each being a multilayer stack of the type studied in Section 6. Strip $i \in \{1, \dots, N_a\}$ has width w_a , total thickness t_a , cross-sectional area $A_a = w_a t_a$, equivalent axial modulus $\bar{Y}_a$, and angular location θ_i . Its contribution is added by area and parallel-axis homogenization, with the strip centroid placed at the carrier radius r_o ; the offset $t_a/2$ of the strip mid-plane is neglected in the moment arm. The strip width is fixed for the whole robot at 80% of each strip's angular share of the mean carrier circumference, $w_a = 0.8 (2\pi \bar{r}_o)/N_a$, where $\bar{r}_o$ is the arc-length average of $r_o(s)$; the remaining 20% insulates adjacent, independently driven strips. A constant width keeps the actuation force per unit active strain identical along the robot. It is geometrically admissible only where $r_o(s) \geq 0.8 \bar{r}_o$; with a strong taper this condition fails near the tip, where A_a is treated as an equivalent active area and a fabricated robot would use narrower distal strips. This thin-strip idealization gives

$$I_{a,i,2} = \frac{w_a t_a^3}{12} + w_a t_a (r_o \sin \theta_i)^2, \quad I_{a,i,3} = \frac{w_a t_a^3}{12} + w_a t_a (r_o \cos \theta_i)^2, \quad (\text{S4})$$

and the effective bending and torsional rigidities

$$(YI_2)_{\text{eff}} = Y_s I_{s,2} + \bar{Y}_a \sum_{i=1}^{N_a} I_{a,i,2}, \quad (YI_3)_{\text{eff}} = Y_s I_{s,3} + \bar{Y}_a \sum_{i=1}^{N_a} I_{a,i,3}, \quad (\text{S5})$$

$$(GJ)_{\text{eff}} \simeq G_s J_s,$$

where Y_s and G_s are the Young and shear moduli of the carrier. The torsional contribution of the strips is neglected because the stack is dominated by the terpolymer layers, whose modulus (171 MPa) is more than one order of magnitude below $Y_s = 5$ GPa.

The full sectional law remains Eq. (26), including the off-diagonal blocks $\mathbf{K}_{\text{se},\text{bt}}$ and $\mathbf{K}_{\text{bt},\text{se}}$. At robot scale, the equivalent-beam homogenization assembles $\mathbf{K}_{\text{se}}(s)$ and $\mathbf{K}_{\text{bt}}(s)$ separately, which yields a block-diagonal law. This reduction is appropriate for the symmetric strip layout considered here; if a lumen arrangement shifts the stiffness centroid appreciably, the full coupled 6×6 law must be restored before loaded predictions are interpreted quantitatively.

S2.3 Serial sections

Serial composition introduces N independently driven sections, indexed by $j \in \{1, \dots, N\}$, so that gross reach, backbone posture, and distal camera orientation can be adjusted without requiring identical lengths or radii. Let $\bar{s}_j \in [0, L_j]$ be the local arc coordinate of section j and L_j its length. At the interface between sections j and $j+1$, kinematic and static continuity require, for $j = 1, \dots, N-1$,

$$\begin{aligned} \mathbf{r}^j(L_j) &= \mathbf{r}^{j+1}(0), & \mathbf{R}^j(L_j) &= \mathbf{R}^{j+1}(0), \\ \mathbf{n}^j(L_j) &= \mathbf{n}^{j+1}(0), & \mathbf{m}^j(L_j) &= \mathbf{m}^{j+1}(0). \end{aligned} \quad (\text{S6})$$

A distal section is therefore solved in the pose delivered by all preceding sections and under the resultants transmitted through the chain: an actuation command applied

to one section can change the global pose of every downstream section even when the downstream actuation state is unchanged. For a clamped-free robot, with $\mathbf{I}$ the 3×3 identity matrix, the boundary conditions are

$$\mathbf{r}^1(0) = \mathbf{0}, \quad \mathbf{R}^1(0) = \mathbf{I}, \quad \mathbf{n}^N(L_N) = \mathbf{n}_{\text{tip}}, \quad \mathbf{m}^N(L_N) = \mathbf{m}_{\text{tip}}, \quad (\text{S7})$$

and the free-tip design simulations use $\mathbf{n}_{\text{tip}} = \mathbf{m}_{\text{tip}} = \mathbf{0}$. Contact or tool forces can be introduced through these terminal resultants or as distributed external resultants in Eq. (28). The boundary-value problem is solved section by section with interface states propagated along the chain, which produces the complete three-dimensional centerline and the tip material frame, needed because endpoint reachability alone does not determine the orientation of an embedded camera or tool.

S2.4 Taper

The carrier radius decreases along the robot. Within section j , the outer radius decreases linearly from its base value $r_{o,\text{base}}^j$ to its tip value $r_{o,\text{tip}}^j$ (Fig. S7). The taper ratio

$$\rho = \frac{r_{o,\text{tip}}^j}{r_{o,\text{base}}^j} \quad (\text{S8})$$

is common to all sections, so that $\rho < 1$ represents a progressively narrower distal body. Geometrical continuity makes the base radius of each section equal to the tip radius of the preceding one, and the radius along section j is

$$r_o^j(\bar{s}_j) = \rho^{j-1} r_{o,\text{base}} \left[1 - (1 - \rho) \frac{\bar{s}_j}{L_j} \right], \quad (\text{S9})$$

where $r_{o,\text{base}} = r_{o,\text{base}}^1$ is the proximal radius of the robot. A single proximal radius and one taper parameter thus determine the complete radius cascade. Because the bending rigidity of the carrier scales approximately with r_o^4 , modest changes in ρ produce a strong proximal-to-distal compliance gradient, and Eq. (S9) also makes the strip moment arm vary along each section.

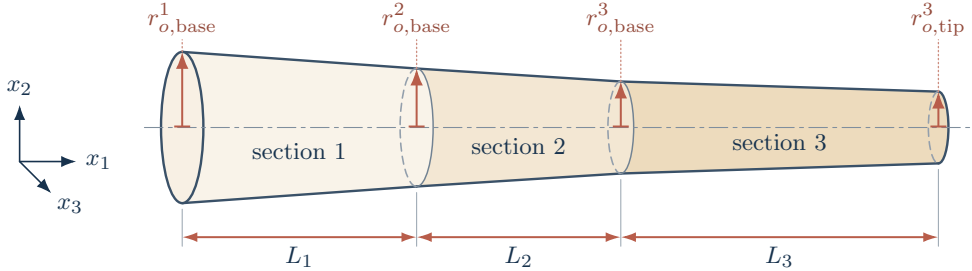


Fig. S7: Tapered tubular robot with three serial sections.

S2.5 Helical actuator routing

Routing determines how the longitudinal action generated by each strip is transmitted to the tubular backbone. In the parallel configuration (Fig. S8(a)), the strip tangent is aligned with the insertion axis; the helical configuration in panel (b) rotates this tangent through the routing angle α_j of section j , so that part of the same actuator action is redirected around the circumference.

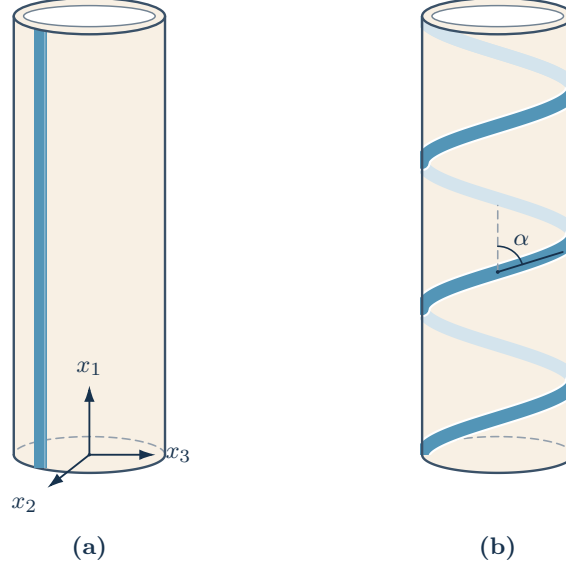


Fig. S8: Actuator routing: (a) parallel; (b) helical.

The longitudinal free strain of strip i in section j is the eigenstrain of Eq. (15) evaluated at that strip state, $\varepsilon_{a,i}^j = \varepsilon_p^*(E_{3,i}^j, D_{3,i}^j)$, which carries the piezoelectric, electrostrictive, and Maxwell terms into the robot scale. The strip transmits this action along its local unit tangent $\mathbf{t}_i^j$ as the force

$$\mathbf{F}_{a,i}^j = F_{a,i}^j \mathbf{t}_i^j, \quad F_{a,i}^j = \bar{Y}_a A_a \varepsilon_{a,i}^j. \quad (\text{S10})$$

In the material frame $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$, with $\mathbf{d}_1$ tangent to the centerline, a strip located at the circumferential angle θ_i has the moment arm $\mathbf{p}_i(s) = r_o(s) (\cos \theta_i \mathbf{d}_2 + \sin \theta_i \mathbf{d}_3)$, which does not depend on routing. With the local circumferential unit vector $\mathbf{e}_{\text{circ},i} = -\sin \theta_i \mathbf{d}_2 + \cos \theta_i \mathbf{d}_3$, the tangent of a helically routed strip is

$$\mathbf{t}_i^j = \cos \alpha_j \mathbf{d}_1 + \sin \alpha_j \mathbf{e}_{\text{circ},i}. \quad (\text{S11})$$

The section force and moment resultants are

$$\mathbf{f}_a^j(s) = \sum_{i=1}^{N_a} F_{a,i}^j (\cos \alpha_j \mathbf{d}_1 + \sin \alpha_j \mathbf{e}_{\text{circ},i}), \quad (\text{S12})$$

$$\mathbf{l}_a^j(s) = \sum_{i=1}^{N_a} \mathbf{p}_i(s) \times \mathbf{F}_{a,i}^j = r_o(s) \sum_{i=1}^{N_a} F_{a,i}^j \begin{bmatrix} \sin \alpha_j \\ \cos \alpha_j \sin \theta_i \\ -\cos \alpha_j \cos \theta_i \end{bmatrix}. \quad (\text{S13})$$

The first component of Eq. (S13), proportional to $\sin \alpha_j$, is the torsional moment introduced by the helical path; the two bending components are reduced by $\cos \alpha_j$. For equally spaced strips under equal activation, the circumferential force components cancel by symmetry while the torsional moments add for a common helix handedness; under differential activation, both bending direction and twist can be modulated. Setting $\alpha_j = 0$ gives parallel routing, $\mathbf{t}_i^j = \mathbf{d}_1$, for which the torsional component vanishes and equal activation cancels the two bending components. This holds within the uniaxial strip model, which transmits only the longitudinal strip force; the transverse active strain of the strips is not represented in the robot-scale section. The resultants of Eqs. (S12)–(S13) are expressed in the material frame, in which the internal force $\mathbf{n}$ and moment $\mathbf{m}$ of Eq. (28) are also written, so they enter equilibrium without further transformation; routing enters only through the direction of the actuator resultants, and the identified strain law is unchanged.

S2.6 Numerical verification of the spatial extension

The laminate transfer and the single-section rod were verified in Section S1 and are not tested again. The tests below prescribe the intrinsic curvatures or the strip strains directly, instead of computing them from the electrical law, so that a discrepancy can only come from the new geometrical and assembly operations. Each result is compared with a reference obtained independently of the implemented model:

- for sections of constant curvature, the exact solution of the rod kinematics (a chain of circular arcs, obtained by the matrix exponential);
- for helical and tapered sections, whose curvature varies along the arc length, a separate high-accuracy integration of the same continuous equations by an adaptive eighth-order scheme, with relative and absolute tolerances of 2×10^{-12} and 2×10^{-14} ;
- for loads and routed moments, closed-form statics.

The implemented model uses a first-order update along the arc length, so its error must decrease in proportion to the node spacing Δs ; the observed convergence order p_{conv} is measured by halving Δs several times. Table S3 lists the nine tests, what each one feeds to the model, and what it checks. The taper parameter uses $\varphi = (1 + \sqrt{5})/2$. The static reference is $\mathbf{m}(s) = (\mathbf{r}(L) - \mathbf{r}(s)) \times \mathbf{n}_{\text{tip}}$, expressed in a common spatial frame.

Table S3: Multisection verification cases and reference checks.

Test	Prescribed case	Check
Single section	Same section in single- and multisection models	Reduction to the planar implementation
Serial planar pose	Lengths (20, 15, 25) mm; curvatures (28, -20, 16) m ⁻¹ ; 41 nodes/section, 121 total	Pose continuity and curvature signs
Subdivision	Uniform rod; 2–10 sections; fixed Δs	Invariance to added interfaces
Routed strip	One uniformly active strip; $\alpha = 0\text{--}30^\circ$	Bending/torsion decomposition, Eq. (S13)
Static transport	Straight three-section chain; transverse tip force 0.25 N	Moment transport and interface continuity
Tubular planar limit	Outer radius 1.6 mm; parallel strips	Planar reduction
Three-strip symmetry	120° spacing; equal strains and permutations of (2, -1, -1)%	Equal-input cancellation; equal differential bending at 120°
Helical field	Routing 25°; pitch 45 mm	Spatial bending and torsion
Tapered helix	$L = 64.2$ mm; linear taper $\rho = 1/\varphi$; 100 nodes	Moment-arm and stiffness gradients

Tests whose reference is an exact identity (single-section consistency, subdivision, routed moment, static transport, symmetry) must agree to working precision: $10^{-12}L$ and 10^{-12} rad for poses, 10^{-12} relative for the routed moment, and $10^{-12}F$ and $10^{-12}FL$ for forces and moments, where L is the test length and F the tip-force magnitude. Tests whose reference is a continuous solution approximated by the first-order update must satisfy a tip-position error below $5 \times 10^{-3}L$, a tip-orientation error below 3 mrad, and an observed order $p_{\text{conv}} \geq 0.9$. The taper ratio $\rho = 1/\varphi$ reproduces the radius cascade of the prospective telecystoscope architecture; it is not fitted to the verification results.

All tests pass (Table S4); limits are upper bounds unless marked otherwise. The tubular and helical cases use 121 nodes. In the table, κ_1 denotes the torsional curvature and $\kappa_b = (\kappa_2^2 + \kappa_3^2)^{1/2}$ the bending-curvature magnitude.

Figure S9(a) shows the planar chain crossing the two curvature interfaces (circles) continuously. Its remaining endpoint discrepancy reflects discretization, while the interface continuity errors are near machine precision (Table S4). Panels (b,c) show the helical and tapered cases, respectively, with x_3 in blue and x_2 in coral. The small separation from the reference decreases under mesh refinement. In panel (d), halving the maximum integration step Δs approximately halves the tip-position error, consistent with the first-order update; the guide line has unit slope. Here $x_{k,\text{tip}}$ denotes the tip coordinate along x_k .

Figure S10(a) compares the closed-form curves with the numerical markers for routed actuation. Here the bending and torsional curvatures produced by one routed

Table S4: Multisection verification results.

Test	Metric	Obtained	Limit
Single-section consistency	Position difference	0	$10^{-12}L$
	Rotation difference	0	10^{-12} rad
Serial planar pose	Tip-position error	$3.42 \times 10^{-3}L$	$5 \times 10^{-3}L$
	Tip-orientation error	0 mrad	3 mrad
	Maximum interface gap	$< 10^{-12}L$	diagnostic
Routed-strip magnitude	Curvature-map relative error at 25°	2.20×10^{-16}	10^{-12}
Loaded static transport	Force error	0	$10^{-12}F$
	Moment error	1.19×10^{-17} N m	$10^{-12}FL$
	Interface resultant gaps	0	diagnostic
Section subdivision	Maximum tip-position change	4.6×10^{-17} m	$10^{-12}L$
	Rotation change	0	10^{-12} rad
Tubular planar limit	Tip-position error	$2.98 \times 10^{-3}L$	$5 \times 10^{-3}L$
	Tip-orientation error	0.009 mrad	3 mrad
Three-strip symmetry	Equal-input residual curvature	$1.99 \times 10^{-15} \text{ m}^{-1}$	10^{-10} m^{-1}
	Relative radial-tip spread	3.39×10^{-16}	10^{-10}
Helical spatial field	Tip-position error	$1.22 \times 10^{-3}L$	$5 \times 10^{-3}L$
	Tip-orientation error	2.464 mrad	3 mrad
	Observed order	0.993	≥ 0.9
Tapered helical field	Tip-position error	$9.48 \times 10^{-5}L$	$5 \times 10^{-3}L$
	Tip-orientation error	0.381 mrad	3 mrad
	Observed order	0.985	≥ 0.9

strip are shown in blue (κ_b) and coral (κ_1); the model coincides with the closed form at every angle, and the dashed line marks the 25° routing used in this verification case. In panel (b), the magnitude of the internal bending moment under the tip force decreases linearly from the base to the tip and is continuous at the two interfaces (dashed lines). In panel (c), the three differential commands place the tip at the same radius, 20.977 mm, in three directions 120° apart, while equal activation leaves the tip on the axis (cross).

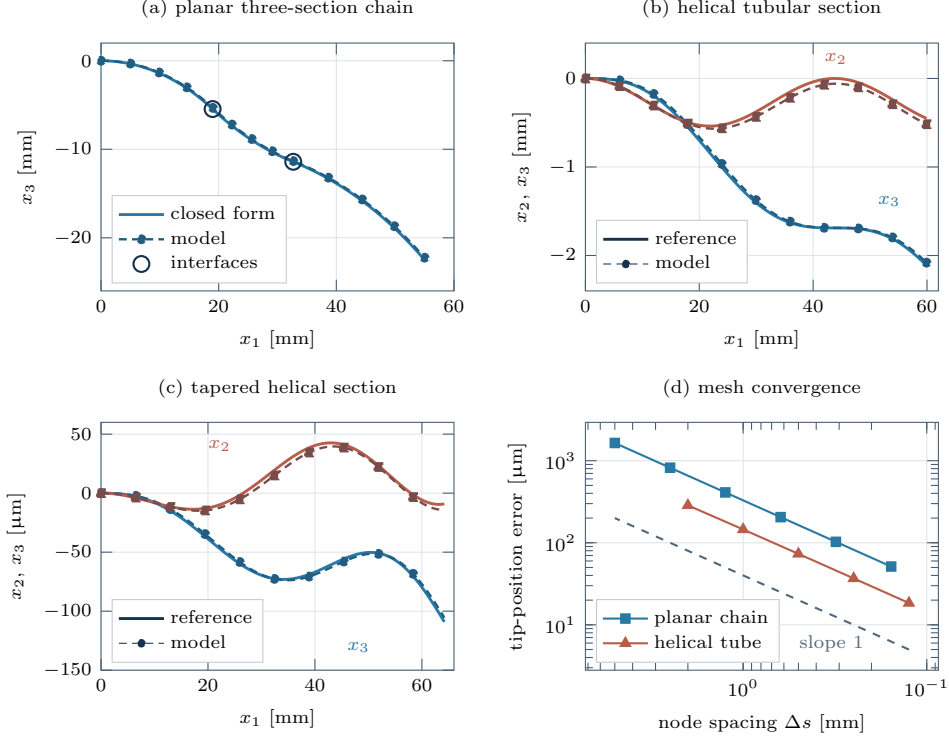


Fig. S9: Spatial verification: (a) serial chain; (b) helical section; (c) tapered helix; (d) convergence. Solid: reference; dashed with markers: model.

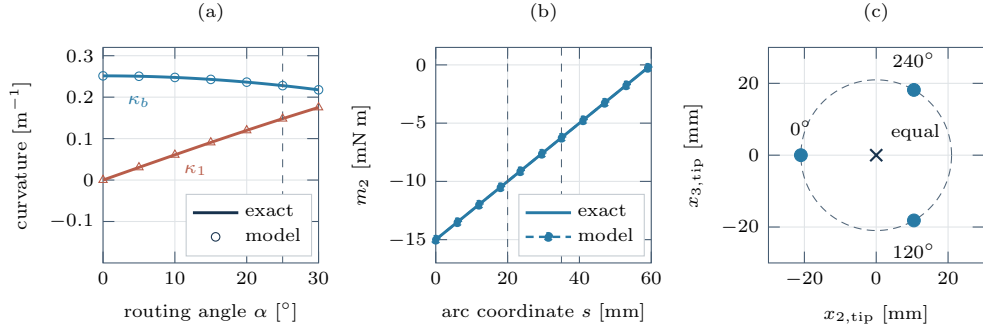


Fig. S10: Additional checks: (a) routed actuation; (b) moment transport; (c) command symmetry.

These tests verify the serial assembly, the tubular section, the helical routing, the taper, and the transport of loads. Multisection shape, tubular blocking force, contact load, and helical torsion remain to be measured on a prototype.

S3 Image-space comparison for Sample 1, test 1 (SV1-a)

Figure S11 shows the six physical states of Fig. 9 for cycle 5 of Sample 1, test 1 (SV1-a in the main article), on which $\Gamma_{\kappa, \text{SV1}}^{\text{eff}}$ was calibrated (cycle 5 was excluded from stage 2). Panels (a)–(f) follow the state order of the main article and report their individual RMSEs. The largest discrepancies occur at the descending zero crossing (d) and negative half-amplitude (e).

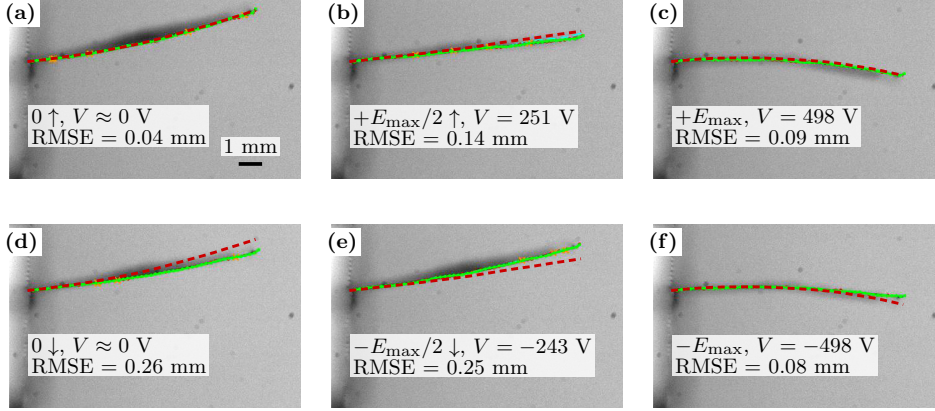


Fig. S11: Image-space comparison for Sample 1, test 1 (SV1-a). Green: tracked centerline; red dashed: model. Panels (a)–(f) use the same state order as the main article.

S4 Experimental processing and numerical settings

Sample 1 and Sample 2 correspond to SV1 and SV2, respectively, in the main article. The measured reference centerline of each sample remains fixed during evaluation; the processing and calibration conventions are given below.

S4.1 Sample identifiers and optical preprocessing

Table S5 maps the test identifiers to the main-article records.

All eight acquisitions use a recorded visible length of 10 mm and 51-point processed centerlines. Reference arc lengths of 1431.9–1529.1 px imply acquisition-specific scales of 143.2–152.9 px/mm, distinct from the illustrative 113.4 px/mm checkerboard configuration in the main article. The recorded length settings have not been independently cross-checked against the original calibration target.

Centerlines were tracked by correlation with spatial regularization and base-translation removal. Only Sample 1, test 2 (SV1-b), has manual-correction flags: frames 1060 and 1151 in evaluation cycle 4. Subsequent comparisons retain the fixed reference registration without per-frame translation, rotation, scale optimization, or further frame rejection.

Table S5: Acquisition identifiers and corresponding main-article records.

Specimen	Test identifier	Main-article record
Sample 1	<code>sample1.sine1</code>	SV1-a
Sample 1	<code>sample1.sine2</code>	SV1-b
Sample 1	<code>sample1.sine3</code>	SV1-c
Sample 2	<code>sample2.370_test1</code>	SV2-a
Sample 2	<code>sample2.370_test2</code>	SV2-b
Sample 2	<code>sample2.500_test3</code>	SV2-c
Sample 2	<code>sample2.500_test4</code>	SV2-d
Sample 2	<code>sample2.500_0p1Hz</code>	SV2-r

Electrical data were sampled at 2048 Hz; median camera intervals were 0.0617–0.0644 s. Recorded timestamps set the cycle phase, and processed frame-associated voltages are used directly. A separate camera-delay uncertainty was not measured.

S4.2 Calibration and reference-state conventions

Following the fitting objectives and cycle partitions of the main article, Stage 1 minimizes the tip-loop residual over $\Gamma_g \in [-2000, 0] \text{ m}^{-1}$ with bounded scalar search (tolerance 10^{-5} m^{-1} , 500-iteration limit). The resulting geometric transfers are -662.848375 and $-81.996506 \text{ m}^{-1}$ for Samples 1 and 2, with calibration RMSEs of 0.167654 and 0.009885 mm. The article reports the corresponding laminate transfers with opposite signs.

For Stage 2, the reference centerline is interpolated to 501 arc points by piecewise cubic Hermite interpolation; the unwrapped tangent is smoothed with a third-degree, 51-point Savitzky–Golay filter. Trapezoidal integration maps curvature increments $[-100, 20] \text{ m}^{-1}$ (spacing 0.1 m^{-1}) to tip-motion magnitude. Bounded trust-region reflective least squares begins at $(p^0, 0)$ with parameter scaling, quadratic residuals, a first-order finite-difference Jacobian, a 5000-evaluation limit, and convergence tolerances of 10^{-12} .

The sum-of-squares regularization weights of Eq. (33) are 25 for relative $\bar{Q}_{13}$ and 625 for δ_Q , corresponding to a δ_Q prior width of 0.04; the final objective is 9.090599062. The shared parameters remain fixed during evaluation.

The processed tip-loop observable subtracts each cycle’s minimum modeled eigenstrain, whereas shape reconstruction subtracts the eigenstrain at the optical reference frame, retaining its voltage and branch. The first voltage increment determines branch initialization (zero selects ascending). Loop closure uses the maximum absolute field of the acquisition or cycle; no separate polarization transient is identified.

Reference eigenstrains are about 0.00175–0.00177 for Sample 1, 0.001217 for Sample 2 at 369 V, and 0.00162–0.00164 at 498 V, yielding peak geometric-curvature increments of -54.63 to -54.65 , -4.365 , and -6.31 to -6.32 m^{-1} , respectively.

S4.3 Reference-preserving shape integration and error metrics

The experimental reconstruction neglects axial extension. To recover the measured reference despite tangent smoothing and numerical integration, the integrated change in unit tangent is added to the original centerline:

$$\mathbf{r}(s, t) = \mathbf{r}^0(s) + \int_0^s \left\{ \begin{bmatrix} \cos[\theta^0(\xi) + k(t)\xi] \\ \sin[\theta^0(\xi) + k(t)\xi] \end{bmatrix} - \begin{bmatrix} \cos \theta^0(\xi) \\ \sin \theta^0(\xi) \end{bmatrix} \right\} d\xi, \quad k(t) = \Gamma_g \Delta \varepsilon_p^*(t). \quad (\text{S14})$$

The integration uses 501 arc coordinates with a trapezoidal rule; the predicted centerline is then evaluated at the 51 recorded points. In the consistent-reference limit, this construction gives the inextensible, unloaded specialization of the rod kinematics. The general loaded and multilayer formulations retain coupled axial strain.

Across the 14 evaluation cycles, the mean processed-loop RMSE is 0.0965384 mm. The median of the 14 per-cycle median shape errors is 0.121781 mm, whereas pooling all evaluation frames before computing the median gives 0.118486 mm. For the ten cycles excluding structural-calibration records, the corresponding mean loop error and median of the per-cycle shape medians are 0.0905264 and 0.128216 mm. The main article’s record table averages per-cycle shape medians; its model-comparison table uses 84 standardized states.

For signed transverse-tip evaluation, the error is the predicted minus measured transverse tip coordinate on each native camera frame, without symmetrization or minimum removal. The mean cycle RMSE is 0.175656 mm (Sample 1: 0.343137 mm; Sample 2: 0.050044 mm). The mean ratios of processed-loop RMSE to loop span are 6.31% for Sample 1 and 18.36% for Sample 2, including test 4 (SV2-d).

For the reduced-model comparison in the main article, the quadratic geometric coefficient is constrained to $[-0.02, 0] \text{ m}^{-1}/(\text{MV m}^{-1})^2$, with a scalar-search tolerance of 10^{-12} . The (processed-loop, shape) RMSE pairs are (0.096387, 0.378366) mm for the straight-reference variant, (0.198964, 0.102667) mm for the quadratic variant, and (0.096538, 0.115395) mm for the complete model.

S4.4 Sensitivity of the spatial interpretation

The spatial sensitivity analysis compares cubic and quartic global fits, Savitzky–Golay windows of 9, 11, and 15 points, and three smoothing-spline settings. For each profile, the distal/proximal ratio is the absolute value of the mean curvature increment in the distal fifth ($s/L_v \in [0.8, 1]$) divided by that in the proximal fifth ($s/L_v \in [0, 0.2]$). For Sample 2, the median ratios at 369 and 498 V are 0.325 and 0.380 with cubic fits, but 0.113 and 1.781 with quartic fits. Only ten of the 16 estimator–amplitude combinations have a median ratio below 0.75, limiting confidence in the precise spatial localization of the reduced response. The interquartile band in Fig. 10 shows between-profile variation for the chosen estimator, rather than total measurement and differentiation uncertainty. Predictions for both samples use the same fixed visible-origin tangent convention.