

## Online Resource 3

### Supplementary methods for:

Steinarsson, A., Jakobsdóttir, G., Benediktsson, S. S., & Jóhannsson, E. (2026). *Aerobic endurance and physiological thresholds estimated from submaximal heart-rate kinetics: Comparison with cardiopulmonary exercise and sprint testing* [Unpublished manuscript]. Faculty of Health Promotion, Sport and Leisure Studies, University of Iceland.

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## TrueZone heart rate kinetics model (TZM) description

### A dynamic three-component log-drift heart rate model

#### Overview

The heart-rate response to exercise is modeled as the sum of three physiologically distinct components — a baseline shift out of the resting state, an immediate response to instantaneous intensity, and a slow drift driven by an accumulating metabolic debt. All components share a common normalized intensity axis and a small parameter set derived from the athlete's model profile. The model operates on 1-second power input for both running and cycling and predicts continuous HR through activity and recovery.

#### Intensity axis

Let  $W(t)$  denote instantaneous mechanical power (converted from speed via a fixed polynomial). The normalized intensity is

$$V(t) = \frac{[W(t) - V_0]}{V_i}$$

where  $V_0$  (floor) and  $V_i$  (scale) are athlete-specific model parameters. By construction,  $V = 1.5$  locates an aerobic threshold ( $V_{1.5}$ , equivalent to  $VT_1$ ),  $V = 2$  locates the second ventilatory threshold ( $V_2$ , equivalent to  $VT_2$ ), and  $V = 3$  locates an anaerobic threshold ( $V_3$ , equivalent to  $vVO_{2max}$ ). All internal dynamics operate on  $V$ .

#### The three components

**Baseline.** The resting-to-working baseline transition is captured by a softplus in the ratio  $W/V_0$ :

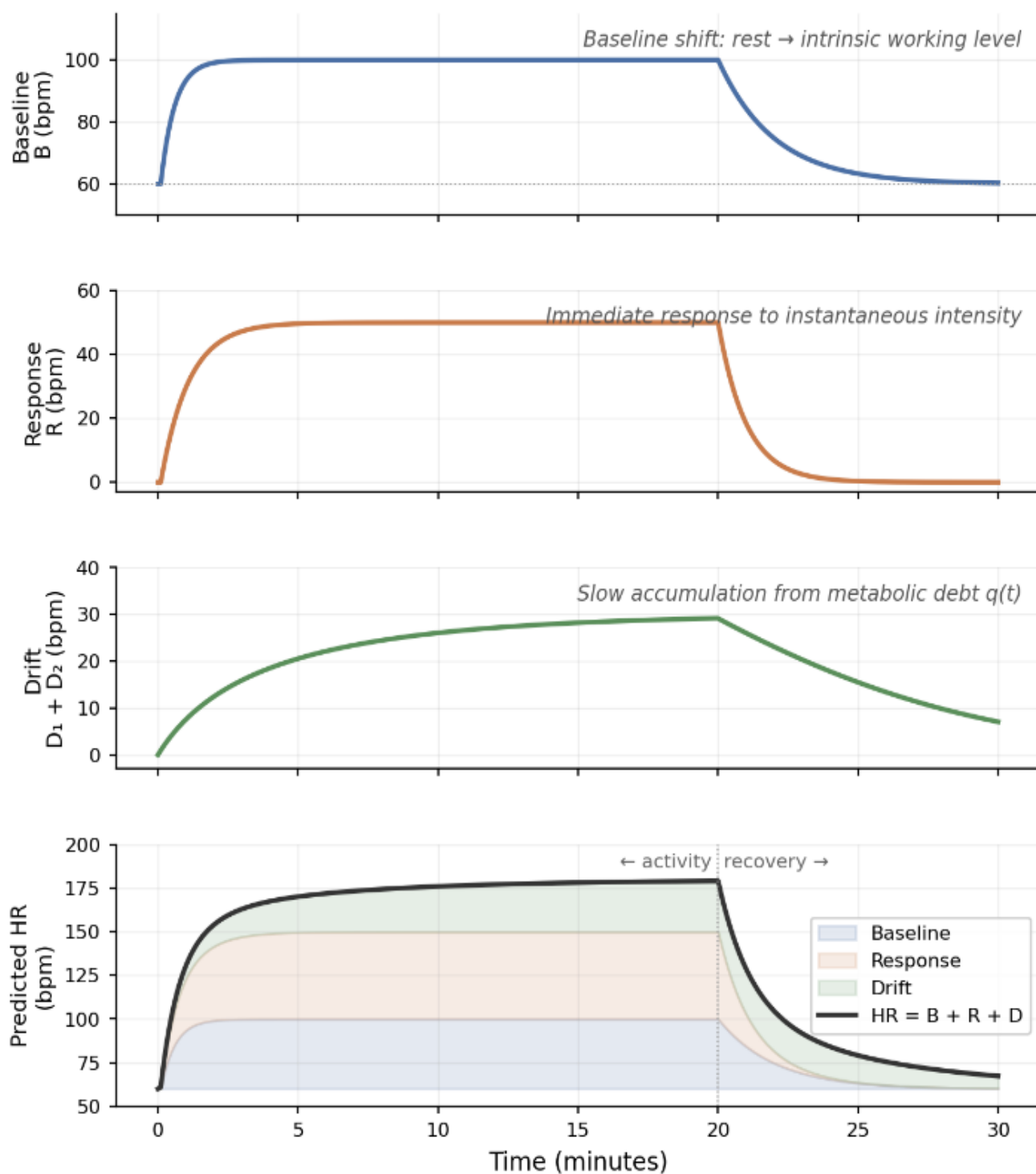
$$G(V) = \left[ \frac{1}{\ln(1 + e^k)} \right] \cdot \ln \left( 1 + \exp \left[ k \left( 1 - \frac{W}{V_0} \right) \right] \right)$$
$$B(V) = P_i + (\Pi - P_i) \cdot (1 - G(V))$$

with sharpness  $k$  (default 10).  $G$  transitions smoothly from 1 at rest to 0 above threshold, so  $B$  rises from resting HR ( $P_i$ ) toward an intrinsic working baseline ( $\Pi$ ).

**Response.** The immediate intensity-driven contribution is a saturating softplus integral:

$$R(V) = O(f(V)), \text{ where } f(V) = \left( \frac{m_{eff}}{m} \right) \cdot \left( \frac{2}{3} V + \frac{1}{6} \right) \text{ for } V > 1$$
$$\text{and } f(V) = f_1 - f_{bB} \cdot (1 - V) \text{ for } V \leq 1$$

- 31 The piecewise form of  $f$  reflects a shallower response gradient below  $V_1$  than above; the two pieces meet continuously at  $V =$   
 32 1.  $R$  saturates near  $m_{eff}\Pi$  at high  $V$ .



- 33  
 34 **Fig. S4.** Schematic of the three-component model. Baseline, response, and drift each contribute independently to the total predicted HR  
 35 (bottom panel), which sums the components (shown as translucent bands) and follows first-order smoothing kinetics.

**Drift.** The drift is governed by an internal debt variable  $q(t)$  that integrates the metabolic cost of sustained intensity and is drained by refill dynamics during recovery:

$$\frac{dq}{dt} = D(V) \cdot \left[ 1 - \left( \frac{q}{q^*} \right)^n \right]_{\pm} G(V) \cdot \left( \frac{q}{\tau_r} \right)$$

$$D(V) = 16^V \text{ for } V \geq 1; \quad D(V) = \max(0, (V - V_{base}) \cdot D_b) \text{ for } V < 1$$

$$q^*(V) = D(V) \cdot 2X \cdot 60$$

Here  $D(V)$  is the per-second drain (piecewise: exponential above  $V_1$ , linear below),  $q^*(V)$  is an intensity-dependent target, and the saturation exponent  $n$  (default 4) controls how sharply the accumulator approaches its plateau.  $\tau_r$  is a recovery time constant scaling inversely with the athlete's endurance parameter  $E$  via  $X = 6 \cdot 10^E$  minutes. The drift contribution to HR is a log-of-debt:

$$D_1(t) = \left( \frac{\Pi}{a_1} \right) \cdot \ln \left( 1 + \frac{q(t)}{b_1} \right)$$

A second, empirically derived slow component tracks the primary:

$$q_2 = \frac{q}{32}, \text{ contributing } D_2 = \left( \frac{\Pi}{a_2} \right) \cdot \ln \left( 1 + \frac{q_2}{b_2} \right) \cdot \eta$$

### HR assembly and smoothing

The instantaneous target HR is  $T(t) = B + R + D_1 + D_2$ . Measured HR follows this target with first-order kinetics:

$$HR(t + \Delta t) = HR(t) + [T(t) - HR(t)] \cdot c(t)$$

where  $c(t) = c_a$  during activity and  $c(t) = c_a/\rho$  during recovery, with  $\rho$  the recovery-slowdown factor.

### Modality

Running and cycling share the model structure and parameter set. Running input is speed, converted to power by a fixed polynomial; cycling uses measured power directly. Both modalities use the same drain base (16), reflecting that the  $V$  axis is designed to normalize modality differences at the intensity level.

### Fitting

For a measured HR session, the parameters  $E$ ,  $P$  (threshold power equivalent), and  $V$ -scale calibration are fit jointly by minimizing the squared residual between predicted and measured HR across both activity and recovery. Figure S5 illustrates the fitting workflow: successive parameter adjustments improve the fit by exploiting the distinct signatures of  $V$  (intensity),  $E$  (drift steepness), and true  $HR_{\max}$  on different phases of the curve.

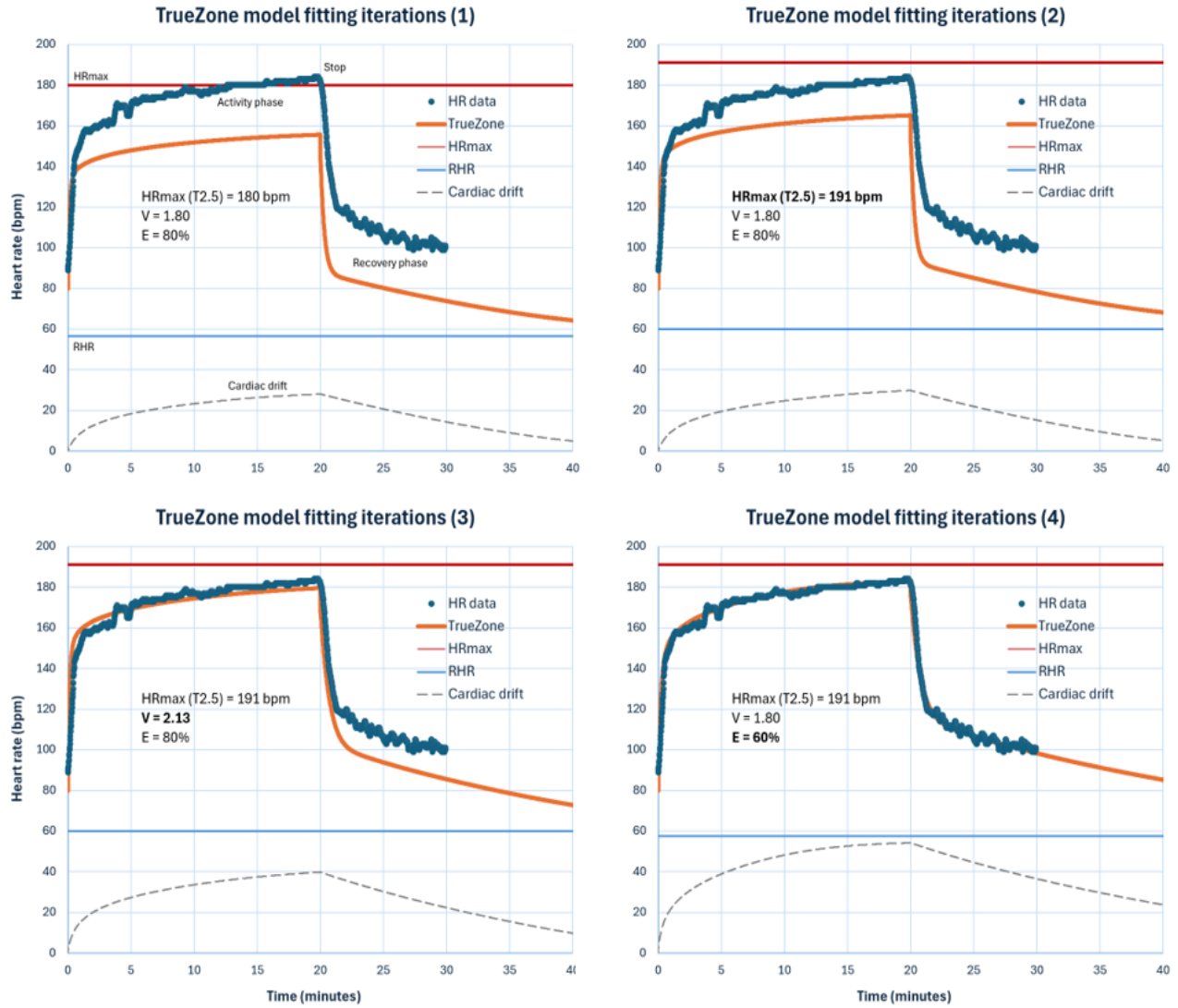


**Fig. S5.** TrueZone model-fitting process for a representative participant. Each panel compares the observed heart rate data (blue) during a submaximal run and recovery against the TrueZone model fit (orange) for a different combination of the three fitted parameters — maximum heart rate ( $HR_{max}$ , at heart rate  $P_2 + P_i/2$ ), intensity ( $V$ ), and endurance ( $E$ ). Panel 1 shows an initial fit with randomized parameters, while panels 2-4 show steps of adjusting in order  $HR_{max}$ ,  $V$  and  $E$  (all bolded) until an optimized fit is achieved in panel 4, illustrating how each parameter independently shapes the curve.

### Diagnostic parameters

Two optional parameters extend the model for athlete phenotypes or session states not captured by the standard three-parameter fit: Drift scaling factor  $\phi$  (default 1). Scales the displayed drift contribution during activity via

$$D_{displayed} = ((D_1 + D_2)) / [1 + (\phi - 1) \cdot ((1 - G)) / \max(1, V)]$$

without modifying the internal accumulator  $q$ . Values  $\phi > 1$  suppress displayed drift (common in endurance-trained athletes and cases of pre-exercise fatigue);  $\phi < 1$  amplifies it. The suppression releases smoothly during recovery via the  $(1 - G)$  gating,

so the recovery curve follows the un-scaled trajectory — a pattern consistent with observations in which measured recovery matches an athlete's un-blunted parameters despite blunted activity HR. Load-dependent recovery slowdown  $\alpha$  (default 0). The recovery smoothing coefficient is scaled by  $(1 + \alpha \cdot q/b_i)$ , slowing initial HR descent when accumulated debt is high (as observed in interval sessions with cumulative load) and returning to nominal speed as  $q$  drains. Both defaults ( $\phi = 1$ ,  $\alpha = 0$ ) reduce the model to the three-component core.

#### Derivation of endurance ( $E$ ) from the model framework.

The geometry of the V-scale also defines a theoretical relationship between aerobic endurance ( $E$ ) and the ratio of maximum running speed to the second threshold ( $V_{\max}/V_2$ ). Because this relationship follows directly from the model's evenly spaced recruitment thresholds, it contains no empirically fitted coefficients; Table S4 summarizes the derivation.

**Table S4** Theoretical derivation of the closed-form relationship between aerobic endurance ( $E$ ) and the ratio of maximum running speed to the second threshold ( $V_{\max}/V_2$ ), used to calculate the independent endurance estimate ( $E_r$ ).

| Derivation step                                                   | Result                                     |
|-------------------------------------------------------------------|--------------------------------------------|
| Definition of endurance                                           | $E = 2 V_0/V_3$                            |
| Lower intercept of the recruitment scale                          | $V_0 = E/2 V_3$                            |
| Constant intensity increment between thresholds                   | $V_i = (V_3 (2 - E))/6$                    |
| Second threshold (VT <sub>2</sub> -equivalent) position           | $V_2 = (V_3 (E + 4))/6$                    |
| Maximum running speed ( $V_{\max}$ )                              | $V_{\max} = V_2 + (V_3 (2 - E)^2)/2$       |
| Ratio of maximum speed to second threshold ( $R$ )                | $V_{\max}/V_2 = (3E^2 - 11E + 16)/(E + 4)$ |
| Endurance recovered from an observed $V_{\max}/V_2$ ratio ( $R$ ) | $E = (R + 11 - \sqrt{(R - 1)(R + 71)})/6$  |

The relationship is monotonic over the physiological endurance range  $0 \leq E \leq 1$ : at  $E = 0$ ,  $V_{\max}/V_2 = 4.0$ , and at  $E = 1$ ,  $V_{\max}/V_2 = 1.6$  — greater aerobic endurance corresponds to a progressively smaller speed reserve between the second threshold and maximum sprint speed. Inverting the relationship (final row of Table S4) provides an independent route for evaluating  $E$  from external measurements. In the present validation,  $R$  was calculated independently as

$$R_{ind} = \frac{V_{\max}}{V_{T2v}}$$

and the corresponding independent endurance estimate ( $E_{ratio}$ ) as

$$E_r = \frac{R_{ind} + 11 - \sqrt{(R_{ind} - 1)(R_{ind} + 71)}}{6}$$

Because neither sprint-derived maximum speed nor CPET-derived VT<sub>2v</sub> is used in fitting  $E$  from the submaximal heart-rate response, agreement between  $E_r$  and model-derived  $E$  provides an external test of the model-predicted relationship between aerobic endurance, threshold speed, and maximum sprint capacity.

94 **Abbreviations and symbols**

|                        |                                                                                          |
|------------------------|------------------------------------------------------------------------------------------|
| $t$                    | Time                                                                                     |
| $\Delta t$             | Time step used to update predicted heart rate                                            |
| $W(t)$                 | Instantaneous exercise power, measured directly or calculated from running speed         |
| $V(t)$                 | Normalized exercise intensity                                                            |
| $V_0$                  | Lower intercept of the individualized intensity scale                                    |
| $V_1$                  | Intensity increment defining the spacing of the model landmarks                          |
| $V_2$                  | Intensity or running speed at the second model threshold                                 |
| $V_3$                  | Intensity or running speed at the maximal aerobic landmark                               |
| $V_{\max}$             | Model-derived maximum running speed                                                      |
| $E$                    | Model-derived endurance parameter, expressed on a 0–1 scale                              |
| $X$                    | Endurance-dependent time scale, defined as $X = 6 \times 10^E \text{ min}$               |
| $G(V)$                 | Gating function controlling the transition between resting and working states            |
| $k$                    | Sharpness parameter of the gating function                                               |
| $P_i$                  | Resting heart rate                                                                       |
| $\Pi$                  | Intrinsic working heart-rate baseline                                                    |
| $B(V)$                 | Baseline contribution to predicted heart rate                                            |
| $R(V)$                 | Immediate intensity-dependent contribution to predicted heart rate                       |
| $O$                    | Saturating softplus-integral function used to calculate the immediate response           |
| $f(V)$                 | Piecewise intensity function governing the immediate response                            |
| $m_{\text{eff}}, m$    | Scaling parameters entering the immediate-response function                              |
| $f_i$                  | Value of the immediate-response intensity function at $V = 1$                            |
| $f_{\beta}^B$          | Coefficient governing the response gradient below $V = 1$                                |
| $q(t)$                 | Internal debt accumulator                                                                |
| $D(V)$                 | Intensity-dependent rate of accumulation in the debt variable                            |
| $q^*(V)$               | Intensity-dependent target or plateau of the debt accumulator                            |
| $n$                    | Saturation exponent governing the approach to the debt plateau                           |
| $V_{\text{base}}$      | Lower intensity boundary used in the below-threshold accumulation function               |
| $D_{\beta}$            | Scaling coefficient for below-threshold debt accumulation                                |
| $\tau_r$               | Recovery time constant governing depletion of the debt accumulator                       |
| $D_1(t)$               | Primary slow-drift contribution to heart rate                                            |
| $a_1, b_1$             | Scaling coefficients in the primary log-of-debt function                                 |
| $q_2$                  | Scaled debt variable used for the secondary drift component                              |
| $D_2(t)$               | Secondary slow-drift contribution to heart rate                                          |
| $a_2, b_2$             | Scaling coefficients in the secondary log-of-debt function                               |
| $\eta$                 | Multiplier controlling the secondary drift contribution                                  |
| $T(t)$                 | Target heart rate, calculated as $B + R + D_1 + D_2$                                     |
| $HR(t)$                | Predicted heart rate at time $t$                                                         |
| $c(t)$                 | Smoothing coefficient used to update predicted heart rate                                |
| $c_a$                  | Smoothing coefficient during activity                                                    |
| $\rho$                 | Recovery-slowdown factor applied to the smoothing coefficient                            |
| $\phi$                 | Optional factor modifying displayed drift without changing the internal debt accumulator |
| $D_{\text{displayed}}$ | Drift contribution after applying the optional scaling factor                            |
| $\alpha$               | Optional coefficient controlling load-dependent slowing of initial heart-rate recovery   |
| $R_{\text{ind}}$       | Ratio of independently measured maximum sprint speed to CPET-derived VT2 speed           |
| $E_r$                  | Endurance value calculated from $R_{\text{ind}}$ using the TZM relationship              |