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Regularly varying relaxation laws and critical thickness scaling in nonisothermal shock profiles

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Abstract

We study monotone traveling-wave profiles in a four-field nonisothermal model of rational extended thermodynamics for a non-Newtonian fluid. The quadratic nonequilibrium entropy and conservative shock geometry are retained, while the relaxation source is generalized from a power law to an entropy-compatible law that is regularly varying at equilibrium with index $\alpha > 0$. We derive a necessary and sufficient integral criterion for finite spatial reach at strictly subcritical equilibrium endpoints, an exact weak-shock thickness equivalent, and an exact near-characteristic equivalent with localization of the maximal gradient for $0 < \alpha < 1$. The known algebraic thresholds are recovered, but their equality cases require additional information: at the weak threshold $\alpha = 1/2$, slowly varying factors select broadening, a plateau or collapse; at the near-characteristic threshold $\alpha = 1$, a bounded upper envelope yields a positive thickness plateau, whereas an unbounded upper envelope forces collapse. At the exact characteristic encounter, the critical reduced trajectory has a distinct upstream-reach threshold at $\alpha = 2$. Its endpoint regularity is classified separately from its spatial reach. Synthetic calculations for a specified Tait–Murnaghan constitutive law check the asymptotic constants and locations.

Keywords: Nonisothermal shock profiles; Rational extended thermodynamics; Non-Newtonian fluids; Regular variation; Critical asymptotics; Endpoint reach.

1 Introduction

Rational extended thermodynamics describes finite-relaxation processes through hyperbolic balance laws [1]. Classical shock-structure theory relates smooth-profile obstructions to characteristic encounters [2]. Power-law relaxation models provide a tractable setting for shock-thickness classifications in rational extended thermodynamics. In the unpublished preprint *Nonisothermal Shock Structure and Universal Flow-Index Thresholds in a Hyperbolic Power-Law Fluid*, T. Ruggeri extends the two power-law thresholds to nonisothermal shocks and raises the question of whether the classifications persist for more general equilibrium production laws (arXiv:2608.06547v1, 6 August 2026; <https://arxiv.org/abs/2608.06547>).

We extend this classification to regularly varying production laws, retaining the quadratic nonequilibrium entropy, conservative reduction and characteristic geometry of that model. General invertible dissipative stress–rate relations are also available in the isothermal RET framework [3]. Here we determine the shock-thickness asymptotics and equilibrium endpoint behavior for the nonisothermal system.

Away from a threshold, the algebraic order fixes the scaling. At a threshold, subpower corrections enter the leading asymptotics. Regular variation is the natural language for separating an algebraic index from such corrections [4]. We take

$$q(r) = r^\alpha L(r), \quad r \downarrow 0,$$

where L is slowly varying. This includes powers multiplied by logarithms, iterated logarithms, and nonconvergent slowly varying factors. These corrections have a direct constitutive interpretation: if a differentiable production law is measured through its local log–log slope,

$$\alpha_{\text{eff}}(r) := \frac{d \log q(r)}{d \log r},$$

then, for $q(r) = r^\alpha [\log(r_c/r)]^\beta$,

$$\alpha_{\text{eff}}(r) = \alpha - \frac{\beta}{\log(r_c/r)} \longrightarrow \alpha.$$

Thus a logarithmic factor describes an effective flow index that approaches its limiting value slowly as equilibrium is approached.

The main conclusions are as follows. First, a subcritical profile is obtained by quadrature for every positive regularly varying production. Its equilibrium endpoints are reached at finite spatial coordinates precisely when $\int_0^\varepsilon q(r)^{-1} dr < \infty$. Second, with $\delta_F = 1 - F_-$ and $B = \mathcal{P}''(0)/2$, the weak-shock thickness satisfies

$$\Delta \sim \frac{4^\alpha c_0}{\ell_0 a_0} \frac{\delta_F}{q(B\delta_F^2)}.$$

Consequently the weak threshold is $\alpha = 1/2$, but at equality the limit is proportional to $1/L(B\delta_F^2)$ and is not determined by α alone. Third, if $\delta = M_* - M$ and $0 < \alpha < 1$, then

$$\Delta \sim C_c \frac{\delta}{q(\kappa\delta)},$$

with an explicit $C_c > 0$, and the maximal gradient lies at a distance asymptotic to a known multiple of δ from the upstream state. For $\alpha > 1$ the thickness has a positive limit. At $\alpha = 1$, a finite upper envelope of L gives a plateau, whereas an unbounded upper envelope forces collapse. Finally, at $M = M_*$ the upstream spatial reach is controlled by $\int_0^\varepsilon u/q(\kappa u) du$, producing a third threshold $\alpha = 2$.

For the original power law, $\alpha = 1/m$ and $L \equiv 1$. Our formulas reduce exactly to the $m = 2$ and $m = 1$ classifications of [5] and the nonisothermal preprint discussed in Section 1. The regular-variation formulation resolves the marginal cases, locates the maximal gradient and gives the exact-critical reach criterion.

2 Model, source class, and geometric hypotheses

2.1 Nonisothermal RET system

In one-dimensional Lagrangian coordinates (X, t) , consider

$$\rho_0 v_t + [p(F, \theta) - \sigma]_X = 0, \tag{1a}$$

$$F_t - v_X = 0, \tag{1b}$$

$$\rho_0 \left(\frac{v^2}{2} + \varepsilon(F, \theta) \right)_t + [(p(F, \theta) - \sigma)v]_X = 0, \tag{1c}$$

$$\omega(\sigma/\theta)_t - v_X = P(F, \sigma, \theta), \tag{1d}$$

where

$$\omega = \frac{\tau_0 \theta_0}{\mu_0} > 0.$$

We retain the quadratic nonequilibrium entropy used in the power-law shock model discussed in Section 1; it is a specialization of the broader thermodynamic closure in [6], which also admits nonquadratic nonequilibrium entropies. We change only the production:

$$P(F, \sigma, \theta) = -a(F, \theta, |\sigma|) \operatorname{sgn}(\sigma) q(|\sigma|), \quad \operatorname{sgn}(0) = 0. \quad (2)$$

Stress may be nondimensionalized by any fixed positive reference stress; that scale is absorbed into a and the argument of q .

Assumption 2.1 (Source). *On a compact neighborhood of all states considered below, a is continuous and*

$$0 < a_{\min} \leq a(F, \theta, r) \leq a_{\max} < \infty.$$

The function $q : [0, \infty) \rightarrow [0, \infty)$ is continuous, $q(0) = 0$, $q(r) > 0$ for $r > 0$, and is regularly varying at zero with index $\alpha > 0$:

$$\lim_{r \downarrow 0} \frac{q(cr)}{q(r)} = c^\alpha \quad (c > 0). \quad (3)$$

We write $q(r) = r^\alpha L(r)$, where L is slowly varying at zero, and set

$$a_0 := a(1, \theta_0, 0).$$

The original power-law production is recovered by taking

$$\alpha = \frac{1}{m}, \quad L \equiv 1, \quad a(F, \theta, r) = F 2^{\alpha-1} k(\theta)^{-\alpha}.$$

Indeed, $|\sigma|^{(1-m)/m} \sigma = \operatorname{sgn}(\sigma) |\sigma|^{1/m}$. In the formal fast-relaxation balance, the strain rate is proportional to $q(|\sigma|)$. Whenever q is eventually strictly increasing, its local inverse is regularly varying with index $1/\alpha$ by the inversion theorem [4]. Thus the class (2) contains generalized power-law rheologies with subpower corrections. Eventual monotonicity is not required in the profile theorems below.

The entropy production is automatically nonnegative:

$$-\frac{\sigma}{\theta} P(F, \sigma, \theta) = \frac{a(F, \theta, |\sigma|)}{\theta} |\sigma| q(|\sigma|) \geq 0. \quad (4)$$

Since the source contains no derivatives, it does not alter the principal symbol, symmetric-hyperbolic structure, equilibrium sound speed, or subcharacteristic geometry.

2.2 Standing thermodynamic and Hugoniot assumptions

We use the same geometric hypotheses as the nonisothermal power-law theory, stated here in a form sufficient for the proofs.

Assumption 2.2 (Thermodynamic geometry). *The functions $p(F, \theta)$ and $\varepsilon(F, \theta)$ are C^3 on a neighborhood of the selected compressive branch and satisfy*

$$\varepsilon_F = \frac{\theta p_\theta - p}{\rho_0}, \quad p_F < 0, \quad \varepsilon_\theta > 0, \quad p_\theta \geq 0.$$

The integrated energy equation uniquely defines a jointly C^2 function $\theta = \Theta(F; s)$ on $F \in [F_-(s), 1]$, with

$$\theta_0 \leq \Theta(F; s) \leq \theta_-(s).$$

The reduced pressure $P_s(F) := p(F, \Theta(F; s))$ satisfies

$$P_s''(F) > 0$$

on the compact closure of the selected branch family, including its weak and critical limiting states. The selected right-going equilibrium shock is the classical genuinely nonlinear Lax compressive branch.

Assumption 2.3 (Critical continuation). *The downstream states remain in a compact subset of the admissible thermodynamic region as $M \nearrow M_*$. We assume that the selected downstream branch $F_-(M)$ and $\theta_-(M)$ extends C^2 to $M = M_*$, with $F_-^* \in (0, 1)$. This continuation is verified explicitly for the constitutive example in Section 10.*

Here

$$c_0 := c_S(1, \theta_0), \quad M := \frac{s}{c_0}, \quad \bar{\mu}_0 := \frac{\mu_0}{\tau_0 \rho_0 c_0^2}, \quad M_* := \sqrt{1 + \bar{\mu}_0}.$$

The equilibrium adiabatic sound speed is defined by

$$c_0^2 = -\frac{p_F(1, \theta_0)}{\rho_0} + \frac{\theta_0 p_\theta(1, \theta_0)^2}{\rho_0^2 \varepsilon_\theta(1, \theta_0)}.$$

3 Traveling-wave reduction and structural expansions

Let $\varphi = X - st$ with $s > 0$, and normalize the upstream state by

$$(F, v, \theta, \sigma) \rightarrow (1, 0, \theta_0, 0) \quad \text{as } \varphi \rightarrow +\infty.$$

Integration of the three conservation laws gives

$$v = s(1 - F), \tag{5}$$

$$\sigma = \Sigma(F; s) := p(F, \Theta(F; s)) - p_0 - \rho_0 s^2(1 - F), \tag{6}$$

$$\varepsilon(F, \Theta(F; s)) - \varepsilon_0 = \frac{p_0}{\rho_0}(1 - F) + \frac{s^2}{2}(1 - F)^2. \tag{7}$$

Set

$$\Pi(F; s) := \frac{\Sigma(F; s)}{\Theta(F; s)}, \quad D(F; s) := 1 - \omega \Pi_F(F; s).$$

Substitution into (1d) yields the scalar equation

$$F_\varphi = R_M(F) := \frac{a(F, \Theta(F; s), |\Sigma(F; s)|)q(|\Sigma(F; s)|)}{sD(F; s)} \quad (F_- < F < 1), \tag{8}$$

where we used $\Sigma < 0$ on the compressive branch.

The conservative reduction gives the following stress and characteristic identities.

Proposition 3.1 (Reduced stress and characteristic denominator). *Under assumptions 2.2 and 2.3, the following hold.*

(i) $\Sigma(F_-; s) = \Sigma(1; s) = 0$, $\Sigma(F; s) < 0$ for $F \in (F_-, 1)$,

$$\Sigma_F(1; s) = \rho_0(s^2 - c_0^2) > 0, \quad \Sigma_F(F_-; s) < 0.$$

(ii) For $1 < M < M_*$,

$$D(F; M) > 0 \quad (F_- \leq F \leq 1).$$

At $M = M_*$, $D(F; M_*) > 0$ for $F < 1$ and $D(1; M_*) = 0$.

(iii) With $u = 1 - F$ and $\delta = M_* - M$,

$$D(1 - u; M) = d_M \delta + d_F u + o(\delta + u), \quad (9)$$

where

$$d_M = \frac{2M_*}{\bar{\mu}_0} > 0, \quad d_F = \frac{\tau_0}{\mu_0} \Sigma_{FF}(1; M_*) + \frac{2p_\theta(1, \theta_0)}{\rho_0 \varepsilon_\theta(1, \theta_0)} > 0. \quad (10)$$

Moreover,

$$\kappa := \Sigma_F(1; M_*) = \rho_0 c_0^2 (M_*^2 - 1) = \frac{\mu_0}{\tau_0} > 0. \quad (11)$$

Proof. The endpoint identities follow from the Rankine–Hugoniot relations. Strict convexity of Σ and its two zero endpoint values imply $\Sigma < 0$ in the interior and the stated endpoint derivative signs. Differentiating (7) and using the Gibbs identity gives

$$\Theta_F = \frac{\Sigma - \Theta p_\theta}{\rho_0 \varepsilon_\theta} \leq 0. \quad (12)$$

The inequality is strict for $F_- < F < 1$; equality can occur at an equilibrium endpoint when $p_\theta = 0$. At $F = 1$, this identity and the definition of the adiabatic sound speed yield $\Sigma_F(1; s) = \rho_0(s^2 - c_0^2)$.

A direct calculation of the positive acoustic characteristic speed of the homogeneous four-field system gives

$$\lambda_0^2 = -\frac{p_F}{\rho_0} + \frac{(\Sigma - \Theta p_\theta)^2}{\rho_0^2 \Theta \varepsilon_\theta} + \frac{\mu_0 \Theta}{\rho_0 \tau_0 \theta_0}.$$

Using (12), one obtains the exact identity

$$D(F; s) = \frac{\rho_0 \tau_0 \theta_0}{\mu_0 \Theta(F; s)} (\lambda_0^2(F; s) - s^2). \quad (13)$$

Equivalently,

$$\lambda_0^2(F; s) = -\frac{P'_s(F)}{\rho_0} + \frac{\Sigma(F; s) \Theta_F(F; s)}{\rho_0 \Theta(F; s)} + \frac{\mu_0 \Theta(F; s)}{\rho_0 \tau_0 \theta_0}.$$

After subtracting its value at $F = 1$, all three resulting terms are nonnegative for $F_- \leq F < 1$, and the pressure term is strictly positive: strict convexity gives $P'_s(F) < P'_s(1)$, while $\Sigma \Theta_F \geq 0$ and $\Theta \geq \theta_0$. Hence $\lambda_0(F; s) > \lambda_0(1; s)$ for $F_- \leq F < 1$. At the upstream state,

$$D(1; M) = 1 - \frac{M^2 - 1}{\bar{\mu}_0}.$$

This proves the sign assertions and $M_* = \sqrt{1 + \bar{\mu}_0}$. Taylor expansion at $(u, M) = (0, M_*)$ gives (9); differentiating $D = 1 - \omega(\Sigma/\Theta)_F$ at the upstream state gives (10). Here $\Sigma_{FF} = P''_s > 0$, so the displayed expression indeed has $d_F > 0$. Finally (11) follows from the upstream stress derivative and the definition of $\bar{\mu}_0$. $\square$

For weak shocks, let

$$\delta_F := 1 - F_-, \quad \delta_M := M - 1.$$

Let $\theta_-(\delta_F)$ be the equilibrium Hugoniot temperature and define

$$\mathcal{P}(\delta_F) := p(1 - \delta_F, \theta_-(\delta_F)).$$

Lemma 3.2 (Weak Hugoniot parametrization). *The equilibrium Hugoniot relation defines $\theta_-(\delta_F)$ as a C^2 function for all sufficiently small $\delta_F \geq 0$. Moreover,*

$$\mathcal{P}'(0) = \rho_0 c_0^2, \quad \mathcal{P}''(0) = P''_{c_0}(1) > 0. \quad (14)$$

Writing

$$B := \frac{\mathcal{P}''(0)}{2},$$

one has

$$\delta_F = A\delta_M + o(\delta_M), \quad A := \frac{4\rho_0 c_0^2}{\mathcal{P}''(0)} = \frac{2\rho_0 c_0^2}{B}. \quad (15)$$

Proof. Eliminating the shock speed between the equilibrium momentum and energy Rankine–Hugoniot relations gives

$$\mathcal{H}(F, \theta) := \varepsilon(F, \theta) - \varepsilon_0 - \frac{1 - F}{2\rho_0} (p(F, \theta) + p_0) = 0.$$

At $(F, \theta) = (1, \theta_0)$, $\mathcal{H}_\theta = \varepsilon_\theta > 0$; hence the implicit-function theorem gives a C^2 temperature branch. Differentiating at the base state, or equivalently using the Gibbs identity, yields

$$\left. \frac{d\theta_-}{d\delta_F} \right|_0 = \frac{\theta_0 p_\theta(1, \theta_0)}{\rho_0 \varepsilon_\theta(1, \theta_0)}.$$

The definition of the adiabatic sound speed therefore gives $\mathcal{P}'(0) = \rho_0 c_0^2$.

At the downstream state, $\mathcal{P}(\delta_F) - p_0 = \rho_0 s^2 \delta_F$. The momentum secant relation together with the preceding derivative shows $s \rightarrow c_0$. Since $\mathcal{P}(\delta_F) = P_{s(\delta_F)}(1 - \delta_F)$, while $P_s(1) = p_0$ and $P'_s(1) = -\rho_0 c_0^2$ independently of s , Taylor's theorem gives

$$\mathcal{P}(\delta_F) = p_0 + \rho_0 c_0^2 \delta_F + \frac{1}{2} P''_{c_0}(1) \delta_F^2 + o(\delta_F^2).$$

Strict Hugoniot convexity proves the second identity and positivity in (14). Dividing the momentum relation by δ_F then gives

$$\rho_0 c_0^2 (M^2 - 1) = B\delta_F + o(\delta_F).$$

Since $M^2 - 1 = 2\delta_M + o(\delta_M)$, formula (15) follows. $\square$

Lemma 3.3 (Uniform weak-shock stress shape). *As $\delta_F \downarrow 0$,*

$$\frac{\Sigma(1 - \delta_F z; s(\delta_F))}{\delta_F^2} \longrightarrow -Bz(1 - z) \quad (16)$$

uniformly for $z \in [0, 1]$. Moreover, for all sufficiently small δ_F there are constants $0 < c < C < \infty$, independent of z , such that

$$c\delta_F^2 z(1 - z) \leq |\Sigma(1 - \delta_F z; s(\delta_F))| \leq C\delta_F^2 z(1 - z). \quad (17)$$

Proof. Set $\widehat{\Sigma}_{\delta_F}(z) = \delta_F^{-2} \Sigma(1 - \delta_F z; s(\delta_F))$. Both endpoint values vanish and

$$\widehat{\Sigma}_{\delta_F}''(z) = \Sigma_{FF}(1 - \delta_F z; s(\delta_F)) \longrightarrow \mathcal{P}''(0) = 2B$$

uniformly. The unique solution of $f'' = 2B$ with $f(0) = f(1) = 0$ is $f = -Bz(1 - z)$, which proves uniform convergence. For the two-sided bound, choose $0 < b < C$ so that $2b \leq \widehat{\Sigma}_{\delta_F}'' \leq 2C$ for all sufficiently small δ_F . Comparing the Dirichlet solutions with $-bz(1 - z)$ and $-Cz(1 - z)$, or applying the one-dimensional maximum principle to their differences, gives $-Cz(1 - z) \leq \widehat{\Sigma}_{\delta_F}(z) \leq -bz(1 - z)$. This is (17). $\square$

4 Regular-variation tools

We use the uniform convergence theorem, Potter bounds, and Karamata's integral theorem for regularly varying functions [4]. The following composition lemma applies these estimates to the reduced stress.

Lemma 4.1 (Uniform composition through a vanishing profile). *Let $q \in \text{RV}_\alpha(0+)$ with $\alpha > 0$ and $q(0) = 0$. Let $t_n \downarrow 0$, and let x_n, x be nonnegative functions on a compact set K such that $x_n \rightarrow x$ uniformly and $\sup_n \|x_n\|_\infty < \infty$. Then*

$$\sup_{y \in K} \left| \frac{q(t_n x_n(y))}{q(t_n)} - x(y)^\alpha \right| \longrightarrow 0. \quad (18)$$

Proof. Fix $\eta > 0$. On the set where $x \geq 2\eta$, uniform convergence gives $x_n \geq \eta$ for large n , and the uniform convergence theorem for regular variation applies on a compact subinterval of $(0, \infty)$. On the complementary set, $x_n \leq 3\eta$ for large n . Choose $0 < \epsilon < \alpha$. Potter's bound gives, uniformly for small t_n and $0 < y \leq 3\eta < 1$,

$$\frac{q(t_n y)}{q(t_n)} \leq C y^{\alpha - \epsilon}.$$

Thus the contribution of this set is bounded by $C(3\eta)^{\alpha - \epsilon} + (2\eta)^\alpha$. First let $n \rightarrow \infty$ and then $\eta \downarrow 0$. $\square$

We shall also use the standard consequences

$$\int_0^\epsilon \frac{dr}{q(r)} < \infty \quad \text{if } 0 < \alpha < 1, \quad (19)$$

$$\int_0^\epsilon \frac{dr}{q(r)} = \infty \quad \text{if } \alpha > 1, \quad (20)$$

while at $\alpha = 1$ convergence is equivalent to that of $\int_0^\epsilon [rL(r)]^{-1} dr$.

5 Subcritical profiles and equilibrium endpoints

Theorem 5.1 (Subcritical existence and endpoint reach). *Assume [assumptions 2.1](#) and [2.2](#). For every $1 < M < M_*$, (8) has a unique monotone profile up to translation on its maximal active interval, where $F \in (F_-, 1)$, joining F_- to 1. At either equilibrium endpoint $F_e \in \{F_-, 1\}$, the spatial reach is finite if and only if*

$$\int_0^\epsilon \frac{dr}{q(r)} < \infty. \quad (21)$$

Consequently:

- (i) if $0 < \alpha < 1$, both endpoints occur at finite spatial coordinates and constant continuation gives a global C^1 profile;
- (ii) if $\alpha > 1$, both endpoints are approached through infinite spatial tails;
- (iii) if $\alpha = 1$, the alternative is decided by $\int_0^\varepsilon [rL(r)]^{-1} dr$.

Proof. By [proposition 3.1](#), $D > 0$ and $\Sigma < 0$ in the interior, so $R_M(F) > 0$. Fix $F_0 \in (F_-, 1)$ and define

$$\varphi(F) = \int_{F_0}^F \frac{sD(y; s)}{a(y, \Theta(y; s), |\Sigma(y; s)|)q(|\Sigma(y; s)|)} dy.$$

This is strictly increasing and its inverse solves (8). At each endpoint, Σ has a simple zero, while sD and a tend to positive finite limits. Hence, with $x = |F - F_e|$ and some $c_e > 0$,

$$\frac{d\varphi}{dF} \asymp \frac{1}{q(c_e x)}.$$

Regular variation gives $q(c_e x) \sim c_e^\alpha q(x)$, proving (21). The three cases follow from Karamata's integral theorem. If an endpoint is finite, then $F_\varphi \asymp q(c_e x) \rightarrow 0$, so constant continuation is C^1 . $\square$

Corollary 5.2 (Sharp subcritical edge law). *Fix $1 < M < M_*$ and an equilibrium endpoint $F_e \in \{F_-, 1\}$. Put*

$$x := |F - F_e|, \quad c_e := |\Sigma_F(F_e; s)|, \quad D_e := D(F_e; s), \quad a_e := a(F_e, \Theta(F_e; s), 0).$$

If $0 < \alpha < 1$ and φ_e is the finite endpoint coordinate, then

$$|\varphi - \varphi_e| \sim \frac{sD_e}{a_e c_e^\alpha (1 - \alpha)} \frac{x^{1-\alpha}}{L(x)}. \quad (22)$$

If $\alpha > 1$, the endpoint is at infinity and, after fixing any interior origin for φ ,

$$|\varphi| \sim \frac{sD_e}{a_e c_e^\alpha (\alpha - 1)} \frac{x^{1-\alpha}}{L(x)}. \quad (23)$$

Consequently, in the compact-profile case the inverse edge distance is regularly varying with index $1/(1 - \alpha)$.

Proof. The endpoint expansions used in [theorem 5.1](#) sharpen to

$$\left| \frac{d\varphi}{dx} \right| \sim \frac{sD_e}{a_e q(c_e x)} \sim \frac{sD_e}{a_e c_e^\alpha} \frac{x^{-\alpha}}{L(x)}.$$

Karamata's integral theorem at zero gives (22) when $\alpha < 1$ and (23) when $\alpha > 1$. The inverse regular-variation statement follows from the monotone inversion theorem. $\square$

6 Weak-shock thickness and the marginal regime

Let

$$\ell_0 := \tau_0 c_0, \quad \Delta := \frac{1 - F_-}{\ell_0 \max_\varphi F_\varphi}.$$

Theorem 6.1 (Exact weak-shock equivalent). *Under [assumptions 2.1](#) and [2.2](#), as $M \downarrow 1$,*

$$\Delta \sim \frac{4^\alpha c_0}{\ell_0 a_0} \frac{\delta_F}{q(B\delta_F^2)}. \quad (24)$$

Equivalently, writing $q(r) = r^\alpha L(r)$,

$$\Delta \sim \frac{4^\alpha c_0}{\ell_0 a_0 B^\alpha} \frac{\delta_F^{1-2\alpha}}{L(B\delta_F^2)}. \quad (25)$$

Using [\(15\)](#),

$$\Delta \sim \frac{4^\alpha c_0 A^{1-2\alpha}}{\ell_0 a_0 B^\alpha} \frac{(M-1)^{1-2\alpha}}{L(BA^2(M-1)^2)}. \quad (26)$$

Therefore

$$\alpha > \frac{1}{2} \Rightarrow \Delta \rightarrow \infty, \quad \alpha < \frac{1}{2} \Rightarrow \Delta \rightarrow 0,$$

and at $\alpha = 1/2$,

$$\Delta \sim \frac{2c_0}{\ell_0 a_0 \sqrt{B}} \frac{1}{L(B\delta_F^2)}. \quad (27)$$

Proof. Set $z = (1 - F)/\delta_F$. By [lemmas 4.1](#) and [3.3](#),

$$\frac{q(|\Sigma(1 - \delta_F z; s)|)}{q(B\delta_F^2)} \rightarrow [z(1 - z)]^\alpha$$

uniformly on $[0, 1]$. Also $a \rightarrow a_0$, $s \rightarrow c_0$, and $D \rightarrow 1$ uniformly. Hence

$$\frac{F_\varphi(1 - \delta_F z)}{q(B\delta_F^2)} \rightarrow \frac{a_0}{c_0} [z(1 - z)]^\alpha \quad (28)$$

uniformly. The limiting function has its unique maximum at $z = 1/2$, with value $4^{-\alpha} a_0/c_0$. Therefore

$$\max_\varphi F_\varphi \sim \frac{a_0}{c_0} 4^{-\alpha} q(B\delta_F^2),$$

which proves [\(24\)](#). The remaining formulas follow from regular variation, [\(15\)](#), and the fact that a nonzero power dominates every slowly varying factor. $\square$

Corollary 6.2 (Logarithmic trichotomy at $\alpha = 1/2$). *For small r , let*

$$q_\beta(r) = r^{1/2} \left[\log \left(\frac{r_c}{r} \right) \right]^\beta, \quad r_c > 0.$$

Then, up to an explicit positive constant,

$$\Delta \sim [\log(1/(M-1))]^{-\beta}.$$

Thus sources with the same regular-variation index $1/2$ produce respectively broadening, a positive plateau, and collapse when $\beta < 0$, $\beta = 0$, and $\beta > 0$.

The limiting shape also becomes universal after the same rescaling.

Corollary 6.3 (Weak-profile shape). *Let $\varphi_{\delta_F}(z)$ be the inverse profile defined by $F(\varphi_{\delta_F}(z)) = 1 - \delta_F z$, with translation chosen so that $\varphi_{\delta_F}(1/2) = 0$. Define*

$$Y_{\delta_F}(z) := \frac{a_0}{c_0} \frac{q(B\delta_F^2)}{\delta_F} \varphi_{\delta_F}(z).$$

Then, locally uniformly for $z \in (0, 1)$,

$$Y_{\delta_F}(z) \longrightarrow - \int_{1/2}^z [r(1-r)]^{-\alpha} dr. \quad (29)$$

If $0 < \alpha < 1$, the convergence extends uniformly to $[0, 1]$.

Proof. Differentiate the inverse relation:

$$Y'_{\delta_F}(z) = - \frac{a_0}{c_0} \frac{q(B\delta_F^2)}{F_\varphi(1 - \delta_F z)}.$$

Equation (28) gives local uniform convergence of the derivative to $-[z(1-z)]^{-\alpha}$. For $\alpha < 1$, (17) and Potter bounds supply an integrable majorant $C[z(1-z)]^{-\alpha-\eta}$ with $\alpha + \eta < 1$, so dominated convergence reaches the endpoints. $\square$

7 Approach to the characteristic threshold

Set

$$\delta := M_* - M, \quad u := 1 - F, \quad J_* := 1 - F_-^*, \quad s_* := M_* c_0.$$

From [proposition 3.1](#), uniformly as $(u, \delta) \rightarrow (0, 0)$,

$$|\Sigma(1-u; M)| = \kappa u[1 + o(1)], \quad (30)$$

$$D(1-u; M) = d_M \delta + d_F u + o(\delta + u). \quad (31)$$

Lemma 7.1 (Uniform local comparability). *There are $\varepsilon_0, \delta_0, c, C > 0$ such that, whenever $0 \leq u \leq \varepsilon_0$ and $0 \leq \delta \leq \delta_0$,*

$$cu \leq |\Sigma(1-u; M)| \leq Cu, \quad c(\delta + u) \leq D(1-u; M) \leq C(\delta + u). \quad (32)$$

For every fixed $\varepsilon > 0$, D also has a uniform positive lower bound on $u \geq \varepsilon$ for all sufficiently small $\delta \geq 0$.

Proof. The first pair follows from $\Sigma(1; M) = 0$ and $\Sigma_F(1; M) \rightarrow \kappa > 0$; the second follows from (31) and $d_M, d_F > 0$. The away-from-endpoint statement is compactness combined with $D(F; M_*) > 0$ for $F < 1$. $\square$

By the uniform convergence theorem for regular variation, the stress comparability also gives $q(|\Sigma(1-u; M)|) \asymp q(u)$ uniformly in this neighborhood. Thus the estimates below do not require monotonicity of q .

Theorem 7.2 (Near-critical collapse for $0 < \alpha < 1$). *Suppose [assumptions 2.1 to 2.3](#) hold and $0 < \alpha < 1$. Define*

$$H_\alpha := \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{d_M^{1-\alpha} d_F^\alpha}, \quad y_\alpha := \frac{\alpha d_M}{(1-\alpha) d_F}. \quad (33)$$

Then

$$\max_{\varphi} F_{\varphi} \sim \frac{a_0}{s_*} H_{\alpha} \frac{q(\kappa\delta)}{\delta}, \quad (34)$$

$$\Delta \sim \frac{J_* s_*}{\ell_0 a_0 H_{\alpha}} \frac{\delta}{q(\kappa\delta)}. \quad (35)$$

Moreover, every maximizer $F_{\max}(M)$ satisfies

$$\frac{1 - F_{\max}(M)}{M_* - M} \longrightarrow y_{\alpha}. \quad (36)$$

In particular,

$$\Delta \sim \frac{J_* s_* d_M^{1-\alpha} d_F^{\alpha}}{\ell_0 a_0 \kappa^{\alpha} \alpha^{\alpha} (1-\alpha)^{1-\alpha}} \frac{\delta^{1-\alpha}}{L(\kappa\delta)} \longrightarrow 0. \quad (37)$$

Proof. On the boundary-layer scale $u = \delta y$, (30), (31), and regular variation give, locally uniformly for $y \geq 0$,

$$\frac{\delta}{q(\kappa\delta)} F_{\varphi}(1 - \delta y; M) \longrightarrow G_{\alpha}(y) := \frac{a_0}{s_*} \frac{y^{\alpha}}{d_M + d_F y}. \quad (38)$$

The scalar factor $y^{\alpha}/(d_M + d_F y)$ has a unique maximizer y_{α} and maximum H_{α} .

It remains to exclude a maximum outside the boundary layer. Choose $\eta \in (0, 1 - \alpha)$. For $R\delta \leq u \leq \varepsilon$, lemma 7.1 implies

$$F_{\varphi} \leq C \frac{q(Cu)}{\delta + u}.$$

Potter's bound, with $y = u/\delta \geq R$, yields

$$\frac{\delta F_{\varphi}}{q(\kappa\delta)} \leq C \frac{y^{\alpha+\eta}}{1+y} \leq C R^{-(1-\alpha-\eta)}.$$

For $u \geq \varepsilon$, D is uniformly positive and F_{φ} is bounded, whereas $\delta/q(\kappa\delta) \rightarrow 0$. Letting first $\delta \downarrow 0$, then $R \rightarrow \infty$, proves global localization and (34). The strict limiting maximizer also gives (36). Since $1 - F_{-}(M) \rightarrow J_*$, the thickness formulas follow. $\square$

Theorem 7.3 (Plateau and marginal alternatives). *Under assumptions 2.1 to 2.3:*

(i) *If $\alpha > 1$, then R_* extends continuously to $F = 1$ by $R_*(1) = 0$, and*

$$\Delta \longrightarrow \frac{J_*}{\ell_0 \max_{F \in [F_*^*, 1]} R_*(F)} > 0, \quad (39)$$

where, for $F < 1$,

$$R_*(F) = \frac{a(F, \Theta(F; s_*), |\Sigma(F; s_*)|) q(|\Sigma(F; s_*)|)}{s_* D(F; s_*)}. \quad (40)$$

(ii) *Let $\alpha = 1$ and put*

$$\Lambda_+ := \limsup_{r \downarrow 0} L(r).$$

If $\Lambda_+ < \infty$, then

$$\max_{\varphi} F_{\varphi} \longrightarrow \mathcal{M}_* := \sup_{F \in [F_*^*, 1)} R_*(F) \in (0, \infty), \quad \Delta \longrightarrow \frac{J_*}{\ell_0 \mathcal{M}_*} > 0. \quad (41)$$

If $\Lambda_+ = \infty$, then

$$\max_{\varphi} F_{\varphi} \longrightarrow \infty, \quad \Delta \longrightarrow 0. \quad (42)$$

(iii) In the special case $\alpha = 1$ and $L(r) \rightarrow \ell \in [0, \infty)$, R_* extends continuously to

$$R_*(1) = \frac{a_0 \kappa \ell}{s_* d_F}, \quad (43)$$

and $\mathcal{M}_* = \max_{F \in [F_*, 1]} R_*(F)$.

Proof. For $\alpha > 1$, $q(r)/r \rightarrow 0$. In a small upstream neighborhood, (30)–(31) imply

$$F_\varphi \leq C \frac{q(Cu)}{\delta + u} \leq C \frac{q(Cu)}{u}.$$

The right-hand side is uniformly small when the neighborhood is small. Away from the endpoint, all coefficients converge uniformly to their critical values. Hence $R_M \rightarrow R_*$ uniformly after defining $R_*(1) = 0$, and the maxima converge. The critical function is positive in the interior, which proves part (i).

Now let $\alpha = 1$. Suppose first that $\Lambda_+ < \infty$. Then L is bounded near zero and R_* is bounded on $[F_*, 1)$. The lower bound in (41) follows by fixing any $F < 1$, passing to the limit $R_M(F) \rightarrow R_*(F)$, and then taking the supremum over such F . For the upper bound, take an arbitrary sequence $M_n \nearrow M_*$ and maximizers F_n , and write $u_n = 1 - F_n$. If a subsequence satisfies $u_n \rightarrow u_0 > 0$, uniform convergence away from the upstream endpoint gives a limit no larger than $\mathcal{M}_*$. If $u_n \rightarrow 0$, then

$$R_{M_n}(1 - u_n) = \frac{a_0 \kappa}{s_*} \frac{u_n}{d_M \delta_n + d_F u_n} L(|\Sigma(1 - u_n; M_n)|) [1 + o(1)].$$

Since $u_n/(d_M \delta_n + d_F u_n) \leq d_F^{-1}$ and $\limsup_{u \downarrow 0} R_*(1 - u) = a_0 \kappa \Lambda_+ / (s_* d_F)$, the corresponding limsup is again at most $\mathcal{M}_*$. Thus the maxima converge to $\mathcal{M}_*$. Positivity follows from the positive interior critical rate.

If $\Lambda_+ = \infty$, the critical local asymptotics and the strictly increasing stress coordinate imply that R_* is unbounded near $F = 1$. For any $K > 0$, choose a fixed interior point F_K with $R_*(F_K) > 2K$. Continuity at this fixed point gives $R_M(F_K) > K$ for every sufficiently near-critical $M < M_*$. Hence $\max_F R_M(F) \rightarrow \infty$, proving (42), even when L has arbitrarily deep troughs. Finally, if $L \rightarrow \ell$, the local asymptotics immediately give the continuous endpoint value (43). $\square$

Corollary 7.4 (Explicit logarithmic marginal law). *Let $\alpha = 1$ and, for $0 < r < r_1 < r_c$,*

$$q_\beta(r) = r \left[\log \left(\frac{r_c}{r} \right) \right]^\beta, \quad \beta > 0.$$

Assume in addition that a is C^1 near $(1, \theta_0, 0)$ and that the reduced functions Σ and D are C^2 in (F, M) near $(1, M_)$ (for example, it is enough to strengthen the constitutive data by one derivative). Then every maximizer satisfies*

$$1 - F_{\max}(M) \sim \frac{d_M}{\beta d_F} \delta \log(1/\delta), \quad (44)$$

$$\Delta \sim \frac{J_* s_* d_F}{\ell_0 a_0 \kappa} [\log(1/\delta)]^{-\beta}. \quad (45)$$

Proof. Write $S(u, \delta) = |\Sigma(1 - u; M)|$ and let u_δ be a maximizer. The slope at $u = \sqrt{\delta}$ diverges like a positive constant times $[\log(1/\delta)]^\beta$, whereas the slope is uniformly bounded away from the upstream endpoint. Hence $u_\delta \rightarrow 0$, and the maximizer is interior. The C^1 hypothesis and the C^2 geometric expansions give

$$\frac{S_u}{S} = \frac{1}{u} + O(1), \quad \frac{D_u}{D} = \frac{d_F}{d_M \delta + d_F u} + O(1), \quad \partial_u \log \left(\frac{a}{s} \right) = O(1)$$

uniformly in the local region. Indeed, writing $A = d_M \delta + d_F u$ and $D = A + E$, joint C^2 regularity gives $E = O((\delta + u)^2)$ and $E_u = O(\delta + u)$. Since $D \asymp A \asymp \delta + u$, $D_u/D - d_F/A = (AE_u - d_F E)/(AD) = O(1)$ uniformly. Likewise, $S(0, \delta) = 0$ and bounded S_{uu} give $uS_u - S = O(u^2)$, which proves the first estimate. Put $\Lambda_S = \log(r_c/S(u_\delta, \delta))$. Logarithmic differentiation of $aq_\beta(S)/(sD)$ at u_δ yields

$$\frac{d_M \delta}{d_M \delta + d_F u_\delta} - \frac{\beta}{\Lambda_S} + O(u_\delta) = 0. \quad (46)$$

Since $u_\delta \Lambda_S \rightarrow 0$, the error is $o(1/\Lambda_S)$. Equation (46) first implies $u_\delta/\delta \rightarrow \infty$ and then

$$\frac{u_\delta}{\delta} \sim \frac{d_M}{\beta d_F} \Lambda_S.$$

Moreover $S(u_\delta, \delta) \sim \kappa u_\delta$, so $\Lambda_S = \log(1/\delta) - \log(u_\delta/\delta) + O(1)$. The preceding relation makes the second term only $O(\log \log(1/\delta))$; hence $\Lambda_S \sim \log(1/\delta)$, proving (44). At this maximizer $u_\delta/\delta \rightarrow \infty$, and therefore

$$\max F_\varphi \sim \frac{a_0 \kappa}{s_* d_F} [\log(1/\delta)]^\beta.$$

Together with $1 - F_-(M) \rightarrow J_*$, this gives (45). $\square$

8 Exact-critical upstream reach

At $M = M_*$, the numerator and denominator of (8) both vanish at $F = 1$. Their relative orders determine the spatial reach of the reduced branch.

Theorem 8.1 (Exact-critical upstream reach). *Under assumptions 2.1 to 2.3, the critical reduced branch on $(F_*^*, 1)$ satisfies, as $u = 1 - F \downarrow 0$,*

$$F_\varphi(1 - u; M_*) \sim \frac{a_0}{s_* d_F} \frac{q(\kappa u)}{u}. \quad (47)$$

Its upstream state lies at a finite spatial coordinate if and only if

$$\int_0^\varepsilon \frac{u}{q(\kappa u)} du < \infty. \quad (48)$$

Thus:

- (i) *if $0 < \alpha < 2$, the upstream reach is finite;*
- (ii) *if $\alpha > 2$, the upstream state is approached through an infinite tail;*
- (iii) *if $\alpha = 2$, finiteness is equivalent to*

$$\int_0^\varepsilon \frac{du}{uL(u)} < \infty.$$

At a finite critical endpoint, constant continuation is C^1 whenever $\alpha > 1$. For $0 < \alpha < 1$ the one-sided slope diverges. At $\alpha = 1$, constant continuation is C^1 exactly when $L(r) \rightarrow 0$; if $L(r) \rightarrow \ell \in (0, \infty)$ the active branch has a finite nonzero one-sided slope, and in the remaining cases the slope inherits the limiting or oscillatory behavior of L through (47).

Proof. Set $\delta = 0$ in (30)–(31); continuity of a gives (47). Therefore

$$-\frac{d\varphi}{du} \sim \frac{s_* d_F}{a_0} \frac{u}{q(\kappa u)},$$

which proves the integral criterion. Since $u/q(\kappa u)$ is regularly varying with index $1 - \alpha$, Karamata's theorem yields the threshold $\alpha = 2$, with the stated marginal integral. Finally, the critical slope is regularly varying with index $\alpha - 1$: it tends to zero for $\alpha > 1$, diverges for $\alpha < 1$, and is proportional to L at $\alpha = 1$. $\square$

At the downstream endpoint, $D(F_-^*; M_*) > 0$, so the reach criterion is $\int_0^\varepsilon q(r)^{-1} dr < \infty$. The transition interval is finite when both this criterion and the upstream criterion (48) hold.

Remark 8.2 (Singular change of endpoint geometry). *The limit $M \nearrow M_*$ does not preserve the spatial type of the upstream endpoint. For $1 < \alpha < 2$, every strictly subcritical profile has an infinite upstream tail by theorem 5.1, whereas the exact-critical reduced branch reaches the upstream state at a finite coordinate. At $\alpha = 1$, this infinite-to-finite change occurs only if $\int_0^\varepsilon [rL(r)]^{-1} dr = \infty$; if that integral converges, the strictly subcritical endpoints are already finite. For example, $q(r) = r[\log(r_c/r)]^2$ has finite subcritical reach. The reason is the additional linear zero of D : it changes the inverse-profile integrand from $1/q(u)$ to $u/q(u)$. For $0 < \alpha < 1$ the endpoint remains finite but its one-sided slope changes from zero to infinity at criticality. The threshold $\alpha = 2$ therefore reflects the additional zero of the characteristic denominator at the upstream state.*

Corollary 8.3 (Sharp exact-critical edge law). *Let $u = 1 - F$. If $0 < \alpha < 2$ and φ_+ is the finite upstream coordinate, then*

$$\varphi_+ - \varphi \sim \frac{s_* d_F}{a_0 \kappa^\alpha (2 - \alpha)} \frac{u^{2-\alpha}}{L(u)}. \quad (49)$$

If $\alpha > 2$, after fixing an interior origin,

$$|\varphi| \sim \frac{s_* d_F}{a_0 \kappa^\alpha (\alpha - 2)} \frac{u^{2-\alpha}}{L(u)}. \quad (50)$$

Thus the inverse critical edge distance is regularly varying with index $1/(2 - \alpha)$ when $0 < \alpha < 2$.

Proof. Integrate the reciprocal of (47) and apply Karamata's integral theorem to the regularly varying function $u/q(\kappa u)$. Slow variation gives $L(\kappa u) \sim L(u)$, producing the displayed constants. Monotone inversion gives the final statement. $\square$

For the logarithmic family

$$q(r) = r^2 [\log(r_c/r)]^\beta,$$

theorem 8.1 gives finite critical reach exactly for $\beta > 1$. When $\beta > 1$, the sharper edge law is

$$\varphi_+ - \varphi \sim \frac{s_* d_F}{a_0 \kappa^2 (\beta - 1)} \left[\log \left(\frac{r_c}{\kappa u} \right) \right]^{1-\beta}.$$

This spatial-reach threshold complements the two maximum-slope thickness thresholds.

9 Recovery of the power-law theory

For $q(r) = r^{1/m}$, one has $\alpha = 1/m$ and $L \equiv 1$. The weak exponent becomes

$$1 - 2\alpha = 1 - \frac{2}{m},$$

while the near-critical collapse exponent for $m > 1$ becomes

$$1 - \alpha = 1 - \frac{1}{m} = \frac{m-1}{m}.$$

Thus [theorems 6.1, 7.2](#) and [7.3](#) recover exactly the known thresholds $m = 2$ and $m = 1$ [\[5\]](#) and the nonisothermal preprint discussed in [Section 1](#). The subcritical endpoint criterion in [theorem 5.1](#) also recovers finite support for $m > 1$ and infinite tails for $0 < m \leq 1$.

For the exact-critical branch, the upstream-reach threshold $\alpha = 2$ corresponds to $m = 1/2$.

10 Numerical examples

We use the Tait–Murnaghan constitutive example

$$p(F, \theta) = p_\infty(F^{-n} - 1) + a_T(\theta - \theta_0), \quad (51)$$

$$\varepsilon(F, \theta) - \varepsilon(1, \theta_0) = C_v(\theta - \theta_0) + \frac{a_T\theta_0 + p_\infty}{\rho_0}(F - 1) + \frac{p_\infty}{\rho_0(n-1)}(F^{1-n} - 1), \quad (52)$$

where, for $n = 1$, the last quotient is interpreted as $-\log F$. The energy constraint gives

$$\Theta(F; s) = \theta_0 + \frac{1}{C_v} \left[\frac{s^2}{2}(1 - F)^2 + \frac{a_T\theta_0 + p_\infty}{\rho_0}(1 - F) - \frac{p_\infty}{\rho_0(n-1)}(F^{1-n} - 1) \right]. \quad (53)$$

Proposition 10.1 (Verification for the Tait–Murnaghan example). *Let $p_\infty, C_v > 0$, $n \geq 1$, $a_T \geq 0$, and suppose*

$$\frac{a_T}{\rho_0 C_v} \leq n + 1. \quad (54)$$

For every selected positive-temperature right-going compressive branch, the constitutive, temperature-compatibility, and Hugoniot-convercity parts of [assumption 2.2](#) hold. In particular,

$$P_s''(F) = np_\infty F^{-n-2} \left[(n+1) - \frac{a_T F}{\rho_0 C_v} \right] + \frac{a_T s^2}{C_v} > 0. \quad (55)$$

On bounded Mach intervals the downstream states remain in a compact admissible set, so [assumption 2.3](#) also holds. Furthermore,

$$B = \frac{1}{2} \left[n(n+1)p_\infty + \frac{a_T^3 \theta_0}{\rho_0^2 C_v^2} \right]. \quad (56)$$

Proof. The displayed internal energy satisfies the Gibbs identity. Its constitutive derivatives are

$$p_F = -np_\infty F^{-n-1} < 0, \quad p_\theta = a_T \geq 0, \quad \varepsilon_\theta = C_v > 0.$$

Formula [\(53\)](#) therefore gives the unique smooth compatible temperature. With $u = 1 - F$,

$$C_v \frac{d\Theta}{du} = s^2 u + \frac{a_T \theta_0}{\rho_0} - \frac{p_\infty}{\rho_0} [(1-u)^{-n} - 1].$$

This function is strictly concave in u . At the upstream endpoint it equals $a_T\theta_0/\rho_0$, while the downstream Rankine–Hugoniot relation makes it equal to $a_T\theta_-/\rho_0$ at $u = 1 - F_-$. It is therefore nonnegative throughout, and Θ increases monotonically from θ_0 to θ_- . Two differentiations give (55), whose positivity follows from (54). Also $\Theta(F; s) \rightarrow -\infty$ as $F \downarrow 0$, uniformly for bounded s . A positive-temperature branch is therefore bounded away from $F = 0$, and (53) then bounds its temperature. Along a bounded Mach interval, every sequence of downstream roots consequently has a convergent subsequence. The nontrivial roots cannot converge to $F = 1$, because $\Sigma_F(1; M_*) > 0$ makes the upstream root locally isolated. Strict convexity leaves only one nontrivial limiting root, and its derivative satisfies $\Sigma_F(F_-; s) < 0$. The implicit-function theorem then yields the required C^2 continuation and proves the compactness part of [assumption 2.3](#).

For completeness, the equilibrium Hugoniot relation can be solved explicitly. Writing $u = 1 - F$ and $T(u) = \theta_-(u) - \theta_0$, one obtains

$$\left(C_v - \frac{a_T u}{2\rho_0}\right) T(u) = \frac{a_T\theta_0 + p_\infty}{\rho_0} u - \frac{p_\infty}{\rho_0(n-1)} ((1-u)^{1-n} - 1) + \frac{p_\infty u}{2\rho_0} ((1-u)^{-n} - 1), \quad (57)$$

with the continuous logarithmic interpretation at $n = 1$. Substitution into $p(1-u, \theta_0 + T(u))$ and a second-order expansion gives

$$p(1-u, \theta_-(u)) = \rho_0 c_0^2 u + \frac{1}{2} \left[n(n+1)p_\infty + \frac{a_T^3 \theta_0}{\rho_0^2 C_v^2} \right] u^2 + O(u^3),$$

which proves (56). □

The normalized parameters are

$$\rho_0 = \theta_0 = C_v = k_0 = \tau_0 = c_0 = 1, \quad n = 5, \quad a_T = 0.02, \quad \gamma = 0.5, \quad p_\infty = 0.19992,$$

with $k(\Theta) = k_0 e^{-\gamma(\Theta - \theta_0)}$. We set $\bar{\mu}_0 = 0.5$ and use

$$a(F, \Theta, r) = F 2^{\alpha-1} k(\Theta)^{-\alpha}, \quad q(r) = r^\alpha [\log(e + 1/r)]^\beta.$$

The resulting constants are

M_*	1.22474487139159
J_*	0.121527966929536
$B = \mathcal{P}''(0)/2$	2.998804
A	0.666932550443443
d_M	4.89897948556636
d_F	12.055216
κ	0.5

The nontrivial downstream root is computed by Brent’s method after dividing out the persistent upstream root. The maximum slope is found on a combined linear and logarithmic grid and refined by bounded scalar optimization. Cancellation-safe formulas based on $\log(1-u)$ and $\expm1$ are used near $F = 1$. Online Resource 1 contains the computational scripts and data tables. All numerical data in this section are synthetic outputs of the stated constitutive law.

At $M-1 = 10^{-7}$, the ratio of the numerical weak thickness to (24) is unity to within approximately 10^{-6} for $\alpha = 0.35, 0.5, 0.75$, and the maximizing normalized position converges to $z = 1/2$. At

$M_* - M = 10^{-7}$ and $\alpha = 3/4$, the numerical-to-asymptotic ratio in (35) is approximately 1.00000090, while

$$\frac{1 - F_{\max}}{M_* - M} \approx 1.21913, \quad y_{3/4} \approx 1.21914.$$

Using the unrounded values, the absolute difference between the computed scaled location and $y_{3/4}$ is approximately 2.04×10^{-6} .

High-precision calculations and marginal convergence. A separate implementation uses 80-digit arithmetic, reconstructs temperature from the energy integral and stress from momentum conservation, and locates stationary rates by bracketed scalar root finding for the selected smooth source laws. Online Resource 1 records twelve weak, near-critical and logarithmic cases and three constitutive boundary cases. At the smallest tested non-marginal scale, 10^{-10} , the thickness ratios agree with their leading equivalents to relative error below 10^{-8} . Convergence is slower at a logarithmic margin. For the same constitutive example with $\alpha = 1$ and $q(r) = r \log(e + 1/r)$, asymptotic to $r \log(1/r)$, the numerical-to-leading thickness ratio is approximately 1.1581 at $\delta = 10^{-6}$ and 1.0738 at $\delta = 10^{-20}$. The $\log \log(1/\delta)$ correction in (46) contributes to this slow approach to the limiting location scale.

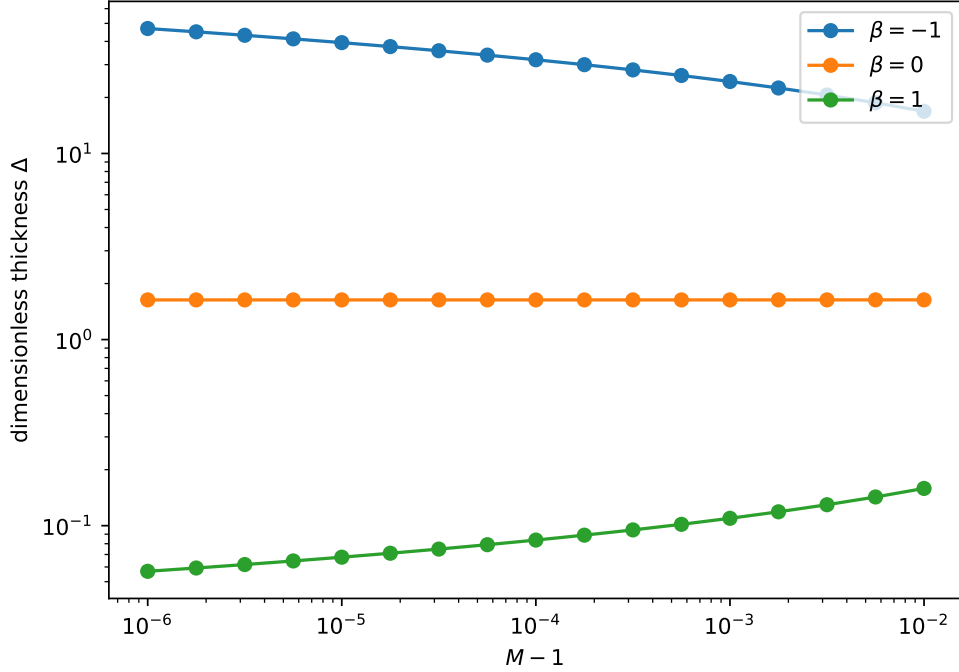


Figure 1: At the weak marginal index $\alpha = 1/2$, the slowly varying exponent β selects broadening, a plateau, or collapse.

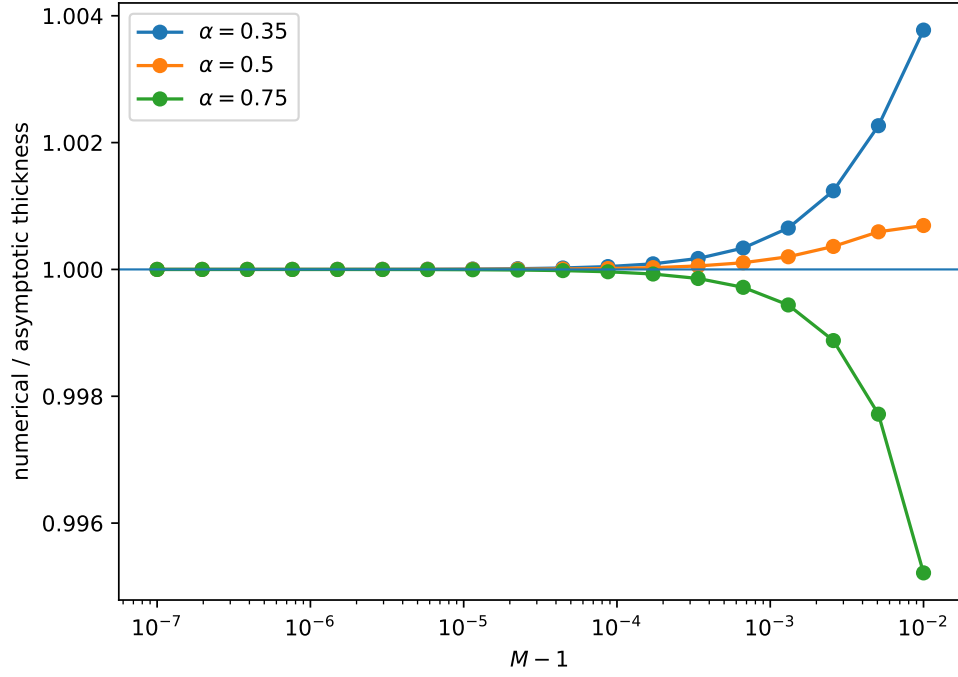


Figure 2: Ratio between numerical thickness and the exact weak-shock equivalent (24).

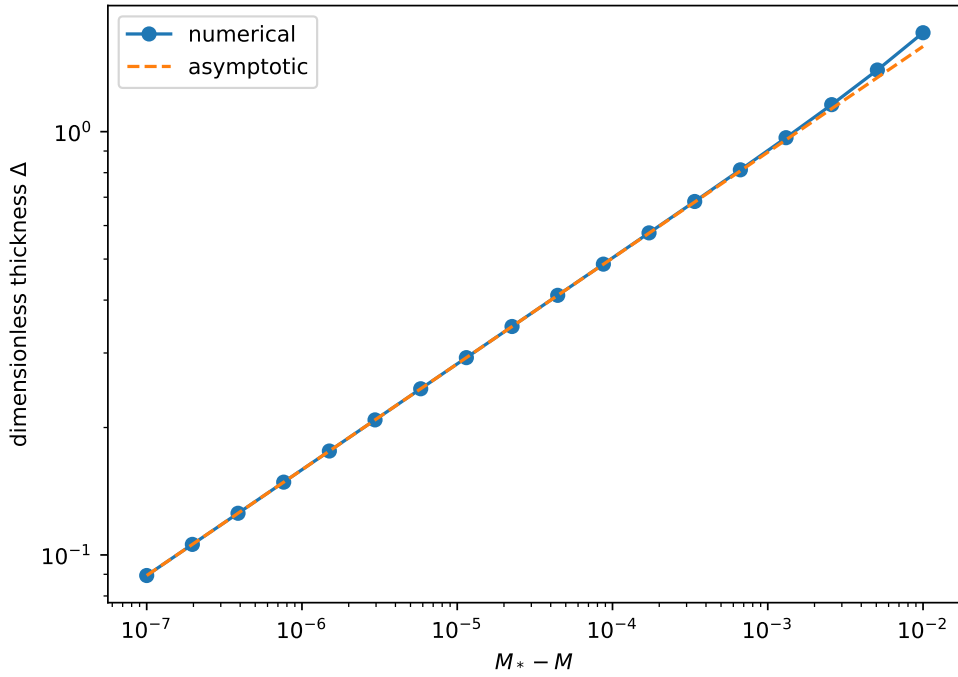


Figure 3: Near-critical numerical thickness and the prediction (35) for $\alpha = 3/4$.

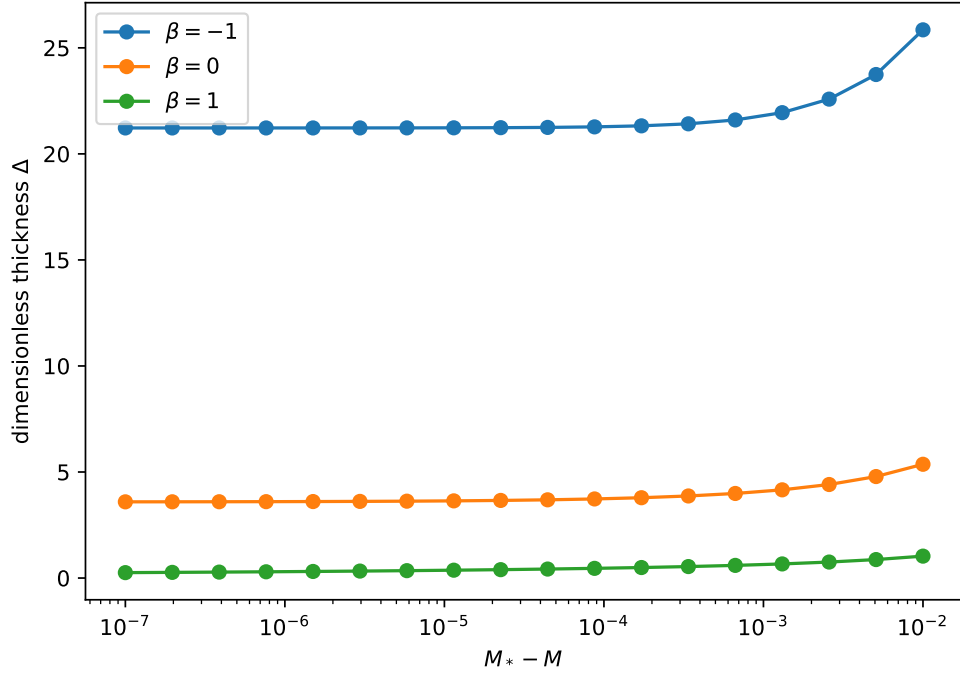


Figure 4: The near-critical marginal case $\alpha = 1$. A growing logarithmic factor causes collapse, while bounded or vanishing factors lead to plateaus.

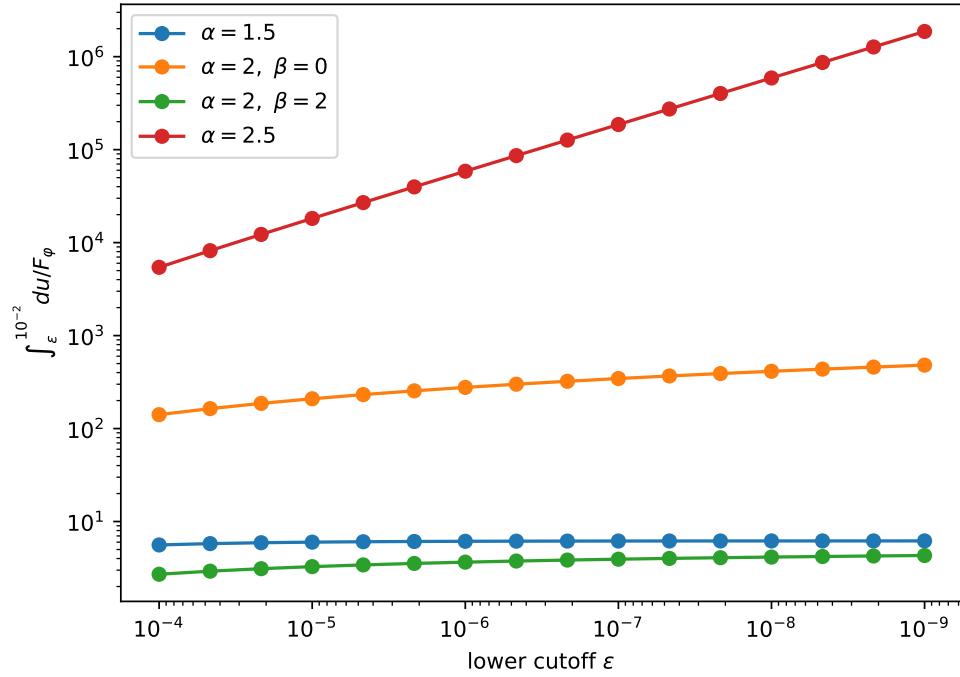


Figure 5: Truncated exact-critical upstream length. The index $\alpha = 2$ is marginal; for $\alpha = 2$, a logarithmic factor with $\beta = 2$ restores finite reach.

11 Discussion

The four asymptotic mechanisms can be read from the following phase map.

Regime	Governing quantity	Threshold	Marginal discriminator
Subcritical endpoint	$\int_0^\varepsilon q(r)^{-1} dr$	$\alpha = 1$	$\int_0^\varepsilon [rL(r)]^{-1} dr$
Weak-shock thickness	$\delta_F/q(B\delta_F^2)$	$\alpha = 1/2$	$L(B\delta_F^2)^{-1}$
Near-critical thickness	$\delta/q(\kappa\delta)$ for $0 < \alpha < 1$; critical-rate supremum for $\alpha \geq 1$	$\alpha = 1$	upper envelope of L
Exact-critical reach	$\int_0^\varepsilon u/q(\kappa u) du$	$\alpha = 2$	$\int_0^\varepsilon [uL(u)]^{-1} du$

The regular-variation index α determines the algebraic exponent away from a threshold, with slowly varying factors entering multiplicatively. At a marginal index, those factors determine the qualitative alternative between broadening, a plateau and collapse.

Three separate local balances are responsible for the indices $1/2$, 1 , and 2 . In a weak shock, the deformation jump is $O(\delta_F)$ but the reduced stress is $O(\delta_F^2)$, giving $\Delta \asymp \delta_F/q(\delta_F^2)$ and the index $1/2$. Near the characteristic encounter, stress and the characteristic denominator are both linear in the upstream distance, but the denominator contains the additional scale $M_* - M$, giving $\Delta \asymp \delta/q(\delta)$ for $0 < \alpha < 1$. At $\alpha = 1$ the upper envelope of L decides between collapse and a plateau, while $\alpha > 1$ yields a positive plateau. These alternatives identify the index 1 . At the exact encounter, spatial reach requires integrating the inverse slope $u/q(u)$, giving the index 2 .

The quadratic nonequilibrium entropy gives a constant ω and preserves the entropy symmetrizer and characteristic geometry of the four-field RET model. A state-dependent $\omega(\sigma/\theta)$ would change this geometry and calls for a corresponding convexity and characteristic analysis. Rheological calibration of slowly varying corrections, dynamic stability and continuation beyond M_* are further directions for this model.

For $0 < \alpha < 1$, the weak and near-critical thickness functions have regular-variation indices

$$\beta_w = 1 - 2\alpha, \quad \beta_c = 1 - \alpha, \quad \beta_w = 2\beta_c - 1,$$

with respect to $M - 1$ and $M_* - M$, respectively. Slowly varying factors affect slopes fitted over finite observation windows, so comparison with rheological data also requires estimating these corrections.

Reproducibility. Online Resource 1 provides the symbolic and numerical scripts, synthetic data tables, plotting outputs, dependency versions and execution records, including the separate high-precision implementation.

12 Conclusion

Regularly varying production laws extend the two power-law shock-thickness classifications to sources with slowly varying corrections. The algebraic index determines the non-marginal scaling, while the slowly varying factor resolves the equality cases. At the exact characteristic encounter, the upstream-reach criterion yields the additional threshold $\alpha = 2$. These results combine the scalar traveling-wave reduction, characteristic geometry, uniform regular-variation estimates and global localization of the maximum slope.

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Data and code availability. All numerical data are synthetic calculations for the model specified in Section 10. The symbolic and numerical scripts, CSV tables, plotting outputs, dependency information, execution records and separate numerical implementation are provided in the accompanying supplementary archive, Online Resource 1.

AI assistance and human verification. OpenAI Codex with the GPT-6 Astra model assisted with problem exploration, derivation and proof drafting, symbolic and numerical code, plotting, mathematical and reference checks, and drafting and revising the English manuscript. The author reviewed and verified the core mathematical derivations and computational code, approved the final manuscript, and takes responsibility for its content.

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References

- [1] I. Müller, T. Ruggeri: *Rational Extended Thermodynamics*, 2nd edn. Springer, New York (1998). <https://doi.org/10.1007/978-1-4612-2210-1>
- [2] G. Boillat, T. Ruggeri: On the shock structure problem for hyperbolic system of balance laws and convex entropy. *Continuum Mechanics and Thermodynamics* **10**, 285–292 (1998). <https://doi.org/10.1007/s001610050094>
- [3] T. Ruggeri: Rational extended thermodynamics for non-Newtonian fluids with finite relaxation time. *Physics of Fluids* **37**, 111702 (2025). <https://doi.org/10.1063/5.0302652>
- [4] N. H. Bingham, C. M. Goldie, J. L. Teugels: *Regular Variation*. Cambridge University Press, Cambridge (1987). <https://doi.org/10.1017/CB09780511721434>
- [5] T. Ruggeri, S. Taniguchi: Flow-index universality and critical regime transitions of shock thickness in a power-law fluid model within rational extended thermodynamics. *Physics of Fluids* **38**, 083105 (2026). <https://doi.org/10.1063/5.0348589>
- [6] T. Arima, T. Ruggeri: A symmetric hyperbolic non-isothermal model for viscoelastic solids and non-Newtonian fluids. *International Journal of Engineering Science* **222**, 104491 (2026). <https://doi.org/10.1016/j.ijengsci.2026.104491>