

Beyond Entanglement Entropy: Spectral Entropy of Reduced States in Random Quantum Circuits

Jim Dorsey

jimdorsey4521@gmail.com

Miami University of Science and Technology

Research Article

Keywords: random quantum circuits; spectral entropy; reduced density matrix; entanglement entropy; quantum scrambling

Posted Date: October 13th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-11330759/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: The authors declare no competing interests.

Beyond Entanglement Entropy: Spectral Entropy of Reduced States in Random Quantum Circuits

Jim Dorsey

Miami University of Science and Technology

August 2026

Abstract:

Entanglement entropy has long served as a central diagnostic of complexity and information scrambling in quantum many-body systems, and its behavior in random quantum circuits is by now well characterized. However, as a single scalar, entanglement entropy captures only the total uncertainty of the reduced density matrix spectrum while discarding the detailed structure of that spectrum — including the shape of the eigenvalue distribution, level repulsion signatures, and the transition from integrable to chaotic behavior. In this work, we propose **spectral entropy** — the Shannon entropy of the eigenvalue distribution of the reduced density matrix — as a structured diagnostic that goes beyond standard entanglement entropy. Using numerical simulations, we compute the spectral entropy of subsystem reduced states in Haar-random and Clifford circuits as a function of circuit depth, and compare its behavior against von Neumann entanglement entropy, Rényi-2 entropy, and entanglement spectrum level-spacing statistics. We expect to find three key results: (i) spectral entropy exhibits a growth timescale distinct from that of entanglement entropy, potentially reflecting spectral restructuring more sensitively; (ii) in the depth regime where entanglement entropy has already saturated, spectral entropy may continue to evolve, revealing late-time complexification that standard entanglement measures cannot resolve; (iii) spectral entropy may display a distinctive response to measurement-induced or noise-induced disentanglement that differs qualitatively from entanglement entropy. This work positions spectral entropy as a complementary — not replacement — tool to entanglement entropy, offering a finer spectral lens on information scrambling, thermalization, and complexity growth in random quantum circuits.

Keywords: random quantum circuits; spectral entropy; reduced density matrix; entanglement entropy; quantum scrambling

1. Introduction

The study of complexity in quantum many-body systems has emerged as a central theme at the intersection of quantum information theory, condensed matter physics, and quantum computation. At the heart of this endeavor lies a deceptively simple question: how can one quantify the degree to which a quantum state or the evolution that produced it is genuinely complex? For isolated quantum systems undergoing unitary dynamics, entanglement entropy

has served as the workhorse diagnostic. Its growth under random quantum circuits, its scaling with subsystem size in ground states of local Hamiltonians, and its saturation at values dictated by the Page curve have all become standard tools for characterizing quantum correlations (Page, 1993; Hayden and Preskill, 2007). Yet entanglement entropy, for all its utility, is a coarse-grained measure. It collapses the full eigenvalue spectrum of the reduced density matrix into a single scalar, discarding precisely the structural information that might distinguish one form of complexity from another.

This thesis takes as its starting point the observation that the eigenvalue spectrum of a reduced density matrix—the entanglement spectrum—contains far more information than its entropy alone. The distribution of these eigenvalues, their spacing statistics, and their functional form encode details about the nature of quantum correlations that a single number cannot capture (Li and Haldane, 2008). Recent work has demonstrated that the entanglement spectrum exhibits characteristic signatures of quantum chaos: in systems that thermalize, the spacing statistics of the entanglement spectrum follow Wigner-Dyson distributions, whereas integrable systems display Poissonian statistics (Chen and Grover, 2014). This connection between spectral statistics and dynamical complexity suggests that spectral properties of reduced states might serve as more refined diagnostics than entanglement entropy alone.

We propose spectral entropy—the Shannon entropy of the eigenvalue distribution of the reduced density matrix—as a complementary diagnostic to standard entanglement entropy. Formally, for a reduced density matrix ρ_A with eigenvalues $\{\lambda_i\}$, the spectral entropy is defined as $S_{\text{spec}} = -\sum_i \lambda_i \log \lambda_i$. This quantity shares with the von Neumann entropy the property of vanishing for pure states, but it differs in a crucial respect: it is sensitive not only to the number of significant eigenvalues but also to the shape of their distribution. Two states with identical entanglement entropy may have markedly different spectral entropies, and conversely, the spectral entropy may evolve on timescales distinct from those of entanglement entropy under unitary dynamics.

The motivation for this proposal comes from several converging lines of inquiry. First, studies of quantum chaos in Hamiltonian systems have shown that spectral measures of the energy spectrum—level-spacing statistics, the spectral form factor, and the distribution of matrix elements—carry substantial diagnostic and predictive power for nonequilibrium dynamics (Kos et al., 2018). These spectral measures discriminate integrable from chaotic regimes and indicate the timescales of entanglement spreading, scrambling onset, and operator-complexity growth. By analogy, one might expect that spectral measures of the entanglement spectrum carry similar diagnostic power for the dynamics of quantum correlations.

Second, the entanglement spectrum statistics have been shown to provide a more general connection to irreversibility than energy-level statistics alone. In stochastic quantum circuits, where energy levels are not well-defined, the appearance of Wigner-Dyson fluctuations in the entanglement spectrum precedes the failure of disentangling algorithms, marking the onset of irreversibility (Chen and Grover, 2014). This suggests that spectral properties of reduced states may serve as early indicators of complexification, potentially preceding saturation of entanglement entropy.

Third, the practical challenges of characterizing quantum states on near-term quantum devices have motivated the development of randomized measurement protocols that can estimate nonlinear functionals of the density matrix (Elben et al., 2023). Spectral entropy, being a polynomial functional of the eigenvalues, is in principle estimable through such protocols, though the practical challenges of high-order moments remain substantial. Recent work on classical shadows and randomized measurements has demonstrated that Rényi entropies of low order can be efficiently estimated, and machine learning approaches have been developed to infer von Neumann entropy from finite-order Rényi entropies (Elben et al., 2023). Spectral entropy, however, remains relatively unexplored in this context.

The central questions addressed in this research are as follows. How does spectral entropy of reduced states evolve under random quantum circuits, and how does this evolution compare to that of standard entanglement entropy? Does spectral entropy continue to evolve after entanglement entropy has saturated, thereby revealing late-time complexification invisible to standard measures? How does spectral entropy respond to measurement-induced disentanglement, and does it exhibit signatures of measurement-induced phase transitions? These questions are investigated through numerical simulation of Haar-random and Clifford circuits, with careful attention to finite-size effects and the distinction between typical and averaged behavior.

The structure of this document is as follows. Section 2 reviews the relevant literature on entanglement entropy, random quantum circuits, the entanglement spectrum, and spectral measures of quantum chaos. Section 3 describes the methodology, including circuit models, numerical techniques, and the definition of spectral entropy. Section 4 presents results from numerical simulations. Section 5 discusses the findings in the context of existing literature and identifies implications for quantum information theory. Section 6 concludes with a summary and directions for future work.

2. Literature Review

2.1 Entanglement Entropy in Quantum Many-Body Systems

Entanglement entropy has been the dominant paradigm for quantifying quantum correlations in many-body systems for nearly three decades. The von Neumann entanglement entropy, defined for a pure state $|\psi\rangle_{AB}$ as $S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A)$ where $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$, measures the degree to which a bipartite state deviates from a product state. Its scaling with subsystem size has become an organizing principle in quantum many-body physics, distinguishing gapped ground states (area law), critical ground states (logarithmic corrections), and highly excited states (volume law) (Eisert et al., 2010; Amico et al., 2008).

The volume-law scaling of entanglement entropy in typical excited states was first understood through random matrix theory and the Page curve. Page (1993) showed that for a random pure state drawn from the Haar measure on the full Hilbert space, the average entanglement entropy of a subsystem of dimension d_A is approximately $\log d_A$ minus a correction that vanishes as the subsystem dimension becomes small relative to the total dimension. This result

established the notion that typical states are nearly maximally entangled, with entanglement entropy saturating at the maximum value allowed by the subsystem size.

For dynamical systems, the growth of entanglement entropy following a quench has been studied extensively. In integrable systems, entanglement entropy typically grows linearly at early times before saturating, while in chaotic systems the growth can be faster, with the saturation value approaching the Page value (Calabrese and Cardy, 2005; Kim and Huse, 2013). The rate of entanglement growth has been connected to the Lieb-Robinson velocity and the operator spreading dynamics (Nahum et al., 2018).

2.2 Random Quantum Circuits

Random quantum circuits have emerged as a paradigmatic model for studying the dynamics of entanglement and information scrambling. These circuits consist of local unitary gates drawn from a random ensemble—typically Haar-random or Clifford—arranged in a fixed geometry, most commonly a brickwork architecture in one dimension. The appeal of random circuits lies in their tractability: while individual circuits are complicated, ensemble averages can often be computed analytically using techniques from random matrix theory and representation theory of the unitary group.

A central result in the theory of random circuits is the connection between circuit depth and the formation of unitary designs. A random circuit of sufficient depth approximates a Haar-random unitary on the full Hilbert space, and the depth required for this approximation scales linearly with the number of qubits (Brandão et al., 2016; Harrow and Mehraban, 2023). This connection to unitary designs underpins the use of random circuits in proposals for quantum advantage (Arute et al., 2019) and in the study of black hole dynamics (Hayden and Preskill, 2007).

The entanglement dynamics of random circuits have been studied in detail. For a one-dimensional brickwork circuit of depth t , the entanglement entropy of a contiguous subsystem of size $|A|$ grows linearly with t until it reaches the Page value, after which it saturates and exhibits fluctuations around this value (Nahum et al., 2017; von Keyserlingk et al., 2018). The saturation time scales as the subsystem size, reflecting the light-cone propagation of entanglement. The approach to saturation is characterized by a universal scaling form, with the entanglement entropy growing as a function of $(t - |A|)/|A|^{1/2}$ in the vicinity of the saturation transition.

More recent work has focused on the role of measurements in random circuits. When projective measurements are interspersed with unitary gates at a rate p , the system undergoes a measurement-induced phase transition between a volume-law entangled phase at low p and an area-law entangled phase at high p (Li et al., 2018; Skinner et al., 2019; Chan et al., 2019; Bao et al., 2020). This transition, which occurs in the trajectory-averaged entanglement entropy, has been characterized through finite-size scaling and conformal field theory techniques. The critical properties depend on the measurement rate, the gate set (Clifford versus Haar), and the presence of symmetries.

2.3 Entanglement Spectrum and Spectral Statistics

The entanglement spectrum—the set of eigenvalues of the reduced density matrix—was introduced by Li and Haldane (2008) as a probe of topological order in fractional quantum Hall states. They observed that the low-lying entanglement spectrum exhibits a structure that mirrors the edge excitation spectrum, providing a fingerprint of the underlying topological phase. This observation spawned a large body of work using the entanglement spectrum to diagnose topological phases, symmetry-protected topological phases, and quantum criticality.

The statistics of entanglement spectra have since been connected to quantum chaos. Chen and Grover (2014) studied random quantum circuits with stochastic evolution and showed that the onset of irreversibility—the failure of algorithms designed to disentangle the system—is preceded by the appearance of Wigner-Dyson statistics in the entanglement spectrum. This result established the entanglement spectrum as a more general diagnostic of chaos than the energy spectrum, applicable even when energy is not conserved.

More recently, the spectral form factor, a Fourier transform of the two-point correlation function of the density of states, has been used to diagnose quantum chaos in many-body systems (Kos et al., 2018; Liu, 2018). The spectral form factor exhibits a characteristic dip-ramp-plateau structure in chaotic systems, reflecting the underlying random matrix statistics. Analogous structures have been observed in the entanglement spectrum of chaotic systems, suggesting that spectral measures of reduced states carry information about the dynamical complexity of the full system.

2.4 Spectral Entropy and Related Measures

Spectral entropy, as defined in this work, is the Shannon entropy of the eigenvalue distribution of the reduced density matrix. While this specific quantity has not been extensively studied in the quantum many-body context, related measures have appeared in the literature. The participation entropy, defined as the Rényi entropy of the eigenvalue distribution, has been used to quantify localization in Hilbert space (Luitz et al., 2015). The inverse participation ratio, which is the second moment of the eigenvalue distribution, is a standard measure of localization in single-particle systems.

In the context of quantum chaos, the concept of spectral complexity has been introduced as a measure of the spread of an initial state over the eigenbasis of the Hamiltonian (Balasubramanian et al., 2022). Spectral complexity and its connection to the spectral form factor have been explored in integrable and chaotic spin chains (Balasubramanian et al., 2022). These works suggest that spectral measures of state complexity, rather than just the energy spectrum itself, are sensitive to dynamical regimes.

The relationship between entanglement entropy and the full entanglement spectrum has been explored in the context of tensor network methods. The success of matrix product state approximations relies on the rapid decay of the entanglement spectrum, which determines the truncation error (Schollwöck, 2011). States with slowly decaying spectra require larger bond dimensions and are more "complex" in the sense of being harder to represent classically. This connection suggests that the shape of the eigenvalue distribution, not just its entropy, is a relevant measure of state complexity.

2.5 Randomized Measurements and Entropy Estimation

Estimating nonlinear functionals of the density matrix, such as Rényi entropies and spectral entropy, from experimental data is a significant challenge. Full state tomography requires exponentially many measurements in the number of qubits. Randomized measurement protocols, including classical shadows and the randomized measurement toolbox, have emerged as resource-efficient alternatives (Elben et al., 2023; Huang et al., 2020).

Classical shadows allow for the estimation of expectation values of observables and polynomial functionals of the density matrix from a modest number of measurements. Rényi entropies of low order can be estimated from classical shadow data, and machine learning techniques have been developed to infer von Neumann entropy from finite-order Rényi entropies (Elben et al., 2023). However, estimating spectral entropy directly from experimental data remains an open challenge, as it requires knowledge of the full eigenvalue spectrum.

From a theoretical perspective, the copy complexity of estimating entropies has been characterized. Estimating Rényi entropies of order k requires at least k copies of the state, and estimating von Neumann entropy requires exponentially many copies in the worst case (Acharya et al., 2019). This fundamental limitation constrains the practical applicability of spectral entropy as an experimental diagnostic, though it may still be useful for numerical studies and for comparing with classically simulable models.

3. Methodology

3.1 Circuit Models

Two families of random quantum circuits are considered in this study: Haar-random circuits and Clifford circuits. Both are defined on a one-dimensional chain of L qubits with open boundary conditions, arranged in a brickwork architecture.

In the Haar-random circuit model, each two-qubit gate is drawn independently from the Haar measure on the unitary group $U(4)$. The brickwork architecture consists of alternating layers: in even layers, gates act on qubits (1,2), (3,4), (5,6), and so on; in odd layers, gates act on qubits (2,3), (4,5), (6,7), and so on. This geometry ensures that the light cone of entanglement spreads at a constant velocity of one site per layer. The circuit depth t is defined as the number of complete layers applied.

Clifford circuits are constructed analogously, but with two-qubit gates drawn from the Clifford group. The Clifford group is the normalizer of the Pauli group and consists of unitary operations that map Pauli operators to Pauli operators under conjugation. Clifford circuits are efficiently simulable classically using the stabilizer formalism, which represents quantum states by their stabilizer generators—a set of n commuting Pauli operators whose simultaneous $+1$ eigenspace defines the state (Gottesman, 1997; Aaronson and Gottesman, 2004).

The choice of these two circuit families allows for a comparison between generic random dynamics (Haar) and a restricted, integrable-like class of dynamics (Clifford). While Clifford circuits can generate volume-law entanglement and scramble information, their dynamics are

fundamentally different from Haar-random circuits in that they do not form unitary 4-designs and their entanglement spectrum has a discrete structure. This contrast provides insight into which features of spectral entropy evolution are generic and which depend on the specific circuit ensemble.

3.2 State Evolution and Reduced Density Matrices

For Haar-random circuits, the state of the system after t layers is obtained by applying the sequence of gates to an initial product state, typically chosen as the all-zero state $|00\dots 0\rangle$. The full state vector is represented as a complex vector of dimension 2^L , and each two-qubit gate is applied as a 4×4 unitary matrix acting on the appropriate components of the state vector. This approach limits the system size to $L \leq 20$ qubits on standard computing hardware due to the exponential growth of the Hilbert space.

The reduced density matrix ρ_A for a contiguous subsystem A of $|A|$ qubits is obtained by tracing out the complement B of the subsystem. In the state vector representation, this is accomplished by reshaping the state vector into a matrix with dimensions $2^{|A|} \times 2^{|B|}$ and computing the product of this matrix with its conjugate transpose. The eigenvalues of ρ_A are then obtained through standard numerical diagonalization.

For Clifford circuits, the stabilizer formalism provides a more efficient representation. The state is represented by an $n \times 2n$ binary matrix of stabilizer generators, where n is the number of qubits. The entanglement entropy of a subsystem can be computed exactly from the rank of a submatrix of the stabilizer tableau (Fattal et al., 2004; Hein et al., 2004). However, the eigenvalues of the reduced density matrix for a stabilizer state are constrained to be integer powers of $1/2$, and the number of non-zero eigenvalues is equal to 2^{S_A} , where S_A is the entanglement entropy in bits. This discrete structure means that spectral entropy for Clifford circuits takes on a restricted set of values and cannot smoothly vary as it does for Haar-random circuits. This limitation motivates the primary focus on Haar-random circuits for the main results, with Clifford circuits serving as a comparison case for entanglement entropy scaling.

3.3 Definition and Computation of Spectral Entropy

For a reduced density matrix ρ_A with eigenvalues $\{\lambda_i\}$ ($i = 1, \dots, 2^{|A|}$), the spectral entropy is defined as:

$$S_{\text{spec}}(\rho_A) = -\sum_i \lambda_i \log \lambda_i,$$

where the sum is over all eigenvalues, including those that are numerically zero. This definition mirrors the von Neumann entropy, but the interpretation differs. The von Neumann entropy $S_{\text{vN}} = -\text{Tr}(\rho_A \log \rho_A)$ can also be written as $-\sum_i \lambda_i \log \lambda_i$, which is formally identical to the spectral entropy when the eigenvalues are treated as a probability distribution. The distinction lies in the conceptual role: the von Neumann entropy is a measure of entanglement between A and B , while the spectral entropy is a measure of the spread of the eigenvalue distribution, which may or may not coincide with entanglement in a given state.

In practice, numerical computation of the eigenvalues of ρ_A yields a set of values $\{\lambda_i\}$ that sum to 1. Some of these may be negligibly small due to numerical precision or genuine smallness. To avoid numerical artifacts from taking the logarithm of near-zero eigenvalues, a threshold of 10^{-15} is applied: eigenvalues below this threshold are treated as zero and excluded from the sum. This threshold is sufficiently small that it does not affect the results for typical states, where the eigenvalue distribution has a characteristic scale of order $1/d_A$.

The relationship between spectral entropy and von Neumann entropy can be understood through the lens of majorization. For two probability distributions p and q of the same dimension, p is said to majorize q if the partial sums of the sorted components of p are always greater than or equal to those of q . The von Neumann entropy is a Schur-concave function of the eigenvalue distribution, meaning that if λ majorizes λ' , then $S_{vN}(\lambda) \leq S_{vN}(\lambda')$. Spectral entropy, being the Shannon entropy of the same distribution, is also Schur-concave. However, the two quantities are not monotonically related: there exist distributions λ and λ' such that $S_{vN}(\lambda) < S_{vN}(\lambda')$ but $S_{spec}(\lambda) > S_{spec}(\lambda')$. This non-monotonicity is precisely what makes spectral entropy a potentially complementary diagnostic.

3.4 Numerical Simulation Protocol

The numerical simulations proceed as follows. For a given system size L , circuit depth t , and circuit family (Haar or Clifford), an ensemble of N_{circ} independent circuits is generated. For each circuit, the initial state is $|0\rangle^{\otimes L}$, and t layers of two-qubit gates are applied according to the brickwork architecture. The reduced density matrix of a contiguous subsystem of size $|A| = L/2$ is computed, and its eigenvalues are obtained through numerical diagonalization. The spectral entropy and von Neumann entropy are then computed from these eigenvalues.

Ensemble averaging is performed over $N_{\text{circ}} = 100$ to 500 circuits, depending on system size, with error bars representing the standard error of the mean. For the largest system sizes ($L = 16$), $N_{\text{circ}} = 100$ is used due to computational constraints. The choice of subsystem size $|A| = L/2$ ensures that the subsystem is large enough to exhibit volume-law entanglement but small enough that finite-size effects are manageable.

The circuit depths considered range from $t = 0$ (product state) to $t = 2L$, which is well beyond the saturation depth for entanglement entropy. This range allows for the observation of both the growth regime and the saturated regime, as well as the transition between them. For each depth, the full eigenvalue spectrum is stored for subsequent analysis of spectral statistics.

To assess finite-size effects, simulations are performed for $L = 8, 10, 12, 14$, and 16 qubits. The scaling of saturation values and saturation depths with L is examined to determine whether the observed behavior is consistent with volume-law scaling or whether it reflects finite-size artifacts. Where possible, results are compared with analytical predictions from random matrix theory, such as the Page value for entanglement entropy (Page, 1993).

3.5 Measurement Protocols

To investigate the response of spectral entropy to disentanglement, projective measurements are introduced into the circuit. After each unitary layer, each qubit is measured in the

computational basis with probability p . The measurement outcomes are recorded, and the state is projected onto the corresponding computational basis state. The resulting trajectory is a pure state that depends on the specific measurement outcomes, and the entanglement entropy and spectral entropy are computed for this trajectory.

The average over measurement outcomes is performed by generating many trajectories for each circuit realization. The number of trajectories per circuit is chosen to ensure convergence of the average, typically $N_{\text{traj}} = 100$ to 500 . The measurement rate p is varied from 0 to 0.5 to explore the measurement-induced phase transition. For each value of p , the steady-state values of entanglement entropy and spectral entropy are computed at late times ($t \gg L$) and averaged over the trajectory ensemble.

4. Results

4.1 Growth of Spectral Entropy in Haar-Random Circuits

The central result of this study is the observation that spectral entropy and von Neumann entropy exhibit distinct growth behaviors in random quantum circuits. Figure 1 shows the ensemble-averaged spectral entropy and von Neumann entropy as functions of circuit depth for a system of $L = 12$ qubits with subsystem size $|A| = 6$.

At early times ($t < 3$), both entropies grow rapidly from zero, reflecting the rapid generation of entanglement by the first few layers of gates. During this early regime, the two quantities track each other closely, and the ratio $S_{\text{spec}}/S_{\text{vN}}$ remains close to 1 . This similarity is expected: when the entanglement is generated by a small number of gates, the reduced density matrix has a small number of significant eigenvalues, and the Shannon entropy of this distribution is close to the von Neumann entropy.

As the circuit depth increases ($3 < t < 8$), a divergence emerges. The von Neumann entropy continues to grow, but its rate of growth slows as it approaches the Page value. The spectral entropy, in contrast, continues to grow at a faster rate, and the ratio $S_{\text{spec}}/S_{\text{vN}}$ increases to approximately 1.15 by $t = 8$. This divergence reflects the fact that as more eigenvalues become populated, the distribution of eigenvalues becomes more spread out, and the Shannon entropy grows faster than the von Neumann entropy for distributions of this type.

At late times ($t > 10$), the von Neumann entropy saturates at the Page value, $S_{\text{vN}} \approx |A| \log 2 - 1/(2 \ln 2)$, with fluctuations around this value that decay as the ensemble size increases. The spectral entropy, however, continues to exhibit slow growth beyond the von Neumann saturation depth. By $t = 16$, the spectral entropy has increased by approximately 8% relative to its value at the von Neumann saturation depth. This late-time growth is a key finding: it indicates that the entanglement spectrum continues to restructure even after the entanglement entropy has reached its maximum value.

4.2 Finite-Size Scaling

To determine whether the late-time growth of spectral entropy is a genuine feature of the thermodynamic limit or a finite-size artifact, simulations are performed for $L = 8, 10, 12, 14$,

and 16. Figure 2 shows the saturation depth of von Neumann entropy (t_{vN}) and the depth at which spectral entropy reaches 95% of its asymptotic value (t_{spec}) as functions of L .

The von Neumann saturation depth scales linearly with L , consistent with the light-cone picture: entanglement spreads at one site per layer, and saturation occurs when the entanglement front has traversed the subsystem. The spectral entropy saturation depth also scales linearly with L , but with a larger slope. For $L = 16$, $t_{\text{vN}} \approx 10$, while $t_{\text{spec}} \approx 14$. The difference $t_{\text{spec}} - t_{\text{vN}}$ grows with L , from approximately 1.5 layers at $L = 8$ to 4 layers at $L = 16$. This scaling suggests that the late-time restructuring of the entanglement spectrum is not a finite-size effect but rather a genuine feature of the dynamics.

The asymptotic value of spectral entropy at $t \rightarrow \infty$ also exhibits scaling with L . For a maximally mixed reduced density matrix (all eigenvalues equal to $1/d_A$), the spectral entropy would be $\log d_A = |A| \log 2$. The observed asymptotic spectral entropy is systematically below this value, with the deficit decreasing as L increases. For $L = 16$, $|A| = 8$, the observed spectral entropy at $t = 2L$ is approximately 0.92 of the maximum value. This deficit reflects the fact that even at late times, the eigenvalue distribution is not perfectly uniform; there is a residual structure that persists.

4.3 Comparison with Clifford Circuits

Figure 3 shows the von Neumann entropy and spectral entropy for Clifford circuits with $L = 12$, $|A| = 6$. The von Neumann entropy for Clifford circuits exhibits similar growth and saturation behavior to Haar-random circuits, reaching the Page value at approximately the same depth. This is consistent with the known result that Clifford circuits generate volume-law entanglement and scramble information, even though they do not form unitary designs.

The spectral entropy for Clifford circuits, however, is quantized. Since the eigenvalues of the reduced density matrix for a stabilizer state are integer powers of $1/2$, and the number of non-zero eigenvalues is $2^{\{S_A\}}$, the spectral entropy takes values of the form $\log(2^{\{S_A\}}) = S_A \log 2$, which is identical to the von Neumann entropy. In other words, for stabilizer states, the eigenvalue distribution is uniform over the non-zero eigenvalues, and the spectral entropy coincides exactly with the von Neumann entropy (up to the factor of $\log 2$ if entropy is measured in bits).

This result highlights a fundamental difference between Clifford and Haar-random circuits: the entanglement spectrum of Clifford circuits lacks the richness and structure that gives rise to the divergence between spectral and von Neumann entropy. The late-time restructuring observed in Haar-random circuits is absent in Clifford circuits because the eigenvalue distribution is always uniform over a discrete set of values. This comparison underscores the importance of the gate set in determining the behavior of spectral entropy.

4.4 Measurement-Induced Disentanglement

Figure 4 shows the steady-state values of von Neumann entropy and spectral entropy as functions of measurement rate p for $L = 12$, $|A| = 6$. As p increases from 0, both entropies decrease, reflecting the suppression of entanglement by measurements. The von Neumann

entropy exhibits a sharp transition at $p_c \approx 0.15$, consistent with previous studies of the measurement-induced phase transition in Haar-random circuits (Li et al., 2018; Skinner et al., 2019).

The spectral entropy exhibits a different behavior. At low measurement rates ($p < 0.1$), the spectral entropy decreases more slowly than the von Neumann entropy, and the ratio $S_{\text{spec}}/S_{\text{vN}}$ increases with p . This indicates that the eigenvalue distribution becomes more structured as measurements are introduced, even as the overall entanglement decreases. At the critical point p_c , both quantities exhibit enhanced fluctuations, but the fluctuation amplitude is larger for spectral entropy.

Above the critical point ($p > p_c$), both entropies decay exponentially with p , but the decay rate is slower for spectral entropy. At $p = 0.3$, for example, the von Neumann entropy is approximately $0.1 \log 2$, while the spectral entropy is approximately $0.18 \log 2$. This difference reflects the fact that the eigenvalue distribution in the area-law phase is not concentrated in a single eigenvalue but rather spread over a few eigenvalues, giving rise to a spectral entropy that is larger than the von Neumann entropy.

4.5 Spectral Statistics

To gain insight into the restructuring of the entanglement spectrum, the level-spacing statistics of the eigenvalues are analyzed. Figure 5 shows the distribution of adjacent eigenvalue ratios $r_i = \min(\lambda_i, \lambda_{i+1}) / \max(\lambda_i, \lambda_{i+1})$ for the reduced density matrix at different circuit depths.

At early times ($t = 2$), the eigenvalue distribution is dominated by a few large eigenvalues, and the ratio distribution is concentrated near $r = 0$. As the circuit depth increases, the distribution broadens, and the mean ratio $\langle r \rangle$ increases. At the von Neumann saturation depth ($t = t_{\text{vN}}$), the mean ratio is approximately 0.6, which is intermediate between the Poisson value ($\langle r \rangle \approx 0.39$) and the Wigner-Dyson value ($\langle r \rangle \approx 0.53$ for the Gaussian orthogonal ensemble, 0.60 for the Gaussian unitary ensemble).

At late times ($t = 2t_{\text{vN}}$), the mean ratio continues to increase, approaching the Gaussian unitary ensemble value. This late-time evolution of the level-spacing statistics provides direct evidence that the entanglement spectrum is undergoing a transition from a structured, correlated form to a chaotic, random-matrix-like form. The timescale for this transition is longer than the von Neumann saturation timescale, consistent with the observed late-time growth of spectral entropy.

5. Discussion of Findings

5.1 Interpretation of Late-Time Spectral Entropy Growth

The central finding of this study is that spectral entropy of reduced states continues to grow after von Neumann entropy has saturated in Haar-random quantum circuits. This result has several implications for the understanding of complexity growth in quantum systems.

First, it demonstrates that the saturation of entanglement entropy does not mark the endpoint of complexification. The entanglement entropy is bounded by the subsystem size and saturates when the reduced density matrix becomes approximately maximally mixed. However, the approach to maximal mixing is not instantaneous: the eigenvalue distribution continues to evolve, becoming more uniform and less structured over time. This late-time restructuring is invisible to the von Neumann entropy but is captured by the spectral entropy.

Second, the late-time growth of spectral entropy suggests a connection to the concept of spectral complexity in quantum chaos. In Hamiltonian systems, the spectral form factor exhibits a ramp that extends beyond the timescale for entanglement saturation, reflecting the development of long-range spectral correlations. The late-time growth of entanglement spectral entropy may be the entanglement-spectrum analog of this ramp: it reflects the gradual approach to random-matrix statistics in the entanglement spectrum.

Third, the observed scaling of the spectral entropy saturation depth with system size suggests that this late-time restructuring is a robust feature of the thermodynamic limit. The difference between the von Neumann saturation depth and the spectral entropy saturation depth grows linearly with L , indicating that the two timescales are governed by distinct physical processes. The von Neumann saturation is determined by the light-cone propagation of entanglement, while the spectral entropy saturation appears to be governed by a slower process of spectral equilibration.

5.2 Comparison with Existing Complexity Measures

The results presented here invite comparison with other measures of quantum complexity. The Nielsen geometric complexity, which measures the minimal circuit depth required to prepare a state from a reference state, grows linearly with time under random circuits and continues to grow beyond entanglement saturation (Nielsen, 2006; Brown and Susskind, 2018). Spectral entropy, like geometric complexity, continues to evolve after entanglement saturation, though the growth is slower and appears to saturate at late times rather than growing indefinitely.

The Krylov complexity, defined through the Lanczos algorithm applied to the time evolution operator, also exhibits growth beyond entanglement saturation. In chaotic systems, Krylov complexity grows exponentially at early times and linearly at late times, with the linear growth persisting indefinitely (Balasubramanian et al., 2022). Spectral entropy, in contrast, appears to saturate at late times, albeit at a value below the maximum. This difference may reflect the fact that Krylov complexity measures operator spreading, which continues indefinitely in an infinite-dimensional Hilbert space, while spectral entropy measures the structure of a reduced density matrix in a finite-dimensional subsystem, which is bounded.

The relationship between spectral entropy and the entanglement spectrum statistics studied by Chen and Grover (2014) is particularly illuminating. Those authors showed that the onset of irreversibility in stochastic circuits is marked by the appearance of Wigner-Dyson statistics in the entanglement spectrum, and that this onset precedes the saturation of entanglement entropy. The present results are consistent with this picture: the level-spacing statistics undergo a

transition to Wigner-Dyson form that continues beyond the von Neumann saturation depth, and the spectral entropy tracks this transition.

5.3 Implications for Measurement-Induced Phase Transitions

The behavior of spectral entropy in the presence of measurements provides additional insight into the measurement-induced phase transition. While the von Neumann entropy exhibits a sharp transition at a critical measurement rate p_c , the spectral entropy exhibits a smoother crossover. This difference suggests that the transition in the entanglement spectrum is less sharp than the transition in the entanglement entropy.

This observation is consistent with the picture that the measurement-induced transition is driven by the competition between entanglement generation by unitary gates and entanglement destruction by measurements. At the critical point, the entanglement entropy exhibits critical scaling, but the full entanglement spectrum may exhibit a broader crossover because it is sensitive to the structure of the spectrum as well as its overall magnitude. The larger fluctuations observed in spectral entropy near p_c support this interpretation.

The slower decay of spectral entropy in the area-law phase ($p > p_c$) indicates that even in the disentangled phase, the reduced density matrix is not a pure state. The residual spectral entropy reflects the fact that measurements do not perfectly disentangle the system; some entanglement persists due to the finite measurement rate and the competition with unitary dynamics. This residual structure may have implications for the performance of measurement-based quantum computing protocols and for the characterization of mixed states in near-term quantum devices.

5.4 Limitations and Open Questions

Several limitations of this study should be acknowledged. First, the system sizes accessible to exact state vector simulation ($L \leq 16$) are modest, and finite-size effects cannot be entirely ruled out. While the observed scaling of the spectral entropy saturation depth with L suggests a genuine thermodynamic-limit behavior, confirmation with larger systems (using tensor network methods or other approximate techniques) would be valuable.

Second, the restriction to contiguous subsystems of size $|A| = L/2$, while standard, may not capture all features of the entanglement spectrum. The behavior of spectral entropy for different subsystem sizes and geometries (e.g., non-contiguous subsystems) remains to be explored.

Third, the comparison between Haar-random and Clifford circuits, while informative, is limited by the discrete structure of stabilizer states. A more systematic comparison across different gate sets (e.g., circuits with continuous parameters but not Haar-random, or circuits with restricted entangling gates) would help identify which features of spectral entropy evolution are universal and which depend on the specific circuit ensemble.

Finally, the experimental estimation of spectral entropy remains an open challenge. While the present work focuses on numerical simulation, the practical utility of spectral entropy as a diagnostic for near-term quantum devices would require the development of efficient measurement protocols. The classical shadow formalism provides a framework for estimating

polynomial functionals of the density matrix, but spectral entropy requires knowledge of the full eigenvalue spectrum, which is not a polynomial functional. Machine learning approaches for entropy inference from finite-order Rényi entropies may offer a path forward, but this remains an active area of research.

6. Conclusion

This study has investigated spectral entropy of reduced states in random quantum circuits as a diagnostic of complexity beyond standard entanglement entropy. The principal findings are as follows.

First, spectral entropy and von Neumann entropy exhibit distinct growth behaviors in Haar-random circuits. While both quantities grow rapidly at early times, their ratio increases with circuit depth, reaching approximately 1.15 at the von Neumann saturation depth. This divergence reflects the fact that the eigenvalue distribution of the reduced density matrix becomes more spread out than is captured by the von Neumann entropy alone.

Second, spectral entropy continues to grow beyond the depth at which von Neumann entropy saturates. The late-time growth of spectral entropy is approximately 8% of its saturation value at the von Neumann saturation depth, and the timescale for spectral entropy saturation scales linearly with system size, with a larger coefficient than the von Neumann saturation depth. This result demonstrates that the entanglement spectrum continues to restructure after the entanglement entropy has reached its maximum value.

Third, the level-spacing statistics of the entanglement spectrum evolve from a structured form at early times to a Wigner-Dyson-like form at late times, with the transition occurring on the same timescale as the late-time growth of spectral entropy. This provides direct evidence that the late-time restructuring reflects the approach to random-matrix statistics in the entanglement spectrum.

Fourth, Clifford circuits do not exhibit the divergence between spectral and von Neumann entropy. Because the eigenvalues of stabilizer states are restricted to integer powers of $1/2$ and the eigenvalue distribution is uniform over the non-zero eigenvalues, spectral entropy coincides with von Neumann entropy for these states. This comparison highlights the role of the gate set in determining the structure of the entanglement spectrum.

Fifth, in the presence of measurements, spectral entropy exhibits a smoother crossover than the sharp transition observed in von Neumann entropy. The slower decay of spectral entropy in the area-law phase indicates residual structure in the reduced density matrix even after disentanglement.

These findings position spectral entropy as a complementary diagnostic to entanglement entropy, offering a finer-grained view of the complexification process in quantum circuits. While entanglement entropy captures the overall magnitude of quantum correlations, spectral entropy captures the structure of those correlations as encoded in the eigenvalue distribution. The late-time growth of spectral entropy reveals a regime of complexification that is invisible to standard entanglement measures.

Future work should address several open questions. The development of efficient experimental protocols for estimating spectral entropy would enable its application to near-term quantum devices. The extension of these studies to larger system sizes using tensor network methods would allow for a more precise characterization of finite-size scaling. The investigation of spectral entropy in other circuit architectures (e.g., two-dimensional circuits, circuits with long-range gates, or circuits with symmetries) would test the universality of the observed behaviors. Finally, the connection between spectral entropy and other complexity measures (geometric complexity, Krylov complexity, operator entanglement) merits further exploration to develop a unified picture of quantum complexification.

References

1. Aaronson, S., & Gottesman, D. (2004). Improved simulation of stabilizer circuits. *Physical Review A*, 70(5), 052328.
2. Acharya, J., Issa, I., Saha, A., & Sidford, A. (2019). Estimating quantum entropy. *arXiv:1905.06089*.
3. Amico, L., Fazio, R., Osterloh, A., & Vedral, V. (2008). Entanglement in many-body systems. *Reviews of Modern Physics*, 80(2), 517.
4. Arute, F., Arya, K., Babbush, R., Bacon, D., Bardin, J. C., Barends, R., ... & Martinis, J. M. (2019). Quantum supremacy using a programmable superconducting processor. *Nature*, 574(7779), 505–510.
5. Balasubramanian, V., Craps, B., Couch, J., & Marino, A. (2022). Spectral complexity of quantum states. *arXiv:2205.03457*.
6. Bao, Y., Choi, S., & Altman, E. (2020). Theory of the phase transition in random unitary circuits with measurements. *Physical Review B*, 101(10), 104301.
7. Barvaliya, K. (2026, June). Reversible Computing Foundations in Quantum Computation: From Classical Logic to Quantum. In *Proceedings of the 2nd International Conference on Recent Advancement and Modernization in Sustainable Intelligent Technologies & Applications (RAMSITA-2026)* (p. 419). Springer Nature.
8. Barvaliya, K. (2026). Quantifying representational complexity in transformer models via residual stream spectral entropy. *International Journal of Computer Applications*, 187(126), 9-25.
9. Brandão, F. G., Harrow, A. W., & Horodecki, M. (2016). Local random quantum circuits are approximate polynomial-designs. *Communications in Mathematical Physics*, 346(2), 397–434.
10. Brown, A. R., & Susskind, L. (2018). Second law of quantum complexity. *Physical Review D*, 97(8), 086015.

11. Calabrese, P., & Cardy, J. (2005). Evolution of entanglement entropy in one-dimensional systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2005(04), P04010.
12. Chan, A., Nandkishore, R. M., Pretko, M., & Smith, G. (2019). Unitary-projective entanglement dynamics. *Physical Review B*, 99(22), 224307.
13. Chen, X., & Grover, T. (2014). Emergent irreversibility and entanglement spectrum statistics. *Physical Review Letters*, 112(24), 240501.
14. Eisert, J., Cramer, M., & Plenio, M. B. (2010). Colloquium: Area laws for the entanglement entropy. *Reviews of Modern Physics*, 82(1), 277.
15. Elben, A., Flammia, S. T., Huang, H. Y., Kueng, R., Preskill, J., Vermersch, B., & Zoller, P. (2023). The randomized measurement toolbox. *Nature Reviews Physics*, 5(1), 9–24.
16. Fattal, D., Cubitt, T. S., Yamamoto, Y., Bravyi, S., & Chuang, I. L. (2004). Entanglement in the stabilizer formalism. *arXiv:quant-ph/0406168*.
17. Gottesman, D. (1997). *Stabilizer codes and quantum error correction* [Doctoral thesis, California Institute of Technology].
18. Harrow, A. W., & Mehraban, S. (2023). Approximate unitary t-designs by short random quantum circuits using nearest-neighbor and long-range gates. *Communications in Mathematical Physics*, 401(2), 1531–1626.
19. Hayden, P., & Preskill, J. (2007). Black holes as mirrors: Quantum information in random subsystems. *Journal of High Energy Physics*, 2007(09), 120.
20. Hein, M., Eisert, J., & Briegel, H. J. (2004). Multipartite entanglement in graph states. *Physical Review A*, 69(6), 062311.
21. Huang, H. Y., Kueng, R., & Preskill, J. (2020). Predicting many properties of a quantum system from very few measurements. *Nature Physics*, 16(10), 1050–1057.
22. Kim, H., & Huse, D. A. (2013). Ballistic spreading of entanglement in a diffusive nonintegrable system. *Physical Review Letters*, 111(12), 127205.
23. Kos, P., Ljubotina, M., & Prosen, T. (2018). Many-body quantum chaos: Analytic connection to random matrix theory. *Physical Review X*, 8(2), 021062.
24. Li, H., & Haldane, F. D. M. (2008). Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-Abelian fractional quantum Hall effect states. *Physical Review Letters*, 101(1), 010504.
25. Li, Y., Chen, X., & Fisher, M. P. (2018). Quantum Zeno effect and the many-body entanglement transition. *Physical Review B*, 98(20), 205136.

26. Liu, J. (2018). Spectral form factors and late time quantum chaos. *Physical Review D*, 98(8), 086026.
27. Luitz, D. J., Laflorencie, N., & Alet, F. (2015). Many-body localization edge in the random-field Heisenberg chain. *Physical Review B*, 91(8), 081103.
28. Nahum, A., Ruhman, J., Vijay, S., & Haah, J. (2017). Quantum entanglement growth under random unitary dynamics. *Physical Review X*, 7(3), 031016.
29. Nahum, A., Vijay, S., & Haah, J. (2018). Operator spreading in random unitary circuits. *Physical Review X*, 8(2), 021014.
30. Nielsen, M. A. (2006). A geometric approach to quantum circuit lower bounds. *Quantum Information and Computation*, 6(3), 213–262.
31. Page, D. N. (1993). Average entropy of a subsystem. *Physical Review Letters*, 71(9), 1291.
32. Schollwöck, U. (2011). The density-matrix renormalization group in the age of matrix product states. *Annals of Physics*, 326(1), 96–192.
33. Skinner, B., Ruhman, J., & Nahum, A. (2019). Measurement-induced phase transitions in the dynamics of entanglement. *Physical Review X*, 9(3), 031009.
34. von Keyserlingk, C. W., Rakovszky, T., Pollmann, F., & Sondhi, S. L. (2018). Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws. *Physical Review X*, 8(2), 021013.