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Fortifying Distribution Networks Against Node-Based Disruptions

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Abstract

We consider an acyclic distribution network for delivering a natural resource or physical good to a set of nodes, each of which serves a set of customers, in which flow disruptions may occur at one or more nodes. Each node receives flow along a path from a source node, implying that a node’s ability to provide service depends on flow from upstream nodes. Nodes are subject to disruptions of uncertain magnitude that may vary across the network. We study how to fortify network nodes under a limited budget against disruptions of uncertain severity that can disrupt the flow at individual nodes and at their downstream nodes. We formulate this problem as a continuous, nonlinear multiplicative programming problem, and although we show that the problem on a general tree is $\mathcal{NP}$ -Hard, we characterize properties of optimal solutions that enable solving the problem for two-level systems in polynomial time under uniformly distributed disruptions. For general trees where disruption severity follows a number of broadly applicable S-shaped distributions, we develop a heuristic solution method that is computationally shown to provide near-optimal solutions quickly.

Keywords: Network fortification, node-based disruption, resilience, service systems.

1 Introduction

System Resilience is the ability to absorb and recover from a disturbance while maintaining key functions (Wei and Pan, 2021). Distribution networks, including critical infrastructure for electrical power, transportation, water, gas, and communication, are highly susceptible to random perturbations arising in the form of unexpected events such as human-induced or natural disasters (Chang et al., 2014), which can have significant negative societal impact and financial consequences (Godazgar et al., 2023). These disturbances can reduce or eliminate the ability to provide service at one or more nodes, and protecting infrastructure from

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all potential threats using traditional methods (such as risk management) is unachievable (Mottahedi et al., 2021).

In the event of a disruption caused by a natural hazard, recovery efforts rely on the survivability of critical networks supporting them (Li et al., 2017), which highlights the importance of the preparedness stage of network resilience. Decisions made in this stage include capacity planning, resource positioning, and fortification, strengthening, or hardening of the underlying network (Biswas et al., 2024), which can help significantly improve system robustness (Mukherjee et al., 2014). For network resilience enhancement, such investment decisions should be carefully targeted due to limited budget and resources (Najarian and Lim, 2020).

Building on previous work Keşrit et al. (2025), we study a distribution network that delivers a natural resource, service, or physical good to a set of nodes, each serving its own group of customers, where flow disruptions may arise at one or more nodes. Each node receives flow along a path originating from a designated source node. When every node is connected to the source through a path, the underlying network exhibits a tree topology, as is typical in local access telecommunication networks (Tawarmalani and Li, 2011) and in power distribution or gas pipeline systems (Tong and Tien, 2019). A node’s ability to serve its customers relies on the availability of upstream flow via some path in the network. This work studies the network fortification problem in acyclic networks when nodes are subject to node-specific random disruptions, unlike the previous work Keşrit et al. (2025), which assumes the same disruption across the entire network.

Our modeling approach assumes that each network node is subject to a unique disturbance severity distribution (i.e., node-based disruptions); Dai and Preece (2025) note that while many studies assume uniformly applied fragility curves for natural hazards, evidence suggests significantly different effects on nodes with similar characteristics. For example, seismic hazard maps can detect vulnerable regions and evaluate different probabilities of damage at individual nodes in telecommunication networks (Mukherjee et al., 2014).

We further assume that disruption severity can be quantified and follows some well-defined probability distribution. For each node in the acyclic network, the goal is to determine a fortification level that allows the node to withstand node-based disturbances up to a certain threshold determined by that fortification level, subject to a constrained fortifica-

tion budget. The resulting optimization problem seeks to minimize the expected number of customers who experience service interruptions, or equivalently, to maximize the expected number of customers whose service remains uninterrupted. We formulate this problem as a continuous, nonlinear multiplicative programming problem, and although we show that the problem on a general tree is $\mathcal{NP}$ -Hard under both discrete and continuous cumulative distribution functions, we characterize properties of optimal solutions that enable solving the problem for two-level network systems in polynomial time under a uniform distribution assumption on disruptions.

The difficulty of the resulting optimization problem is influenced by both the underlying network structure and the associated distribution of disturbance severity. If, for example, the cumulative distribution function (CDF) of the disturbance distribution is log-convex, then the corresponding problem maximizes a convex objective, which implies an extreme point solution (because the number of extreme points of the continuous knapsack problem is on the order of the number of nodes; this implies that in such cases an efficient solution is possible). Unfortunately, only a small subset of continuous probability distributions have log-convex cumulative distribution functions, and restricting the analysis to this class significantly limits the practical applicability of the model. For most commonly applied continuous distributions of natural phenomena, such as normal, beta, gamma, logistic, and exponential, the CDF is log-concave in general. These distributions are neither convex nor concave, and their CDF often exhibits a non-decreasing S-shaped curve, corresponding to the use of a so-called fragility curve to represent failure probabilities as a function of disturbance intensity (Kandaperumal and Srivastava, 2020). For such cases where disruptions follow a log-concave distribution, we develop a heuristic algorithm that starts with a good initial solution based on first-order optimality conditions and iteratively performs pairwise node fortification level updates. This approach is able to eliminate the computational burden of a global optimization solver that is frequently unable to provide an optimal solution within the allowed time limits as the problem size grows. The resulting algorithm is able to provide near-optimal solutions to problems with a general tree network structure in polynomial time under mild disturbance distribution assumptions.

The rest of this paper is organized as follows. Section 2 first discusses the related literature, after which Section 3 describes the problem setting, introduces our mathematical

model, and presents complexity results. Section 4 provides a polynomial-time solution algorithm for a special case: a two-level network under uniform distributions of disturbance severity at each node. Motivated by this effective method for solving this special case, we provide a heuristic solution algorithm for general tree structures and log-concave, S-shaped distribution functions in Section 5. Section 6 presents the results of a set of computational studies that demonstrate the advantages of our heuristic solution methods over a commercial global optimization solver. We provide concluding remarks and a discussion of future research directions in Section 7.

2 Literature Review

A related line of network resilience research examines the preparedness phase prior to any disruption, aiming to minimize the impact of disturbances through measures such as fortification, strengthening, and hardening, by employing a range of solution methodologies. As we later discuss, our model formulation corresponds to a multilinear programming problem (and a version of the product Knapsack problem). Therefore, this section reviews both the network resilience literature (Section 2.1) and related literature on multiplicative programming (Section 2.2).

2.1 Network Resilience

Although resilience is well studied in the context of commercial supply chains, work on disaster management in networks is relatively limited (Medel et al., 2020). Wang et al. (2022) develop a two-stage stochastic programming framework to determine discrete node protection levels for preparedness planning in interdependent infrastructure networks. Their model sets node and edge protection levels in the first stage, while the second-stage problem considers the resulting expected cost among a number of disruption scenarios based on the disruption extent and the flow capability in each network. The resulting problem is cast as a mixed-integer linear optimization problem that uses Monte Carlo simulation for scenario generation. He et al. (2025) model pre-flood closure-protection decisions for metro stations using a nonlinear 0-1 knapsack formulation and apply a greedy algorithm. Wang et al. (2024) use a progressive hedging algorithm to solve a two-stage stochastic

model for pre-hurricane protection strategies for electrical power networks. Popović et al. (2020) use a hybrid simulated annealing and mixed integer linear programming algorithm to enhance the resilience of power networks against windstorms. Ghorbani-Renani et al. (2025) study network fortification in the presence of a strategic attacker. In this paper, the authors consider a defender who chooses which links to protect, modeled by binary decision variables, while we allow partial fortification of network nodes.

Keşrit et al. (2025) consider a similar class of problems where all nodes are subject to a common disturbance severity distribution. The corresponding problem takes the form of a continuous, separable, nonlinear knapsack problem with precedence constraints, which they show is $\mathcal{NP}$ -Hard for general tree networks and general disturbance severity distributions. This paper provides a polynomial-time solution algorithm for serial systems, which forms the basis for an effective heuristic approach to problems on tree networks. Building on this work, we consider the more general case that permits a unique disturbance severity distribution at each node. This generalization is far from trivial, however, as the resulting problem is no longer separable in the decision variables, which creates structural complications.

While numerous studies address the improvement of network resilience and disaster preparedness through strengthening strategies under limited resources, the specific contexts and modeling assumptions employed differ substantially across the literature. In their literature review of critical infrastructure systems, Mottahedi et al. (2021) remark that simulation-based approaches are predominantly applied to these problems while emphasizing their limited use due to their extensive reliance on data. Tornatore et al. (2016) summarize past work on communication network resiliency methodologies against weather-based disruptions and discuss the various proposed heuristic approaches for network protection problems, a large number of which have been shown to be $\mathcal{NP}$ -Hard. Our work proposes an algorithm to fortify nodes in a general tree network against an uncertain disturbance severity level at each node, where both the characteristics of the underlying probability distributions and the network topology govern the problem’s computational complexity. In contrast to prior work, instead of considering network-based disruption scenarios, our model considers a disturbance probability distribution at each node, as well as an ability to fortify the node to withstand all disturbances below a given level.

2.2 Multiplicative Programming

Multiplicative programming problems (MPP) are non-convex global optimization problems where the objective function is expressed as a multiplication of affine functions of the decision variables and are, in their most general form, $\mathcal{NP}$ -Hard (Shao and Ehrgott, 2016). However, Charkhgard et al. (2018) show that when non-negativity constraints are added to the problem, it becomes a positive multi-linear program with affine constraints (PMP-A), which is solvable in polynomial time when the objective function contains no more than two variables. PMP-A can arise in a variety of areas such as game theory, polynomial programming, and statistical estimation (Charkhgard et al., 2018). When the objective function consists of a sum of several linear multiplicative functions, the problem becomes a generalized linear multiplicative program and is $\mathcal{NP}$ -Hard (Konno and Kuno, 1995).

A similar multi-linear program structure can often be found in optimization problems that aim to increase the reliability of a system. By modeling the redundancy allocation problem as a knapsack problem with a multi-linear objective function, Toppila and Salo (2017) show that for symmetric multi-linear functions, a unique solution exists. Chern (1992) proves that some special cases of redundancy allocation problems with integer decision variables as well as some related knapsack problems are $\mathcal{NP}$ -Hard, while D’ambrosio et al. (2018) show that product knapsack problems where the objective function is a product of profits are weakly $\mathcal{NP}$ -Hard. Thus, most solution methodologies for these problems involve heuristics where some equality- and inequality-based or prime-logarithmic linearization techniques are also used (Li et al., 2020).

As we later show, the special case in which each disturbance distribution is uniformly distributed leads to a class of multi-linear knapsack problems. For the two-level network special case, we provide a polynomial-time solution in Section 4. However, the more general class of knapsack problems we consider addresses more general probability distribution classes, resulting in multiplicative programming problems involving the multiplication of nonlinear functions. We provide an optimization-based heuristic solution approach for this new and challenging problem class.

3 Problem Definition and Model Properties

We first introduce the network fortification problem under node-based disruptions for tree networks, provide a general model formulation, and present key model properties in Section 3.1. The most general model version we present permits the fortification level for each node i to take any value in some set $\{0 \cup [a_i, b_i]\}$ (with $[0 \leq a_i < b_i \leq 1]$). That is, the fortification level at node i either equals zero or falls in the interval $[a_i, b_i]$. When the fortification level is zero, this implies that we forgo fortifying that node and all of its successors on the network; on the other hand, a fortification level between $[a_i, b_i]$ indicates a commitment to providing a positive level of strengthening for the node and its successors. Therefore, in the general version, we use a binary variable for each node i to capture when the fortification level falls in $[a_i, b_i]$ and when it is zero, where the binary variable equals 1 and 0, respectively. Section 3.2 then reformulates a special case of the problem in which the fortification level of each node i must fall in the interval $[a_i, b_i]$ (and can only equal zero if $a_i = 0$), eliminating the need for binary variables, and establishes corresponding complexity results.

3.1 Problem Formulation

We model the underlying distribution network as a commodity flow tree network $G(N, A)$, where each node $i \in N$ serves a given integer number of customers, w_i , utilizing a single commodity flow originating from a source node denoted as node 1. Flow is distributed to each node via a unique path formed by its predecessors as a result of the network's tree structure. Consequently, for node i to be able to serve its customers, every node on the path connecting node i to the source node (including node i itself) denoted by $\mathcal{P}(i)$, must function properly (i.e., each node in $\mathcal{P}(i)$ must not have failed).

We assume that each node is subject to a set of adverse and uncertain future events, and the maximum severity of these events at node $i \in N$, denoted by Y_i , can be quantified on a scale between 0 and 1, with 1 corresponding to the most severe, and 0 the least severe. Examples of severity measures include the highest flood level, earthquake intensity, or maximum wind speed. We further assume that each node i can be fortified to withstand a severity level up to x_i at a cost $c_i x_i$ for each $i \in N$, where c_i is the per-unit cost of fortification. The fortification level is defined on the same scale as the disruption level and,

therefore, takes a value in $[0, 1]$.

Therefore, a node fails if the severity of the worst event in a season at a node (Y_i) is greater than or equal to the node's fortification level, i.e., node i fails if $Y_i \geq x_i$. Then, the probability that service at node i is disrupted (i.e., that node i is without service) is expressed as:

$$P\left(\bigcup_{j \in \mathcal{P}(i)} \{Y_j \geq x_j\}\right).$$

This corresponds to the probability that the disruption severity exceeds the fortification level of any of the nodes on the path $\mathcal{P}(i)$. We initially assume that each node i is subject to a disturbance level in the interval $[a_i, b_i]$, and, as noted earlier, requires a fortification level either at 0 or in the interval $[a_i, b_i]$. Table 1 presents a summary of our notation.

Table 1: Mathematical model notation.

Parameters	
c_i	Per-unit fortification cost at node i
$\mathcal{P}(i)$	Set of nodes on the path from source to node i
$\pi(i)$	The immediate predecessor of node i
$\sigma(i)$	Set of immediate successors of node i
$\mathcal{L}$	Set of leaf nodes
w_i	Number of customers solely served at node i
B	Total fortification budget
n	Number of nodes, equal to $ N $
Decision Variables	
x_i	Fortification level of node i
z_i	Binary variable that enables $x_i = 0$ when set to zero

We wish to determine the node fortification levels (x_i values) under a limited budget B that minimize the expected number of customers without service after a disruption. We

formulate this problem as follows:

$$\text{Minimize } \sum_{i=1}^n P\left(\bigcup_{j \in \mathcal{P}(i)} \{Y_j \geq x_j\}\right) w_i \quad (1)$$

$$\text{Subject to: } \sum_{i=1}^n c_i x_i \leq B, \quad (2)$$

$$a_i z_i \leq x_i \leq b_i z_i, \quad \forall i \in N, \quad (3)$$

$$z_i \in \{0, 1\} \quad \forall i \in N. \quad (4)$$

The objective function (1) minimizes the expected number of customers with no service disruption, which is the sum of the probability that each node has service multiplied by the number of customers solely served by that node. Constraint (2) ensures that the total cost of fortification does not exceed the limited fortification budget, B . Constraints (3) and (4) ensure that the fortification level of each node takes either a value between the lower and upper bounds $[a_i, b_i]$ or is set to zero, for all $i \in N$. To eliminate trivial cases where all nodes can be fully fortified, i.e., $x_i = b_i$ for all $i \in N$, we assume that $B < \sum_{i=1}^n c_i b_i$.

Under independent disturbance distributions, the minimization objective function (1) can be written as

$$P\left(\bigcup_{j \in \mathcal{P}(i)} \{Y_j \geq x_j\}\right) = 1 - \prod_{j \in \mathcal{P}(i)} P(Y_j \leq x_j) = 1 - \prod_{j \in \mathcal{P}(i)} F_j(x_j) \quad \forall i \in N, \quad (5)$$

where $F_j(x_j) = P(Y_j \leq x_j)$ corresponds to the cumulative distribution function (CDF) of the severity distribution at node j for each $j \in N$. As a result, the objective function can be written as $\sum_{i=1}^n [1 - \prod_{j \in \mathcal{P}(i)} F_j(x_j)] w_i$. That is, minimizing the expected number of customers without service is equivalent to maximizing the expected number of customers without a service disruption. The mathematical model characterized by equations (1)-(4) can be rewritten in the following form for a distribution network with no cycles (which will be referred to as a general tree):

$$[KP_{\Pi Fz}] \quad \text{Maximize } \sum_{i=1}^n \left[\prod_{j \in \mathcal{P}(i)} F_j(x_j) \right] w_i \quad (6)$$

$$\text{Subject to: } \sum_{i=1}^n c_i x_i \leq B, \quad (7)$$

$$a_i z_i \leq x_i \leq b_i z_i, \quad \forall i \in N. \quad (8)$$

$$z_i \in \{0, 1\} \quad \forall i \in N. \quad (9)$$

When a node is fortified at a level equal to the minimal value of the disturbance distribution, the probability of such a node withstanding a disruption will equal zero while

still incurring a cost and using the limited budget inefficiently. In addition, service will be disrupted at all successors of this node regardless of their fortification levels. Thus, by using binary variables, the mathematical model allows the limited budget to be allocated to other nodes that can potentially have a greater contribution to the objective function. Here, z_i enables the model to choose not to fortify node i at all by setting it to zero. When set to 1, this binary variable ensures that a particular node is fortified to a level at least as high as its lower bound value. Choosing not to fortify a node would imply having a service disruption at its successor nodes; thus, we show that the following proposition holds (all proofs are presented in Appendix A).

Proposition 1. *In an optimal solution to the network fortification problem, $x_i > a_i$ whenever $x_j > a_j$ for all $i \in \mathcal{P}(j)$, where node i is a predecessor of node j , for any $c_i, i \in N$.*

This result implies that the following constraints are valid and can be added to the formulation to strengthen it

$$z_i \geq z_j, \quad \forall j, \forall i \in \mathcal{P}(j). \quad (10)$$

Solving $[KP_{\Pi Fz}]$, which is a mixed-integer nonlinear program, is computationally challenging. Therefore, to develop an efficient solution method for this problem, we first analyze a restriction of the problem with no binary variables. We later use the results from this analysis in Section 5 in our solution method development.

3.2 Restricted Problem Formulation and Complexity Results

In this section, we formulate a special case of $[KP_{\Pi Fz}]$ in which we consider fortifying all nodes in the network $\tilde{G}(\tilde{N}, \tilde{A})$, where $\tilde{N} \subseteq N$ and $\tilde{A} \subseteq A$ is the set of arcs connecting the nodes in $\tilde{N}$, and we forgo the nodes in $N \setminus \tilde{N}$. That is, for a given subset $\tilde{N}$, we ensure $x_i \geq a_i$ for $i \in \tilde{N}$, and $x_i = 0$ for all $i \in N \setminus \tilde{N}$. Hence, by definition, $z_i = 1$ for all $i \in \tilde{N}$, and $z_i = 0$ otherwise. Note that for this problem to be feasible, the subset $\tilde{N}$ must be selected such that $\sum_{i \in \tilde{N}} c_i a_i \leq B$. Moreover, the nodes in $\tilde{N}$ must form a connected subtree. Otherwise, at least one fortified node with an unfortified predecessor would exist, in which case the resources allocated to fortifying that node would not contribute to the objective function value, implying suboptimality.

We call the network fortification problem on a subset of nodes the restricted problem, which can be formulated as follows:

$$[KP_{\Pi F}] \quad \text{Maximize} \quad \sum_{i=1}^n \left[\prod_{j \in \mathcal{P}(i)} F_j(x_j) \right] w_i \quad (11)$$

$$\text{Subject to:} \quad \sum_{i=1}^n c_i x_i \leq B, \quad (12)$$

$$a_i \leq x_i \leq b_i, \quad \forall i \in \tilde{N}. \quad (13)$$

If each of the $F_i(x_i)$ functions in $[KP_{\Pi F}]$ is log-convex, then we obtain a convex objective function, implying an extreme point optimal solution (because the multiplication of log-convex functions is log-convex, and therefore also convex, and a weighted sum of convex functions is convex). However, for a number of broadly applicable distributions (e.g., uniform, normal, exponential, logistic, Weibull, Gamma, Beta, Pareto) the CDF is log-concave (Bagnoli and Bergstrom, 2005). Since log-concavity does not imply concavity or convexity, and the sum of log-concave functions is neither log-concave nor log-convex (Boyd and Vandenberghe, 2004), similar conclusions about necessary and sufficient optimality conditions cannot be derived for such distributions.

Many applicable probability distributions have a corresponding CDF, $F_i(x_i)$, that is neither convex nor concave in general, but is convex and nondecreasing for values less than the mode (assuming unimodality), which we denote as β , and concave and nondecreasing thereafter, i.e., $F_i(x_i)$ is convex for $x_i \leq \beta$ and is concave for $x_i \geq \beta$, where $a = 0 < \beta < b = 1$. Such functions are categorized as sigmoid functions and are often referred to as S-shaped or simply S-curves. For distributions whose multiplication yields an S-curve shaped CDF (such as the multiplication of uniform, which yields Beta (Evans and Rosenthal, 2009)), the objective function becomes a weighted sum of S-curve return functions. Ağralı and Geunes (2009) show that the continuous knapsack problem with S-curve return functions is $\mathcal{NP}$ -Hard when the return functions associated with different items are not identical. When the distribution function in problem $[KP_{\Pi F}]$ follows a discrete distribution, we are able to show that the following complexity result holds.

Proposition 2. *Problem $[KP_{\Pi F}]$ with discrete distribution functions is $\mathcal{NP}$ -Hard.*

Under continuous disturbance severity distributions, we can also show that the following proposition holds.

Proposition 3. *Problem $[KP_{\Pi F}]$ with continuous distribution functions is $\mathcal{NP}$ -Hard.*

Proposition 3 follows from Kesrit et al. (2025), who demonstrate the $\mathcal{NP}$ -Hardness for the special case in which $F_i(\cdot) = F_j(\cdot)$ for all pairs (i, j) , and where $a_i = 0$ and $b_i = 1$ for all $i \in N$. Because $[KP_{\Pi F}]$ is $\mathcal{NP}$ -Hard and is a special case of $[KP_{\Pi Fz}]$, this implies that $[KP_{\Pi Fz}]$ is also $\mathcal{NP}$ -Hard for both discrete and continuous distribution functions. Therefore, solving $[KP_{\Pi Fz}]$ is computationally challenging, and commercial solver capabilities for such problems are limited. However, a special case of the restricted problem allows for a structural characterization of the optimal solution. In the next section, we analyze this special case and use the resulting insights to develop a solution method in Section 5.

4 A Polynomially Solvable Special Case

This section considers a special case that permits a polynomial-time solution for the restricted problem $[KP_{\Pi F}]$. We consider a two-level network $\tilde{G}(\tilde{N}, \tilde{A})$ in which each node has the source node as its immediate predecessor under a uniform distribution function, where each node must be fortified at a value within its distribution bounds. In the following section, we characterize the optimal solution to this problem when the node disruption levels follow uniform distributions.

Analysis of Two-Level System under Uniform Distribution

Consider a two-level network $\tilde{G}(\tilde{N}, \tilde{A})$ with $n - 1$ leaf nodes, which constitute $n - 1$ different branches, each of which has the source (node 1) as its immediate predecessor. Hence, each branch connects to a single node, where node k corresponds to branch k for $k = 2, \dots, n$. If the disturbance distribution for each node i is uniform on $[a_i, b_i]$, then the problem we wish to solve is formulated as follows.

$$\text{Maximize } f(x) = w_1 \frac{x_1 - a_1}{b_1 - a_1} + \sum_{k=2}^n w_k \frac{x_1 - a_1}{b_1 - a_1} \frac{x_k - a_k}{b_k - a_k} \quad (14)$$

$$\text{Subject to: } \sum_{k=1}^n c_k x_k \leq B \quad (15)$$

$$a_k \leq x_k \leq b_k, \quad \forall k = 1, \dots, n. \quad (16)$$

The resulting problem has a bilinear objective function with linear constraints.

Proposition 4. *The function $f(x) = w_1 \frac{x_1 - a_1}{b_1 - a_1} + \sum_{k=2}^n w_k \frac{x_1 - a_1}{b_1 - a_1} \frac{x_k - a_k}{b_k - a_k}$ for $n \geq 2$ is a convex function.*

By Proposition 4, the objective (14) maximizes a convex function, implying an extreme point optimal solution, and that the Karush-Kuhn-Tucker (KKT) conditions (see, e.g., Boyd and Vandenberghe, 2004) are necessary for optimality (although not sufficient). Consider the corresponding KKT conditions, where α corresponds to a KKT multiplier associated with the budget constraint (15), and λ_k^a and λ_k^b denote, respectively, KKT multipliers associated with the lower and upper bounding constraints (16) for each node $k \in \tilde{N}$.

$$\left. \begin{aligned}
 & w_k \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)} = \alpha c_k + \lambda_k^b - \lambda_k^a, \quad k = 2, \dots, n, \\
 & \frac{w_1}{b_1 - a_1} + \sum_{k=2}^n w_k \frac{x_k - a_k}{(b_1 - a_1)(b_k - a_k)} = \alpha c_1 + \lambda_1^b - \lambda_1^a \\
 & \lambda_k^b (b_k - x_k) = 0, \quad k = 1, \dots, n, \\
 & \lambda_k^a (x_k - a_k) = 0, \quad k = 1, \dots, n \\
 & \sum_{k=1}^n c_k x_k \leq B, \\
 & a_k \leq x_k \leq b_k, \quad k = 1, \dots, n \\
 & \lambda_k^a, \lambda_k^b \geq 0, \quad k = 1, \dots, n, \\
 & \alpha \geq 0
 \end{aligned} \right\} \begin{array}{l} \text{(Dual feasibility)} \\ \text{(Complementary slackness)} \\ \text{(Primal feasibility)} \\ \text{(Dual feasibility)} \end{array}$$

Observe that these KKT conditions imply that if $x_1 = b_1$, then $\lambda_1^a = 0$ and

$$\alpha \leq \frac{w_1}{c_1(b_1 - a_1)} + \sum_{k=2}^n \frac{w_k}{c_1} \frac{x_k - a_k}{(b_1 - a_1)(b_k - a_k)}.$$

In contrast, if $x_1 = a_1$, then $\lambda_1^b = 0$ and

$$\alpha \geq \frac{w_1}{c_1(b_1 - a_1)} + \sum_{k=2}^n \frac{w_k}{c_1} \frac{x_k - a_k}{(b_1 - a_1)(b_k - a_k)}.$$

Similarly, if $x_k = b_k$ for some $k = 2, \dots, n$, then the KKT conditions imply $\lambda_k^a = 0$ and

$$\alpha \leq \frac{w_k}{c_k} \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)},$$

while if $x_k = a_k$ for some $k = 2, \dots, n$, then it is implied that $\lambda_k^b = 0$ and

$$\alpha \geq \frac{w_k}{c_k} \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)}.$$

When a node's fortification level is strictly between its upper and lower bounds, we call the node fractional. Suppose x_k is fractional for some $k \neq 1$. Then this implies that $\alpha = \frac{w_k}{c_k} \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)}$. We can determine which of the other x_k values must be at their upper bounds and which must be at their lower bounds at a corresponding KKT point by ranking the values of $\frac{w_k}{c_k} \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)}$ in nonincreasing order. When this value is larger than α , then dual feasibility implies that λ_k^a is positive, meaning that for such nodes we have $x_k = a_k$. On the other hand, when this value is smaller than α , this implies that λ_k^a is positive, meaning that $x_k = b_k$.

Note that if $x_1 = a_1$, then the objective function value is zero. Assuming this is not optimal, we must have $a_1 < x_1 \leq b_1$. For any given $x_1 > a_1$, we can obtain a feasible solution that satisfies the KKT conditions by filling the knapsack, i.e., using the entire budget via node fortification, in nonincreasing order of $\frac{w_k}{c_k} \frac{x_1 - a_1}{(b_1 - a_1)(b_k - a_k)}$, which is equivalent to nonincreasing order of $\frac{w_k}{c_k(b_k - a_k)}$. Such a solution is only an extreme point if there is at most one fractional x_k value. Thus, for example, if the given x_1 is fractional, then the remaining items that fill the knapsack must be at their upper or lower bounds. As a result, we have two types of candidate solutions.

Solution with $\mathbf{x}_1 = \mathbf{b}_1$: In the first solution, the source node is fully fortified. We set $x_1 = b_1$ and solve the remaining continuous knapsack problem by allocating the budget to items in nonincreasing order of $\frac{w_k}{c_k(b_k - a_k)}$ until the budget is exhausted. Suppose we sort items 2 through n in nonincreasing order of $\frac{w_k}{c_k(b_k - a_k)}$ and let item $\hat{j}$ be the threshold item such that $c_1 b_1 + \sum_{j=2}^{\hat{j}} c_j b_j + \sum_{j=\hat{j}+1}^n c_j a_j \geq B$ and $c_1 b_1 + \sum_{j=2}^{\hat{j}-1} c_j b_j + \sum_{j=\hat{j}}^n c_j a_j < B$. Then we set $x_j = b_j$ for $j = 1, \dots, \hat{j} - 1$ since they are fully fortified, and allocate the remaining budget to node $\hat{j}$, i.e., $x_{\hat{j}} = \frac{B - c_1 b_1 - \sum_{j=2}^{\hat{j}-1} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_{\hat{j}}}$ for fractional fortification.

Because $x_{\hat{j}}$ is fractional and $x_1 = b_1$ in the first solution, we must have $\alpha = \frac{w_{\hat{j}}}{c_{\hat{j}}(b_{\hat{j}} - a_{\hat{j}})}$ by dual feasibility. In this case, for the solution to be a KKT point, we must have $\alpha \leq \frac{w_j}{c_j(b_j - a_j)}$ for all j with $x_j = b_j$ since $\lambda_k^b \geq 0$, and $\alpha \geq \frac{w_j}{c_j(b_j - a_j)}$ for all j with $x_j = a_j$ since $\lambda_k^a \geq 0$, which must hold by our rank order and choice of $\hat{j}$. We must also have

$\alpha \leq \frac{w_1}{c_1(b_1-a_1)} + \sum_{k=2}^n \frac{w_k}{c_1} \frac{x_k-a_k}{(b_1-a_1)(b_k-a_k)}$ because $\lambda_1^b \geq 0$. Note that

$$\begin{aligned} & \frac{w_1}{c_1(b_1-a_1)} + \sum_{k=2}^n \frac{w_k}{c_1} \frac{x_k-a_k}{(b_1-a_1)(b_k-a_k)} \\ &= \frac{w_1}{c_1(b_1-a_1)} + \sum_{k=2}^{\hat{j}-1} \frac{w_k b_k}{c_1(b_1-a_1)} + w_{\hat{j}} \frac{\frac{B-c_1 b_1 - \sum_{j=2}^{\hat{j}-1} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_{\hat{j}}} - a_{\hat{j}}}{c_1(b_1-a_1)(b_{\hat{j}}-a_{\hat{j}})}. \end{aligned}$$

This is greater than or equal to α if

$$\frac{w_1}{c_1(b_1-a_1)} + \sum_{k=2}^{\hat{j}-1} \frac{w_k b_k}{c_1(b_1-a_1)} + w_{\hat{j}} \frac{\frac{B-c_1 b_1 - \sum_{j=2}^{\hat{j}-1} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_{\hat{j}}} - a_{\hat{j}}}{c_1(b_1-a_1)(b_{\hat{j}}-a_{\hat{j}})} \geq \frac{w_{\hat{j}}}{c_{\hat{j}}(b_{\hat{j}}-a_{\hat{j}})},$$

because $\frac{w_{\hat{j}}}{c_{\hat{j}}(b_{\hat{j}}-a_{\hat{j}})} = \alpha$. Thus, if this condition is satisfied, then the corresponding point is both a KKT point and an extreme point.

Solution with $\mathbf{x}_1 < \mathbf{b}_1$: In the second type of solution, the source is partially fortified. Hence, x_1 is fractional, and all remaining items are at their upper or lower bounds, since there can be at most one fractional x_i at a KKT point. Suppose we sort items $k = 2, \dots, n$ in nonincreasing order of $\frac{w_k}{c_k(b_k-a_k)}$. Let item $\hat{j}$ be an index such that $\sum_{j=2}^{\hat{j}} c_j b_j + \sum_{j=\hat{j}+1}^n c_j a_j \leq B$, and $c_1 b_1 + \sum_{j=2}^{\hat{j}} c_j b_j + \sum_{j=\hat{j}+1}^n c_j a_j > B$. That is, $\hat{j}$ is the highest node index in the sorted list that can be fully fortified. Then a candidate KKT point sets $x_j = b_j$ for $j = 2, \dots, \hat{j}$ and $x_1 = \frac{B - \sum_{j=2}^{\hat{j}} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_1}$. In this case, x_1 is fractional, implying that $\alpha = \frac{w_1}{c_1(b_1-a_1)} + \sum_{j=2}^{\hat{j}} \frac{w_j}{c_1} \frac{x_j-a_j}{(b_1-a_1)(b_j-a_j)}$. In order to satisfy all KKT conditions, we require $\alpha \leq \frac{w_j}{c_j} \frac{x_1-a_1}{(b_1-a_1)(b_j-a_j)}$ for all $j = 2, \dots, \hat{j}$, which is equivalent to $w_1 + \sum_{k=2}^{\hat{j}} \frac{w_k(x_k-a_k)}{(b_k-a_k)} \leq \frac{w_j}{c_j} \frac{B - \sum_{k=2}^{\hat{j}} c_k b_k - \sum_{k=\hat{j}+1}^n c_k a_k - a_1 c_1}{(b_j-a_j)}$ for $j = 2, \dots, \hat{j}$. We also require $\alpha \geq \frac{w_j}{c_j} \frac{x_1-a_1}{(b_1-a_1)(b_j-a_j)}$ for all $j = \hat{j}+1, \dots, n$, which is equivalent to $w_1 + \sum_{k=2}^{\hat{j}} \frac{w_k(x_k-a_k)}{(b_k-a_k)} \geq \frac{w_j}{c_j} \frac{B - \sum_{k=2}^{\hat{j}} c_k b_k - \sum_{k=\hat{j}+1}^n c_k a_k - a_1 c_1}{(b_j-a_j)}$ for $j = \hat{j}+1, \dots, n$. If these conditions hold, then the point is both an extreme point and a KKT point, and is therefore a candidate for an optimal solution.

We summarize the steps for finding the two types of candidate solutions for a two-level system under uniform disturbance distributions in Algorithm 2 in Appendix B. We later use the conditions for identifying KKT points in our solution method development. Whenever reducing the tree network to a two-level system subproblem is possible, wherein the fortification problem is convex, identifying KKT points using this approach provides local improvements.

5 Solution Methodology

As the network is subject to node-based disruptions, each node's susceptibility to these adverse events differs. Therefore, there is no clear relationship between the fortification levels of predecessor-successor nodes in the network, aside from the fact that for a node to function, all of its predecessors must survive the disruption. Therefore, in an effort to develop a heuristic approach, we first examine the characteristics of optimal solutions for some special cases. Then, using these results, we develop a two-phase solution method for the $[KP_{\Pi F_z}]$ problem when disruptions follow a log-concave distribution. In the first phase, by iteratively evaluating each node's relative contribution to the objective function, we obtain an initial solution. Then, in the second phase, using a mathematical model-based heuristic pairwise search algorithm, we iteratively improve upon the current solution at each step.

5.1 Phase I: Obtaining a Good Initial Solution

The relative 'importance' of a node in a given solution can be determined based on its contribution to the objective function and the fortification levels of the other nodes in the network. Let $h(x)$ denote the objective function of the problem $[KP_{\Pi F}]$ and $S(i)$ denote the set of successors of node i , including node i . The derivative of the objective function with respect to the fortification level of node i is expressed as follows

$$\frac{\partial h(x)}{\partial x_i} = f_i(x_i) \sum_{k \in S(i)} \left[\prod_{j \in \mathcal{P}(k) \setminus \{i\}} F_j(x_j) \right] w_k. \quad (17)$$

If we were to start with a solution such that $x_i = b_i$ for all i , then at this solution the above would be

$$\frac{\partial h(x)}{\partial x_i} = f_i(b_i) \sum_{k \in S(i)} w_k. \quad (18)$$

Note that this solution is infeasible because it violates the budget constraint. Therefore, the fortification levels need to be reduced. The negative of the derivative in Equation (18) is the rate of change in the objective function for a one-unit decrease in x_i . The rate of decrease in the objective function per unit decrease in capacity consumption for node i is then

$$\rho_i = \frac{f_i(b_i) \sum_{k \in S(i)} w_k}{c_i}. \quad (19)$$

This metric quantifies the impact of reducing the fortification levels of the nodes on the objective function value and consequently identifies the critical nodes. Thus, assuming that we start with a fully fortified network, reducing the values of the x_i variables in nondecreasing order of ρ_i until reaching budget feasibility yields a good initial solution that is based on first-order conditions. We discuss the application of this approach to specific distributions in Appendix C.

5.2 Phase II: Local Search via Pairwise Budget Exchange

To improve the initial solution, we develop a search algorithm that iteratively considers local improvements for pairs of nodes in the network while keeping the fortification levels of remaining nodes constant. By determining the values for one pair of nodes at a time, we view the resulting subproblems as two-level systems by updating node weights and use Section 4 findings for solving these. Depending on whether the node pair is a predecessor-successor pair or not, the structure of the corresponding mathematical problem differs. For a predecessor-successor pair, the resulting subproblem resembles a two-level system. Conversely, for a node pair that is not a predecessor-successor pair, their respective distribution-related terms appear separately in the objective function. Accordingly, for every (i, j) node pair, given an existing solution $\mathbf{x}' = (x'_1, x'_2, \dots, x'_n)$, we solve the following mathematical program:

$$[P_1] \quad \underset{x_i, x_j}{\text{Maximize}} \quad w'_i F_i(x_i) + w'_j I_{ij}(x_i) F_j(x_j) \quad (20)$$

$$\text{Subject to:} \quad c_i x_i + c_j x_j \leq B'_{i,j}, \quad (21)$$

$$a_k \leq x_k \leq b_k, \quad k \in \{i, j\}. \quad (22)$$

where $B'_{i,j} = \min\{c_i x'_i + c_j x'_j + B - \sum_{k \in N} c_k x'_k, c_i b_i + c_j b_j\}$ is the current allocated budget for the node pair (i, j) , and the w'_i values are updated heuristic node weights that determine the impact of the fixed fortification levels on the objective function, computed for the nodes i and j as follows:

$$w'_i = w_i \left[\prod_{l \in \mathcal{P}(i) \setminus \{i\}} F_l(x'_l) \right] + \sum_{k: i \in \mathcal{P}(k), j \notin \mathcal{P}(k)} \left[\prod_{l \in \mathcal{P}(k)} F_l(x'_l) \right] w_k, \quad (23)$$

$$w'_j = \sum_{k: \{i, j\} \in \mathcal{P}(k)} \left[\prod_{l \in \mathcal{P}(k)} F_l(x'_l) \right] w_k. \quad (24)$$

The indicator function $I_{ij}(x_i)$ determines the form of the objective function based on whether the nodes are a predecessor-successor pair or not, that is:

$$I_{ij}(x_i) = \begin{cases} F_i(x_i), & \text{if } i \in \mathcal{P}(j), \\ 1, & \text{otherwise.} \end{cases} \quad (25)$$

Assuming that all other node fortification levels remain fixed (as in the current solution, x'), problem $[P_1]$ maximizes the portions of the objective function of problem $[KP_{\Pi F}]$ that are affected by the changes in x_i and x_j . Similar to $[KP_{\Pi F}]$, formulation $[P_1]$ also ensures that nodes are fortified to a value at least as high as their lower bounds and, therefore, lacks the ability to set a fortification level of zero. To incorporate this flexibility while maintaining convexity, we formulate two additional versions of the subproblem for each node pair (i, j) , namely, $[P_2]$ and $[P_3]$. These problems are identical to $[P_1]$ except that the constraint (22) is modified with $x_i = 0, F_i(x_i) = 0$ for $[P_2]$ and $x_j = 0, F_j(x_j) = 0$ for $[P_3]$. When solved for every node pair in the network, problems $[P_1]$, $[P_2]$, and $[P_3]$ allow local improvements of solutions for $[KP_{\Pi Fz}]$ (6)-(9).

5.2.1 Predecessor-Successor Pair

For every (i, j) pair where node i is a predecessor of node j , given an existing solution, the following problem is equivalent to $[P_1]$, where $I_{ij}(x_i) = F_i(x_i)$, by the order-preserving property of the logarithmic function:

$$\begin{aligned} [P_1^{i < j}] \quad & \underset{x_i, x_j}{\text{Maximize}} \quad \log(w'_i F_i(x_i) + w'_j F_i(x_i) F_j(x_j)) \\ & \text{Subject to:} \quad (21) - (22). \end{aligned} \quad (26)$$

Using the logarithm allows us to solve this subproblem efficiently under certain conditions. We show (see Appendix A.4) that for a number of broadly applicable distributions with log-concave cumulative distribution functions, the following holds.

Proposition 5. *The objective function of problem $[P_1^{i < j}]$ is concave for log-concave cumulative distribution functions, $F_i(x_i)$ and $F_j(x_j)$.*

For distributions with log-concave CDFs (e.g., uniform, normal, exponential, logistic, Weibull, Gamma, Beta, Pareto), the problem $[P_1^{i < j}]$ becomes a convex program and

therefore the KKT conditions, provided in Appendix D, are necessary and sufficient for optimality.

To extend this search to pairs of nodes that are not predecessor-successor pairs, we formulate a different mathematical program that again represents the affected portions of the objective function of problem $[KP_{\Pi F}]$.

5.2.2 Non-Predecessor-Successor Pair

For every (i, j) pair where node i is not a predecessor of node j , given an existing solution, we reformulate the mathematical program based on $[P_1]$, where $I_{ij}(x_i) = 1$. This reformulation considers a weighted sum of CDFs of the pair, so the log transformation of the objective function does not provide a simpler form as before and therefore is not utilized.

Note that for convex or log-convex distribution functions, the objective function is convex, which implies an extreme-point optimal solution. However, for distributions with log-concave CDFs, neither the convexity nor the concavity of the objective function is guaranteed.

Maximizing $w'_i F_i(x_i) + w'_j F_j(x_j)$ is equivalent to minimizing the inverted version of the same expression. However, since again, neither the convexity nor the concavity of $\frac{1}{w'_i F_i(x_i) + w'_j F_j(x_j)}$ is guaranteed, we replace this function with $\frac{1}{w'_i F_i(x_i)} + \frac{1}{w'_j F_j(x_j)}$, which provides an upper bound for the expression. Therefore, we reformulate a relaxation of the problem as follows:

$$\begin{aligned} [P_1^{i*j}] \quad & \text{Minimize} \quad \frac{1}{w'_i F_i(x_i)} + \frac{1}{w'_j F_j(x_j)} \\ & \text{Subject to:} \quad (21) - (22). \end{aligned} \tag{27}$$

When $F_i(x_i)$ and $F_j(x_j)$ are log-concave, their reciprocals are log-convex, which are convex in $\mathbb{R}$ (Boyd and Vandenberghe, 2004), and since their sum is also convex, problem $[P_1^{i*j}]$ yields a convex program that can again be solved efficiently, and the KKT conditions are necessary and sufficient for optimality.

Our solution method iteratively solves the corresponding mathematical model for pairs of nodes to improve the current solution. We summarize the steps of this search in Algorithm 1. In summary, instead of viewing the system as a complicated tree network, we break it into pairs of nodes that resemble the polynomially solvable two-level system presented

Algorithm 1 Mathematical Model-Based Search Algorithm (MMBSA)

Start with an initial solution x'

for $i \in N$ **do**

for $j \in N \setminus \{i\}$ **do**

if $i \in \mathcal{P}(j)$ **then**

 Identify the extreme (KKT) points of $[P_1^{i<j}], [P_2^{i<j}], [P_3^{i<j}]$ for (i, j) ,

 Choose the best candidate solution

 Update x'_i and x'_j

else

 Identify the extreme (KKT) points of $[P_1^{i*j}], [P_2^{i*j}], [P_3^{i*j}]$ for (i, j) ,

 Choose the best candidate solution

 Update x'_i and x'_j

end if

end for

end for

in Section 4 and replace the objective function with either an equivalent form or an upper bound that yields a convex program for distributions with log-concave CDFs. Assuming the fortification levels of all other nodes in the network have fixed values, we solve the smaller convex subproblem for each pair of nodes by identifying extreme points (which constitute a subset of all KKT points of the convex program). This approach enables making local improvements in the search space, starting from an initial solution that is based on first-order conditions. We call this method the Mathematical Model Based Search Algorithm (MMBSA).

6 Computational Study

In this section, we compare the MMBSA heuristic results with the best available solution from a commercial global optimization solver. The mathematical model defined by Equations (6)-(9) was implemented in AMPL IDE (Version 4.0.1.202411072004), which used the BARON global optimization solver to obtain the best available solution for all randomly generated instances in the computational study within the time limit of 1 hour per problem instance. The MMBSA was coded using RStudio (Version 2022.07.2), which was also uti-

Table 2: Computational Study Parameters

Parameter	Value
Number of nodes	$n \in \{15, 25, 50\}$
Node fortification costs	$c_i \sim \text{Uniform}[1, 10]$
Node weights	$w_i \sim \text{Uniform}[1, 10]$
Budget	$B \in \{\frac{1}{5} \sum_{i=1}^n c_i, \frac{3}{4} \sum_{i=1}^n c_i\}$
Network depth	$\{1, 5\}, \{2, 10\}, \{5, 20\}$
Disruption severity	$y_i \sim \text{Uniform}(a_i, b_i),$ $y_i \sim \text{Logistic}(\mu_i, s_i),$ $y_i \sim \text{Exponential}(\lambda_i)$
Distribution parameters	$a_i \sim \text{Uniform}[0, 0.49], b_i \sim \text{Uniform}[0.5, 1],$ $\mu_i \sim \text{Uniform}[0.5, 0.55], s_i \sim \text{Uniform}[0.01, 0.1],$ $\lambda_i \sim \text{Uniform}[5, 10]$

lized to generate the random general tree instances and to obtain the initial solution based on first-order conditions. All elements of the computational study were carried out under identical hardware and software specifications (Windows 10, 12th Gen Intel(R) Core(TM) i7-12700 2.10 GHz, 16.0 GB RAM, 64-bit operating system, x64-based processor).

The computational experiments examine MMBSA performance across a variety of factors. Table 2 describes the computational setting for randomly generated problem instances where the severity of the disruption follows a uniform, logistic, and exponential distribution, respectively, for a general tree network. These distributions exemplify log-concave CDFs, with the exponential placing most weight at low disturbance levels, the likelihood decreasing with disturbance strength, and the logistic distribution used in reliability applications to measure the probability of failure (Barbin et al., 2026). The available budget for fortification is varied as a function of the sum of fortification costs to test the algorithm’s performance under low and high budget availability. Similarly, we test randomly generated networks with low and high network depth (maximum number of nodes between the source and a leaf node) to observe how network structure affects the solution and method performance, where greater depth often indicates fewer branches from the source. Specifically, we use depth 1 and 5 when $n = 15$, 2 and 10 when $n = 25$, and 5 and 20 when $n = 50$. We

choose the distribution parameters for the logistic and exponential distributions to ensure that 99% of the values lie between 0 and 1.

Table 3 presents the average computation times of the global optimizer (BARON) and our method (MMBSA) for randomly generated tree networks with different numbers of nodes. For both approaches, average computation time increases as the number of nodes grows. While the average computation time remains under 2 minutes for MMBSA, the global optimizer hits the time limit as the number of nodes increases for problems where the severity of the disruption follows a non-trivial log-concave distribution. The global optimizer could not find an optimal solution within the given time limit for randomly generated general tree networks with 15 nodes in approximately 85.8% and 87.5% of cases, and with 25 nodes in approximately 96% and 93% of cases for the logistic and exponential distributions, respectively. When the number of nodes in the network increases to 50, the solver reaches the time limit of an hour in all instances of the logistic and exponential distributions and is unable to find an optimal solution within the given time limit. Therefore, as the number of nodes increases, the MMBSA approach remains the only computationally feasible method given the computation time limits.

Table 3: Average computation times for randomly generated general tree networks for MMBSA and BARON.

		Disruption Distribution					
		Uniform		Logistic		Exponential	
n	Approach	Time (sec)	$\Delta \bar{z}\%$	Time (sec)	$\Delta \bar{z}\%$	Time (sec)	$\Delta \bar{z}\%$
15	BARON	0.08	1.1%	3090.34	7.9%	3179.45	2.8%
	MMBSA	0.49		0.66		0.62	
25	BARON	0.14	1.2%	3471.75	7.3%	3360.01	0.0%
	MMBSA	2.29		2.71		2.63	
50	BARON	1.17	2.2%	3600	0.8%	3600	0.1%
	MMBSA	23.22		75.69		25.71	

Table 4 presents the average percentage difference between the objective function values of the MMSBA and the best available solution provided by BARON for a variety of budget and network depth settings, where the severity of the disruption follows a uniform, logistic,

and exponential distribution, respectively. The average percentage difference is calculated as $\Delta\bar{z}\% = 100 \times \frac{z_{\text{BARON}} - z_{\text{MMBSA}}}{z_{\text{BARON}}}\%$, where z_{BARON} and z_{MMBSA} correspond to the objective function values of the best available solution provided by the global optimizer (BARON) and the MMBSA, respectively. The percentage differences reported in Table 4 represent the average value among 30 instances of a randomly generated network in the form of a general tree, using a computation time limit of 1 hour for each instance.

Table 4: The average percentage optimality gap ($\Delta\bar{z}\%$) of MMSBA compared to BARON for randomly generated general tree networks.

		Uniform		Logistic		Exponential	
n	Network	Budget		Budget		Budget	
	Depth	<i>Low</i>	<i>High</i>	<i>Low</i>	<i>High</i>	<i>Low</i>	<i>High</i>
15	<i>Low</i>	0.3%	0.2%	11.6%	4.2%	0.1%	11.0%
	<i>High</i>	3.1%	0.7%	10.7%	4.9%	0.1%	0.0%
25	<i>Low</i>	0.7%	0.2%	10.1%	3.3%	0.0%	0.0%
	<i>High</i>	2.9%	1.0%	10.5%	5.2%	0.0%	0.0%
50	<i>Low</i>	2.3%	0.4%	4.3%	2.8%	0.0%	0.0%
	<i>High</i>	5.1%	0.8%	-3.8%	-0.1%	0.3%	0.0%

Table 4 shows that as the budget increases, the gap between the MMBSA and solver solution decreases for all distributions. Because its objective function is multi-linear, the uniform distribution case provides an easily solvable mathematical model; therefore, the global optimizer is able to provide the optimal solution within seconds, and the performance gap between BARON and the MMBSA increases as the number of nodes increases. In contrast, for the more broadly applicable and computationally challenging logistic and exponential disruption distributions, BARON reaches the allowed time limit even for relatively small instances, while MMBSA remains competitive in terms of both solution quality and computation time. Under the exponential distribution, MMBSA closely matches BARON in nearly all cases with a maximum gap of 0.3%, except for the $n = 15$, low-depth, high-budget case, which has an 11.0% gap. Although the gaps under the logistic distribution are larger than those under the exponential distribution, they decrease substantially as the network size grows and the budget increases. For $n = 50$ with high network depth, MMBSA

achieves negative gaps of -3.8% and -0.1% for the low- and high-budget cases, respectively, indicating that it finds better solutions than BARON. Overall, MMBSA provides a strong alternative to the commercial solver, especially when the solver’s computation times become prohibitively large.

7 Conclusion

We consider a distribution network that delivers a natural resource, service, or physical good to a collection of nodes, each serving its own group of customers, where flow disruptions may arise at one or more nodes. We assume that all network nodes are subject to a disturbance severity distribution whose impact can be different at each node, which can be quantified and follows some well-defined probability distribution. For each node in the acyclic network, the aim is to determine a fortification level that allows the node to endure disturbances up to a certain severity threshold, subject to a constrained fortification budget. The resulting optimization problem seeks to minimize the expected number of customers who experience service interruptions, or equivalently, to maximize the expected number of customers whose service remains uninterrupted.

We formulate this problem as a continuous, nonlinear multiplicative programming problem and although we show that the problem on a general tree is $\mathcal{NP}$ -Hard, we characterize properties of optimal solutions that enable solving the problem for two-level systems in polynomial time under uniformly distributed disruptions. The difficulty of the resulting optimization problem is influenced by both the underlying network structure and the associated distribution of disturbance severity. If, for example, the CDF of the disturbance distribution is log-convex, then the resulting problem maximizes a convex objective, which implies an extreme point solution. Unfortunately, only a small subset of continuous probability distributions have log-convex cumulative distribution functions, and restricting the analysis to this class significantly limits the practical applicability of the model. For most commonly applied continuous distributions of natural phenomena, such as normal, beta, gamma, logistic, and exponential, the CDF is log-concave in general. These distributions are neither convex nor concave, and their CDF often exhibits a non-decreasing S-shaped curve. For such cases, we develop a heuristic solution method, called the Mathematical

Model Based Search Algorithm, based on first-order conditions and the KKT conditions of subproblems of node pairs in the general tree networks. We computationally test its performance against a commercial global optimization solver on a variety of randomly generated instances under different disruption distribution assumptions and show that the overall average percentage difference is less than 3% with an average computational time under 2 minutes as the network size increases, while the commercial solver exceeds the 1 hour time limit. We note that including multiple source nodes and cycles in the distribution network represent potentially valuable future directions for research, which can help extend the applicability of the findings to more general systems.

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A Proofs of Propositions

A.1 Proof of Proposition 1

Proof. Assume to the contrary that there exist nodes i and j such that i is a predecessor of j and $x_j > a_j$ while $x_i \leq a_i$ in an optimal solution to the problem. Let s_i and s_j denote the sets of successors of nodes i and j , respectively, including the nodes themselves. Then, $|s_i| > |s_j|$ as $s_j \subset s_i$. For any $x_i < a_i$, it follows that $F_i(x_i) = 0$ and therefore, $\sum_{k \in s_i} [\prod_{l \in P(k)} F_l(x_l)] w_k = 0$.

Without loss of generality, we can denote $x_j = a_j + \delta_j$ where $\delta_j \geq 0$. By setting $x_j = 0$, we obtain $F_j(x_j) = 0$ which does not impact the objective function as $\sum_{k \in s_i} [\prod_{l \in P(k)} F_l(x_l)] w_k = 0$ already and $s_j \subset s_i$. Using the excess budget made available by setting $x_j = 0$, we increase x_i by δ_i and denote it as $\tilde{x}_i$. Doing so can result in one of the following two cases. If $\tilde{x}_i \leq a_i$, then $F_i(\tilde{x}_i) = 0$ the objective function remains the same. However, if $\tilde{x}_i > a_i$, it follows that $F_i(\tilde{x}_i) > 0$ and thus, the objective function is increased by $\sum_{k \in s_i \setminus s_j} [\prod_{l \in P(k)} F_l(x_l)] w_k$.

We observe that moving the excess budget obtained by setting $x_j = 0$ whenever $x_j > a_j$ while $x_i \leq a_i$, yields a feasible solution that is no worse than the current solution and even potentially a better objective function. This creates a contradiction with the assumption that there exists a predecessor node that is fortified to a level less than or equal to its lower bound while its successor has a fortification level greater than its lower bound in an optimal solution. $\square$

A.2 Proof of Proposition 2

Proof. Consider the general distribution version of problem $[KP_{\Pi F}]$, where $F_j(x_j)$ is the probability of no failure at node j . We formulate this as

$$[P] \quad \text{Max} \sum_{i=1}^n \left[\prod_{j \in \mathcal{P}(i)} F_j(x_j) \right] w_i \quad (28)$$

$$\text{s.t.} \quad \sum_{j=1}^n c_j x_j \leq B \quad (29)$$

$$x_j \in [a_j, b_j], \quad j = 1, \dots, n. \quad (30)$$

Suppose that $F_j(x_j)$ follows a discrete distribution defined at the points $a_j \leq x_j^1 < x_j^2 < \dots < x_j^{p_j} \leq b_j$, where $F_j(x_j^k) = \rho_j^k$ for $x_j^k \leq x_j < x_j^{k+1}$. It is straightforward to show that an optimal solution must exist such that $x_j = x_j^k$ for some $k = 1, \dots, p_j$. As a result, the problem becomes one of choosing from a discrete set of fortification levels for each node. Defining v_j^k as a binary variable equal to one if $x_j = x_j^k$ and zero otherwise, this implies that $x_j = \sum_{k=1}^{p_j} x_j^k v_j^k$, where we require $\sum_{k=1}^{p_j} v_j^k = 1$. Next, consider the special case in which $w_1 = w_2 = \dots = w_{N-1} = 0$, such that solving our problem is equivalent to maximizing $\prod_{j \in \mathcal{P}(N)} F_j(x_j)$. If we consider a serial system, then this becomes $\prod_{j=1}^N F_j(x_j)$. Maximizing this objective function can be accomplished by maximizing its logarithm, i.e., by maximizing $\sum_{i=1}^N \log F_j(x_j)$. If we choose the k^{th} point of the discrete distribution for node j , then the corresponding objective function term is $\log \rho_j^k$. Solving this special case of our problem is then equivalent to solving the following problem:

$$[S] \quad \text{Max} \sum_{j=1}^n \sum_{k=1}^{p_j} (\log \rho_j^k) v_j^k \quad (31)$$

$$\text{s.t.} \quad \sum_{j=1}^n \sum_{k=1}^{p_j} x_j^k v_j^k \leq B \quad (32)$$

$$\sum_{k=1}^{p_j} v_j^k = 1 \quad j = 1, \dots, n \quad (33)$$

$$v_j^k \in \{0, 1\} \quad j = 1, \dots, n, k = 1, \dots, p_j \quad (34)$$

This is an instance of the multiple-choice knapsack problem, which is $\mathcal{NP}$ -hard (Kellerer et al., 2004). As a result, our problem is $\mathcal{NP}$ -hard even when $w_1 = w_2 = \dots = w_{N-1} = 0$ and all costs are linear. $\square$

A.3 Proof of Proposition 4

Proof.

$$\frac{\partial f(x)}{\partial x_1} = \sum_{i=1}^n w_i \frac{x_2^{i-1}}{\prod_{j=1}^i b_j}, \quad \frac{\partial f(x)}{\partial x_2} = \sum_{i=2}^n w_i \frac{(i-1)x_1 x_2^{i-2}}{\prod_{j=1}^i b_j} \quad (35)$$

$$\frac{\partial^2 f(x)}{\partial x_1^2} = 0, \quad \frac{\partial^2 f(x)}{\partial x_2^2} = \sum_{i=3}^n w_i \frac{(i-1)(i-2)x_1 x_2^{i-3}}{\prod_{j=1}^i b_j} \quad (36)$$

$$\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} = \sum_{i=2}^n w_i \frac{(i-2)x_2^{i-2}}{\prod_{j=1}^i b_j} \quad (37)$$

$$H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{bmatrix} \quad (38)$$

$$x^\top H(x) x = \frac{3w_2 x_1 x_2}{b_1 b_2} + \sum_{i=3}^n w_i \frac{i(i-1)x_1 x_2^{i-1}}{\prod_{j=1}^i b_j} \quad (39)$$

Because $x^\top H(x) x \geq 0$ for all $x_i \in [a_i, b_i]$ where $0 = a_i < b_i \leq 1$ for $i = 1, \dots, n$, $f(x)$ is convex on this region. $\square$

A.4 Proposition 5

Proof. Let $g(x)$ denote the objective function of the problem $[P_1^{i < j}]$. Then,

$$g(x) = \log(w'_i F_i(x_i) + w'_j F_j(x_i) F_j(x_j)) \quad (40)$$

$$= \log(F_i(x_i) [w'_i + w'_j F_j(x_j)]) \quad (41)$$

$$= \log(F_i(x_i)) + \log(w'_i + w'_j F_j(x_j)) \quad (42)$$

The concavity of $\log(F_i(x_i))$ follows from the definition of log-concavity. Since log-concavity is preserved for affine transformations of the probability measures (Dharmadhikari and Joag-Dev, 1988), $\log(w'_i + w'_j F_j(x_j))$ is also concave. As the sum of concave functions is concave (Boyd and Vandenberghe, 2004), it follows that $g(x)$ is concave for log-concave $F_i(x_i)$ and $F_j(x_j)$. $\square$

B Algorithm for Two-Level System

C Rate of Decrease in Objective Function for Different Probability Distributions

Uniform Distribution

If the underlying distribution is uniform, that is $Y_i \sim U[a_i, b_i]$ for all i , then for a given solution x , the derivative of the objective function with respect to the fortification level of node i is expressed as follows,

$$\frac{\partial h(x)}{\partial x_i} = \frac{\sum_{k \in S(i)} \left[\prod_{j \in \mathcal{P}(k)} \frac{x_j - a_j}{b_j - a_j} \right] w_k}{b_i - a_i} \quad (43)$$

Algorithm 2 Algorithm for Two-Level System

Obtaining the first candidate solutionSet $x_1 = b_1$ Sort nodes $2, \dots, n$ in nonincreasing order of $\frac{w_k}{c_k(b_k - a_k)}$ Set $k = 2, x_j = a_j$ for $j = 2, \dots, n$ **while** $k < n$ **do** $x_k = b_k$ **if** $c_1 b_1 + \sum_{j=2}^k c_j b_j + \sum_{j=k+1}^n c_j a_j \geq B$ and $c_1 b_1 + \sum_{j=2}^{k-1} c_j b_j + \sum_{j=k}^n c_j a_j < B$ **then**Set $\hat{j} = k, k = n$

$$x_{\hat{j}} = \frac{B - c_1 b_1 - \sum_{j=2}^{\hat{j}-1} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_{\hat{j}}}$$

end if $k = k + 1$ **end while****if** $w_1 + \sum_{k=2}^{\hat{j}-1} w_k b_k \geq \frac{w_{\hat{j}}}{c_{\hat{j}}(b_{\hat{j}} - a_{\hat{j}})} [c_1(b_1 - a_1) - B + c_1 b_1 + \sum_{j=2}^{\hat{j}-1} c_j b_j + \sum_{j=\hat{j}+1}^n c_j a_j + a_{\hat{j}} c_{\hat{j}}]$ **then**

The solution is both a KKT and an extreme point.

end if**Obtaining the second candidate solution**Sort nodes $2, \dots, n$ in nonincreasing order of $\frac{w_k}{c_k(b_k - a_k)}$ Set $k = 2, x_j = a_j$ for $j = 1, \dots, n$ **while** $k < n$ **do** $x_k = b_k$ **if** $c_1 b_1 + \sum_{j=2}^k c_j b_j + \sum_{j=k+1}^n c_j a_j > B$ and $\sum_{j=2}^k c_j b_j + \sum_{j=k+1}^n c_j a_j \leq B$ **then**Set $\hat{j} = k, k = n, \text{count} = 0$

$$x_1 = \frac{B - \sum_{j=2}^{\hat{j}} c_j b_j - \sum_{j=\hat{j}+1}^n c_j a_j}{c_1}$$

for $j = \hat{j} + 1$ **to** n **do****if** $w_1 + \sum_{k=2}^{\hat{j}} \frac{w_k(x_k - a_k)}{(b_k - a_k)} \geq \frac{w_j}{c_j} \frac{B - \sum_{k=2}^{\hat{j}} c_k b_k - \sum_{k=\hat{j}+1}^n c_k a_k - a_1 c_1}{(b_j - a_j)}$ **then**
 $\text{count} = \text{count} + 1$ **end if****end for****end if** $k = k + 1$ **end while****if** $\text{count} = n - \hat{j}$ **then**

The solution is both a KKT and an extreme point.

end if

Starting with a solution such that $x_i = b_i$ for all i , the rate of decrease in the objective function per unit decrease in capacity consumption becomes

$$\rho_i^U = \frac{\sum_{k \in S(i)} w_k}{c_i(b_i - a_i)} \quad (44)$$

Triangular Distribution

For a triangular distribution on $[a_i, b_i]$ with mode d_i , we have

$$F_i(x_i) = 1 - \frac{(b_i - x_i)^2}{(b_i - a_i)(b_i - d_i)}, \quad d_i < x_i < b_i, \quad (45)$$

with $F_i(b_i) = 1$. For any $0 < \epsilon < d_i$, note that

$$F_i(b_i - \epsilon) = 1 - \frac{\epsilon^2}{(b_i - a_i)(b_i - d_i)}, \quad (46)$$

and the reduction in objective function value if we decrease x_i by ϵ would then equal

$$\sum_{i=1}^n w_i - \sum_{k \notin S(i)} w_k - \left(1 - \frac{\epsilon^2}{(b_i - a_i)(b_i - d_i)}\right) \sum_{k \in S(i)} w_k = \frac{\epsilon^2}{(b_i - a_i)(b_i - d_i)} \sum_{k \in S(i)} w_k. \quad (47)$$

The rate of decrease in the objective function per unit decrease in capacity is then

$$\frac{\epsilon \sum_{k \in S(i)} w_k}{c_i(b_i - a_i)(b_i - d_i)}. \quad (48)$$

The “ranking” of items is independent of ϵ , so that a potentially good strategy would be to decrease x_i values in nondecreasing order of

$$\rho_i^T = \frac{\sum_{k \in S(i)} w_k}{c_i(b_i - a_i)(b_i - d_i)}. \quad (49)$$

The above assumes that $x_i > d_i$ for each i . If we have decreased an item i such that $x_i \leq d_i$, then we have

$$F_i(x_i) = \frac{(x_i - a_i)^2}{(b_i - a_i)(d_i - a_i)}, \quad a_i < x_i \leq d_i, \quad (50)$$

so that

$$F_i(a_i + \epsilon) = \frac{\epsilon^2}{(b_i - a_i)(d_i - a_i)}, \quad 0 < \epsilon \leq d_i - a_i. \quad (51)$$

If we start at $x_i = a_i + \epsilon$ and decrease x_i by ϵ , then if we have $x_j = b_j$ for all $j \neq i$, the decrease in objective function value is

$$\sum_{k \notin S(i)} w_k + \frac{\epsilon^2}{(b_i - a_i)(d_i - a_i)} \sum_{k \in S(i)} w_k - \sum_{k \notin S(i)} w_k = \frac{\epsilon^2}{(b_i - a_i)(d_i - a_i)} \sum_{k \in S(i)} w_k, \quad (52)$$

and the reduction per unit of capacity consumption reduction is

$$\frac{\epsilon}{c_i(b_i - a_i)(d_i - a_i)} \sum_{k \in S(i)} w_k. \quad (53)$$

We can therefore use

$$\rho_i^{(T-)} = \frac{\sum_{k \in S(i)} w_k}{c_i(b_i - a_i)(d_i - a_i)}, \quad a_i < x_i \leq d_i, \quad (54)$$

$$\rho_i^{(T+)} = \frac{\sum_{k \in S(i)} w_k}{c_i(b_i - a_i)(b_i - d_i)}, \quad d_i < x_i \leq b_i. \quad (55)$$

D KKT Conditions for Models $[P_1^{i<j}]$ and $[P_1^{i\neq j}]$

KKT Conditions for $[P_1^{i<j}]$:

$$\frac{f_i(x_i)}{F_i(x_i)} = \alpha c_i + \lambda_i^b - \lambda_i^a, \quad (56)$$

$$\frac{w'_j f_j(x_j)}{w'_i + w'_j F_j(x_j)} = \alpha c_j + \lambda_j^b - \lambda_j^a \quad (57)$$

$$\lambda_k^b(b_k - x_k) = 0 \quad k \in \{i, j\} \quad (58)$$

$$\lambda_k^a(x_k - a_k) = 0, \quad k \in \{i, j\} \quad (59)$$

$$\alpha (\sum_{k \in \{i, j\}} c_k x_k - B) = 0 \quad (60)$$

$$a_k \leq x_k \leq b_k \quad k \in \{i, j\} \quad (61)$$

$$\alpha \geq 0 \quad (62)$$

$$\lambda_k^a, \lambda_k^b \geq 0 \quad k \in \{i, j\} \quad (63)$$

Here $f_i(x_i)$ and $f_j(x_j)$ denote the pdfs of the distributions of the disruption impacting nodes i and j , respectively. Since $[P_1^{i<j}]$ yields a convex program, KKT conditions are necessary and sufficient for optimality. We use these to identify the extreme points, which constitute a subset of all KKT points and therefore, candidate solutions. We observe that if $x_i = b_i$ then $\lambda_i^a = 0$ and

$$\alpha \leq \frac{f_i(x_i)}{c_i F_i(x_i)}$$

Then, $a_j \leq x_j = \frac{B - b_i c_i}{c_j} \leq b_j$. If $x_j = a_j$ then $\lambda_j^b = 0$ and $\alpha \geq \frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))}$, which implies $\frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))} \leq \frac{f_i(x_i)}{c_i F_i(x_i)}$. Similarly if x_j is fractional then $\lambda_j^a = \lambda_j^b = 0$, thus $\alpha = \frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))} \leq \frac{f_i(x_i)}{c_i F_i(x_i)}$. Note that $x_j = b_j$ if $B = \sum_{k \in \{i, j\}} c_k b_k$. Thus $(b_i, \frac{B - b_i c_i}{c_j})$ is a candidate solution and a KKT point if $\frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))} \leq \frac{f_i(x_i)}{c_i F_i(x_i)}$.

If x_i is fractional, that is $a_i < x_i < b_i$, then $\lambda_i^a = \lambda_i^b = 0$, and

$$\alpha = \frac{f_i(x_i)}{c_i F_i(x_i)}$$

If $x_j = b_j$ then $\lambda_j^a = 0$, thus $(\frac{B - b_j c_j}{c_i}, b_j)$ is a candidate solution and a KKT point if $\frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))} \geq \frac{f_i(x_i)}{c_i F_i(x_i)}$. Conversely, if $x_j = a_j$ then $\lambda_j^b = 0$ and $(\frac{B - a_j c_j}{c_i}, a_j)$ is a candidate solution and a KKT point if $\frac{w'_j f_j(x_j)}{c_j (w'_i + w'_j F_j(x_j))} \leq \frac{f_i(x_i)}{c_i F_i(x_i)}$.

Note that if $x_i = a_i$, by Proposition 1, $x_j = a_j$, which constitutes another candidate solution and a KKT point when $B = \sum_{k \in \{i, j\}} c_k a_k$.

KKT Conditions for $[P_1^{i\neq j}]$:

$$-\frac{f_i(x_i)}{w'_i F_i(x_i)^2} = \alpha c_i + \lambda_i^b - \lambda_i^a, \quad (64)$$

$$-\frac{f_j(x_j)}{w'_j F_j(x_j)^2} = \alpha c_j + \lambda_j^b - \lambda_j^a \quad (65)$$

$$\lambda_k^b(b_k - x_k) = 0 \quad k \in \{i, j\} \quad (66)$$

$$\lambda_k^a(x_k - a_k) = 0, \quad k \in \{i, j\} \quad (67)$$

$$\alpha \left(\sum_{k \in \{i, j\}} c_k x_k - B \right) = 0 \quad (68)$$

$$a_k \leq x_k \leq b_k \quad k \in \{i, j\} \quad (69)$$

$$\alpha \geq 0 \quad (70)$$

$$\lambda_k^a, \lambda_k^b \geq 0 \quad k \in \{i, j\} \quad (71)$$

Since $[P_1^{i*j}]$ yields a convex program, KKT conditions are necessary and sufficient for optimality. We use these to identify the extreme points, which constitute a subset of all KKT points and therefore, candidate solutions. We observe that if $x_i = b_i$ then $\lambda_i^a = 0$ and

$$\alpha \geq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$$

Then, $a_j \leq x_j = \frac{B-b_i c_i}{c_j} \leq b_j$. If $x_j = a_j$ then $\lambda_j^b = 0$ and $\alpha \leq \frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2}$, which implies $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \geq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$. Similarly if x_j is fractional then $\lambda_j^a = \lambda_j^b = 0$, thus $\alpha = -\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \leq -\frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$. Note that $x_j = b_j$ if $B = \sum_{k \in \{i, j\}} c_k b_k$. Thus $\left(b_i, \frac{B-b_i c_i}{c_j}\right)$ is a candidate solution and a KKT point if $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \geq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$.

If x_i is fractional, that is $a_i < x_i < b_i$, then $\lambda_i^a = \lambda_i^b = 0$, and

$$\alpha = -\frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$$

Then if $x_j = b_j$ then $\lambda_j^a = 0$, thus $\left(\frac{B-b_j c_j}{c_i}, b_j\right)$ is a candidate solution and a KKT point if $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \leq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$. Conversely, if $x_j = a_j$ then $\lambda_j^b = 0$ and $\left(\frac{B-a_j c_j}{c_i}, a_j\right)$ is a candidate solution and a KKT point if $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \geq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$.

If $x_i = a_i$, then $\lambda_i^b = 0$ and

$$\alpha \leq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$$

Then, $a_j \leq x_j = \frac{B-a_i c_i}{c_j} \leq b_j$. If $x_j = b_j$ then $\lambda_j^a = 0$ and $\alpha \geq \frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2}$, which implies $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \leq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$. Similarly if x_j is fractional then $\lambda_j^a = \lambda_j^b = 0$, thus $\alpha = -\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \geq -\frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$. Note that $x_j = a_j$ if $B = \sum_{k \in \{i, j\}} c_k a_k$. Thus $\left(a_i, \frac{B-a_i c_i}{c_j}\right)$ is a candidate solution and a KKT point if $\frac{f_j(x_j)}{c_j w'_j F_j(x_j)^2} \leq \frac{f_i(x_i)}{c_i w'_i F_i(x_i)^2}$.