

A compensated coupling bound for the average sensitivity of polynomial threshold functions

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Abstract

We sharpen the coupling step in the weighted Fourier-grading argument for average sensitivity of polynomial threshold functions. For every real polynomial threshold function of degree at most d on the uniform n -dimensional Boolean cube, we obtain $\text{AS}(f) \leq (8/\sqrt{2\pi})d\sqrt{n}$, uniformly in dimension and degree. More precisely, if $b_n = \mathbb{E}|B_n - n/2|$ for $B_n \sim \text{Bin}(n, 1/2)$, then $\text{AS}(f) \leq \min\{n, 8db_n(n+d)/(n+2d)\}$. A linear correction to the coupling potential removes the degree-squared loss in the earlier 3.83 bound, and a finite-range estimate makes its former asymptotic coefficient a uniform coefficient. We also construct binomial couplings showing that $8/\sqrt{2\pi}$ is asymptotically sharp for the relaxation that uses only the two marginal laws, the one-sided displacement constraint, and the existing factor-two grading comparison. Optimality for polynomial threshold functions is not asserted.

Keywords: polynomial threshold functions; average sensitivity; Boolean functions; compensated coupling; Gotsman–Linial conjecture; Fourier grading; binomial coupling.

1 Introduction and main result

Write $\Omega_n = \{-1, 1\}^n$ and let X be uniform on Ω_n . For $x \in \Omega_n$, let $x^{\oplus i}$ denote x with coordinate i reversed. For $f : \Omega_n \rightarrow \{-1, 1\}$, its average sensitivity is

$$\text{AS}(f) = \sum_{i=1}^n \mathbb{P}\{f(X) \neq f(X^{\oplus i})\}. \quad (1)$$

We use $\text{sgn}(t) = 1$ for $t \geq 0$ and $\text{sgn}(t) = -1$ for $t < 0$.

OpenAI [1] establishes $\text{AS}(f) \leq 8d\sqrt{n}$ for a real degree-at-most- d polynomial threshold function. Its weighted-grading argument reduces the estimate to a coupling of binomial random variables. The earlier note of Lagdive, Aglave, and Damre [11] refines this reduction to a uniform coefficient 3.83. The present revision further sharpens the probability step. We give the finite-dimensional reduction in Section 3, following [1, Lemma 3 and Section 5], to make the dependence of the numerical coefficient explicit.

Earlier general estimates include those of Harsha et al. [2], Diaconikolas et al. [3], and Kane [4]. Kane’s bound has the form $\sqrt{n}(\log n)^{O(d \log d)} 2^{O(d^2 \log d)}$. Kabanets, Kane, and Lu [5] give another

proof of the optimal dimension exponent. The stronger conjecture identifying exact symmetric maximizers is false [6, 7]; the result below concerns a uniform upper bound, not exact maximizers. Recent results of Chang, Slote, Volberg, and Zhang [8] and of Tseng and Volberg [9] concern fractional sensitivity moments and retain logarithmic factors at moment order one. Grebennikov [10] proves related influence bounds for low-degree rational functions and polynomial nonvanishing indicators. Those classes differ from arbitrary polynomial sign functions.

Theorem 1. *Let $n \geq 1$ and $0 \leq d \leq n$ be integers, let p be a real polynomial of degree at most d , and put $f = \text{sgn } p$ on Ω_n . Then*

$$\text{AS}(f) \leq \frac{8}{\sqrt{2\pi}} d\sqrt{n}. \quad (2)$$

With

$$b_n = \frac{n}{2^n} \binom{n-1}{\lfloor (n-1)/2 \rfloor} = \mathbb{E}|B_n - n/2|, \quad B_n \sim \text{Bin}(n, 1/2), \quad (3)$$

the stronger finite-parameter estimate is

$$\text{AS}(f) \leq \min \left\{ n, 8db_n \frac{n+d}{n+2d} \right\}. \quad (4)$$

2 A compensated coupling estimate

Lemma 2. *Let U, V have the same distribution, finite second moment, and common mean m . Suppose $V - U \leq a$ almost surely, where $a \geq 0$. Set $b = \mathbb{E}|U - m|$, $w = \mathbb{E}(U - V)_+ = \mathbb{E}(V - U)_+$, and $e = \mathbb{E}(U - V)^2$. Then*

$$e + aw \leq 4ab. \quad (5)$$

If both variables lie in a common interval of length $L > 0$, then

$$e \leq 4ab \frac{L+a}{L+2a}. \quad (6)$$

Proof. Put $g(t) = \frac{1}{2}(t - m)|t - m|$ and $\Phi(t) = 4g(t) - a(t - m)$. The interval estimates from the earlier argument are, for $u < v$,

$$\frac{(v - u)^2}{4} \leq g(v) - g(u) \leq \frac{v - u}{2}(|u - m| + |v - m|). \quad (7)$$

They follow respectively by minimizing the integral of $|t - m|$ over an interval of prescribed length, and integrating its convex chord bound. We claim that, for $v - u \leq a$,

$$(u - v)^2 + a(u - v)_+ \leq 2a(|u - m| + |v - m|) + \Phi(u) - \Phi(v). \quad (8)$$

If $v \geq u$, set $s = v - u \in [0, a]$ and $A = |u - m| + |v - m|$. The right side is at least $2(a - s)A + as \geq s^2$, and the positive-part term on the left is zero. If $u > v$, set $\ell = u - v$. The right side is at least $\ell^2 + 2aA - a\ell \geq \ell^2 + a\ell$, since $A \geq \ell$. Thus (8) holds in both cases.

The potential has quadratic growth, so it is integrable under either marginal. Their expectations agree. Taking expectations in (8) gives (5); equality of means gives $\mathbb{E}(U - V)_+ = \mathbb{E}(V - U)_+$. If the common support lies in an interval of length L , then

$$e \leq L\mathbb{E}(U - V)_+ + a\mathbb{E}(V - U)_+ = (L + a)w.$$

Multiply (5) by $L + a$ and multiply $e - (L + a)w \leq 0$ by a . Adding cancels w and yields $(L + 2a)e \leq 4ab(L + a)$. The proof uses neither independence nor exchangeability. $\square$

3 The weighted-grading reduction

This section records the reduction from [1] with the normalization in (1). All matrix norms and dimensions below are finite-dimensional.

Let $N = 2^n$ and use the counting inner product on $\mathbb{C}^{\Omega_n}$. For $S \subseteq [n]$, put $\chi_S(x) = \prod_{i \in S} x_i$ and let V_k be the span of the characters with $|S| \leq k$. Fix a positive weight w , let D_w be multiplication by $\sqrt{w}$, and set

$$K_k = D_w V_k, \quad E_k = K_k \cap K_{k-1}^\perp, \quad K_{-1} = \{0\}.$$

Let Π_k be the projection onto E_k . Since D_w is invertible, $\dim E_k = \binom{n}{k}$, and the centered grading operator

$$M = \sum_{k=0}^n (k - n/2) \Pi_k$$

is self-adjoint and has $\|M\|_{\text{HS}}^2 = nN/4$. If Z_i is multiplication by x_i , then

$$\Pi_s Z_i \Pi_r = 0 \quad \text{for } |s - r| > 1. \quad (9)$$

Indeed, Z_i commutes with D_w and raises polynomial degree by at most one; self-adjointness gives the opposite triangular vanishing.

Proposition 3 (Edge recovery, following OpenAI). *For $h : \Omega_n \rightarrow \{-1, 1\}$, let H be multiplication by h and let E_h be the set of ordered neighboring pairs (x, y) with $h(x) = h(y)$. Then*

$$|E_h| \leq 2\|MH + HM\|_{\text{HS}}^2. \quad (10)$$

Proof. Put $C_i = [M, Z_i]$. Define the unitaries $J = \sum_k (-1)^k \Pi_k$ and $W = \sum_k i^k \Pi_k$. By (9), $O_i = (Z_i - JZ_iJ)/2$ consists of the adjacent-grade blocks of Z_i . Thus $WO_iW^* = iC_i$ and $\|C_i\|_{\text{op}} \leq 1$. In the point basis $(C_i)_{xy} = (y_i - x_i)M_{xy}$, so $|M_{xy}| \leq 1/2$ for $x \neq y$. Also, (9) gives $\|C_i\|_{\text{HS}}^2 \leq N$. Writing ρ for Hamming distance,

$$\sum_{x,y} \rho(x,y) |M_{xy}|^2 = \frac{1}{4} \sum_i \|C_i\|_{\text{HS}}^2 \leq \frac{nN}{4} = \sum_{x,y} |M_{xy}|^2.$$

Let $A_0 = \sum_x |M_{xx}|^2$ and $F = \sum_{\rho(x,y) \geq 2} |M_{xy}|^2$. The preceding inequality gives $F \leq A_0$. Since there are nN ordered neighboring pairs,

$$\sum_{\rho(x,y)=1} (1/4 - |M_{xy}|^2) = A_0 + F \leq 2A_0.$$

Every summand is nonnegative. With $A_h = \sum_{(x,y) \in E_h} |M_{xy}|^2$, we therefore have $|E_h|/4 \leq A_h + 2A_0 \leq 2(A_h + A_0)$. The entries of $T = MH + HM$ are $T_{xy} = (h(x) + h(y))M_{xy}$. Its squared norm is at least $4(A_h + A_0)$, proving (10). $\square$

Proposition 4 (Binomial coupling, following OpenAI). *For $f = \text{sgn } p$ of degree at most d , there is a coupling U, V of $\text{Bin}(n, 1/2)$ such that $V - U \leq d$ and*

$$\text{AS}(f) \leq 2\mathbb{E}(U - V)^2. \quad (11)$$

Proof. Multilinearization preserves the values of p on the cube without raising its degree. By adding a sufficiently small positive constant, one can also ensure that p never vanishes while preserving f under the stated sign convention. Put $\chi = \chi_{[n]}$, $w = |p|$, and $h = \chi f$. Equal-sign neighbors for h are precisely the sensitive neighbors for f , so $\text{AS}(f) = |E_h|/N$.

The Fourier support of $q = \chi p$ lies in degrees at least $n - d$. For $a \in V_r$ and $b \in V_s$,

$$\langle D_w b, H D_w a \rangle = \sum_x \overline{b(x)} a(x) q(x) = 0 \quad \text{if } r + s < n - d.$$

Here $\bar{b}a$ has Fourier degree at most $r + s$. Hence $\Pi_s H \Pi_r = 0$ in this range. Since H is a self-adjoint unitary, the nonnegative weights

$$\mathbb{P}(R = r, S = s) = N^{-1} \|\Pi_s H \Pi_r\|_{\text{HS}}^2$$

have both marginals $2^{-n} \binom{n}{r}$. The blocks of $T = MH + HM$ satisfy $\Pi_s T \Pi_r = (s + r - n) \Pi_s H \Pi_r$, giving

$$N^{-1} \|T\|_{\text{HS}}^2 = \mathbb{E}(R + S - n)^2.$$

Now put $U = S$ and $V = n - R$. Binomial symmetry preserves the marginal law, and $R + S \geq n - d$ gives $V - U \leq d$. Applying Proposition 3 proves (11). $\square$

4 A uniform coefficient

The binomial variables in Proposition 4 lie in $[0, n]$. Apply Lemma 2 with $a = d$ and $L = n$ to obtain $\text{AS}(f) \leq 8db_n(n + d)/(n + 2d)$. Together with $\text{AS}(f) \leq n$, this proves (4).

Put $c_k = 4^{-k} \binom{2k}{k}$. Summing the upper half of the binomial law, using $j \binom{n}{j} = n \binom{n-1}{j-1}$, gives

$$b_{2k} = k c_k, \quad b_{2k+1} = (2k + 1) c_k / 2.$$

For $k \geq 1$, integration by parts and the Gaussian integral give

$$c_k = \frac{2}{\pi} \int_0^{\pi/2} \cos^{2k} t \, dt \leq \frac{2}{\pi} \int_0^\infty e^{-kt^2} \, dt = \frac{1}{\sqrt{\pi k}}. \quad (12)$$

Indeed, $\tan t \geq t$ implies $\log \cos t \leq -t^2/2$ on $[0, \pi/2)$, proving the integrand comparison. For even $n = 2k \geq 2$, this immediately implies

$$\frac{8b_n}{\sqrt{n}} \frac{n + d}{n + 2d} \leq \frac{8}{\sqrt{2\pi}}.$$

For odd $n = 2k + 1 \geq 3$ and $d \geq 1$, the factor $(n + d)/(n + 2d)$ decreases with d and is at most $(n + 1)/(n + 2)$. By (12), the ratio to $8/\sqrt{2\pi}$ is therefore at most

$$\sqrt{\frac{n}{n-1}} \frac{n+1}{n+2} \leq 1.$$

The last squared comparison follows from

$$(n-1)(n+2)^2 - n(n+1)^2 = n^2 - n - 4 \geq 0 \quad (n \geq 3).$$

The case $n = 1$ follows from $\text{AS}(f) \leq 1$, and $d = 0$ gives a constant function. This completes the proof of Theorem 1.

5 A limitation of the coupling relaxation

Let $\mu_j = 2^{-n} \binom{n}{j}$, with $\mu_{-1} = 0$, and put $a_j = \mu_j - \mu_{j-1}$ for $0 \leq j \leq \lfloor n/2 \rfloor$. These numbers are nonnegative. On each integer interval $I_j = \{j, \dots, n-j\}$, assign mass a_j to every edge of the cyclic map that sends u to $u+1$ below the last point and sends $n-j$ back to j . The singleton interval, when present, contributes a self-loop.

Each layer has the same row and column masses. Summing the layers gives row and column mass $\sum_{j \leq \min(u, n-u)} a_j = \mu_u$ at u . Thus the resulting joint law has binomial marginals and $V - U \leq 1$. Its second moment is

$$e_n = \sum_j a_j (n-2j)(n-2j+1) = 4b_n - 1 + \mathbf{1}_{2|n} \mu_{n/2}. \quad (13)$$

For odd n , summation by parts gives $e_n = \sum_{j \leq \lfloor n/2 \rfloor} \mu_j (4(n-2j) - 2) = 4b_n - 1$. For even n , the same sum excludes the central point and the left-tail mass is $(1 - \mu_{n/2})/2$, giving the extra term in (13). Since $b_n \sim \sqrt{n/(2\pi)}$, this feasible second moment satisfies

$$\frac{e_n}{\sqrt{n}} \longrightarrow \frac{4}{\sqrt{2\pi}}.$$

Consequently, the uniform coefficient $8/\sqrt{2\pi}$ cannot be reduced by first bounding the second moment over all couplings with binomial marginals and $V - U \leq d$, and then applying only $\text{AS}(f) \leq 2\mathbb{E}(U - V)^2$. This is a limitation of that relaxation. The constructed couplings need not arise from polynomial threshold functions, and no matching lower bound for actual average sensitivity is asserted. The barrier also holds after excluding degree one. For even n , apply symmetric unimodal layering separately on the even and odd lattices, and use cycles with upward step two. This gives binomial marginals, $V - U \leq 2$, and

$$e_n^{(2)} = 8b_n - 4 + 4\mu_{n/2}.$$

Indeed, successive layer lengths differ by four, and the squared-displacement sum in a layer of length L is $L(L+2)$. Summation by parts in each parity class gives the formula above; their total masses sum to one. Thus $e_n^{(2)}/(2\sqrt{n}) \rightarrow 4/\sqrt{2\pi}$. Further progress should exploit additional structure of the grading coupling or strengthen the edge-recovery comparison.

6 Verification, discussion, and conclusions

The universal result follows from the analytic inequalities above. The supplementary exact checker verifies 12,110 pointwise compensated potential inequalities, 5,315 finite transport-dual inequalities, 2,000 dimensions at the worst normalized degree, all 8,256 dimension-degree pairs through dimension 128, and 96 exact layered couplings. Bounds on π in these checks use alternating rational series and Machin's identity, rather than a floating-point approximation. The normalized finite-parameter bound decreases with d , so degree one is the largest normalized case at each fixed dimension.

Supplemental experiments check 11 coupling linear programs and 63 weighted-gradings. Small marginal residuals from the numerical solver are accounted for by explicit dual potentials; the solver output is not assumed to be an exactly feasible coupling. None of these finite or floating-point calculations is used as a premise in the universal proof. The coefficients and formulas are available with the standard-library checker `verify_range_bound.py` and optional numerical script `explore_range_bound.py` in Supplementary Material S1 (`supplementary-verification.zip`).

This revision makes the former asymptotic coefficient valid uniformly, including degrees that grow with dimension. It does not establish optimality for polynomial threshold functions, exhaustive historical priority, independent human peer review, or a Lean formalization.

Use of artificial intelligence. OpenAI Codex assisted with deriving the probability refinement, preparing the exposition, conducting literature searches, and generating and checking exact-arithmetic and numerical verification scripts. The separate proof audit was also AI-assisted and is not an independent human referee report. The weighted-grading construction and edge-recovery argument are due to OpenAI [1]. The full argument is provided so that the refinement can be examined independently.

Declarations

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Data and code availability. No empirical datasets were collected. The verification scripts and recorded exact and numerical outputs are included in Supplementary Material S1. The exact script uses only the Python standard library; optional numerical checks use NumPy and SciPy. Reproduction instructions and software versions are provided in the supplementary archive.

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