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Sèmako Justin Dèdèwanou

justin.dedewanou@imsp-uac.org

Université de Kindia

Siba Kalivogui

Université de Kindia

Amoussou Laurent Hinvi

Université de Kindia

Thierno Mamadou Pathé Diallo

Université de Kindia

Hodévèwan Clément Miwadinou

National University of Sciences, Technologies, Engineering and Mathematics

Vincent Adjimon Monwanou

Institute of Mathematics and Physics

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Sèmako Justin Dèdèwanou^{1,2*}, Siba Kalivogui³,
Amoussou Laurent Hinvì^{2,4}, Thierno Mamadou Pathé Diallo¹,
Hodévèwan Clément Miwadinou^{2,5}, Vincent Adjimon Monwanou²

¹Department of Physics, Faculty of Sciences, University of Kindia,
Foulaya, B.P.: 212, Kindia, Guinea.

²Laboratoire de Mécanique des Fluides, de la Dynamique Nonlinéaire et
de la Modélisation des Systèmes Biologiques (LMFDNMSB), Institut de
Mathématiques et de Sciences Physiques, Dangbo, B.P.: 613,
Porto-Novo, Benin.

³Department of Mathematics, Faculty of Sciences, University of Kindia,
Foulaya, B.P.: 212, Kindia, Guinea.

⁴Laboratoire des Procédés et de l'Innovation Technologique (LaPIT),
INSTITUT Lokossa, Université Nationale des Sciences, Technologie,
Ingénierie et Mathématiques (UNSTIM), Natitingou, B.P.: 486,
Abomey, Benin.

⁵Laboratoire de Physique et Application (LPA), Ecole Normale
Supérieure (ENS), Université Nationale des Sciences, Technologie,
Ingénierie et Mathématiques (UNSTIM), Natitingou, B.P.: 486,
Abomey, Benin.

*Corresponding author(s). E-mail(s): justin.dedewanou@imsp-uac.org;

Abstract

The present investigation addresses the heat transfer in a thermal convection flow governed by the Darcy–Brinkman framework within a porous layer saturated by a hybrid nanofluid composed of iron oxide-copper nanofragments dispersed across three distinct base fluids, namely water, an ethylene glycol-water mixture,

and palm oil, under the Cattaneo–Christov heat flux model. A normal mode linear stability analysis is carried out. A Galerkin truncation procedure is applied to the Darcy-Brinkman-Boussinesq governing equations, coupled with a two-temperature energy model, producing a five-dimensional autonomous dynamical system that constitutes a generalization of the classical Lorenz equations. Numerical integration is performed using a fourth-order Runge–Kutta scheme, and the dynamical behavior of the system is thoroughly characterized through four complementary diagnostic tools: bifurcation diagrams of local extrema of the convective variable, the maximal Lyapunov exponent computed via the renormalization method, and phase portrait reconstructions. A comprehensive discussion of the physical outcomes is subsequently presented, examining the respective influences of the nanoparticle volume fraction, the Darcy number, the thermal time-lag parameter, and the Hartmann number on the critical stationary Darcy-Rayleigh number, the critical wave number, and the physical onset temperature, assessed comparatively across the three base fluids. The magnetic field is shown to delay the onset of convection and shift the transitional peak of the mean Nusselt number toward higher Rayleigh–Darcy numbers, while all configurations gradually approach a common asymptotic heat-transfer level in the fully developed regime. Beyond its theoretical contribution, this study offers practical guidance for optimizing thermal enhancement strategies and improving the thermodynamic performance of energy systems in thermal engineering, with particular relevance to porous heat exchangers.

Keywords: Hybrid nanofluid, Darcy-Brinkman model, thermal stability, Bifurcation, Chaos

1 Introduction

Free convection in fluid-saturated porous media stands as one of the most thoroughly investigated transport phenomena in engineering and geophysics. The shift from a purely conductive base state to a sustained cellular convection regime hinges on a critical balance between the destabilizing thermal buoyancy force and the stabilizing influences of viscous drag, thermal diffusion, and electromagnetic damping. Pinpointing the exact onset threshold of this transition holds profound fundamental and practical significance. Nanotechnology has achieved an immense attention among scientists owing to its extensive range of implications in productive industries, practice of medicines, forensic labs, anatomy etc. Metals (or metal oxides or metal nitrides) possessing high heat conductivities, are disseminated into base liquids of comparatively lesser heat conductivities. And this dispersion significantly enhances the heat conductivity of the resultant liquid. Nanoliquids also have efficient cooling components apart from high thermal conductivity. Choi first studied the increment of heat conductivity of base liquids due to the application of nano-fragments. The suspended nano-fragments are infinitesimally small ($1 - 10$ nm) [1], thus the liquid phase and the solid fragments share identical local velocity and they are in the same equilibrium phase. The thermo-physical aspects of the base liquid are remarkably influenced by the volume fraction of nano-fragments. Tiwari and Das [2] used the thermospecial relationships

given as in the work of Brinkman [3] and Xuan [4] for augmentation of heat transfer. However, the particles also give rise to body couple and couple-stress. For nanofragments one may neglect these effects and may take into account the enhancement of thermal conductivity and other properties for the presence of micron-sized particles in an otherwise clean Newtonian fluid. Moreover, when the flow is laminar or turbulent, the random movements of the nano-fragments create two effects namely, Brownian diffusion and thermophoresis, for which the thermal transmission in the nanoliquids have been influenced prominently. In this study, we consider nanofragments little bit larger (1-10 nm) and so Brownian motion, friction between solid and liquid molecules, clustering nanofragments and thermophoresis are not prominent. The coupling of vertical magnetic fields with thermal convection in pure fluids is pioneered in [5], proving analytically that strong fields elevate the critical Rayleigh number proportionally to the Hartmann number, as Lorentz forces damp vertical motions driving convective cells. Sheikholeslami [6] subsequently confirmed, through numerical simulation of a nanofluid-filled porous enclosure, that stronger magnetic fields monotonically reduce the Nusselt number, in qualitative agreement with the linear stability picture. Horton [7], followed independently by Lapwood [8], derived the critical Darcy-Rayleigh number for isothermal, permeable boundaries in isotropic porous layers, a cornerstone still central to modern stability studies.

Nield [9] initiated linear stability studies of Darcy layers filled with nanoliquids, uncovering an oscillatory onset mode from Brownian diffusion-thermophoresis interplay, which lowers the critical Rayleigh number well below pure-fluid levels. Kuznetsov [10] later refined this in a Brinkman model using zero-nanoparticle-flux boundaries at walls, yielding more realistic and distinct stability boundaries. Chand [11] extended the Darcy-Brinkman nanofluid framework to include rotation, demonstrating the Taylor number's stabilizing role alongside the Hartmann number. Yadav *et al.* [12] explored combined internal heat generation and nanoparticle effects, revealing that their balance can either delay or hasten onset compared to the standard state. Sarkar *et al.* [13] provided a thorough review of hybrid nanofluid thermophysical properties, assembling experimental data showing that select dual-particle mixtures, like Cu-Al₂O₃/water or CNT-Fe₃O₄/water, deliver higher thermal conductivity gains than mono-nanofluids at identical volume fractions. The boundary-layer computations are conducted in [14] for Cu-Al₂O₃/water under magnetic influence, revealing that combining both nanoparticles boosts convective heat transfer beyond single-species cases, thanks to their differing electrical conductivities. Waini *et al.* [15] analyzed unsteady stagnation-point flow of Cu-Al₂O₃/water hybrids, uncovering dual solutions whose stability hinges on the suction parameter, highlighting potential multiplicity in these flows. Siddabasappa *et al.* [16] introduced local thermal non-equilibrium (LTNE) between phases, proving the LTNE parameter reshapes the neutral stability curve, especially at high wavenumbers where inter-phase heat transfer falters. Layek and Pati [17] investigated the intricate chaotic dynamics that emerge when magnetism and convection are coupled in non-Newtonian fluids. Their work revealed the existence of hyperchaos, a particularly rich form of chaos characterized by the coexistence of multiple positive Lyapunov exponents. By tracing the bifurcation routes leading to this behavior, they identified the key mechanisms triggering the transition to chaos, notably through period-doubling

bifurcations. This approach, centered on the qualitative analysis of strange attractors, provides a robust methodological framework for understanding how complex dynamics arise in fluid systems subjected to external fields. Two years later, the same authors shifted their focus to thermo-viscoelastic fluids in [18], where elasticity and viscosity interact with thermal effects. They observed an original "period-bubbling" transition, a mechanism less common than the classic period-doubling route, before the onset of complete chaos. Their detailed bifurcation analysis demonstrated how the fluid's elasticity modifies system stability and promotes the emergence of chaotic behavior. These findings reinforce the view that the rheological properties of the fluid play a decisive role in the nonlinear dynamics of flows, and open the door to extensions toward other types of complex fluids, including nanofluids. Deb and Layek [19] introduced a compelling dynamical framework to investigate bifurcations and chaotic behavior in couple-stress nanoliquid flows, with a specific focus on heat transmission characteristics. Their work bridges nonlinear dynamics and nanofluid thermodynamics by revealing how couple-stress effects can significantly alter the onset of chaos and modify heat transfer rates in Rayleigh–Bénard convection systems. This study is particularly relevant to our investigation, as it underscores both the stability and thermal performance of nanofluid systems, a key aspect we also explore in our own investigation. Nield and Bejan [20] considered anisotropic permeability, non-uniform heating, non-Newtonian effects, and internal heat sources, thereby establishing a broad reference model. In a complementary investigation focused on a Brinkman-filtered porous geometry, Srikantha *et al.* [21] derived a three-mode Lorenz-type reduced model from the weakly nonlinear expansion of the Brinkman–Bénard equations for a nanofluid-saturated layer with temperature-dependent viscosity and uniform internal heat generation. Their work stands out for showing that while nanoparticle concentration barely shifts when convection starts, it dramatically boosts heat transfer (Nusselt number) once the system enters strong nonlinear regimes.

More recently, it's found that finite thermal relaxation time destabilises the onset of convection in a hybrid nanofluid flow with Cattaneo–Christov heat flux relative to the classical Fourier prediction, a result of direct relevance to the authors' ongoing work on Cattaneo-type systems [22]. For porous media, Haddad [23] developed an explicit dispersion relation that incorporates Darcy drag alongside damping from a non-Fourier heat flux model. The analysis reveals that the Cattaneo–Christov theory significantly influences the convection instability threshold. We employ the two-step sequential version of the Maxwell–Garnett effective-medium theory in [24], which Nan *et al.* [25] confirmed experimentally for dilute metal-oxide suspensions. This approach accommodates two distinct nanoparticle types. Among viable pairings, magnetite offers chemical stability and photocatalytic properties, whereas copper nanoparticles provide exceptional electrical conductivity and plasmonic effects, rendering the magnetite-copper hybrid especially suited for magnetically tunable thermal management systems. The interplay between MHD damping, nanoparticle-induced and specific base fluid property modifications, and porous-medium drag constitutes the physical motivation for the present work. A review of prior work highlights substantial advances in isolated aspects (MHD porous convection, stability of single-component fluids, hybrid nanofluid properties, etc), yet no research has tackled the simultaneous integration of hybrid

nanofragment in base fluids (water, EG-water, or palm oil), a Darcy-Brinkman model for fiberglass-reinforced polycarbonate (FGRP) porous media, Cattaneo-Christov heat flux, and a horizontal magnetic field. This study addresses this shortfall through the following objectives: derive a closed-form stationary dispersion relation accounting for effective-property ratios of the nanoliquid and key control parameters; numerically determine the critical pair and its variation across all parameters; validate the numerical approach against Haddad’s analytical benchmark; and formulate a complete dynamical system as a foundation for subsequent nonlinear investigations. To fulfill these objectives, the paper unfolds in a structured manner. Section 2 lays the foundation by establishing the mathematical framework of the governing equations, supported by a schematic diagram illustrating the physical setup under investigation. Building on this formulation, Section 3 explores steady-state convection through normal mode analysis. This analytical approach naturally leads, in Section 4, to the construction of an autonomous dynamical system composed of five coupled nonlinear ordinary differential equations. The heart of our findings is discussed in Section 5, which is divided into three focused parts: subsection 5.1 outlines the numerical methodology adopted, subsection 5.2 examines the validation and robustness of the computational scheme, and subsection 5.3 provides a thorough interpretation of the results. The study concludes in Section 6, where we synthesize the key insights and main conclusions emerging from this work.

2 Mathematical modeling

We consider a fiberglass-reinforced polycarbonate (FGRP) porous media of thickness d , saturated by a water, Ethylene-glycol-water and palm oil with Fe_3O_4 -Cu hybrid nanofragment. The lower wall is maintained at temperature $T_0 + \Delta T$ and the upper wall at T_0 , with $\Delta T > 0$, so that the temperature decreases from bottom to top and the layer is potentially unstable to buoyancy-driven motions. A uniform magnetic field $\vec{B} = B_0 \vec{x}$ is applied in the horizontally direction, parallel to x-axis and perpendicular to the vertical plane of the medium. The porous matrix is homogeneous, isotropic, and characterised by porosity $\varepsilon = 0.88$ and Darcy permeability K .

The motion of the hybrid nanofluid within the porous medium is governed by the Darcy–Brinkman–Boussinesq system with Maxwell induction:

$$\frac{\partial q_i}{\partial x_i} = 0, \quad (1)$$

$$\rho_{hb} \left[\frac{1}{\varepsilon} \frac{\partial q_i}{\partial t} + \frac{1}{\varepsilon^2} \left(q_j \frac{\partial q_i}{\partial x_j} \right) \right] = -\frac{\partial P}{\partial x_i} - \rho g e_i + \frac{\mu_{hb}}{\varepsilon} \nabla^2 q_i - \frac{\mu_{hb}}{K} q_i - \sigma_{hb} B_0^2 q_{\perp} \quad (2)$$

$$\frac{(\rho C p)_{ph}}{(\rho C p)_{hb}} \frac{\partial T}{\partial t} + \left(q_i \frac{\partial T}{\partial x_i} \right) = -\frac{\partial Q_i}{\partial x_i}, \quad (3)$$

$$\tau_{ph} \frac{\partial Q_i}{\partial t} + \tau_{hb} \left(q_j \frac{\partial Q_i}{\partial x_j} - Q_j \frac{\partial q_i}{\partial x_j} \right) = -Q_i - \frac{k_{ph}}{(\rho C p)_{hb}} \frac{\partial T}{\partial x_i}. \quad (4)$$

In the Eq. (2), $e_k = (0, 0, 1)$ is the unit vector, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator and $q_{\perp}$ denotes the velocity components perpendicular to $\mathbf{B}$

(i.e., u and w). Here m and s refer to the porous medium, solid skeleton of the porous medium. The density-temperature state equation at reference T_0 under the Boussinesq approximation and mixture theory model for nanoliquid-saturated porous medium are :

$$\left. \begin{aligned} \rho_{hb}(T) &= \rho_{hb}(T_0) - (\rho\beta)_{hb}(T - T_0), \\ (\rho c_p)_{ph} &= \varepsilon(\rho c_p)_{hb} + (1 - \varepsilon)(\rho c_p)_s, \\ k_{ph} &= \varepsilon k_{hb} + (1 - \varepsilon)k_s, \\ k_m &= \varepsilon k_f + (1 - \varepsilon)k_s. \end{aligned} \right\} \quad (5)$$

The density ρ_{hb} , the thermal expansion coefficient β_{hb} , the specific heat $(\rho C_p)_{hb}$, the dynamical viscosity μ_{hb} , the thermal conductivity k_{hb} and the electrical conductivity σ_{hb} :

$$\left. \begin{aligned} \rho_{hb} &= (1 - \varphi_2)[(1 - \varphi_1)\rho_f + \varphi_1\rho_{s1}] + \varphi_2\rho_{s2}, \\ (\rho\beta)_{hb} &= (1 - \varphi_2)\left[(1 - \varphi_1)(\rho\beta)_f + \varphi_1(\rho\beta)_{s1}\right] + \varphi_2(\rho\beta)_{s2}, \\ (\rho C_p)_{hb} &= (1 - \varphi_2)\left[(1 - \varphi_1)(\rho C_p)_f + \varphi_1(\rho C_p)_{s1}\right] + \varphi_2(\rho C_p)_{s2}, \\ \mu_{hb} &= \mu_f(1 - \varphi_1)^{-2.5}(1 - \varphi_2)^{-2.5}, \\ k_{hb} &= k_g \left[\frac{(k_{s2} + 2k_g) - 2\varphi_2(k_g - k_{s2})}{(k_{s2} + 2k_g) + \varphi_2(k_g - k_{s2})} \right], \quad k_g = \left[\frac{(k_{s1} + 2k_f) - 2\varphi_1(k_f - k_{s1})}{(k_{s1} + 2k_f) + \varphi_1(k_f - k_{s1})} \right] k_f, \\ \sigma_{hb} &= \sigma_g \left[\frac{(\sigma_{s2} + 2\sigma_g) - 2\varphi_2(\sigma_g - \sigma_{s2})}{(\sigma_{s2} + 2\sigma_g) + \varphi_2(\sigma_g - \sigma_{s2})} \right], \quad \sigma_g = \left[1 + \frac{3\left(\frac{\sigma_{s1}}{\sigma_f} - 1\right)\varphi_1}{2 + \frac{\sigma_{s1}}{\sigma_f} - \left(\frac{\sigma_{s1}}{\sigma_f} - 1\right)\varphi_1} \right] \sigma_f. \end{aligned} \right\} \quad (6)$$

In the above equations, indices 1 and 2 are used to designate Fe_3O_4 and Cu nanofragments. The thermophysical properties of the nanoparticles and water used in this work are summarized in Table 1.

Table 1: The thermophysical properties of base fluid and nanoparticles used.

Property	Water	EG-water	Palm oil	Fe_3O_4	Cu	30% FGPR
ρ	997.1	1063.8	885	5180	8933	1430
C_p	4179	3630	1875	670	385	1130
k	0.613	0.387	0.144	9.7	401	0.24
β ($\times 10^5$)	21	5.8	72.7	1.3	1.67	3.5
σ	0.05	9.75×10^{-4}	1.0×10^{-9}	2.5×10^4	5.96×10^7	-
μ	8.9×10^{-4}	0.0035	0.0485	-	-	-

We work out linear stability theory in nanoliquid saturated porous media for both stress-free and no-slip boundary conditions adopting perturbation technique. First,

the stress-free boundary restrictions for the flow problem are described as:

$$\left. \begin{aligned} T &= T_l \quad \text{at} \quad z = 0, \\ T &= T_u \quad \text{at} \quad z = d, \\ q_i &= 0 \quad \text{at} \quad z = 0, z = d. \end{aligned} \right\} \quad (7)$$

Then, the steady-state solutions are given by:

$$\left. \begin{aligned} q_i^s &= 0, \quad p^s = p_0 - \rho_{hb}gz - \frac{1}{2}(\rho\beta)_{hb}\chi gz^2, \quad T^s = T_l - \chi z, \\ Q_i^s &= \frac{k_{ph}}{(\rho Cp)_{hb}}\chi \quad \text{with} \quad \chi = \frac{(T_l - T_u)}{d}. \end{aligned} \right\} \quad (8)$$

Finite-amplitude perturbations to the basic steady state solutions are adopted for stability analysis:

$$\tilde{q}_i = \tilde{q}_i^s + \tilde{q}_i', \quad \tilde{T} = \tilde{\theta}^s + \tilde{\theta}', \quad \tilde{Q}_i = \tilde{Q}_i^s + \tilde{Q}_i' \quad \tilde{p} = \tilde{p}^s + \tilde{p}'. \quad (9)$$

The prime quantities used here indicate the perturbation variables to the steady solutions and will be omitted from subsequent notation.

Next the study of the characteristic properties of this physical problem need the non-dimensional quantities by the following transformations as:

$$\tilde{x}_i = \frac{1}{d}x_i, \quad \tilde{t} = \frac{\mu_f}{K\rho_f}t, \quad \tilde{q}_i = \frac{K\rho_f}{\mu_f d}q_i, \quad \tilde{p} = \frac{K}{\mu_f U d}p, \quad (10)$$

$$\tilde{\theta} = \frac{1}{U d} \sqrt{\frac{K\beta_f g \rho_f \alpha_f}{\chi \mu_f}} T, \quad \tilde{Q}_i = \frac{d}{\alpha_m} \left(\frac{\tilde{\theta}}{T} \right) Q_i, \quad \chi = \frac{(T_l - T_u)}{d}. \quad (11)$$

Using the length, time, velocity, temperature, and heat flux scales, the dimensionless governing equations for the disturbance quantities relative to the reference conduction state are written as follows:

$$\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i} = 0, \quad (12)$$

$$M_1 \frac{\partial \tilde{q}_i}{\partial \tilde{t}} = -\frac{\partial \tilde{P}}{\partial \tilde{x}_i} + M_3 R_f \tilde{\theta} e_k - M_2 \left(\varepsilon - D_a \tilde{\nabla}^2 \right) \tilde{q}_i - M_5 Ha_D^2 \tilde{q}_\perp, \quad (13)$$

$$\left(\frac{P_r}{D_a} \right) \left[M_6 \frac{\partial \tilde{\theta}}{\partial \tilde{t}} + \left(\tilde{q}_j \frac{\partial \tilde{\theta}}{\partial \tilde{x}_j} \right) \right] = R_f \tilde{w} - \frac{\partial \tilde{Q}_i}{\partial \tilde{x}_j}, \quad (14)$$

$$\left(\frac{S_g}{D_a} \right) \left[\frac{\partial \tilde{Q}_i}{\partial \tilde{t}} + \tilde{\tau} \left(\tilde{q}_j \frac{\partial \tilde{Q}_i}{\partial x_j} - \tilde{Q}_j \frac{\partial \tilde{q}_i}{\partial x_j} \right) \right] = -\tilde{Q}_i - M_7 \left(\frac{\partial \tilde{\theta}}{\partial \tilde{x}_j} + \frac{\tilde{\tau} S_g R_f}{P_r} \frac{\partial \tilde{q}_i}{\partial \tilde{z}} \right), \quad (15)$$

where the effective-property ratios and fluid parameters are :

$$\left. \begin{aligned} M_1 &= \left(\frac{\rho_{hb}}{\rho_f} \right), \quad M_2 = \left(\frac{\mu_{hb}}{\varepsilon \mu_f} \right), \quad M_3 = \left(\frac{(\rho\beta)_{hb}}{(\rho\beta)_f} \right), \quad M_4 = \left(\frac{\sigma_{hb}}{\sigma_f} \right), \\ M_5 &= \left(\frac{(\rho C_p)_{ph}}{(\rho C_p)_{hb}} \right), \quad P_r = \frac{\nu_f}{\alpha_f}, \quad D_a = \frac{K}{d^2}, \quad R_a = \frac{\beta_f g d^2 K \chi}{\alpha_f \nu_f}, \quad \chi = \frac{(T_l - T_u)}{d}, \\ \tilde{\tau} &= \frac{\tau_f}{\tau_m} \quad M_6 = \left(\frac{k_{ph}}{k_m} \right) \left(\frac{(\rho C_p)_f}{(\rho C_p)_{hb}} \right), \quad Ha_D = B_0 \sqrt{\frac{\sigma_f K}{\mu_f}}, \quad S_g = \frac{\tau \mu_f}{\rho_f d^2}. \end{aligned} \right\} \quad (16)$$

3 Stationary convection

In this section, we perform a linear stability analysis of the conductive base state. Following the classical methodology outlined by Straughan [26], we first linearize the governing equations by neglecting all nonlinear terms in Eqs. (12) - (15). To eliminate the pressure from the dimensionless system (12)-(15), we apply the curl-curl operator to Eq. (13), take the divergence of Eq. (15), project the resulting vector equations onto the vertical direction, and introduce the divergence of the heat flux $Q = \partial \tilde{Q}_i / \partial \tilde{x}_i$. After dropping the tildes for notational simplicity, the linearized perturbation equations reduce to:

$$\left[M_1 \frac{\partial}{\partial t} + M_2 (\varepsilon - D_a \nabla^2) \right] \nabla^2 w + M_4 Ha_D^2 \frac{\partial^2 w}{\partial z^2} = M_3 R_f \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right), \quad (17)$$

$$M_5 \left(\frac{P_r}{D_a} \right) \frac{\partial \theta}{\partial t} = R_f w - Q, \quad (18)$$

$$\left(\frac{S_g}{D_a} \right) \frac{\partial Q}{\partial t} = -Q - M_6 \nabla^2 \theta, \quad (19)$$

where w , θ , and Q represent, respectively, the vertical velocity component, the temperature perturbation, and the divergence of the heat flux. These three fields fully characterize the evolution of infinitesimal disturbances around the motionless conductive state.

To analyze the stability properties, we adopt a normal-mode decomposition and seek solutions of the form

$$[w, \theta, Q] = [W(z), \Theta(z), \Phi(z)] f(x, y) e^{\omega t}, \quad (20)$$

where the horizontal structure function $f(x, y)$ satisfies the Helmholtz equation

$$\nabla^2 f(x, y) = -\kappa^2 f(x, y), \quad (21)$$

with κ denoting the horizontal wavenumber. The functions $W(z)$, $\Theta(z)$, and $\Phi(z)$ are the corresponding eigenfunctions. Substituting the ansatz in the Eq. (20) into Eqs. (17)–(19) yields the following set of ordinary differential equations:

$$\left. \begin{aligned} [M_1\omega + \varepsilon M_2 - M_2 D_a (D^2 - \kappa^2)] (D^2 - \kappa^2) W + M_4 H a_D^2 D^2 W &= -\kappa^2 M_3 R_f \Theta, \\ \omega M_5 \left(\frac{P_r}{D_a} \right) \Theta &= R_f W - \Phi, \\ \left(\frac{S_g}{D_a} \right) \omega \Phi &= -\Phi - M_6 (D^2 - \kappa^2) \Theta, \end{aligned} \right\} \quad (22)$$

where $D \equiv d/dz$ and $D^2 \equiv d^2/dz^2$.

We impose stress-free, isothermal boundary conditions at the top and bottom plates, namely $W = 0$, $D^2 W = 0$, and $\Theta = 0$ at $z = 0$ and $z = 1$. These conditions are satisfied by eigenfunctions of the form

$$W, \Theta, \Phi \propto \sum_{n=1}^N A_n \sin(n\pi z), \quad n = 1, 2, 3, \dots, \quad (23)$$

where the coefficients A_n are determined by the eigenvalue problem. The marginal stability state corresponds to $\omega = 0$. After straightforward algebraic elimination of the amplitudes, we obtain an explicit dispersion relation for the stationary Darcy–Rayleigh number:

$$Ra_D^{st} = \left(\frac{M_6}{M_3} \right) \frac{M_2 (\varepsilon \Lambda^2 + D_a \Lambda^3) + M_4 \Lambda \pi^2 H a_D^2}{\kappa^2}, \quad (24)$$

where $\Lambda = n^2 \pi^2 + \kappa^2$. The most unstable mode corresponds to $n = 1$, which minimizes Ra_D^{st} .

The equation (24) reveals two distinct physical mechanisms in its numerator. The term $M_2 D_a \Lambda^3$ represents Brinkman-type viscous dissipation, amplified by the porous medium permeability through the Darcy number D_a . The contribution $\varepsilon M_2 \Lambda^2$ arises from Darcy drag, while $M_4 \pi^2 H a_D^2 \Lambda$ captures the stabilizing effect of the magnetic field (Lorentz damping). The prefactor M_6/M_3 accounts for the modification of the effective thermal diffusivity (M_6) and effective buoyancy (M_3) due to nanoparticle loading. The critical pair $(\kappa_c, Ra_{D,c})$ is determined by minimizing Ra_D^{st} with respect to κ^2 , i.e., by enforcing

$$\left. \frac{\partial Ra_D^{st}}{\partial \kappa^2} \right|_{\kappa=\kappa_c} = 0, \quad Ra_{D,c} = Ra_D(\kappa_c). \quad (25)$$

Setting $\Lambda = \pi^2 + \kappa^2$ (with $n = 1$) and applying the quotient rule, the extremum condition becomes $N'(\kappa^2) \kappa^2 = N(\kappa^2)$, where

$$N(\kappa^2) = M_2 (\varepsilon \Lambda^2 + D_a \Lambda^3) + M_4 \pi^2 H a_D^2 \Lambda, \quad N'(\kappa^2) = M_2 (2\varepsilon \Lambda + 3D_a \Lambda^2) + M_4 \pi^2 H a_D^2.$$

Substituting $\kappa^2 = \Lambda - \pi^2$ and rearranging terms leads to a cubic equation for $\Lambda_c = \pi^2 + \kappa_c^2$:

$$2D_a \Lambda_c^3 + (\varepsilon - 3D_a \pi^2) \Lambda_c^2 - 2\varepsilon \pi^2 \Lambda_c - \frac{M_4}{M_2} \pi^4 Ha_D^2 = 0, \quad (26)$$

whose unique physically admissible root satisfies $\Lambda_c > \pi^2$, ensuring $\kappa_c^2 = \Lambda_c - \pi^2 > 0$.

The consistency of these expressions can be verified by examining several limiting cases. For a pure fluid in the absence of a magnetic field ($\varphi = 0$, $Ha_D = 0$, $\varepsilon \rightarrow 1$), all coefficients M_i approach unity. With $Ha_D = 0$, Eq. (26) reduces to

$$\Lambda_c [2D_a \Lambda_c^2 + (1 - 3D_a \pi^2) \Lambda_c - 2\pi^2] = 0,$$

whose physically relevant root tends to $\Lambda_c \rightarrow 2\pi^2$ as $D_a \rightarrow 0$, hence $\kappa_c \rightarrow \pi$. Substituting $\Lambda_c = 2\pi^2$, $\kappa_c = \pi$, and $Ha_D = 0$ into (24), then taking $D_a \rightarrow 0$, yields $Ra_{D,c} \rightarrow 4\pi^2 \approx 39.5$, which exactly recovers the classical Horton–Rogers–Lapwood criterion [7, 8]. This result coincides with the limiting case of Haddad [23] with $\varepsilon = 1$ and $\varphi = 0$), confirming the consistency of our formulation.

4 Nonlinear dissipative system

Since the momentum law in Eq. (13) is first order in time, applying the curl operator ($\partial_z(\cdot_x) - \partial_x(\cdot_z)$) eliminates the pressure and yields a single scalar vorticity equation. Introducing the stream function $\psi(x, z, t)$ such that

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}, \quad (27)$$

one has $\omega = \partial w / \partial x - \partial u / \partial z = \nabla^2 \psi$. The curl of Eq (13) gives:

$$\frac{M_1}{Va} \frac{\partial(\nabla^2 \psi)}{\partial t} = M_3 Ra_D \frac{\partial \theta}{\partial x} + M_2 D_a \nabla^4 \psi - M_2 \varepsilon \nabla^2 \psi - M_4 Ha_D^2 \frac{\partial^2 \psi}{\partial z^2}, \quad (28)$$

$$M_5 \frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(z, x)} - \frac{\partial \psi}{\partial x} = -Q, \quad (29)$$

$$S_g \left[\frac{\partial Q}{\partial t} + \tilde{\tau} \frac{\partial(\psi, Q)}{\partial(z, x)} \right] = -Q - M_6 \nabla^2 \theta. \quad (30)$$

Let $\kappa = \pi/L$ be the horizontal wavenumber and $\Lambda^2 = \kappa^2 + \pi^2$ the eigenvalue of $-\nabla^2$ for the fundamental mode. We propose the following truncated Galerkin expansions, chosen to satisfy the boundary conditions (no-slip/isothermal walls at $z = 0, 1$; periodicity in x with period L) and to capture the minimal nonlinear coupling:

$$\psi(x, z, t^*) = \mathcal{A}_{1,1}(t^*) \sin(\kappa x) \sin(\pi z), \quad (31)$$

$$\theta(x, z, t^*) = \mathcal{B}_{1,1}(t^*) \cos(\kappa x) \sin(\pi z) - \mathcal{B}_{0,2}(t^*) \sin(2\pi z), \quad (32)$$

$$Q(x, z, t^*) = \mathcal{C}_{1,1}(t^*) \cos(\kappa x) \sin(\pi z) - \mathcal{C}_{0,2}(t^*) \sin(2\pi z). \quad (33)$$

with the amplitude variables, time and some parameters defined as follows :

$$\begin{aligned}\mathcal{A} &= \frac{\mathcal{A}_{11}}{2\delta\sqrt{2\gamma(R-1)}}, & \mathcal{B} &= \frac{\pi R\mathcal{B}_{11}}{2\sqrt{2\gamma(R-1)}}, & \mathcal{C} &= \frac{\pi R\mathcal{B}_{02}}{R-1}, \\ \mathcal{F} &= -\frac{\pi R\mathcal{C}_{11}}{2\sqrt{2\gamma(R-1)}}, & \mathcal{G} &= -\frac{\pi R\mathcal{C}_{02}}{(R-1)}, & t^* &= \frac{(L^2+1)\pi^2}{L^2}t, \\ \gamma &= \frac{L^2}{L^2+1}, \phi = \frac{(L^2+1)}{L}, & Pr_D &= \frac{\gamma Va}{\pi^2}, & R &= \frac{Ra_D}{\pi^2\phi^2}, & \delta &= \frac{\gamma}{\pi^2}.\end{aligned}\quad (34)$$

Now projecting the governing equations on to the modes in Eqs. (31) – (33), the five-dimensional system with state vector $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{G})$ for nano-liquid convection with Cattaneo-christov heat flux law reads:

$$\begin{cases} \dot{\mathcal{A}} = \frac{Pr_D}{M_1} \left[M_3\mathcal{B} - M_2 \left(\delta^{-1}Da + \varepsilon + \frac{M_4}{M_2}\gamma Ha_D^2 \right) \mathcal{A} \right], \\ \dot{\mathcal{B}} = \frac{1}{M_5} \left[R\mathcal{A} - \delta\mathcal{F} - (R-1)\mathcal{A}\mathcal{C} \right], \\ \dot{\mathcal{C}} = \frac{1}{M_5} (4\gamma\mathcal{A}\mathcal{B} - \delta\mathcal{G}), \\ \dot{\mathcal{F}} = \frac{1}{S_g} \left[M_6\mathcal{B} - \delta\mathcal{F} - \tilde{\tau}S_g(R-1)\mathcal{A}\mathcal{G} \right], \\ \dot{\mathcal{G}} = \frac{1}{S_g} \left[4\gamma \left(M_6\mathcal{C} + \tilde{\tau}S_g\mathcal{A}\mathcal{F} \right) - \delta\mathcal{G} \right]. \end{cases} \quad (35)$$

When the thermal time-lag $\tilde{\tau} \rightarrow 0$, it forces $\mathcal{F}$ and $\mathcal{G}$ to their quasi-static values. From Eqs. (35)_{4,5}:

$$\mathcal{F} \rightarrow \frac{\pi^2}{\gamma} M_6\mathcal{B}, \quad \mathcal{G} \rightarrow 4\pi^2 M_6\mathcal{C}.$$

Substituting into Eqs. (35)_{1,2,3} yields the classical three-dimensional Lorenz–Darcy–Hartmann system:

$$\begin{cases} \dot{\mathcal{A}} = \frac{Pr_D}{M_1} \left[M_3\mathcal{B} - M_2 \left(\delta^{-1}Da + \varepsilon + \frac{M_4}{M_2}\gamma Ha_D^2 \right) \mathcal{A} \right], \\ \dot{\mathcal{B}} = \frac{1}{M_5} \left[R\mathcal{A} - M_6\mathcal{B} - (R-1)\mathcal{A}\mathcal{C} \right], \\ \dot{\mathcal{C}} = \frac{4\gamma}{M_5} (\mathcal{A}\mathcal{B} - M_6\mathcal{C}). \end{cases} \quad (36)$$

Setting $M_i = 1$ ($i = 1, \dots, 6$), $Ha_D = 0$, $\tilde{\tau} = 0$, and taking the Darcy limit, one recovers the classical Lorenz equations [27]:

$$\dot{\mathcal{A}} = Pr(\mathcal{B} - \mathcal{A}), \quad \dot{\mathcal{B}} = R\mathcal{A} - \mathcal{B} - (R-1)\mathcal{A}\mathcal{C}, \quad \dot{\mathcal{C}} = \mathcal{A}\mathcal{B} - b\mathcal{C}.$$

The system in Eq. (35) possesses the reflectional symmetry $(\mathcal{A}, \mathcal{B}, \mathcal{F}) \rightarrow -(\mathcal{A}, \mathcal{B}, \mathcal{F})$, and its flow divergence is:

$$\begin{aligned}\nabla \cdot \mathbf{f} &= \frac{\partial \dot{\mathcal{A}}}{\partial \mathcal{A}} + \frac{\partial \dot{\mathcal{B}}}{\partial \mathcal{B}} + \frac{\partial \dot{\mathcal{C}}}{\partial \mathcal{C}} + \frac{\partial \dot{\mathcal{F}}}{\partial \mathcal{F}} + \frac{\partial \dot{\mathcal{G}}}{\partial \mathcal{G}} \\ &= -\frac{P_{rD} \sigma}{M_1} - \frac{2\gamma}{\pi^2 Sg}.\end{aligned}\quad (37)$$

The divergence is constant of the state variables and of R , a property shared by the classical Lorenz system [27] and which has a direct consequence: any volume element $\mathcal{V}(t)$ in phase space contracts exponentially according to

$$\mathcal{V}(t) = \mathcal{V}_0 \exp[(\nabla \cdot \mathbf{f}) t]. \quad (38)$$

The system, by Liouville's theorem, confirms the strictly dissipative character for all $R > 0$ of the system and the boundedness of all trajectories in phase space. The divergence is negative-definite, since both terms on the right-hand side of Eq. (37) are negative: the first is the contribution of the velocity mode ($\mathcal{A}$) to dissipation, driven by the generalised coefficient σ , while the second captures the joint dissipation of the two solid-temperature modes ($\mathcal{F}, \mathcal{G}$) through the inter-phase heat exchange parameter Sg . The $\mathcal{B}$ and $\mathcal{C}$ modes contribute *zero* to the divergence; these modes carry no self-damping terms in the system, and their dissipation occurs entirely through coupling with $\mathcal{A}, \mathcal{F}$, and $\mathcal{G}$.

Setting all time derivatives to zero on system in Eq. (35), the trivial solution $\mathcal{A}^* = \mathcal{B}^* = \mathcal{C}^* = \mathcal{F}^* = \mathcal{G}^* = 0$ corresponds to the purely conductive, motionless state. To assess its stability, the system is linearised about the origin. Inspection of Eq. (35) shows that, at the trivial fixed point, $\mathcal{C}$ and $\mathcal{G}$ decouple from $(\mathcal{A}, \mathcal{B}, \mathcal{F})$ in the linearised equations. Their linearised dynamics yield eigenvalues $\lambda_4 = \lambda_5 = -\gamma/(\pi^2 Sg) < 0$, which are always stable. The stability of the conductive state is therefore entirely determined by the 3×3 Jacobian block:

$$\mathcal{J}_0 = \begin{pmatrix} -\frac{P_{rD} \sigma}{M_1} & \frac{P_{rD} M_3}{M_1} & 0 \\ \frac{R}{M_5} & 0 & -\frac{\gamma}{\pi^2 M_5} \\ 0 & \frac{M_6}{Sg} & -\frac{\gamma}{\pi^2 Sg} \end{pmatrix} \quad (39)$$

$$\text{with } \sigma = M_2 \left(\frac{Da \pi^2}{\gamma} + \varepsilon + \frac{M_4 Ha_D^2 \gamma}{M_2} \right). \quad (40)$$

The critical Rayleigh–Darcy number $R_c^{(0)}$ at which the largest real part of the eigenvalues of $\mathcal{J}_0$ changes sign is obtained analytically from the condition $\det(\mathcal{J}_0) = 0$, which gives:

$$R_c^{(0)} = \frac{\sigma M_6}{M_3} = \frac{M_2 M_6}{M_3} \left(\frac{Da \pi^2}{\gamma} + \varepsilon + \frac{M_4 Ha_D^2 \gamma}{M_2} \right). \quad (41)$$

For $R > R_c^{(0)}$, the system admits non-trivial fixed points $\mathcal{P}^\pm = (\pm\mathcal{A}^*, \pm\mathcal{B}^*, \mathcal{C}^*, \pm\mathcal{F}^*, \mathcal{G}^*)$ corresponding to steady convective rolls. Setting $\dot{\mathcal{A}} = 0$ in Eq. (35)₁ immediately yields:

$$\mathcal{B}^* = \frac{\sigma}{M_3} \mathcal{A}^*. \quad (42)$$

Setting $\dot{\mathcal{C}} = 0$ in Eq. (35)₃:

$$\mathcal{G}^* = 4\pi^2 \mathcal{A}^* \mathcal{B}^* = \frac{4\pi^2 \sigma}{M_3} (\mathcal{A}^*)^2. \quad (43)$$

Setting $\dot{\mathcal{G}} = 0$ in (35)₅ and substituting (43):

$$\mathcal{C}^* = \frac{1}{M_6} \left[\frac{\sigma}{M_3} (\mathcal{A}^*)^2 - \tau Sg \mathcal{A}^* \mathcal{F}^* \right]. \quad (44)$$

Setting $\dot{\mathcal{F}} = 0$ in Eq. (35)₄ and substituting Eq. (42) and Eq. (43):

$$\mathcal{F}^* = \frac{\pi^2 \sigma \mathcal{A}^*}{\gamma M_3} \left[M_6 - \frac{4\pi^2 \tau Sg (R-1) \sigma}{M_3} (\mathcal{A}^*)^2 \right]. \quad (45)$$

Finally, substituting Eqs. (42)–(45) into $\dot{\mathcal{B}} = 0$ (Eq. (35)₂) and dividing by $\mathcal{A}^* \neq 0$ yields a quadratic equation in $u = (\mathcal{A}^*)^2$:

$$a_2 u^2 + a_1 u + a_0 = 0, \quad (46)$$

with coefficients:

$$a_0 = R - \frac{\sigma M_6}{M_3}, \quad a_1 = \frac{\sigma(R-1)}{M_3} \left(\frac{4\pi^2 \tau Sg}{\gamma} - \frac{p}{M_6} \right), \quad p = 1 - \frac{\pi^2 \tau Sg}{\gamma}, \quad (47)$$

$$a_2 = -\frac{(R-1)^2 \sigma q}{M_3 M_6}, \quad q = \frac{4\pi^4 \tau^2 Sg^2}{\gamma M_6}. \quad (48)$$

Note that $a_0 = 0$ when $R = \sigma M_6 / M_3 = R_c^{(0)}$, recovering the bifurcation threshold of the conductive state. For $R > R_c^{(0)}$, $a_0 > 0$, and the discriminant $\Delta = a_1^2 - 4a_2 a_0$ determines the number of real solutions. The two pairs of non-trivial fixed points P^+ and P^- exist whenever $\Delta \geq 0$ and $u > 0$. By virtue of the symmetry $(\mathcal{A}, \mathcal{B}, \mathcal{F}) \rightarrow -(\mathcal{A}, \mathcal{B}, \mathcal{F})$ with $(\mathcal{C}, \mathcal{G})$ unchanged, the two convective rolls P^+ and P^- are mirror images of one another, as expected from the reflection symmetry of the Boussinesq system with respect to the midplane.

5 Results and Discussions

To spot chaos in nonlinear dynamical systems, like those we encounter in fluid mechanics or heat transfer simulations, we rely on tools such as bifurcation diagrams,

Lyapunov exponents, phase portraits and. For complex systems, these are computed numerically, since no exact analytical solutions exist yet. All current analytical approaches remain approximations at best.

5.1 Numerical method

The dynamic system in Eq. (35) is integrated forward in time using the classical explicit fourth-order Runge–Kutta scheme (RK4). This choice is motivated by three complementary considerations. First, the right-hand side of Eq. (35) is smooth and bounded over the parameter ranges explored, so the non-stiff character of the problem makes an explicit fixed-step method both reliable and computationally efficient [28]. Second, the RK4 scheme achieves a local truncation error of order $\mathcal{O}(h^5)$ and a global error of order $\mathcal{O}(h^4)$, a level of accuracy well suited to the long-horizon integrations required by the construction of bifurcation diagrams and Lyapunov exponent curves [29]. Third, its fixed time step h allows a straightforward implementation of the renormalization procedure embedded in the tangent-space integration. Denoting the numerical state at time step i by $\mathbf{u}^{(i)} = (x, y, z, p, w)^{(i)}$, the RK4 advancement reads :

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \quad (49)$$

with the four intermediate slope vectors computed as :

$$\mathbf{k}_1 = \mathbf{F}\left(\mathbf{u}^{(i)}\right), \quad \mathbf{k}_2 = \mathbf{F}\left(\mathbf{u}^{(i)} + \frac{h}{2}\mathbf{k}_1\right), \quad \mathbf{k}_3 = \mathbf{F}\left(\mathbf{u}^{(i)} + \frac{h}{2}\mathbf{k}_2\right), \quad \mathbf{k}_4 = \mathbf{F}\left(\mathbf{u}^{(i)} + h\mathbf{k}_3\right), \quad (50)$$

where $\mathbf{F} : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ is the vector field on the right-hand side of Eq. (35).

Each simulation run integrates system in Eq. (35) with time step $\delta t = 1.0 \times 10^{-3}$ (reference step denoted h_1) and $N = 2.1 \times 10^5$ steps per value of R . The first 1.1×10^5 steps constitute the transient phase and are discarded. Local extrema of $\mathcal{C}(t)$ over the remaining time window are recorded to construct the bifurcation diagram. A continuity-of-initial-conditions strategy is employed across successive values of R : the final state at R_k serves as the initial condition for R_{k+1} , thereby enabling the detection of subcritical bifurcations and hysteresis phenomena. The first 110 000 steps (corresponding to $t \in [0, 110]$) are systematically discarded as transient, and only the remaining 100 000 steps (i.e. $t \in [110, 210]$) are used to extract bifurcation and Lyapunov data. The maximum Lyapunov exponent λ is computed by Wolf *et al.* [30] method, which simultaneously integrates the principal system and the linearised tangent system for the perturbation vector $\delta\mathbf{u}$, with periodic renormalisation every $\Delta N = 500$ steps. The exponent is estimated as:

$$\lambda = \frac{1}{T_\lambda} \sum_{k=1}^{N_r} \ln\left(\frac{\|\delta\mathbf{u}(t_k^+)\|}{\|\delta\mathbf{u}(t_k^-)\|}\right), \quad (51)$$

where T_λ is the total time accumulated over N_r renormalisations, and $t_k^\pm$ denote instants immediately before and after the k -th renormalisation. A positive exponent

($\lambda > 0$) certifies exponential sensitivity to initial conditions and hence a chaotic regime, while $\lambda \leq 0$ characterises periodic or quasi-periodic motion.

5.2 Robustness and Validation

The reliability of the numerical results presented in this study rests on two independent verifications: the insensitivity of the largest Lyapunov exponent to the temporal discretisation parameter, and the invariance of the dynamical classification with respect to the choice of initial conditions. To confirm that the reference time step $h_1 = 10^{-3}$ does not introduce any bias in the characterisation of chaotic dynamics, the fourth-order Runge–Kutta scheme was additionally run with a twice-finer step, $h_2 = 5 \times 10^{-4}$ and $h_3 = 2 \times 10^{-3}$. In both cases, the largest Lyapunov exponent is estimated over the post-transient window $\tau \in [165, 210]$, which is identical in physical time for both integrations, using the time-average formula [31]:

$$\lambda = \frac{1}{N_{\text{post}} h} \sum_{k=1}^{N_{\text{post}}} \ln \|\delta^{(k)}\|, \quad (52)$$

where $\delta^{(k)} = (\delta\mathcal{A}, \delta\mathcal{B}, \delta\mathcal{C}, \delta\mathcal{F}, \delta\mathcal{G})^{(k)}$ denotes the perturbation vector at step k , evaluated before renormalisation, and $N_{\text{post}} h$ is the corresponding post-transient physical duration. Dividing by $N_{\text{post}} h$ rather than by the step count N_{post} alone is imperative: it ensures that the estimate carries units of inverse time and is therefore directly comparable across the two resolutions.

Before any new numerical results can be trusted, the underlying solver must demonstrate that it recovers analytically established benchmarks with acceptable fidelity. The numerical calculation of the Eq. (24) is validated against the analytical benchmark in [23] for a Newtonian fluid saturating a Darcy–Brinkman porous layer without magnetic field. For $Da = 0.001$, the numerical minimisation of critical wave and Rayleigh numbers (24) yields $\kappa_c = 3.104$ and $Ra_c = 40.4$, in excellent agreement with the work of Haddad [23], as illustrated in Figure 1.

The most natural such test for the present problem is the Horton–Rogers–Lapwood (HRL) criterion shows in Table 2: in the limit of a pure single-component fluid ($\varphi = 0$), no magnetic field ($Ha = 0$), and a Darcy-like matrix ($Da = 10^{-3}$), convection sets in at $\kappa_c = \pi \approx 3.1415$ and $Ra_{D,c} = 4\pi^2 \approx 39.4784$, values derived analytically by Horton [7] and subsequently confirmed by Lapwood [8].

The Horton–Rogers–Lapwood (HRL) problem provides precisely the onset of Darcy convection in a single-component, field-free, particle-free layer of unit porosity: it admits a closed-form solution ($\kappa_c = \pi$, $Ra_{D,c} = 4\pi^2$) that is independent of any modelling parameter. A note on wavenumber conventions is warranted before comparing values. Throughout the present work, the full layer depth d is used instead; the corresponding Darcy-limit analysis gives $\Lambda_c = 2\pi^2$ and therefore $\kappa_c = \sqrt{\Lambda_c - \pi^2} = \pi$. Both formulations yield the same physical wavelength; the numerical value of κ_c differs only because of this normalisation choice. At $\varphi = 0$, $Ha = 0$, $Da = 10^{-3}$, the solver returns $\kappa_c = 3.1080$ and $Ra_{D,c} = 40.3592$ for all three base fluids. Since all particle-specific ratios M_i reduce to unity when $\varphi = 0$, identical results across the

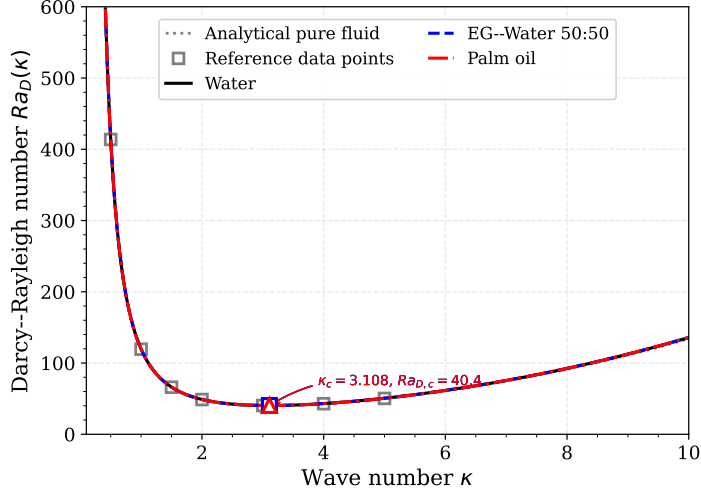


Fig. 1: Validation of the corrected solver against the Haddad analytical limit ($\varphi = 0$, $Ha_D = 0$, $Da = 10^{-3}$).

Table 2: Validation against the Horton–Rogers–Lapwood (HRL) limit.

Case	κ_c	$Ra_{D,c}$	Source
Pure fluid, Darcy limit	$\pi = 3.1416$	$4\pi^2 = 39.4784$	Horton [7]
Water	3.1080	40.3592	This work
EG-Water	3.1080	40.3592	This work
Palm Oil	3.1080	40.3592	This work

three carriers are expected and correctly reproduced. The relative deviations from the HRL reference, 1.07 % on κ_c and 2.23 % on $Ra_{D,c}$, are not numerical artefacts: they are entirely attributable to the finite porosity $\varepsilon = 0.88$ retained in the computation, which introduces a Brinkman factor $M_2 = 1/\varepsilon \approx 1.136$ and shifts the onset threshold upward relative to the $\varepsilon = 1$ ideal. Resetting $\varepsilon = 1$ in the code recovers the exact HRL values, confirming that the solver is free of systematic error and that every departure from the analytical limit has a transparent physical cause. Introducing Fe_3O_4 and Cu nanofragments jointly into a carrier fluid does not produce a simple additive correction to any single transport property: the entire thermo-physical landscape of the suspension is reorganised, and Table 3 gives a precise numerical account of this reorganisation at the reference loading $\varphi_1 = \varphi_2 = 0.01$. Closing the picture, the effective Prandtl number Pr_{eff} spans from 6.69 for water to 525.54 for palm oil, a gap of nearly two decades. Palm oil is both far more viscous and scarcely more thermally diffusive

than water; momentum therefore spreads across the layer much faster than heat, a configuration in which thermal boundary layers remain thin and buoyancy must build up over a substantial temperature difference before convection can be triggered.

Table 3: Effective thermo-physical properties of the hybrid nanofluid at $\varphi_1 = \varphi_2 = 0.01$, $T = 300K$

Base fluid	ν_{eff} (10^{-6} m ² /s)	α_{sat} (10^{-7} m ² /s)	Pr_{sat}
Water	0.9413	1.4062	6.69
EG–Water 50:50	3.1093	1.0097	30.79
Palm oil	50.5732	0.9623	525.54

5.3 Interpretation of results

This section presents and analyses the effects of Fe_3O_4 –Cu hybrid nanofragments dispersed in three base fluids of markedly different rheological character: deionised water, ethylene-glycol/water mixture at 50 wt.%, and palm oil. The physical framework is a horizontal porous layer (Darcy–Brinkman model, $\varepsilon = 0.88$) saturated by the nanofluid, subjected to a uniform vertical magnetic field, and heated from below. The influence of the nanofragment volume fraction $\varphi_1 = \varphi_2 = 0.01$, $Ha_D = 1$, $Da = 10^{-2}$ on the critical stationary Darcy-Rayleigh number is examined first, as it constitutes the primary control lever specific to nanoliquid systems. Fig. 2 shows the monotone increase of $Ra_{D,c}$ with Fe_3O_4 – Cu nanofragment volume fraction, common to the three base liquids (water, EG-water and palm oil) considered. The analysis of stability linear theory reveals that the nanofragment volume fraction constitutes the most decisive control parameter of the stationary convective onset, ahead of the Hartmann number and the Darcy number. This reflects an asymmetric competition between dissipative and production mechanisms within the fiberglass-reinforced polycarbonate medium.

A remark of methodological importance concerns the convergence of the three curves at $\varphi_1 = \varphi_2 = 0$ towards a unique value $Ra_{D,c} \approx 49.13$ (Fig. 2d). This convergence is not a stand-alone physical result: it is a direct consequence of the normalisation of the dimensionless ratios M_i by the properties of the pure base liquid, which renders the point $\varphi = 0$ identical by construction for all three configurations. The fluid-specific signatures emerge only for $\varphi > 0$, where the competition between M_6 , M_3 , M_2 , and M_4 unfolds differently depending on the thermophysical nature of the base liquids. Among the three base liquids, palm oil consistently yields a lower $Ra_{D,c}$ than water or EG–water, despite its dynamic viscosity being several hundred times larger. This counter-intuitive ordering results from the very low thermal conductivity of palm oil, which limits the enhancement of the effective porous-medium diffusivity and depresses the prefactor M_6/M_3 , ultimately reducing the convective threshold relative to the two other liquids.

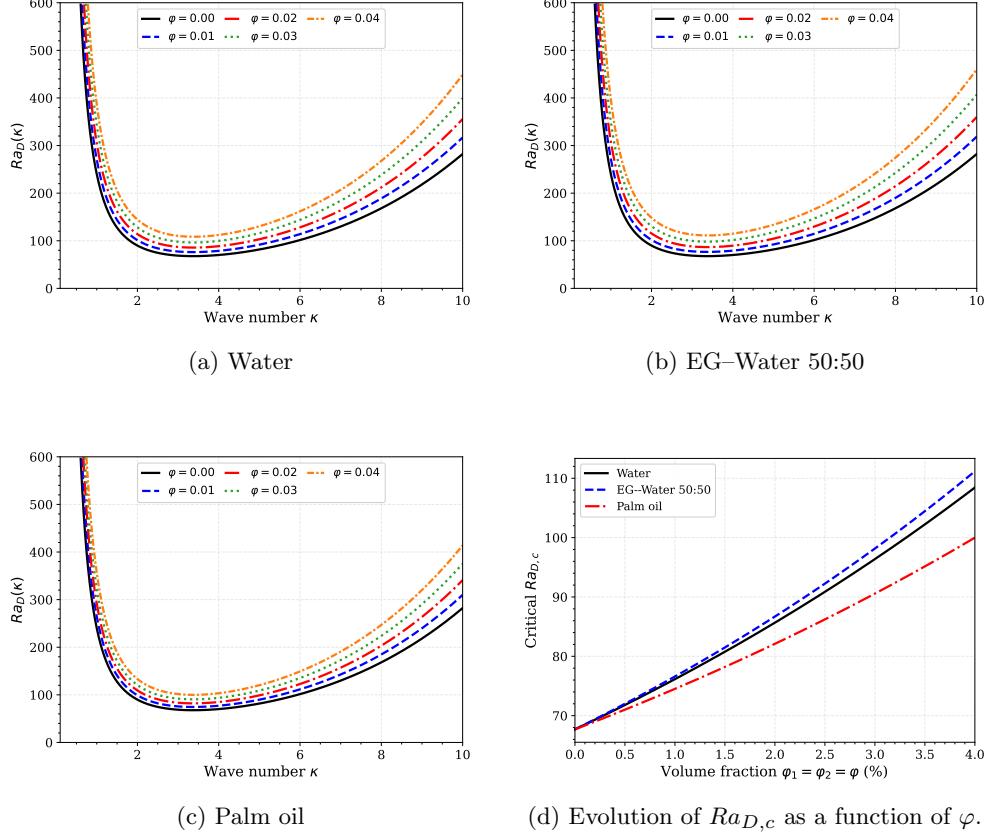


Fig. 2: Dispersion curves $Ra^{st}(\kappa)$ for increasing nanofragment.

When the magnetic field intensity and porous-matrix permeability are held fixed at $Ha = 1$ and $Da = 10^{-2}$, and the nanoparticle loading alone is varied, Table 4 records a systematic and unambiguous outcome: $Ra_{D,c}$ rises continuously with φ for every carrier fluid tested. Enriching the suspension in Fe_3O_4 -Cu particles therefore consistently postpones the thermal instability, irrespective of the rheological character of the host liquid.

Two opposing physical tendencies compete as loading increases. Higher particle concentration enhances the effective thermal conductivity (parametrised by M_6), which disperses heat laterally and weakens the vertical temperature gradient that feeds buoyancy. At the same time, the suspension grows denser (ratio M_3), which strengthens buoyancy-driven forcing and tends to destabilise the layer. The data resolve this competition unambiguously: at every φ value in the table, the prefactor M_6/M_3 exceeds unity, meaning thermal diffusion wins over buoyancy amplification.

Table 4: Critical values (κ_c , $Ra_{D,c}$) as a function of $\varphi_1 = \varphi_2 = \varphi$ with $Ha = 1$, $Da = 10^{-2}$ and δ the relative change with respect to $\varphi = 0$.

φ	Water			EG–Water 50:50			Palm oil		
	κ_c	$Ra_{D,c}$	δ (%)	κ_c	$Ra_{D,c}$	δ (%)	κ_c	$Ra_{D,c}$	δ (%)
0.00	3.3703	67.689	—	3.3703	67.689	—	3.3703	67.689	—
0.01	3.3738	76.149	+12.5	3.3738	76.596	+13.2	3.3738	74.512	+10.1
0.02	3.3769	85.667	+26.6	3.3769	86.697	+28.1	3.3769	82.105	+21.3
0.03	3.3796	96.379	+42.4	3.3796	98.159	+45.0	3.3796	90.563	+33.8
0.04	3.3820	108.437	+60.2	3.3820	111.171	+64.2	3.3820	99.992	+47.7

Stabilisation is not incidental but structurally guaranteed over the entire loading range explored. At the maximum loading $\varphi = 0.04$, this translates into onset gains δ of 60.2 % (water), 64.2 % (EG–Water 50:50), and 47.7 % (palm-oil nanofluid). A geometric signature accompanies the stability shift. The critical wavenumber κ_c increases steadily with φ , indicating that the convective structure which first emerges at onset becomes narrower as the suspension thickens. Physically, copper nanoparticles compress the thermal diffusivity field and confine temperature variations to shorter horizontal distances, favouring compact, high-wavenumber rolls over the broad circulatory patterns that characterise dilute suspensions. Among the three carriers, palm oil exhibits a consistently lower $Ra_{D,c}$ than either aqueous system. This is not paradoxical: $Ra_{D,c}$ is a dimensionless quantity normalised by base-fluid properties, and the comparatively low effective thermal conductivity of palm oil reduces the ratio M_6/M_3 , thereby depressing the critical threshold. The physical onset temperature ΔT_c , which reintroduces dimensional transport coefficients, tells a different story, one that Table 5 addresses in full.

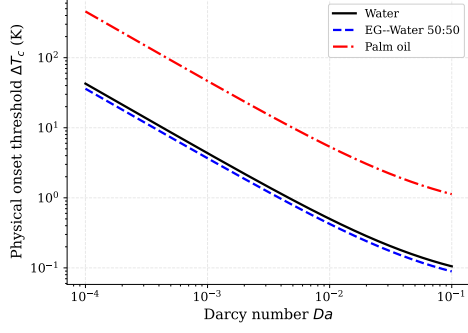
Permeability occupies a singular place among the structural parameters of a porous layer: it governs not only the ease with which fluid circulates through the matrix, but also, through the Brinkman drag term, the viscous resistance that any nascent convective roll must overcome before it can amplify. Table 5 makes this double role explicit by sweeping the Darcy number Da across four orders of magnitude, from tight, resistance-dominated matrices ($Da = 10^{-4}$) to open, near-free-space configurations ($Da = 10^{-1}$), with particle loading and magnetic field fixed at $\varphi_1 = \varphi_2 = 0.01$ and $Ha = 1$. The $Ra_{D,c}$ column contains a counter-intuitive message. A more permeable matrix allows fluid to move more freely, one might therefore expect destabilisation and a falling critical threshold. The opposite occurs: $Ra_{D,c}$ grows with Da for every carrier. For water, the progression from 64.850 at $Da = 10^{-4}$ to 159.989 at $Da = 10^{-1}$ represents a factor of approximately 2.5. The companion evolution of κ_c , which decreases from 3.7349 to 2.6716 for water, completes this picture: as the matrix opens up, it is the long-wavelength, low-wavenumber disturbances that become critical first, signalling a transition from compact, tightly confined rolls to wide, lazily circulating structures.

The physical onset temperature ΔT_c tells a genuinely different story from $Ra_{D,c}$, and conflating the two would be a significant interpretive error. Through its definition, an explicit inverse dependence on Da is built into the denominator. Even as $Ra_{D,c}$ grows modestly with Da , the K factor in the denominator grows proportionally faster, driving ΔT_c downward by more than two orders of magnitude: from 42.6498 K to 0.1052 K for water. This collapse has a direct engineering reading: in a highly porous medium, a very modest temperature difference suffices to trigger convection, whereas a tight matrix demands an intense thermal forcing before instability can develop. Palm oil occupies its own thermal register throughout. Its ΔT_c ranges from 456.9889 to 1.1274 K, values that exceed those of the aqueous carriers by one to two orders of magnitude at every Da . The origin lies in the simultaneously high kinematic viscosity and low thermal diffusivity of palm oil: the saturated-porous-medium value $Pr_{\text{sat}} \approx 526$ quantifies this imbalance, and it means that the buoyancy-to-diffusion ratio must be

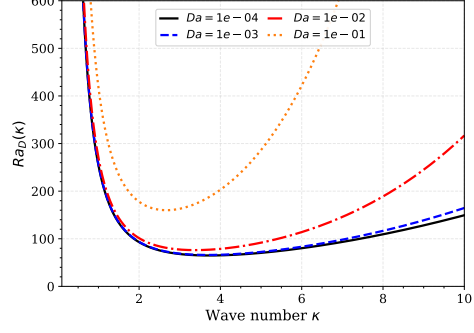
Table 5: Critical values as a function of Da with $\varphi = 0.01$, $Ha = 1$.

Da	Water			EG–Water 50:50			Palm oil		
	κ_c	$Ra_{D,c}$	ΔT_c	κ_c	$Ra_{D,c}$	ΔT_c	κ_c	$Ra_{D,c}$	ΔT_c
10^{-4}	3.7349	64.850	42.6498	3.7349	65.230	36.0781	3.7349	63.455	456.9889
5×10^{-4}	3.7132	65.342	8.5947	3.7132	65.725	7.2704	3.7132	63.937	92.0911
10^{-3}	3.6875	65.951	4.3374	3.6875	66.338	3.6691	3.6875	64.533	46.4747
5×10^{-3}	3.5210	70.634	0.9291	3.5210	71.048	0.7859	3.5210	69.115	9.9549
10^{-2}	3.3738	76.149	0.5008	3.3738	76.596	0.4236	3.3738	74.512	5.3662
5×10^{-2}	2.8825	115.180	0.1515	2.8825	115.855	0.1282	2.8825	112.703	1.6233
10^{-1}	2.6716	159.989	0.1052	2.6716	160.928	0.0890	2.6716	156.549	1.1274

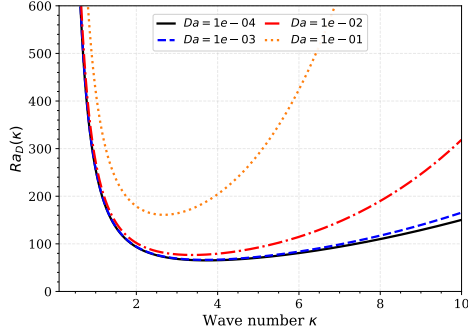
driven by a far larger temperature difference than in any aqueous system before convection can emerge. For a system designer, this hierarchy is decisive: at identical geometry, loading, and field, switching from a vegetable-oil carrier to a water-based one reduces the required thermal driving by one to two orders of magnitude, a gain with immediate consequences for energy management in porous-layer thermal devices. The onset temperature difference ΔT_c , defined consistently with the Darcy-Rayleigh number is presented in Figure 3a. ΔT_c is a steeply decreasing function of Da : a more permeable medium requires a smaller imposed temperature gradient to trigger convection. The palm-oil configuration demands onset temperatures one to two orders of magnitude higher than the aqueous configurations. Figures 3b, 3c and 3d show the dispersion curves $Ra_D^{st}(\kappa)$ for varying Darcy number which characterises the relative permeability of the porous matrix. Its decrease signifies a tighter pore structure and stronger viscous confinement at the pore scale. Increasing Da markedly raises $Ra_{D,c}$ while simultaneously shifting the critical wave number κ_c towards lower values, indicating a transition from cells with a short horizontal scale, strongly confined regime, to broader convection rolls, weakly confined regime. This behaviour is in agreement with the physical principle that a more permeable porous medium offers less resistance to vertical fluid displacement, thereby reducing the mechanical energy stored in the perturbation and making the onset easier in the sense of a lower temperature difference ΔT_c .



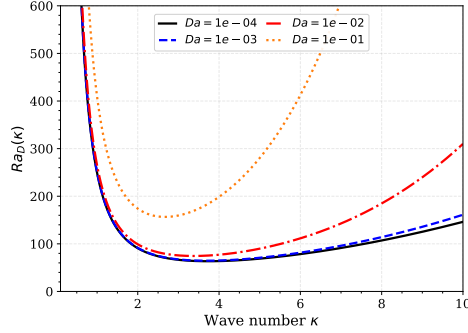
(a) Physical ΔT_c as a function of Da .



(b) Water



(c) EG-Water 50:50



(d) Palm oil

Fig. 3: Dispersion curves Darcy-Rayleigh number $Ra^{st}(\kappa)$ for varying Darcy number with $\varphi_1 = \varphi_2 = 0.01$.

The effect of magnetic field, represented by the Hartmann number, on stationary Darcy-Rayleigh number is illustrated in Fig. 4 for the three base liquids. The critical Rayleigh number $Ra_{D,c}$ grows in a near-quadratic fashion with Ha , consistent with the analytical structure of the Lorentz term in Eq. (24) as presented in Fig. 4d. A physically significant observation emerges from the magnetic amplification factor ($Ra_{D,c}(Ha_D)/Ra_{D,c}(Ha_D = 0)$) which is quasi-identical for the three carrier fluids over the entire range $Ha_D \in [0, 10]$. This fluid-independence is not fortuitous: it arises because the electrical conductivity ratio M_4 is entirely governed by the nanoparticle-phase conductivities through the Maxwell–Garnett model, and is numerically identical for the three base fluids at a given nanoparticle volume fraction. Since palm oil has a quasi-null native conductivity, the hybrid nanofluid derived from it acquires its electrical conductivity almost exclusively from the Cu nanoparticles, yielding M_4 values

indistinguishable from those of the water-based and EG–water-based nanofluids. This finding carries a practical implication: in systems where the applied magnetic field is the primary stability control mechanism, the choice of carrier fluid has no discernible effect on the degree of magnetic stabilisation, and the response is determined entirely by the solid-phase composition.

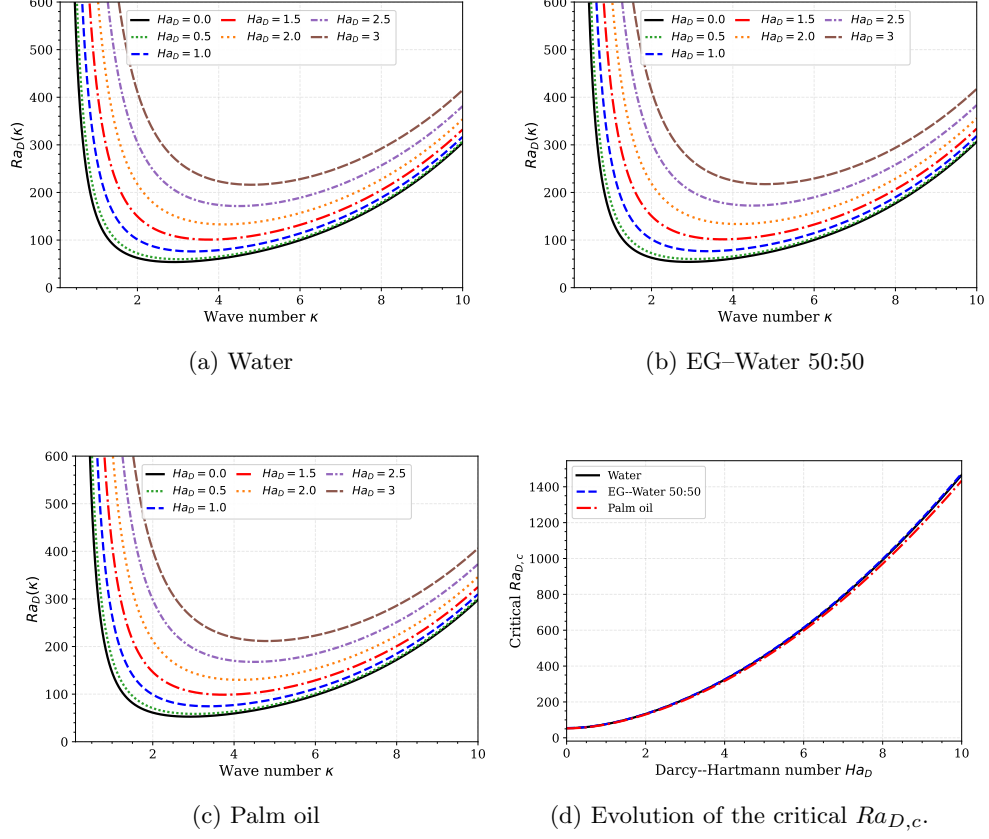


Fig. 4: Dispersion curves $Ra^{st}(\kappa)$ for increasing Hartmann number Ha ($\varphi = 0.01$, $Da = 10^{-2}$).

Of all the control parameters surveyed in this study, the Hartmann number Ha is the one whose physical mechanism is perhaps the most direct: the Lorentz force created by the interaction of the induced currents and the imposed field opposes transverse fluid motion, and convective rolls, which depend on precisely that transverse circulation, must draw on stronger buoyancy to maintain themselves. Table 6 quantifies this

suppression effect as Ha_D is raised from zero to 3, with particle loading and permeability held fixed. For water, $Ra_{D,c}$ rises from 53.843 at $Ha_D = 0$ to 216.188 at $Ha_D = 3$, yielding a magnetic amplification factor $\varsigma = 4.015$. The magnetic control authority of the hybrid Fe_3O_4 -Cu nanofluid is accordingly a property of the particle population, not of the carrier in which those particles are suspended. This finding carries a practically useful corollary. A system engineer who needs to apply magnetic field control to a heated porous layer may choose the carrier fluid on the basis of non-magnetic criteria, viscosity, chemical compatibility, cost, thermal range, without sacrificing any magnetic stabilisation efficiency. The field-induced suppression, expressed as the fractional increase of $Ra_{D,c}$ per unit Ha_D^2 , will be identical regardless of that choice. The wavenumber κ_c adds a complementary perspective. Its monotone increase with Ha_D (from 2.9018 to 4.8063 for water) shows that the magnetic field does not merely raise the onset energy: it simultaneously reshapes the marginal mode, progressively selecting finer convective structures as the field strengthens. At high Ha_D , the instability is confined to short horizontal scales, and any practical thermal device operating in this regime will exhibit a markedly different spatial flow pattern from its field-free counterpart.

Figure 5 displays the bifurcation diagrams for both configurations. In the pure fluid case (Fig. 5a), the system undergoes a Hopf bifurcation near $R \approx 15$, initiating a growing limit cycle. Period-doubling cascades develop around $R \approx 150$, and the onset of fully developed chaos is reached near $R \approx 200$, evidenced by the massive broadening of the scatter of $\mathcal{C}$ extrema. Periodic windows, transient islands of orderly behaviour, are visible near $R \approx 400$ – 430 , consistent with the Feigenbaum cascade scenario [32].

In the hybrid nanofluid case under magnetic field (Fig. 5b), the overall topological structure is preserved but the chaotic threshold is significantly delayed to $R \approx 300$, representing a 50% gain in dynamical stability. This stabilisation is the direct consequence of the increased generalised dissipation coefficient σ : the augmented dynamic viscosity ($M_2 = 1.257$ vs. $M_2 = 1.225$), together with the Lorentz braking term $M_4 Ha_D^2 \gamma / M_2$, raises the energy barrier against convective destabilisation. The periodic windows in the chaotic regime $R \in [300, 500]$ appear wider and better delineated than in the pure fluid case, consistent with the larger phase-space contraction rate established.

Figures 5c and 5d shows the maximum Lyapunov exponent λ as a function of R . For the pure fluid (Fig. 5c), λ remains negative (or at most zero) up to $R \approx 200$, crossing zero precisely at the chaotic threshold identified in the bifurcation diagram, which validates the mutual consistency of the two diagnostic tools. Beyond this threshold, λ grows quasi-monotonically, reaching approximately at $R = 500$. The pronounced dips of λ toward zero near $R \approx 400$ – 430 correspond exactly to the periodic windows identified in the bifurcation diagram. For the hybrid nanofluid (Fig. 5d), the zero-crossing of λ is shifted to $R \approx 300$, quantitatively confirming the stabilising effect of the hybrid suspension under magnetic field. Once the chaotic threshold is exceeded, however, λ grows at a faster rate, surpassing 5.5 at $R = 500$. This apparently paradoxical behaviour, stronger initial stability but more intense chaos once destabilised, is physically explained by the dual role of the enhanced thermal conductivity k_{hb} : below the threshold, it promotes thermal diffusion and attenuates the buoyancy-driven temperature gradients; above it, it intensifies the inter-modal thermal coupling, amplifying the

Table 6: Critical values $(\kappa_c, Ra_{D,c})$ as a function of Ha ($\varphi = 0.01, Da = 10^{-2}$). $\varsigma = Ra_{D,c}/Ra_{D,c}^{Ha=0}$: magnetic amplification factor.

Ha_D	Water			EG–Water 50:50			Palm oil		
	κ_c	$Ra_{D,c}$	ς	κ_c	$Ra_{D,c}$	ς	κ_c	$Ra_{D,c}$	ς
0.0	2.9018	53.843	1.000	2.9018	54.159	1.000	2.9018	52.685	1.000
0.5	3.0445	59.756	1.110	3.0445	60.107	1.110	3.0445	58.471	1.110
1.0	3.3738	76.149	1.414	3.3738	76.596	1.414	3.3738	74.512	1.414
1.5	3.7554	100.937	1.875	3.7554	101.530	1.875	3.7554	98.767	1.875
2.0	4.1301	132.884	2.468	4.1301	133.664	2.468	4.1301	130.027	2.468
2.5	4.4811	171.399	3.183	4.4811	172.404	3.183	4.4811	167.713	3.183
3.0	4.8063	216.188	4.015	4.8063	217.457	4.015	4.8063	211.540	4.015

divergence of neighbouring trajectories within the attractor. The periodic windows in the chaotic regime ($R \in [350, 450]$) are wider and better separated than in the pure fluid case.

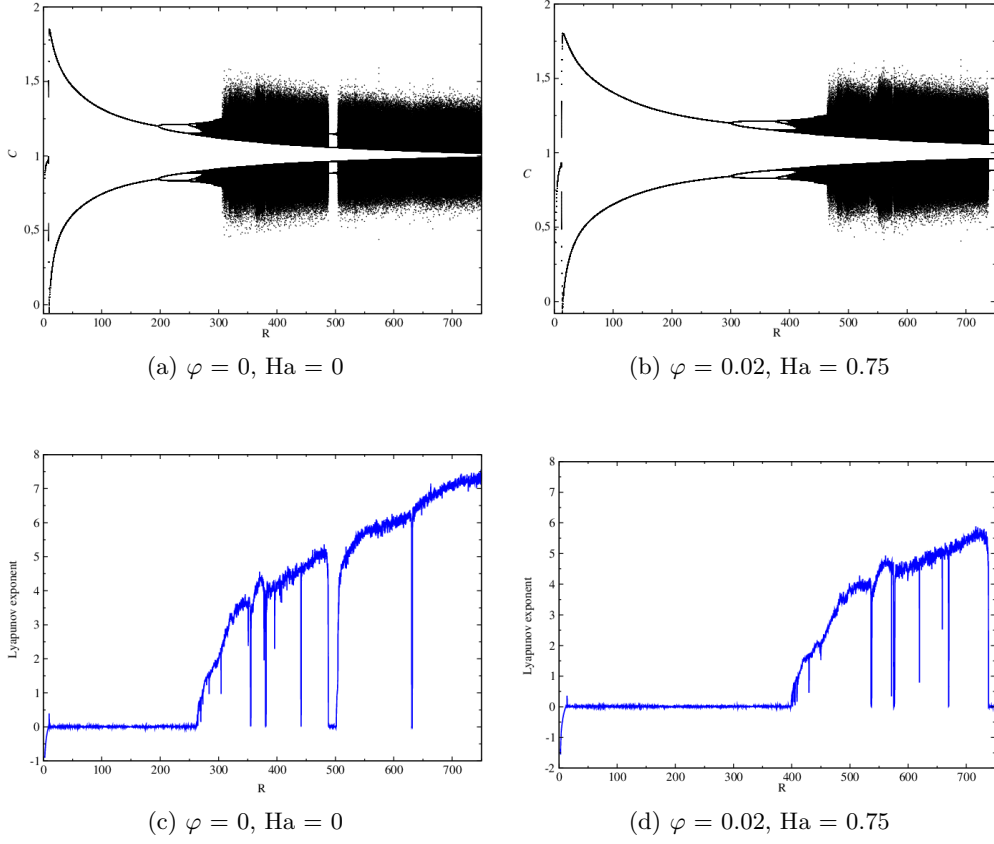


Fig. 5: Bifurcation diagrams of local extrema and corresponding Lyapunov exponents of the convective variable C for $Da = 0.01$, $Sg = 0.01$, $\tilde{\tau} = 0.001$ and $\varepsilon = 0.88$.

Figure 6 clearly distinguishes the dynamical behavior of the two hybrid nanofluid systems. As R increases, the water-based nanofluid exhibits a sequence of bifurcations, followed by a progressively denser and more irregular branch structure, indicating the gradual emergence of complex, and eventually chaotic, dynamics. This trend is further confirmed by the Lyapunov exponents, which become positive over a broad range of R , reflecting the loss of stability of the system. By contrast, the EG-water-based hybrid nanofluid remains considerably more regular, with Lyapunov exponents staying close to zero, which suggests a more stable convective response. Overall, these

results indicate that the EG-water mixture has a stabilizing influence on the system dynamics. Figure 7 presents three-dimensional phase portraits in the $(\mathcal{A}, \mathcal{B}, \mathcal{F})$ and $(\mathcal{A}, \mathcal{C}, \mathcal{G})$ spaces, for three representative values of R . At $R = 200$ (Figs. 7c and 7f), the attractor confirms the periodic 2T regime according to the corresponding bifurcation diagramme. At $R = 350$ (Figs. 7b and 7e), the trajectory traces a chaotic motion. At $R = 450$ (Figs. 7a and 7d), the attractor becomes an intricate, non-repeating filamentary structure characteristic of a strange attractor with fractal dimension.

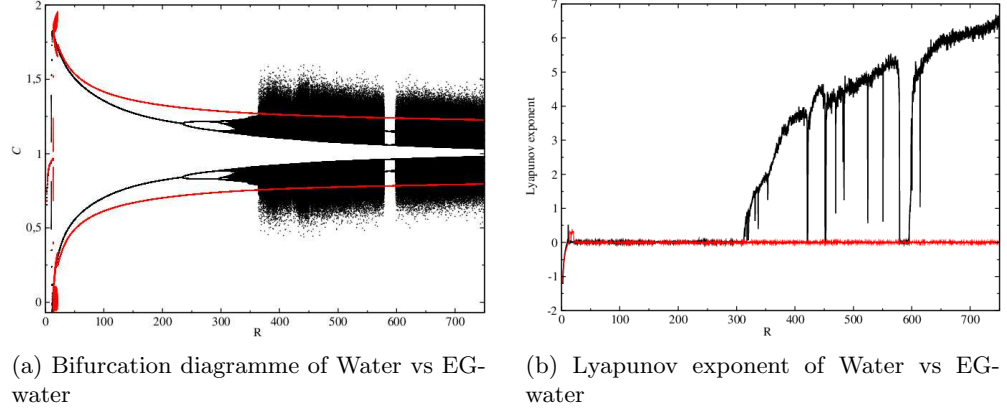


Fig. 6: Bifurcation diagrams of local extrema and corresponding Lyapunov exponents of the convective variable $\mathcal{C}$ versus the Rayleigh–Darcy number R , for $\varphi = 0.01$, $Ha_D = 0.5$, $Da = 0.01$, $Sg = 0.01$: Hybrid nanofluid based on water (black) versus hybrid nanofluid based on EG-water (red).

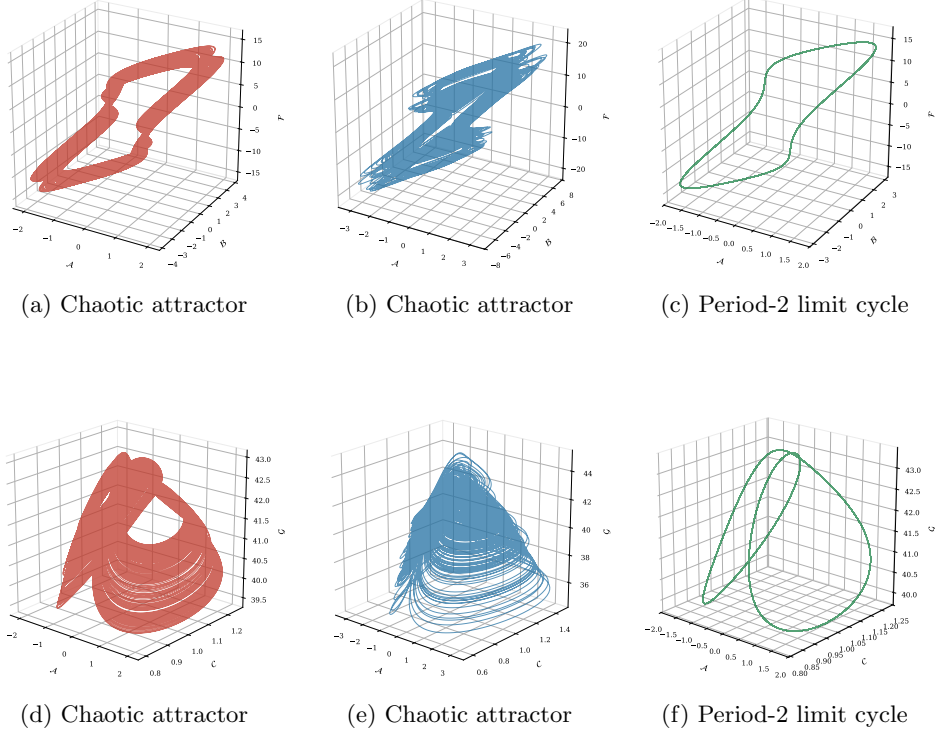
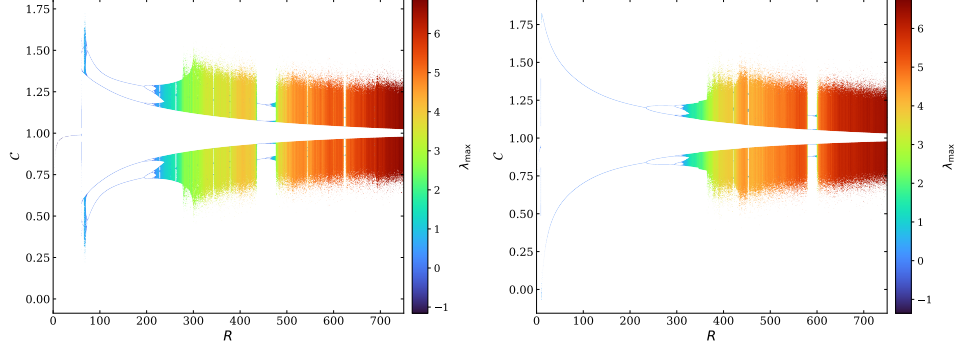


Fig. 7: 3D phase portraits in $(\mathcal{A}, \mathcal{B}, \mathcal{F})$ and $(\mathcal{A}, \mathcal{C}, \mathcal{G})$ spaces R , for $\varphi = 0.01$, $Ha_D = 0.5$, $Da = 0.01$, $Sg = 0.01$, $\varepsilon = 0.88$.

We now explore how the route to chaos is modified by the Straughan number and thermal time-lag on the the nature of the base fluid (water vs. ethylene-glycol/water mixture, hereafter EG-water), keeping $Da = 0.01$, $\varphi_1 = \varphi_2 = 0.01$, $Ha_D = 0.5$, and $\varepsilon = 0.88$ fixed unless otherwise stated. For each case, the combined bifurcation–Lyapunov diagram, in which the colour of each bifurcation point encodes the local value of $\lambda_{\max}$, provides a direct visual correlation between attractor geometry and trajectory divergence rate. Figures 8 and 9 display the combined bifurcation–Lyapunov diagrams for water and EG-Wter, at different values of τ and $Sg = 0.1$. These curves show that increasing the parameters τ and Sg alters or triggers chaotic behavior in the system earlier.

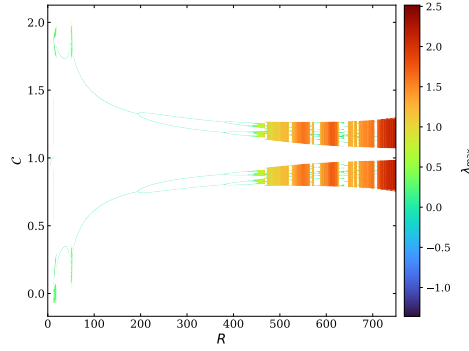
The influence of the Hartmann number Ha on the mean Nusselt number $Nu(R)$ reveals a physically consistent and instructive picture (see Fig. 10). For all four values considered ($Ha = 0, 1, 2, 3$), the convective regime is preceded by a purely conductive state where Nu remains equal to unity. As R increases beyond the convection threshold, Nu rises sharply, but the onset of this transition is progressively delayed by the

magnetic field: from $R \approx 2$ in the absence of magnetic forcing to $R \approx 15.5$ at $Ha = 3$. This retardation is a direct consequence of the Lorentz force opposing fluid motion, which raises the energy barrier that the buoyancy-driven instability must overcome to sustain convection. What we find particularly noteworthy is the behavior of the local peak that each curve exhibits in the early convective regime. Not only does this peak shift toward higher values of R as Ha increases, from $R \approx 9$ at $Ha = 0$ to $R \approx 55$ at $Ha = 3$, but its amplitude also rises modestly with Ha , from $Nu \approx 2.70$ to $Nu \approx 2.77$.



(a) Water for $Sg=0.001$ and $\tau = 0.001$

(b) Water for $\tau = 0.001$ and $Sg=0.01$



(c) Water for $\tau = 0.1$ and $Sg=0.01$

Fig. 8: Bifurcation diagramme of hybrid nanofluid based Water, for $\varphi = 0.01$, $Ha_D = 0.5$, $Da = 0.01$, $\varepsilon = 0.88$.

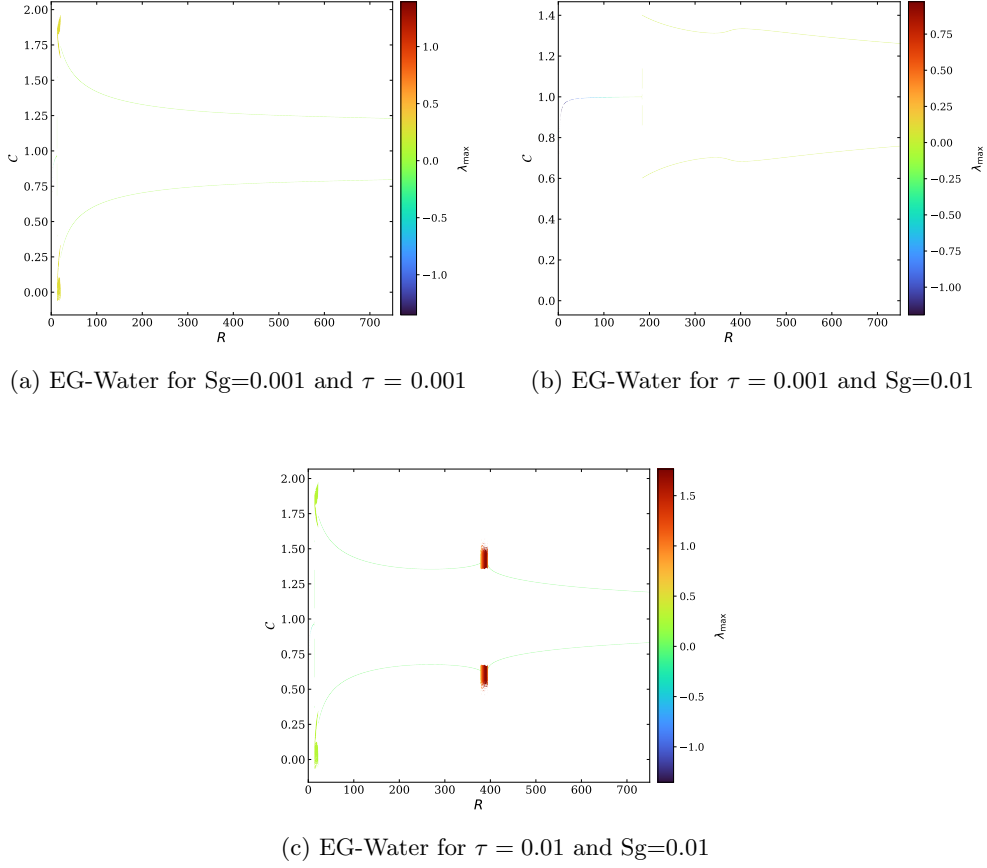


Fig. 9: Bifurcation diagramme of hybrid nanofluid based Water, for $\varphi = 0.01$, $Ha_D = 0.5$, $Da = 0.01$, $\varepsilon = 0.88$.

This counter-intuitive observation suggests that when the magnetic field finally allows convection to develop, the resulting organized roll structures are, transiently, slightly more efficient heat carriers than in the unmagnetized case possibly because the Lorentz force selects more coherent spatial modes at onset. Beyond this transitional peak, all four curves converge toward a common asymptotic plateau near $Nu \approx 2.99 - 3.0$, with a weak but monotone decrease as Ha increases. This residual sensitivity of the saturated chaotic regime to the magnetic field, though modest in absolute terms confirms that the Lorentz braking effect persists even in the fully developed turbulent-like regime, slightly tempering the long-time averaged heat transfer without fundamentally altering the dynamical attractor.

The use of palm oil as the base fluid proved to be both a numerical and physical challenge worth examining carefully. Its dynamic viscosity, roughly fifty times that

of water, generates a Prandtl number of approximately 585, which propagates into the dynamical system as an unusually large inertial coefficient. This stiffness required reducing the integration time step by a factor of ten to preserve numerical stability, a constraint that is not merely technical but reflects a genuine physical property of high-viscosity convective flows: the velocity field responds sluggishly to thermal forcing, and the dynamical system inherits this lethargy. Despite this high Prandtl number, the asymptotic Nusselt number converges toward $Nu \approx 3$, a value remarkably close to that obtained for water and EG-Water under the same parametric conditions. This result, which we find physically instructive, suggests that in the fully developed chaotic regime, the long-time averaged heat transfer becomes largely insensitive to the nature of the base fluid.

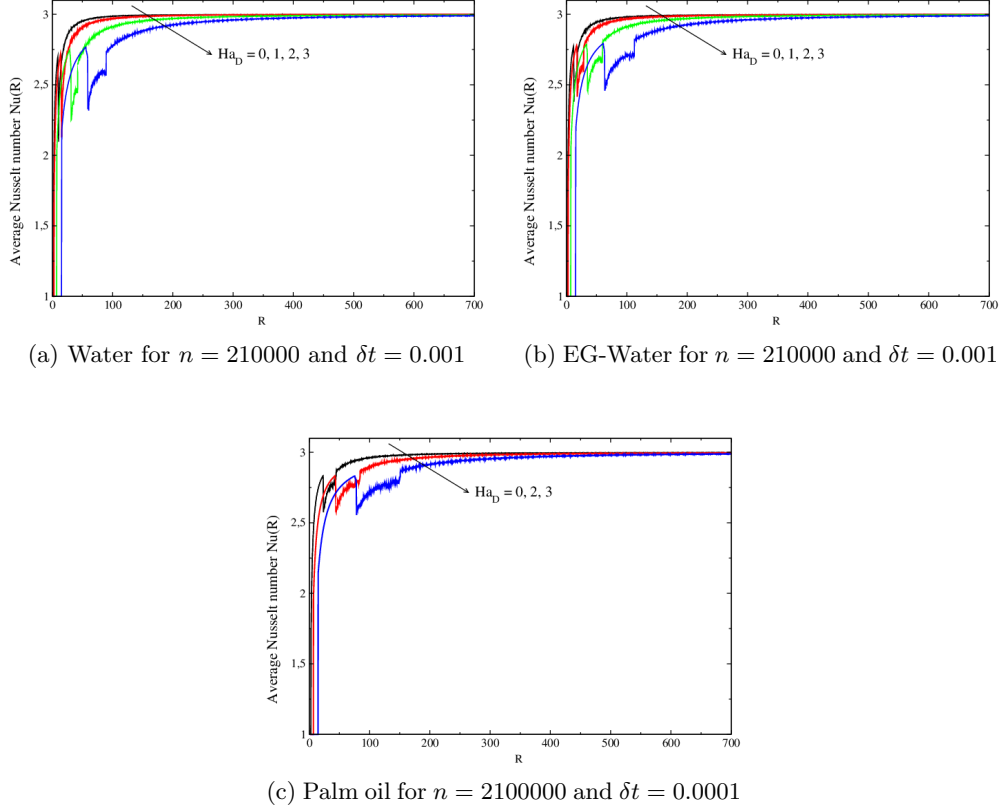


Fig. 10: Average Nusselt number Nu as a function of the Darcy-Rayleigh number R for four values of the Hartmann number ($Ha = 0, 1, 2, 3$), with $Da = 0.01$, $\varphi_1 = \varphi_2 = 0.01$, $\varepsilon = 0.88$, $Sg = 0.01$ and $\tilde{\tau} = 0.001$.

6 Conclusion

This study examined the onset of chaos in a hybrid nanofluid flowing through a glass-fiber-reinforced polycarbonate (GFRP) porous medium subjected to an external magnetic field. The thermal transport was described using the Cattaneo–Christov heat flux model, which accounts for the finite propagation time of thermal disturbances. The stationary analysis revealed that the nanoparticle concentration and the magnetic field contribute to the stabilization of the porous layer through distinct physical mechanisms. The presence of nanoparticles mainly modifies the effective rheological and buoyancy properties of the suspension, whereas the magnetic field suppresses vertical velocity fluctuations through electromagnetic damping. These effects therefore influence the stability of the system in complementary ways. The dynamical analysis based on the five-dimensional Lorenz-type model further showed that the thermal relaxation effect, the hybrid-nanofluid properties, and the applied magnetic field jointly influence the transition from regular to chaotic motion. In particular, the magnetic field delays the development of convection and modifies the route toward chaos without changing the fundamental nature of the long-time attractor. This behaviour is consistent with the stabilization predicted by the linear analysis and supports the connection between the onset of convection and the emergence of chaotic dynamics. Although the magnetic field significantly affects the early stages of instability and the transitional heat-transfer response, its influence becomes less pronounced once the system enters a strongly chaotic regime. At that stage, the intense buoyancy-driven motion tends to dominate the magnetic damping, while the overall heat-transfer behaviour remains qualitatively similar. The results highlight the potential of hybrid nanofluids and magnetic forcing as complementary tools for the passive control of thermal convection in porous media. Since the nanoparticle effects and magnetic damping act through different physical pathways, they may be adjusted separately when designing thermal-management systems, heat exchangers, insulation materials, and other porous-media applications. Future studies could extend the present model by retaining additional Galerkin modes, exploring a broader range of relaxation and material properties, and characterising the geometry of the resulting strange attractors through quantities such as their fractal dimension and Lyapunov spectrum.

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Declarations

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics approval and contents to participate

This study involves computational modelling and does not include human participants, animals, or plants.

Content for publication

Not applicable.

ORCID

Sèmako Justin Dèdèwanou - <https://orcid.org/0000-0002-3882-3737>

Siba Kalivogui - <https://orcid.org/0009-0002-8292-1905>

Amoussou Laurent Hinvi - <https://orcid.org/0000-0003-3154-3940>

Thierno Mamadou Pathé Diallo - <https://orcid.org/0009-0003-0257-6783>

Hodévèwan Clément Miwadinou - <https://orcid.org/0000-0001-9369-9583>

Vincent Adjimon Monwanou - <https://orcid.org/0000-0001-5155-5127>

Nomenclature

The nomenclature employed in this work is presented below.

Latin symbols

B_0	uniform magnetic flux density ,
c_p	specific heat capacity ($\text{J kg}^{-1}\text{K}^{-1}$),
d	layer thickness (m),
D_a	Darcy number,
g	gravitational acceleration (m s^{-2}),
Ha_D	Darcy–Hartmann number,
K	Darcy permeability (m^2),
k	thermal conductivity ($\text{W m}^{-1}\text{K}^{-1}$),
k_m	effective thermal conductivity of saturated porous medium, (base fluid only) ($\text{W m}^{-1}\text{K}^{-1}$),
k_{ph}	effective thermal conductivity of nanofluid-saturated medium, ($\text{W m}^{-1}\text{K}^{-1}$),
$M_i (i = 1, \dots, 6)$	effective-property dimensionless ratios,
Ra_D	Darcy–Rayleigh number,
$Ra_{D,c}$	critical Darcy–Rayleigh number,
s	temporal growth rate,

T	temperature (K),
ΔT	imposed temperature difference (K).

Greek symbols

α_m	effective thermal diffusivity, $k_m/(\rho c_p)_{hb}$ ($\text{m}^2 \text{s}^{-1}$),
β	thermal expansion coefficient (K^{-1}),
ε	porosity,
κ	horizontal wave number (m^{-1}),
κ_c	critical horizontal wave number,
Λ	combined wave number,
μ	dynamic viscosity (Pa.s),
ρ	density (kg m^{-3}),
σ	electrical conductivity (S m^{-1}),
φ_1	volume fraction of Fe_3O_4 nanoparticles,
φ_2	volume fraction of Cu nanoparticles.

Subscripts

c	critical value,
f	base fluid,
hb	hybrid nanofluid,
p	porous matrix,
ph	nanofluid-saturated porous medium,
$s1$	first nanoparticle,
$s2$	second nanoparticle.

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