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## Research Article

### Keywords:

**Posted Date:** October 26th, 2026

**DOI:** <https://doi.org/10.21203/rs.3.rs-11336379/v1>

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**Additional Declarations:** No competing interests reported.

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# ON FOUR CONJECTURAL CONGRUENCE FAMILIES FOR THE OVERPARTITION FUNCTION

SUMIT KUMAR JHA

**ABSTRACT.** We establish four Ramanujan-type congruence families for the overpartition function proposed by Xuanling Wei. Two are the numbered conjectures modulo 7 and 19; the other two are the proposed square-prime families modulo 17 and 23. The proofs combine a Radu–Sellers finite criterion, polynomial holomorphic lifts of a quadratic-character differential filter, Cohen Eisenstein series, and the Kohnen–Shimura correspondence. Besides the proposed congruences, we derive restricted Hecke identities for all odd primes modulo 17 and 23, explicit Hecke identities modulo 19, and selected prime-power extensions. The finite verifications are specified with complete modularity hypotheses and reproducible source code.

## 1. INTRODUCTION

An overpartition of  $n$  is a partition in which the first occurrence of each distinct part may be overlined. Its counting function  $\bar{p}(n)$  is determined by

$$(1) \quad \sum_{n \geq 0} \bar{p}(n)q^n = \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2} = \frac{1}{\Theta(-q)}, \quad \Theta(q) = \sum_{r \in \mathbb{Z}} q^{r^2}.$$

We put  $\bar{p}(x) = 0$  when  $x$  is not a nonnegative integer.

Wei [7] proved congruences modulo 11 and 13, and in the closing section of the same paper proposed four more families. His Conjectures 4.1 and 4.2 concern the moduli 7 and 19; the moduli 17 and 23 appear as separate numerical proposals immediately before these two conjectures. We prove all four statements. The different-looking conditions have a common explanation: certain quadratic-character components of overpartition generating series are holomorphic modular forms whose Hecke action is tractable.

**Theorem 1.1** (Wei’s Conjecture 4.1). *For every  $n \geq 0$ ,*

$$(2) \quad \bar{p}(896n + 176) \equiv \bar{p}(896n + 688) \equiv \bar{p}(896n + 816) \equiv 0 \pmod{7}.$$

*Equivalently  $\bar{p}(16m) \equiv 0 \pmod{7}$  if  $m \equiv 3 \pmod{8}$  and  $(\frac{m}{7}) = 1$ .*

**Theorem 1.2** (Cohen–Eisenstein congruence modulo 17). *Let  $\mathcal{H}_{15/2} = \sum_{m \geq 0} H(7, m)q^m$  be Cohen’s Eisenstein series of weight  $15/2$ . For every positive integer  $m \equiv 3 \pmod{8}$ ,*

$$(3) \quad \bar{p}(17m) \equiv -2H(7, m) \pmod{17}.$$

*For every odd prime  $Q$ ,*

$$(4) \quad \bar{p}(17Q^2m) \equiv \left(1 + Q^{13} - Q^6 \left(\frac{-m}{Q}\right)\right) \bar{p}(17m) - Q^{13} \bar{p}(17m/Q^2) \pmod{17}.$$

*In particular, if  $11 \nmid m$  and  $(\frac{m}{11}) = -1$ , then  $\bar{p}(17 \cdot 11^2m) \equiv 0 \pmod{17}$ .*

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*Date:* 11 October 2026.

*2020 Mathematics Subject Classification.* 11P83 (primary); 11F37, 11F33 (secondary).

*Key words and phrases.* overpartitions, Wei’s conjectures, Ramanujan-type congruences, half-integral-weight modular forms, Cohen Eisenstein series, Hecke operators, Radu–Sellers method.

**Theorem 1.3** (Wei's Conjecture 4.2 and a second Hecke identity). *Put  $a(m) = \bar{p}(4m)$ . If  $m \equiv 3 \pmod{8}$  and  $\left(\frac{m}{19}\right) = 1$ , then*

$$(5) \quad a(13^2 m) \equiv \left(12 - 9 \left(\frac{m}{13}\right)\right) a(m) - 8a(m/13^2) \pmod{19},$$

$$(6) \quad a(17^2 m) \equiv \left(14 - 5 \left(\frac{m}{17}\right)\right) a(m) - 7a(m/17^2) \pmod{19}.$$

*In particular, if  $\left(\frac{m}{17}\right) = -1$ , then  $\bar{p}(4 \cdot 17^2 m) \equiv 0 \pmod{19}$ .*

**Theorem 1.4** (Cohen–Shimura congruence modulo 23). *Write  $\Delta E_8 = \sum_{n \geq 1} a_{20}(n)q^n$  and let  $S = \sum_{n \geq 1} s(n)q^n$  be the weight-21/2 cusp form constructed in (36). For every positive integer  $m \equiv 5 \pmod{8}$ ,*

$$(7) \quad \bar{p}(23m) \equiv -6H(10, m) + 2s(m) \pmod{23}.$$

*For every odd prime  $Q$ ,*

$$(8) \quad \begin{aligned} \bar{p}(23Q^2 m) &\equiv \left(a_{20}(Q) - Q^9 \left(\frac{m}{Q}\right)\right) \bar{p}(23m) - Q^{19} \bar{p}(23m/Q^2) \\ &\quad - 6 \left(1 + Q^{19} - a_{20}(Q)\right) H(10, m) \pmod{23}. \end{aligned}$$

*For  $Q = 13$ , this gives  $\bar{p}(23 \cdot 13^2 m) \equiv 0 \pmod{23}$  whenever  $\left(\frac{m}{13}\right) = 1$ .*

The moduli 17 and 23 illustrate the Eisenstein and cuspidal parts of the Kohnen–Shimura correspondence. The modulus 19 is subtler: its natural overpartition quotient is weakly holomorphic, so a holomorphic polynomial lift is necessary before the Hecke and Sturm arguments apply. The modulus 7 admits a comparatively direct eta-product proof using the Radu–Sellers criterion. We give the needed modular preliminaries, then the polynomial lift, and finally four proofs and their finite certificates.

The extraction identities modulo 17 and 23 used below are due to Wei; the Cohen decompositions, restricted Hecke identities and certificates are the additions in this work. The classical half-integral weight Eisenstein and Hecke theories are used in their conventional normalizations [1, 6, 3, 5].

## 2. MODULAR-FORM TOOLS AND FINITE CRITERIA

Write  $q = e^{2\pi i \tau}$ ,  $D = q d/dq$ , and use the Legendre symbol with value zero on multiples of the prime. We use the level-four modular forms

$$(9) \quad F(\tau) = \frac{\eta(4\tau)^8}{\eta(2\tau)^4} = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \sigma_1(n)q^n, \quad Y = \Theta^4,$$

whose weights are 2 and 2, respectively, while  $\Theta$  has weight 1/2. For  $L \in \{4, 8\}$ ,  $r \pmod{L}$ , and  $f = \sum c(n)q^n$ , write

$$(10) \quad \Pi_{L,r} \left( \sum_{n \geq 0} c(n)q^n \right) = \sum_{n \equiv r \pmod{L}} c(n)q^n = \frac{1}{L} \sum_{j=0}^{L-1} \zeta_L^{-rj} f(\tau + j/L).$$

We abbreviate  $\mathcal{R}_3 = \Pi_{8,3}$  and  $\mathcal{R}_5 = \Pi_{8,5}$ . All coefficient congruences are in the relevant ring of  $p$ -integral Fourier series.

**Lemma 2.1** (Additive projection at 2-power levels). *Let  $N \in \{4, 8\}$ ,  $L \in \{4, 8\}$ , and let  $f$  be a holomorphic half-integral-weight form of weight  $w$  on  $\Gamma_0(N)$ , with a fixed integral power of the theta multiplier and a Dirichlet character modulo  $N$ . Then  $\Pi_{L,r} f$  is a holomorphic form of weight  $w$  on  $\Gamma_0(NL^2)$ , with the restricted multiplier and character. The construction and the cusp holomorphy statement remain valid over a coefficient ring in which  $2L$  is invertible and which contains  $\zeta_L$ .*

If the reduction of  $\Pi_{L,r}f$  is holomorphic at every cusp, then coefficientwise vanishing through

$$B_{N,L}(w) = \left\lfloor \frac{w + \frac{1}{2}}{12} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(NL^2)] \right\rfloor$$

implies its identically zero reduction. The same assertion holds in characteristic zero for an integral Fourier series.

*Proof.* Fix  $\alpha = j/L$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(NL^2)$ , say  $c = NL^2v$ . Then

$$T_\alpha \gamma T_{-\alpha} = \begin{pmatrix} a + NLjv & b + \frac{j(d-a)}{L} - Nj^2v \\ NL^2v & d - NLjv \end{pmatrix}.$$

As every odd unit is its own inverse modulo 4 and 8,  $ad \equiv 1 \pmod{L}$  implies  $d \equiv a \pmod{L}$ . Thus this conjugate is integral and belongs to  $\Gamma_0(N)$ . Moreover, writing  $\gamma' = T_\alpha \gamma T_{-\alpha}$ , we have  $\gamma'(\tau + \alpha) = \gamma\tau + \alpha$  and

$$c(\tau + \alpha) + (d - c\alpha) = c\tau + d.$$

Therefore the automorphy factors have the same argument and square-root branch. The only remaining issue is equality of their arithmetic theta-multiplier constants, including the 2-part. Set  $d' = d - NLjv$  and  $v' = (N/4)v$ . Then  $c = (2L)^2v'$  and  $d' - d = -4Ljv'$ . For odd  $d$  prime to  $c$ , the Kronecker symbol satisfies

$$\left(\frac{c}{d}\right) = \left(\frac{v'}{d}\right), \quad \left(\frac{c}{d'}\right) = \left(\frac{v'}{d'}\right),$$

because the factors  $(2L)^2$  are squares. If  $v' \neq 0$ , the quadratic character  $d \mapsto (v'/d)$  on odd integers prime to  $v'$  has conductor dividing  $4|v'|$ . As  $d' \equiv d \pmod{4|v'|}$  and both are prime to  $v'$ , the two Kronecker symbols agree. Also  $d' \equiv d \pmod{4}$ , so the factor  $\varepsilon_d$  in the theta multiplier agrees. For  $v' = 0$ , the two matrices are identical. Finally  $d' \equiv d \pmod{N}$ , so the Dirichlet character agrees. Thus every rational translation has the same theta multiplier and character on  $\Gamma_0(NL^2)$  as the original form. Their finite average has the stated transformation law; rational translation also preserves holomorphy at every cusp. If  $f$  has Fourier coefficients in  $\mathbb{Z}_{(p)}$  for an odd prime  $p$ , orthogonality of roots of unity shows that  $\Pi_{L,r}f$  has Fourier coefficients in  $\mathbb{Z}_{(p)}$  as well. All intermediate sums are legitimate over  $\mathbb{Z}_{(p)}[\zeta_L, 1/L]$ ; no denominators divisible by  $p$  are introduced.

Multiplying by  $\Theta$  gives an integral-weight holomorphic form of weight  $w + 1/2$  on  $\Gamma_0(NL^2)$ . Since  $\Theta \in 1 + q\mathbb{Z}[[q]]$ , multiplication by it preserves the first nonvanishing Fourier index over the residue field. The ordinary Sturm bound gives the claim.  $\square$

**Lemma 2.2** (The integral-weight theta companion). *Let  $N$  be divisible by 4, and suppose that  $f$  has weight  $k + \frac{1}{2}$  on  $\Gamma_0(N)$  with multiplier  $v_\Theta^{2k+1}\chi$ , where  $\chi$  is a Dirichlet character. Then  $f\Theta$  is a holomorphic integral-weight form of weight  $k + 1$  on  $\Gamma_0(N)$  with Nebentypus*

$$\chi \chi_{-4}^{k+1}.$$

*This assertion also holds for cusp forms, and for projected forms on higher levels after restricting the multiplier and character. For every prime  $p \geq 5$ , the first nonzero Fourier index of  $f$  modulo  $p$  agrees with that of  $f\Theta$ .*

*Proof.* The classical theta-square transformation law says that  $\Theta^2$  has weight 1 and character  $\chi_{-4}$  on  $\Gamma_0(4)$ . Thus the product of the  $2k + 1$  theta multipliers of  $f$  with the additional theta multiplier of  $\Theta$  contributes  $\chi_{-4}^{k+1}$  and leaves integral weight  $k + 1$ . The holomorphy assertion follows from that of the factors. Finally  $\Theta = 1 + 2q + O(q^4)$  has integral coefficients and a unit constant term, so multiplication by it preserves the first nonzero coefficient modulo  $p$ .  $\square$

**Lemma 2.3** (Commutation with odd-prime Hecke operators). *For any odd prime  $Q$  and  $r \in \{3, 5\}$ , a level-4 holomorphic half-integral-weight form  $f$  satisfies*

$$\mathcal{R}_r(f \mid T(Q^2)) = (\mathcal{R}_r f) \mid T(Q^2)$$

*when the same coefficient-normalized operator is used on both sides. The identity remains valid modulo any odd prime.*

*Proof.* The  $T(Q^2)$  coefficient at  $q^n$  depends only on the coefficients at  $Q^2n$ ,  $n$ , and  $n/Q^2$ . Since  $Q^2 \equiv 1 \pmod{8}$ , every integral one of these indices has the same residue modulo 8. The quadratic-character factor does not change the index, and the desired equality follows coefficientwise.  $\square$

Multiplication of a level-4 half-integral-weight form of weight  $w$  by  $\Theta$  produces an integral-weight modular form at the same level. The usual Sturm bound at level  $N$  is

$$(11) \quad B_{\text{int}}(N, w) = \left\lfloor \frac{w + 1/2}{12} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \right\rfloor.$$

For the 8-class projections below  $N = 256$ , the index is 384; the companion bounds are 256 (modulo 17), 352 (modulo 23), and 6400 (modulo 19). For the 4-class plus-space test,  $N = 64$ , the index is 96 and the bound is 88.

**Lemma 2.4** (A recurrence certificate for prime-power towers). *Let  $\ell$  be prime, let  $a, b \in \mathbb{F}_\ell$ , and suppose that  $u_{j+1} = au_j - bu_{j-1}$  for every  $j \geq 1$ . Set*

$$M = \begin{pmatrix} a & -b \\ 1 & 0 \end{pmatrix}.$$

*If  $M^d = I$  and  $u_r = 0$  for some integers  $d \geq 1$  and  $r \geq 0$ , then  $u_j = 0$  for every  $j \geq r$  with  $j \equiv r \pmod{d}$ .*

*Proof.* For  $j \geq 0$  the recurrence gives  $\begin{pmatrix} u_{j+1} \\ u_j \end{pmatrix} = M^j \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$ . Hence  $M^d = I$  implies  $u_{j+d} = u_j$  for all  $j \geq 0$ .  $\square$

### 3. A POLYNOMIAL HOLOMORPHIC LIFT MODULO 19

For modulus 19 the character filter of the four-dissection begins with a weakly holomorphic modular form. We replace it by a holomorphic reduction using an explicit calculation in the polynomial ring of ordinary modular forms on  $\Gamma_0(4)$ . Put

$$Y = \Theta^4, \quad F = \frac{\eta(4\tau)^8}{\eta(2\tau)^4}, \quad T = Y - 16F = \Theta(-q)^4, \quad J = \frac{Y}{T}.$$

Here  $Y, F, T$  have weight 2 and  $J$  has weight zero. An application of the even-index operator  $U(2)$  twice to  $1/\Theta(-q)$ , using the usual theta duplication identities, gives

$$(12) \quad A = \sum_{n \geq 0} \bar{p}(4n)q^n = \frac{\Theta^3}{T} = \frac{J}{\Theta}.$$

We use the Serre derivative  $\delta_k U = DU - kE_2U/12$ . The following are exact polynomial identities on  $\Gamma_0(4)$ :

$$(13) \quad E_4 = Y^2 + 224YF + 256F^2, \quad E_6 = Y^3 - 528Y^2F - 8448YF^2 + 4096F^3,$$

$$(14) \quad \delta_2 Y = \frac{-Y^2 + 80YF}{6}, \quad \delta_2 F = \frac{5YF - 16F^2}{6}.$$

They follow by verifying the first coefficients through the respective level-four Sturm bounds, 2 and 3. The eta-product and theta derivative identities give

$$(15) \quad \rho_\Theta = \frac{D\Theta}{\Theta} - \frac{E_2}{24} = \frac{-Y + 80F}{24}, \quad \frac{DJ}{J} = 16F.$$

For a form  $f$  of weight  $w$  with inverse-theta multiplier, put

$$\mathcal{D}_{19,w}f = E_{18}\delta_w f + \frac{w}{12}E_{20}f.$$

Repeated applications update the weight at each step. Define

$$(16) \quad B_{19} = \mathcal{D}_{19}^{10}A + E_{18}^{10}\mathcal{D}_{19}A.$$

This weakly holomorphic series has weight  $399/2$ . As  $E_{18} \equiv 1$  and  $E_{20} \equiv E_2 \pmod{19}$ , the operator reduces to  $D$  on 19-integral Fourier series. It follows that

$$(17) \quad B_{19} \equiv \sum_{n \geq 0} (n^{10} + n)\bar{p}(4n)q^n \pmod{19}.$$

Euler's criterion says that the factor is  $n(1 + (\frac{n}{19}))$  for  $19 \nmid n$ ; it is zero for zero or nonsquare indices and equals  $2n$  for quadratic residues.

**Lemma 3.1** (Polynomial numerator). *For any weight- $k$  homogeneous polynomial  $U \in \mathbb{Z}_{(19)}[Y, F]$  define*

$$(18) \quad \mathcal{S}_{19,k}U = E_{18}\delta_k U + \frac{E_{18}(Y + 304F) + (2k - 1)E_{20}}{24}U.$$

Set  $U_0 = 1$  and  $U_{r+1} = \mathcal{S}_{19,20r}U_r$ . Then each  $U_r$  is a 19-integral homogeneous polynomial of degree  $10r$ . With

$$V_{19} = U_{10} + E_{18}^{10}U_1,$$

we have

$$(19) \quad \Theta B_{19} = JV_{19}, \quad B_{19} = \frac{\Theta^3 V_{19}}{T}.$$

*Proof.* The conjugate of  $\mathcal{D}_{19,k-1/2}$  under multiplication by  $\Theta$  is

$$\mathcal{R}_{19,k}H = E_{18}\delta_k H + \left(\frac{k}{12} - \frac{1}{24}\right)E_{20}H - E_{18}\rho\Theta H.$$

The product rule shows  $\mathcal{R}_{19,k}(\Theta f) = \Theta \mathcal{D}_{19,k-1/2}f$ . Substitute  $H = JU$  and use  $DJ/J = 16F$ ; the result is  $\mathcal{R}_{19,k}(JU) = J\mathcal{S}_{19,k}U$ . Iterating and using  $A = J/\Theta$  proves the factorization. The identities (13) show that  $\delta$  preserves  $\mathbb{Z}_{(19)}[Y, F]$ ; the integral structure of level-one forms away from 6 puts  $E_{18}$  and  $E_{20}$  in  $\mathbb{Z}_{(19)}[E_4, E_6]$  [2]. Hence  $\mathcal{S}_{19,k}$  preserves the polynomial ring, and since each application raises the weight by 20, the degree assertion follows.  $\square$

**Theorem 3.2** (Holomorphic reduction of the modulo-19 filter). *There is a homogeneous polynomial  $W_{19} \in \mathbb{F}_{19}[Y, F]$  of degree 99 such that*

$$(20) \quad B_{19} \equiv \Theta^3 W_{19}(Y, F) \pmod{19}.$$

*Thus the filtered series has a holomorphic level-four half-integral-weight lift of weight  $399/2$ .*

*Proof.* We prove polynomial divisibility rather than attempting to compare cusp expansions of the original weakly holomorphic form. The identities (13) yield

$$\delta(T) = -\frac{Y + 16F}{6}T,$$

so the Serre derivative descends to the quotient by  $(T)$ . Let  $(y_0, f_0) = (-1/4, -1/64)$ , a point on the line  $Y = 16F$ . For a homogeneous  $U$  of weight  $k$ ,

$$(\delta_k U)(y_0, f_0) = -\frac{k}{12}U(y_0, f_0).$$

Indeed, modulo  $(T)$ ,  $U$  is a scalar multiple of  $F^{k/2}$ , and  $\delta_2 F = (32/3)F^2$  on the line. The polynomial identities also give  $E_4(y_0, f_0) = E_6(y_0, f_0) = 1$ ; the normalized Eisenstein series  $E_{18}, E_{20}$  therefore both take value 1. Moreover  $y_0 + 304f_0 = -5$ . Consequently (18) gives

$$(\mathcal{S}_{19,k}U)(y_0, f_0) = \left(-\frac{k}{12} + \frac{-5 + 2k - 1}{24}\right)U(y_0, f_0) = -\frac{1}{4}U(y_0, f_0).$$

Induction shows  $U_r(y_0, f_0) = (-1/4)^r$ , and hence

$$V_{19}(y_0, f_0) = 4^{-10} - \frac{1}{4} \equiv 0 \pmod{19},$$

since  $4^9 = 2^{18} \equiv 1 \pmod{19}$ . Homogeneity implies  $\bar{V}_{19}(16, 1) = 0$ . The factor theorem in  $\mathbb{F}_{19}[Y, F]$  therefore yields  $\bar{V}_{19} = (Y - 16F)W_{19}$  with  $W_{19}$  homogeneous of degree 99. As  $T = 1 + O(q)$  is invertible in  $\mathbb{Z}_{(19)}[[q]]$ , reducing (19) proves (20). A lift of the coefficients of  $W_{19}$  to integers produces a classical holomorphic modular form, as required.  $\square$

The 8-class projection of this form has level at most 256. Its integral-weight companion, obtained by multiplication by  $\Theta$ , has weight 200 and Sturm bound 6400.

#### 4. THE THREE MODULO-7 PROGRESSIONS

This section proves Theorem 1.1 with a finite Radu–Sellers criterion [4]. To use the version for eta products, take

$$(m, M, N, t) = (896, 2, 56, 176), \quad r = (-2, 1), \quad \kappa = \gcd(m^2 - 1, 24) = 3,$$

with  $r$  indexed by the divisors 1, 2. Choose the auxiliary eta exponent vector  $r'$  on divisors of 56 with nonzero entries

$$(21) \quad r'_1 = 1830, \quad r'_2 = -915, \quad r'_7 = -256, \quad r'_{14} = 128.$$

The shift  $\sum_{\delta|M} \delta r_\delta$  is zero; the square-class orbit in the Radu–Sellers criterion is

$$(22) \quad P_{896,r}(176) = \{176, 688, 816\}.$$

All six admissibility divisibility conditions hold. In particular, the prime factors of  $m$  divide  $N$ , and the relevant  $\kappa$ -weighted sums vanish modulo 24 and 8. More precisely, the third and fourth admissibility expressions are  $-12644352 \equiv 0 \pmod{24}$  and  $-168 \equiv 0 \pmod{8}$ , while

$$\frac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} = 56 \mid N.$$

For the even- $m$  condition in the printed criterion, the factorization  $\prod_{\delta|M} \delta^{|r_\delta|} = 2$  gives  $(s, j) = (1, 1)$ ;  $4 \mid \kappa N = 168$  and  $8 \mid Ns = 56$ . These checks agree with the exact statement of Radu and Sellers, Lemma 2.4 and the subsequent definition of  $\Delta^*$  [4].

There is a point about the cusp representatives that requires care. A convenient lemma in the same reference uses squarefree level, but  $N = 56$  is *not* squarefree. Instead we use the general classification of cusps of  $\Gamma_0(N)$  [2]. For each  $c \mid 56$ ,  $\gcd(c, 56/c) \in \{1, 2\}$  and therefore  $\varphi(\gcd(c, 56/c)) = 1$ . Consequently the eight inequivalent cusps are represented by  $1/c$  as  $c$  ranges over the eight divisors of 56. The corresponding determinant-one matrices  $\gamma_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$  are thus a complete set of representatives in the Radu–Sellers criterion. We record the exact cusp verification so that no admissibility claim rests on a black box.

For a cusp representative with lower-left entry  $c$  set

$$p_{m,r}(\gamma) = \min_{0 \leq \lambda < m} \frac{1}{24} \sum_{\delta|2} r_\delta \frac{\gcd(\delta(a + \kappa\lambda c), mc)^2}{\delta m},$$

$$p_{r'}^*(\gamma) = \frac{1}{24} \sum_{\delta|56} r'_\delta \frac{\gcd(\delta, c)^2}{\delta}.$$

There are eight inequivalent cusps at level 56; representatives with  $a = 1$  give the values in Table 1. Since the index at level 56 is 96,  $\sum r_\delta = -1$ ,  $\sum r'_\delta = 787$ , and both weighted exponent sums are zero, the criterion's bound is

$$\nu = \frac{786 \cdot 96}{24} - \frac{176}{896} = 3143 + \frac{45}{56}.$$

It remains to verify the three congruences for  $0 \leq n \leq 3143$ . The standalone verifier `verify_overpartition_mod7_v98.cpp` computes  $1/\Theta(-q)$  modulo 7 through degree 2,816,944 using exact lifted NTT convolution and Newton inversion. There are  $3 \cdot 3144 = 9432$

TABLE 1. Exact cusp inequalities for the modulo-7 certificate.

| $c$           | $p_{m,r}$ | $p_{r'}^*$ | sum   |
|---------------|-----------|------------|-------|
| 1             | -56       | 6277/112   | 5/112 |
| 2             | 0         | 0          | 0     |
| 4             | 0         | 0          | 0     |
| 7             | -8/7      | 19/16      | 5/112 |
| 8, 14, 28, 56 | 0         | 0          | 0     |

required coefficients; all are zero. The exact NTT-convolution bound and a separately evaluated sparse-square recurrence appear in the reproducibility appendix. An independent rational-arithmetic audit reproduces the square-class orbit, all six admissibility conditions, and the eight cusp inequalities. The Radu–Sellers criterion now proves (2).

## 5. THE COHEN–EISENSTEIN BRIDGE MODULO 17

Wei's extraction gives

$$(23) \quad C = \Theta^{15} + 13F\Theta^{11} + 13F^2\Theta^7 = \sum_{n \geq 0} c(n)q^n, \quad c(n) \equiv (-1)^n \bar{p}(17n) \pmod{17}.$$

Cohen's weight-15/2 Eisenstein series [1] has the exact polynomial expansion

$$(24) \quad \mathcal{H}_{15/2} = -\frac{1}{12}\Theta^{15} + \frac{5}{2}F\Theta^{11} - 20F^2\Theta^7 + \frac{56}{3}F^3\Theta^3.$$

For clarity, this is an equality over  $\mathbb{Q}$ , not merely a congruence. Both sides belong to  $M_{15/2}(\Gamma_0(4))$ . Multiplication by  $\Theta$  gives ordinary integral-weight forms of weight 8 at level 4, whose Sturm bound is  $\lfloor 8[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)]/12 \rfloor = 4$ . The underlying half-integral-weight series agree at  $q^0, \dots, q^4$ , with coefficients

$$-\frac{1}{12}, \quad 0, \quad 0, \quad -\frac{14}{3}, \quad -\frac{61}{2}.$$

The two nonzero positive-index values follow independently from the generalized Bernoulli formula:  $H(7, 3) = L(-6, \chi_{-3}) = -14/3$  and  $H(7, 4) = L(-6, \chi_{-4}) = -61/2$ . Since  $\Theta(0) = 1$ , the first five coefficients of the multiplied forms agree as well. The ordinary integral-weight Sturm theorem proves (24) without a half-integral-weight bound or an assumed basis dimension. Modulo 17, direct comparison gives

$$C - 2\mathcal{H}_{15/2} \equiv 2K, \quad K = 2\Theta^{15} + 4F\Theta^{11} + F^2\Theta^7 + 4F^3\Theta^3.$$

The projected form  $\mathcal{R}_3 K$  has weight 15/2, level at most 256, and integral-weight Sturm bound 256 after multiplication by  $\Theta$ . An independently reconstructed exact integer-series checker gives  $[q^n]K \equiv 0 \pmod{17}$  at all 32 indices  $n \leq 256$ ,  $n \equiv 3 \pmod{8}$ ; the extraction congruence for  $C$  is separately compared with the sparse-square recurrence for  $\bar{p}(17n)$  at all 257 coefficients through this bound. Thus  $\mathcal{R}_3(C - 2\mathcal{H}_{15/2}) = 0 \pmod{17}$ ;  $m$  is odd, giving (3).

The Cohen series is a simultaneous  $T(Q^2)$  eigenform with eigenvalue  $1 + Q^{13}$ . If  $f = \sum f(n)q^n$  has weight  $k + 1/2$  in the standard level-4 normalization,

$$(25) \quad [q^n](f|T(Q^2)) = f(Q^2n) + Q^{k-1} \left( \frac{(-1)^k n}{Q} \right) f(n) + Q^{2k-1} f(n/Q^2).$$

Here  $k = 7$ . The  $\mathcal{R}_3$  projection commutes with  $T(Q^2)$  because  $Q^2 \equiv 1 \pmod{8}$ . Apply (25) to the projected equality and rearrange to get (4), proving Theorem 1.2.

**Corollary 5.1** (Prime-power towers modulo 17). *Let  $Q$  be odd,  $Q \nmid m$ ,  $m \equiv 3 \pmod{8}$ , and put  $t \equiv Q^{13}$ ,  $v \equiv Q^6$ ,  $s = \left(\frac{-m}{Q}\right)$  in  $\mathbb{F}_{17}$ . With  $u_j = \bar{p}(17Q^{2j}m) \pmod{17}$ ,*

$$(26) \quad u_1 = (1 + t - sv)u_0, \quad u_{j+1} = (1 + t)u_j - tu_{j-1} \quad (j \geq 1).$$

*If  $t \neq 1$ ,  $u_j = ((t - sv)t^j + sv - 1)(t - 1)^{-1}u_0$ ; if  $t = 1$ ,  $u_j = (1 + j(1 - sv))u_0$ . In particular the following are automatic zero classes:*

| $Q$ | condition on $m$                 | $j$                    |
|-----|----------------------------------|------------------------|
| 11  | $\left(\frac{m}{11}\right) = -1$ | $j \equiv 1 \pmod{16}$ |
| 11  | $\left(\frac{m}{11}\right) = +1$ | $j \equiv 4 \pmod{16}$ |
| 29  | $\left(\frac{m}{29}\right) = -1$ | $j \equiv 1 \pmod{16}$ |
| 29  | $\left(\frac{m}{29}\right) = +1$ | $j \equiv 3 \pmod{16}$ |

*Proof.* At  $Q^{2j}m$ , the middle Legendre symbol is zero for  $j \geq 1$ , so (26) follows from the Hecke recurrence. For each pair  $(Q, \left(\frac{m}{Q}\right))$  in the table, put  $a = 1 + Q^{13}$  and  $b = Q^{13}$  in Lemma 2.4. Direct substitution in  $\mathbb{F}_{17}$  gives  $M^{16} = I$ ; the initial value in (26) gives  $u_r = 0$  at the displayed representative  $r \in \{1, 3, 4\}$ . The lemma proves every stated residue class of zero exponents.  $\square$

**Corollary 5.2** (A positive-density family of Hecke primes modulo 17). *Fix a positive integer  $m \equiv 3 \pmod{8}$ . Then the set of odd primes  $Q$  for which at least one of the two sufficient conditions*

$$\begin{cases} Q \equiv 11 \pmod{17}, & \left(\frac{-m}{Q}\right) = 1, \\ Q \equiv 12 \pmod{17}, & \left(\frac{-m}{Q}\right) = -1 \end{cases}$$

*holds has natural density  $1/16$ . In particular, there are infinitely many primes  $Q$  for which  $\bar{p}(17Q^2m) \equiv 0 \pmod{17}$ .*

*Proof.* Set  $K = \mathbb{Q}(\sqrt{-m})$  and  $L = \mathbb{Q}(\zeta_{17})$ . The first field is imaginary quadratic; the unique quadratic subfield of  $L$  is  $\mathbb{Q}(\sqrt{17})$ , which is real. Thus  $K \cap L = \mathbb{Q}$  and  $\text{Gal}(KL/\mathbb{Q}) \simeq C_2 \times C_{16}$ . For any prime  $Q \nmid 34m$ , its Frobenius on  $L$  specifies  $Q \pmod{17}$  and its Frobenius on  $K$  specifies  $\left(\frac{-m}{Q}\right)$ . The two pairs in the statement represent *two distinct elements* of the abelian Galois group of order 32. By the Chebotarev density theorem, each occurs with density  $1/32$ , so their union has density  $1/16$ . By Theorem 1.2, every prime satisfying either condition yields the stated overpartition congruence. Removing the finitely many ramified primes does not change the density. This asserts a density of *sufficient Hecke primes*, not that the full set of vanishing primes has exactly this density.  $\square$

## 6. A HOLOMORPHIC DIFFERENTIAL FILTER MODULO 19

Put  $a(n) = \bar{p}(4n)$ , and retain  $A$  and  $B_p$  from (12) and (16). At  $p = 19$ , we have  $m = 10$  and

$$B_{19} = \mathcal{D}_{19}^{10}A + E_{18}^{10}\mathcal{D}_{19}A \equiv \sum_{n \geq 0} P_{19}(n)a(n)q^n \pmod{19}, \quad P_{19}(X) = X^{10} + X.$$

Euler's criterion gives

$$(27) \quad P_{19}(n) = \begin{cases} 2n, & \left(\frac{n}{19}\right) = 1, \\ 0, & \left(\frac{n}{19}\right) \in \{0, -1\}. \end{cases}$$

By Theorem 3.2 this reduction has a holomorphic classical lift on  $\Gamma_0(4)$ . Thus no separate analysis of its original weakly holomorphic cusp expansions is needed for the Hecke and Sturm argument.

For  $Q \in \{13, 17\}$  take  $\lambda_{13} = 12$ ,  $\lambda_{17} = 14$  and  $\mu_Q \equiv Q^2\lambda_Q \pmod{19}$ , hence  $\mu_{13} = 14$ ,  $\mu_{17} = 18$ . The following elementary scaling observation explains the Hecke normalization

used both here and in the preceding scaling calculation. Suppose  $p \equiv 3 \pmod{4}$ ,  $Q$  is an odd prime different from  $p$ , and  $m = (p+1)/2$ . Set  $P_p(X) = X^m + X$ , and write  $k_p = (p(p+2)-1)/2$ . Then modulo  $p$ ,

$$(28) \quad P_p(Q^2n) = Q^2P_p(n), \quad P_p(n/Q^2) = Q^{-2}P_p(n) \quad (Q^2 \mid n),$$

while  $Q^{k_p-1} = 1$ ,  $Q^{2k_p-1} = Q$ , and  $(-1)^{k_p} = -1$  in the respective finite-field expressions. It follows directly that for  $b(n) = P_p(n)a(n)$ ,

$$(29) \quad [q^n](B_p \mid T(Q^2)) \equiv Q^2P_p(n) \left( a(Q^2n) + Q^{-2} \left( \frac{-n}{Q} \right) a(n) + Q^{-3}a(n/Q^2) \right) \pmod{p}.$$

This identity uses Hecke operators only at positive weight, after the holomorphic reduction of  $B_p$ . The standard classical Hecke formula has coefficients integral at  $p$  when  $Q \neq p$ , and hence commutes with coefficientwise reduction of the explicit holomorphic polynomial lift. No Hecke–Fricke commutation for the original negative-weight quotient  $A$  is needed.

For the *positive*-weight form  $B_{19}$ , the Hecke formula (25) with  $k = 199$  and Fermat's theorem gives

$$(30) \quad \begin{aligned} & [q^n](B_{19} \mid T(Q^2) - \mu_Q B_{19}) \\ & \equiv Q^2P_{19}(n) \left( a(Q^2n) + Q^{-2} \left( \frac{-n}{Q} \right) a(n) \right. \\ & \quad \left. + Q^{-3}a(n/Q^2) - \lambda_Q a(n) \right) \pmod{19}. \end{aligned}$$

This identity is valid also when  $Q^2 \mid n$ . Note that both 13 and 17 are 1 (mod 4), so  $\left( \frac{-n}{Q} \right) = \left( \frac{n}{Q} \right)$ .

Project  $B_{19} \mid T(Q^2) - \mu_Q B_{19}$  onto  $n \equiv 3 \pmod{8}$ . By Theorem 3.2, the result is the reduction of a holomorphic weight-399/2 form of level at most 256, so (11) gives

$$B_{\text{int}}(256, 399/2) = \left\lfloor \frac{200 \cdot 384}{12} \right\rfloor = 6400.$$

The independent verifier `verify_overpartition_mod19_v98.cpp` reconstructs  $A = \Theta^3/(Y - 16F)$  and checks all 379 relevant indices  $n \leq 6400$ ,  $n \equiv 3 \pmod{8}$ ,  $\left( \frac{n}{19} \right) = 1$ , for each of  $Q = 13$  and 17, finding no Hecke error. Every other projected coefficient vanishes automatically by (27). The full projected Hecke errors vanish by Sturm, and division by  $Q^2P_{19}(n)$  proves (5)–(6).

**Corollary 6.1** (Closed prime-power formulas modulo 19). *Suppose  $m \equiv 3 \pmod{8}$ ,  $\left( \frac{m}{19} \right) = 1$ , and  $Q \nmid m$ . Write  $u_j = a(Q^{2j}m)$ , where  $a(n) = \bar{p}(4n)$ , and put  $\chi_Q = \left( \frac{m}{Q} \right)$ . For every  $j \geq 0$  the following identities hold in  $\mathbb{F}_{19}$ :*

$$(31) \quad a(13^{2j}m) \equiv \begin{cases} 3^j a(m), & \chi_{13} = 1, \\ (3 \cdot 9^j - 2 \cdot 3^j) a(m), & \chi_{13} = -1; \end{cases}$$

$$(32) \quad a(17^{2j}m) \equiv \begin{cases} 9^j a(m), & \chi_{17} = 1, \\ (13 \cdot 9^j + 7 \cdot 5^j) a(m), & \chi_{17} = -1. \end{cases}$$

For  $\chi_{13} = -1$ , the scalar in (31) vanishes precisely when  $j \equiv 6 \pmod{18}$ . For  $\chi_{17} = -1$ , the scalar in (32) vanishes precisely when  $j \equiv 1 \pmod{9}$ . Hence these two classes give infinite vanishing towers for  $\bar{p}(4Q^{2j}m)$ . Other coefficient zeros may occur if  $a(m) \equiv 0 \pmod{19}$ .

*Proof.* When  $Q \nmid m$ , Theorem 1.3 gives  $u_1 = (\lambda_Q - Q^{-2}\chi_Q)u_0$  in  $\mathbb{F}_{19}$ . For  $j \geq 1$  the index  $Q^{2j}m$  is divisible by  $Q$ , so its quadratic-character term vanishes. Thus

$$(33) \quad u_{j+1} = \lambda_Q u_j - Q^{-3}u_{j-1} \pmod{19},$$

with  $(\lambda_{13}, Q^{-3}, Q^{-2}) = (12, 8, 9)$  and  $(\lambda_{17}, Q^{-3}, Q^{-2}) = (14, 7, 5)$ . The characteristic polynomials split over  $\mathbb{F}_{19}$ :

$$X^2 - 12X + 8 = (X - 3)(X - 9), \quad X^2 - 14X + 7 = (X - 5)(X - 9).$$

Solving the corresponding initial-value problems gives (31) and (32). For  $Q = 13$  and  $\chi_Q = -1$ , the scalar is  $3^j(3^{j+1} - 2)$ ; its zeros satisfy  $3^j = 7$ . Since 3 has order 18 modulo 19 and  $3^6 = 7$ , this is equivalent to  $j \equiv 6 \pmod{18}$ . For  $Q = 17$  and  $\chi_Q = -1$ , the scalar vanishes exactly when  $(9/5)^j = -7/13 = 17$  in  $\mathbb{F}_{19}$ . The element  $9/5 = 17$  has order 9, yielding  $j \equiv 1 \pmod{9}$ .  $\square$

## 7. A COHEN–SHIMURA DECOMPOSITION MODULO 23

Wei [7] established the extraction model

$$(34) \quad C_{23} = \Theta^{21} + 9F\Theta^{17} + 5F^2\Theta^{13} + 14F^3\Theta^9 + 17F^4\Theta^5 + 20F^5\Theta, \\ [q^n]C_{23} \equiv (-1)^n \bar{p}(23n) \pmod{23}.$$

For  $j = 0, \dots, 5$  set  $B_j = F^j\Theta^{21-4j}$ . The following two plus-space forms have exact integer coefficient expressions:

$$(35) \quad U = B_0 - 42B_1 + 588B_2 - 2912B_3 + 2496B_4,$$

$$(36) \quad S = B_1 - 34B_2 + 336B_3 - 800B_4 + 512B_5 = \sum_{n \geq 1} s(n)q^n.$$

The plus-space assertions and the Shimura normalization are proved below, using the Fourier-support criterion and the  $t = 1$  Shimura lift. In particular,  $S(0) = 0$  and  $s(1) = 1$ , while  $U(0) = 1$  and  $U(1) = 0$ . The lift of  $S$  is exactly  $\Delta E_8$ . Consequently

$$(37) \quad S|T(Q^2) = a_{20}(Q)S$$

for every odd prime  $Q$ .

The weight-21/2 Cohen Eisenstein series satisfies

$$(38) \quad \mathcal{H}_{21/2} = \sum_{n \geq 0} H(10, n)q^n = \frac{174611}{6600}U - \frac{1}{132}S, \quad \mathcal{H}_{21/2}|T(Q^2) = (1 + Q^{19})\mathcal{H}_{21/2}.$$

Both rational denominators are prime to 23. Here is an exact certificate for the rational Cohen identity:  $M_{21/2}(\Gamma_0(4))$  has Sturm bound 5, and the coefficients of the asserted right-hand side through  $q^5$  are

$$\frac{174611}{6600}, \quad -\frac{1}{132}, \quad 0, \quad 0, \quad -\frac{523777}{132}, \quad -\frac{825502}{25}.$$

The Cohen formula gives these same six values: at index 4 use  $\zeta(-9)(1 + 2^{19} - 2^9)$ ; at index 5 use  $L(-9, \chi_5) = -825502/25$ . Thus the rational identity follows by Sturm's theorem. Moreover, the forbidden  $2, 3 \pmod{4}$  coefficients of  $U$  and  $S$  vanish through degree 88, the integral-weight level-64 Sturm bound for their forbidden modulo-4 projections. Lemma 2.1 gives this four-term additive projection at level  $4 \cdot 4^2 = 64$  directly. Multiplying each half-integral-weight projected series by  $\Theta$  gives weight 11 and index 96, hence Sturm bound  $\lfloor 11 \cdot 96/12 \rfloor = 88$ . A separate integer-arithmetic certificate checks the full forbidden-coefficient support through degree 88, not merely their residues modulo 23. Their expressions are holomorphic level-4 forms. To justify the cuspidal and Shimura assertions without assuming an eigenform identification, put  $x = F/\Theta^4$ . The defining polynomial factors exactly:

$$(39) \quad S = \Theta^{21}P(x), \quad P(x) = x(2x - 1)(16x - 1)(16x^2 - 16x + 1).$$

At  $\infty$  the constant term vanishes because  $F = q + O(q^3)$ . At cusp  $1/2$ ,  $F$  has order zero and  $\Theta$  has positive order  $1/4$  in the natural local parameter; each monomial  $F^j\Theta^{21-4j}$ ,  $1 \leq j \leq 5$ , therefore vanishes. At cusp 0 the eta-transformation formulas give

$$(F/\Theta^4)|W_4 \longrightarrow \frac{1}{16} \quad (\Im \tau \longrightarrow \infty),$$

and  $P(1/16) = 0$ . Hence  $S$  is a cusp form on  $\Gamma_0(4)$ .

The Kohnen support condition for weight  $21/2$  requires  $s(n) = 0$  whenever  $n \equiv 2, 3 \pmod{4}$ . We apply the exact additive projections  $\Pi_2$  and  $\Pi_3$  of Lemma 2.1 to  $S$ . These are

holomorphic weight-21/2 forms at level 64 with a common theta multiplier. After multiplying by  $\Theta$ , their weight is 11 and the ordinary modular index is 96; hence the integral-weight Sturm bound is  $\lfloor 11 \cdot 96/12 \rfloor = 88$ . The exact integer-coefficient calculation gives zero for all the forbidden indices through degree 88, and therefore  $S \in S_{21/2}^+(4)$ . This argument does not presume the Shimura eigenform identity.

For clarity, we spell out the lift normalization. In Shimura's convention [6, 3], with  $k = 10$  and squarefree parameter  $t = 1$ , the quadratic character is  $\chi_{(-1)^k t} = \chi_1 = 1$ . Thus the Fourier expansion of the lift is

$$(40) \quad \text{Sh}_1(S)(\tau) = \sum_{n \geq 1} \left( \sum_{d|n} d^9 s((n/d)^2) \right) q^n.$$

Here  $S$  is in Kohnen's plus space, so the level-one refinement of Kohnen's correspondence [3] applies: this is a cusp form of weight 20 on  $\text{SL}_2(\mathbb{Z})$ . That space is one-dimensional, generated by  $\Delta E_8$ . Since  $s(1) = 1$ , the coefficient of  $q$  in (40) is 1. This proves the *exact* equality  $\text{Sh}_1(S) = \Delta E_8$ , and not merely equality up to a scalar. Equivariance at all odd primes now proves (37).

For example, direct coefficient extraction gives  $s(4) = -56$ ,  $s(9) = 30969$ ,  $s(25) = -4330535$ , and therefore

$$\begin{aligned} a_{20}(2) &= s(4) + 2^9 = 456, \\ a_{20}(3) &= s(9) + 3^9 = 50652, \\ a_{20}(5) &= s(25) + 5^9 = -2377410. \end{aligned}$$

As an independent reproducibility check, a fresh integer-only implementation calculates the coefficients of  $S$  through degree 520, finds no forbidden  $2, 3 \pmod{4}$  coefficients, and confirms (40) against a separately computed  $\Delta E_8$  at every degree  $1 \leq n \leq 22$ . It also verifies the odd-prime Hecke equation for  $Q = 3, 5, 7, 11, 13$  wherever  $Q^2 n \leq 520$ . These calculations are checks on the normalization, not substitutes for the cited correspondence theorem. Modulo 23,  $\mathcal{H}_{21/2} \equiv 5U + 4S$ , and therefore  $6\mathcal{H}_{21/2} - 2S \equiv 7U - S$ .

The projection onto  $n \equiv 5 \pmod{8}$  of  $C_{23} - (7U - S)$  is zero modulo 23: the independently reconstructed checker verifies all 44 positions through the integral-weight (weight 11) level-256 Sturm bound 352; a separate overpartition recurrence verifies the extraction congruence for  $C_{23}$  at every coefficient through this bound. Since  $m \equiv 5 \pmod{8}$  is odd, (7) follows. Apply  $T(Q^2)$  to the projected identity, use (37), (38), and the  $k = 10$  instance of (25). The projection commutes with  $T(Q^2)$ , which proves (8).

**Corollary 7.1** (Scalar Hecke primes and towers modulo 23). *If an odd prime  $Q$  satisfies  $a_{20}(Q) \equiv 1 + Q^{19} \pmod{23}$ , then for  $m \equiv 5 \pmod{8}$ ,*

$$(41) \quad \bar{p}(23Q^2m) \equiv \left( 1 + Q^{19} - Q^9 \left( \frac{m}{Q} \right) \right) \bar{p}(23m) - Q^{19} \bar{p}(23m/Q^2) \pmod{23}.$$

*In particular  $Q = 5, 13, 17, 31, 181$  satisfy the scalar condition. More generally, suppose  $Q \nmid m$ ,  $m \equiv 5 \pmod{8}$  and put  $b = Q^{19} \pmod{23}$ ,  $c = Q^9 \left( \frac{m}{Q} \right) \pmod{23}$ . For every integer  $j \geq 0$  we have the closed formula*

$$(42) \quad \bar{p}(23Q^{2j}m) \equiv \left( 1 + (b - c) \sum_{i=0}^{j-1} b^i \right) \bar{p}(23m) \pmod{23},$$

*where the empty sum for  $j = 0$  is zero. Thus any  $j$  for which the parenthetical factor vanishes yields an unconditional congruence for all  $m$  in the indicated quadratic-character class. The following automatic zero exponents are examples:*

| $Q$ | condition                         | $j$ with $\bar{p}(23Q^{2j}m) \equiv 0 \pmod{23}$ |
|-----|-----------------------------------|--------------------------------------------------|
| 13  | $\left(\frac{m}{13}\right) = +1$  | $j \equiv 1 \pmod{11}$                           |
| 5   | $\left(\frac{m}{5}\right) = -1$   | $j \equiv 12 \pmod{22}$                          |
| 5   | $\left(\frac{m}{5}\right) = +1$   | $j \equiv 15 \pmod{22}$                          |
| 17  | $\left(\frac{m}{17}\right) = -1$  | $j \equiv 8 \pmod{22}$                           |
| 17  | $\left(\frac{m}{17}\right) = +1$  | $j \equiv 16 \pmod{22}$                          |
| 181 | $\left(\frac{m}{181}\right) = -1$ | $j \equiv 1 \pmod{22}$                           |
| 181 | $\left(\frac{m}{181}\right) = +1$ | $j \equiv 15 \pmod{22}$                          |

*Proof.* The scalar condition cancels the final summand of (8). Write  $u_j = \bar{p}(23Q^{2j}m)$  and set  $b = Q^{19}$  and  $c = Q^9 \left(\frac{m}{Q}\right)$  in  $\mathbb{F}_{23}$ . The Hecke relation gives  $u_{j+1} = (1+b)u_j - bu_{j-1}$  for  $j \geq 1$ , with  $u_1 = (1+b-c)u_0$ . Subtracting  $u_j$  yields  $u_{j+1} - u_j = b(u_j - u_{j-1})$ . By induction,  $u_j - u_{j-1} = b^{j-1}(b-c)u_0$ , and summation proves (42), including the case  $b = 1$ ; no division by  $b-1$  is needed. For the tabulated primes, the zero factors and periods can be checked directly in  $\mathbb{F}_{23}$ . Equivalently, Lemma 2.4 applies with  $a = 1+b$ . The matrix has  $M^{11} = I$  for  $Q = 13$  and  $M^{22} = I$  for the remaining listed primes; substitution in the displayed initial formula verifies  $u_r = 0$  at the table's representative  $r$ . This proves all the stated infinite residue classes. There are no asserted automatic zeros for  $Q = 31$ . The eigenvalue  $a_{20}(181) \equiv 18 \equiv 1 + 181^{19} \equiv -181^9 \pmod{23}$  supplies the new prime-181 first-power family.  $\square$

*Remark 7.2.* An exact coefficient scan of  $\Delta E_8$  also finds the scalar condition for  $Q = 97, 103, 139, 167, 173, 313, 421, 449$  below 500. This is a finite list, not a claim that all primes in any residue class satisfy the criterion; formula (8) applies whether or not it holds.

## 8. REPRODUCIBILITY AND SCOPE OF THE PROOFS

The foregoing deductions combine published modular-form theorems with exact finite certificates. Their roles should not be confused: Sturm's theorem or the Radu–Sellers criterion turns a finite coefficient comparison into an infinite identity only after the weight, level, multiplier and cusp holomorphy have been justified. The projected comparisons used above are summarized here.

| Result            | Finite bound         | Exact check                                                      |
|-------------------|----------------------|------------------------------------------------------------------|
| Modulo 7          | $0 \leq n \leq 3143$ | 9432 coefficients in three progressions; eight cusp inequalities |
| Modulo 17         | $q^{256}$            | 32 character-projected indices; 257 extraction coefficients      |
| Modulo 19         | $q^{6400}$           | 379 comparisons for each of $Q = 13, 17$                         |
| Modulo 23         | $q^{352}$            | 44 projected indices; 353 extraction coefficients                |
| Plus-space of $S$ | $q^{88}$             | All forbidden indices $2, 3 \pmod{4}$ vanish over $\mathbb{Z}$   |

For modulus 7 we use the parameters and cusp inequalities stated in Section 4. The exact C++17 verifier reconstructs  $1/\Theta(-q)$  modulo 7 through degree 2,816,944 by Newton inversion and an exact modular number-theoretic transform. The modulo-19 program reconstructs the four-dissection quotient through degree 1,848,155. In both programs, using the transform prime 998244353, the maximum un-reduced convolution coefficients are bounded by  $2,816,945 \cdot 6^2$  and  $1,848,156 \cdot 18^2$  respectively, both less than 998244353. These bounds ensure that reduction first modulo the transform prime and then modulo 7 or 19 returns the exact finite-field products. Independent rational-arithmetic and polynomial-series checkers verify the cusp data and the projected Cohen identities.

The accompanying source-code supplement contains three C++17 coefficient verifiers (modulo 7, 19, and the combined 17/23 cases), plus two independent Python checks of the

Radu–Sellers cusp data and the Cohen projections, with execution instructions and recorded outputs. This source should accompany journal review, particularly for the largest finite verification.

The results are related to the general Hecke congruence framework of Ryan–Siroli–Villegas–Morales–Zheng [5]. The claims here are specific explicit resolutions of Wei’s proposed families, including their accompanying identities; we do not claim that every congruence consequence is independent of all previously established Hecke constructions. Likewise, the Cohen series and Shimura correspondence are classical inputs, not new constructions.

Author contributions. The named author accepts responsibility for the arguments, computational verifications, and final manuscript.

AI-assisted preparation. Generative-AI tools assisted in mathematical exploration, reorganisation and drafting of this manuscript, and examination of verification code. The named author is responsible for independently examining the arguments, computations, references, and originality of all claims before submission.

Funding. No external funding was received.

Data and code availability. Finite coefficient verification is supplied in the supplementary source-code archive.

Conflict of interest. The author declares no conflict of interest.

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