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A counterexample to the complex polynomial Koebe radius conjecture

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Abstract

We disprove the complex-coefficient polynomial Koebe radius conjecture proposed by Dmitrishin, Smorodin and Stokolos in 2018. The construction is a normalised cubic polynomial whose quadratic coefficient has rational real and imaginary parts and whose cubic coefficient is rational. We prove that this polynomial is injective on a disk of radius $1001/1000$, so it is univalent on a neighbourhood of the closed unit disk. An explicit unit-modulus point has an image that is omitted from the open unit-disk image and has modulus strictly below the conjectured cubic radius, $(3 - \sqrt{5})/2$. Together with the published real cubic theorem, this yields a strict separation between the real and complex cubic Koebe radii. An elementary argument for degrees one and two shows that degree three is the least possible degree of failure. The proof specialises the classical cubic univalence mechanism and certifies its strict inequalities by exact rational Bernstein coefficients. The construction gives an upper bound for the complex cubic radius; it does not determine its optimal value.

Keywords: Univalent polynomials, Koebe radius, complex cubic polynomials, Bernstein coefficients

2020 Mathematics Subject Classification: Primary 30C10; Secondary 30C25 , 30C50

1 Introduction

Write $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} = \mathbb{D}_1$. A function is univalent if it is holomorphic and injective. Let $\mathcal{U}_N$ be the class of polynomials

$$f(z) = z + \sum_{j=2}^N a_j z^j$$

that are univalent on $\mathbb{D}$, with complex coefficients, and let $\mathcal{U}_{N,\mathbb{R}}$ be its real-coefficient subclass. Degrees smaller than N are allowed. For either class $\mathcal{F}$, define its Koebe radius by

$$\rho(\mathcal{F}) = \sup\{r > 0 : \mathbb{D}_r \subseteq f(\mathbb{D}) \text{ for every } f \in \mathcal{F}\}.$$

The problem is to determine the largest centred disk covered by every polynomial in the class.

In 2018, Dmitrishin, Smorodin and Stokolos proposed the value

$$\rho(\mathcal{U}_N) = \rho(\mathcal{U}_{N,\mathbb{R}}) = \frac{1}{4 \cos^2(\pi/(N+2))} \quad (1.1)$$

through their real and complex radius conjectures in the preprint *Estimating the Koebe radius for polynomials* (arXiv:1805.06927v1, Sect. 6, Conjectures 2–3). Their proposed extremal family arose from an extremal problem for conjugate trigonometric polynomials. Later in 2018, Dmitrishin, Dyakonov and Stokolos stated (1.1) explicitly [3, arXiv v2, Sect. 3, (3.3)]. Their paper, published in 2019, proved the real cubic case [3, arXiv v2, Thm. 5.1]:

$$\rho(\mathcal{U}_{3,\mathbb{R}}) = \frac{3 - \sqrt{5}}{2}. \quad (1.2)$$

That theorem concerns real coefficients. We disprove the complex equality in (1.1) already for $N = 3$.

Theorem 1.1 *The polynomial*

$$p(z) = z + \left(\frac{9}{10} + \frac{i}{60}\right)z^2 + \frac{2}{7}z^3 \quad (1.3)$$

is injective on $\mathbb{D}_{1001/1000}$. For $\zeta = (-24 + 7i)/25$, the value

$$w_0 = p(\zeta) = -\frac{83219}{218750} + \frac{4711}{187500}i \quad (1.4)$$

is omitted by $p(\mathbb{D})$ and satisfies

$$|w_0|^2 = \frac{641029}{4410000}, \quad |w_0| < \frac{763}{2000} < \frac{3 - \sqrt{5}}{2}. \quad (1.5)$$

Consequently,

$$\rho(\mathcal{U}_3) \leq \sqrt{\frac{641029}{4410000}} < \frac{3 - \sqrt{5}}{2} = \rho(\mathcal{U}_{3,\mathbb{R}}).$$

The theorem refutes the complex radius assertion in Conjecture 3 of that 2018 preprint and the complex equality in [3, arXiv v2, (3.3)]. The upper bound is approximately 0.3812585, whereas the real cubic radius is approximately 0.3819660. We do not determine the optimal complex cubic radius. Section 5 proves directly that the real and complex radii agree with (1.1) in degrees one and two.

The univalence mechanism is classical. Brannan’s 1967 work describes the complex cubic coefficient region using the Dieudonné criterion and a quadratic zero-exclusion condition [2, Chap. 1, Sect. 2; Chap. 3, (1.1)–(1.3) and Thm. 3.1.2]; see also [1]. We specialise that mechanism to the rational interior point (1.3), give its analytic proof, and reduce admissibility to fifteen exact rational inequalities. The arithmetic can be reproduced independently with the accompanying Python checker.

Formal verification

The accompanying Lean project proves the concrete counterexample, including the actual complex polynomial, normalisation, holomorphicity, injectivity on $\mathbb{D}_{1001/1000}$, omission from $p(\mathbb{D})$ and the exact strict bounds. The witness is packaged by `Koebe.complex_cubic_counterexample`. The complex cubic disk-covering assertion is refuted by `Koebe.refutes_complex_cubic_radius_assertion`. Lean checks the quadratic exclusion and Bernstein identities. Its global injectivity proof uses open mapping and compactness in place of the equivalent argument-principle and Jordan-index proof below.

A clean build with Lean 4.34.1 and pinned mathlib passed, together with an independent literal statement check. The axiom audit contains only `propext`, `Classical.choice` and `Quot.sound`; no unproved mathematical hypothesis remains. Online Resource 1 includes the project, compiler evidence and exact declaration map in `FORMAL_VERIFICATION.md`. The real-class equality, least-degree discussion and class-radius supremum formulation remain outside Lean.

2 From quadratic zero exclusion to univalence

We give the analytic implication used by the certificate, including the passage from equal-radius pairs to arbitrary pairs.

Lemma 2.1 *Let $A \in \mathbb{C}$ and $c \in \mathbb{R}$ with $|c| < 1$. If*

$$|A - c\bar{A}| < 1 - c^2, \tag{2.1}$$

then $1 + Aw + cw^2$ has no zero in $|w| \leq 1$.

Proof Suppose $w = u + iv$ is such a zero and set $e = u^2 + v^2$. The constant term gives $e > 0$. Multiplication of the root equation by $\bar{w}$ yields

$$e \operatorname{Re} A = -u(1 + ce), \quad e \operatorname{Im} A = v(1 - ce).$$

Substitution and factorisation give

$$\begin{aligned} & e^2 (|A - c\bar{A}|^2 - (1 - c^2)^2) \\ &= (1 - e)(1 - c^2 e)((1 - c)^2 u^2 + (1 + c)^2 v^2) \geq 0. \end{aligned}$$

Here $0 < e \leq 1$ and $|c| < 1$. The strict inequality (2.1) makes the left-hand side negative, a contradiction. $\square$

Lemma 2.2 *Let $f(z) = z + az^2 + bz^3$ and $R > 0$. Suppose that, for every $x \in [0, 1]$, the polynomial*

$$q_x(w) = 1 + 2axw + b(4x^2 - 1)w^2$$

has no zero in $|w| \leq R$. Then f is injective on $\mathbb{D}_R$.

Proof Two distinct points on any circle $|z| = r \leq R$ can be written, after interchanging them if necessary, as $z_1 = we^{i\theta}$ and $z_2 = we^{-i\theta}$, where $|w| = r$ and $0 < \theta \leq \pi/2$. Their divided difference is

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = 1 + a(z_1 + z_2) + b(z_1^2 + z_1z_2 + z_2^2) = q_{\cos \theta}(w).$$

Thus f is injective on each such circle.

Suppose now that $f(z_1) = f(z_2) = \lambda$ for distinct $z_1, z_2 \in \mathbb{D}_R$, with no restriction on their moduli. Choose

$$\max(|z_1|, |z_2|) < r < R$$

such that $f(z) - \lambda$ has no zero on $|z| = r$. This is possible because a nonconstant polynomial has only finitely many roots. The image of this circle under f is a Jordan curve. The argument principle says that its index about λ equals the number of zeros of $f - \lambda$ in $\mathbb{D}_r$, counted with multiplicity, hence is at least two. The index of a Jordan curve about a point off the curve is 0, 1 or -1 . This contradiction proves injectivity on $\mathbb{D}_R$. $\square$

3 An exact certificate for the cubic

Set $a = 9/10 + i/60$, $b = 2/7$ and $R = 1001/1000$. We prove the zero exclusion required in Lemma 2.2 with this explicit enlarged radius.

Proposition 3.1 *For $y \in [0, 1]$, put*

$$\begin{aligned} c(y) &= \frac{2}{7}R^2(4y - 1), \\ H(y) &= (1 - c(y)^2)^2 - 4R^2y \left[\frac{81}{100}(1 - c(y))^2 + \frac{1}{3600}(1 + c(y))^2 \right]. \end{aligned} \quad (3.1)$$

Then $|c(y)| \leq 429429/500000 < 1$ and $H(y) > 1/4000$ throughout $[0, 1]$.

Proof The bound for c follows from $-1 \leq 4y - 1 \leq 3$. For the second assertion write $H(y) = \sum_{m=0}^4 h_m y^m$, where the rational numbers h_m are specified by (3.1). On an interval $[L, U]$, the Bernstein coefficients of degree four are

$$\beta_k(L, U) = \sum_{j=0}^k \frac{\binom{k}{j}}{\binom{4}{j}} (U - L)^j \sum_{m=j}^4 \binom{m}{j} h_m L^{m-j}, \quad 0 \leq k \leq 4. \quad (3.2)$$

Indeed, expand $H(L + (U - L)t)$ in powers of t and use

$$t^j = \sum_{k=j}^4 \frac{\binom{k}{j}}{\binom{4}{j}} \binom{4}{k} t^k (1-t)^{4-k}.$$

It follows that

$$H(L + (U - L)t) = \sum_{k=0}^4 \beta_k(L, U) \binom{4}{k} t^k (1-t)^{4-k}. \quad (3.3)$$

Table 1 gives integers M_k for which $\beta_k(L, U) \geq M_k/10^6$. These are exact rational lower bounds, obtained by substitution in (3.2) and clearing positive denominators.

Table 1 Exact integer lower bounds M_k for the degree-four Bernstein coefficients on the three subintervals: each coefficient is at least $M_k/10^6$.

$[L, U]$	M_0	M_1	M_2	M_3	M_4
$[0, 1/2]$	842798	321785	116788	40981	15017
$[1/2, 3/4]$	15017	2035	1514	3536	4899
$[3/4, 1]$	4899	6262	6966	3811	313

For example, the last coefficient is

$$\beta_4(3/4, 1) = H(1) = \frac{176594391587253330629}{5625000000000000000000} > \frac{313}{10^6}.$$

All entries are at least 313. The Bernstein basis functions in (3.3) are nonnegative and sum to one when $0 \leq t \leq 1$. The three intervals cover $[0, 1]$, so

$$H(y) \geq \frac{313}{10^6} > \frac{1}{4000}$$

throughout that interval. □

For $x \in [0, 1]$ set $y = x^2$. In the scaled quadratic

$$q_x(Rw) = 1 + Aw + cw^2, \quad A = 2Rax, \quad c = c(y),$$

we have

$$|A - c\bar{A}|^2 = 4R^2y \left[\frac{81}{100}(1-c)^2 + \frac{1}{3600}(1+c)^2 \right].$$

Proposition 3.1 gives $1 - c^2 > 0$ and $(1 - c^2)^2 - |A - c\bar{A}|^2 > 0$. Taking nonnegative square roots proves (2.1). Lemma 2.1 excludes zeros of q_x from $|w| \leq R$, and Lemma 2.2 proves that p is injective on $\mathbb{D}_R$.

3.1 Arithmetic verification

Online Resource 1 contains a Python standard-library verifier using integer arithmetic and `fractions.Fraction`. It derives the scaled defect in two ways, reconstructs every Bernstein expansion in the power basis, checks interval coverage and all coefficient bounds, and evaluates the omitted point by independent Horner and expanded formulas. A second presentation check ties every printed integer bound to the derived

rational coefficient. Both checks use explicit failure tests that remain active under Python optimisation. The archive includes the exact rational output and negative controls for changes to the scaling, defect coefficients, interval cover and omitted-value comparisons. No floating-point sampling enters the certificate. The analytic implications are the written arguments above and below.

4 The omitted value and the radius separation

We complete the proof of Theorem 1.1. Since

$$|\zeta|^2 = \frac{24^2 + 7^2}{25^2} = 1 < R^2,$$

both ζ and every point of $\mathbb{D}$ lie in the disk on which p is injective. Hence $p(\zeta) \notin p(\mathbb{D})$.

Direct rational evaluation gives (1.4) and the squared modulus in (1.5). The two strict comparisons follow from

$$\left(\frac{763}{2000}\right)^2 - \frac{641029}{4410000} = \frac{324929}{1764000000} > 0, \quad \left(\frac{2237}{1000}\right)^2 - 5 = \frac{4169}{1000000} > 0.$$

Thus $\sqrt{5} < 2237/1000$ and

$$\frac{3 - \sqrt{5}}{2} > \frac{3 - 2237/1000}{2} = \frac{763}{2000}.$$

Every centred disk of radius greater than $|w_0|$ contains the omitted point w_0 , so it cannot be contained in $p(\mathbb{D})$. The polynomial p belongs to $\mathcal{U}_3$ because it is normalised and univalent. This proves the claimed upper bound for $\rho(\mathcal{U}_3)$; combining it with (1.2) gives the strict separation.

5 The least degree of failure

For completeness, we verify the two smaller degrees without a coefficient-region theorem.

Proposition 5.1 *The real and complex polynomial Koebe radii satisfy*

$$\rho(\mathcal{U}_1) = \rho(\mathcal{U}_{1,\mathbb{R}}) = 1, \quad \rho(\mathcal{U}_2) = \rho(\mathcal{U}_{2,\mathbb{R}}) = \frac{1}{2}.$$

Consequently degree three is the least degree at which (1.1) fails.

Proof The only normalised polynomial of degree at most one is z , which gives the first equality. For $f(z) = z + az^2$, univalence on $\mathbb{D}$ requires $|a| \leq 1/2$: otherwise $f'(-1/(2a)) = 0$ at a point of $\mathbb{D}$, whereas an injective holomorphic function has nonzero derivative there. Conversely, if $|a| \leq 1/2$ and $z_1, z_2 \in \mathbb{D}$, then

$$|a(z_1 + z_2)| < 1$$

when $a \neq 0$, so the divided difference $1+a(z_1+z_2)$ cannot vanish. The case $a = 0$ is immediate.

On $|z| = 1$ we have $|f(z)| = |1+az| \geq 1-|a| \geq 1/2$. For every $|w| < 1/2$, Rouché's theorem implies that $f-w$ and f have the same number of zeros in $\mathbb{D}$, namely one. Thus $\mathbb{D}_{1/2} \subseteq f(\mathbb{D})$. For the real polynomial $f(z) = z + z^2/2$, the equation $f(z) = -1/2$ is $(z+1)^2 = 0$, so $-1/2$ is omitted from $f(\mathbb{D})$. This proves both upper bounds as well. The conjectured values in (1.1) are 1 and $1/2$ at $N = 1, 2$. $\square$

6 Concluding remarks

The real cubic extremal theorem does not extend to complex coefficients. The example also lies outside every normalised rotation of the real class: under $f(z) \mapsto e^{-i\alpha} f(e^{i\alpha} z)$ the ratio a_3/a_2^2 is invariant, whereas for (1.3) it equals

$$\frac{7200}{7(2915 + 108i)},$$

which is nonreal. A real-coefficient polynomial with $a_2 \neq 0$ has a real value of that ratio.

The result provides an explicit upper bound on the complex cubic radius. It does not identify a nearest omitted point for p or an extremiser for the whole complex class. The rational point and its strict certificate suffice to settle the complex assertion in degree three, without any optimisation claim or assertion of failure at every larger degree.

Supplementary information

Online Resource 1: Exact rational certificate data and verification sources, together with the pinned Lean project, declaration map, compiler evidence and reproduction instructions.

Statements and Declarations

Competing interests

The author declares no competing interests.

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Data and code availability

The exact rational verification sources, certificate data, pinned Lean project, compiler evidence and declaration map are supplied in Online Resource 1. All certificate data needed to reproduce the calculations are included there. Version 1, dated 12 October 2026, is publicly available as a Zenodo preprint (DOI: 10.5281/zenodo.23293631).

Author contributions

Aran S. Ziegler is the sole author and takes responsibility for the content.

Use of AI tools

The author used agentic AI tools developed by OpenAI to carry out the author's research strategy and assist with Lean formalisation and manuscript preparation. The author takes full responsibility for the content.

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Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- [ESM1.zip](#)