

Online Resource 1: technical companion Population-harmonic crossover under modulation-axis misalignment in a dissipative two-level system

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This companion retains the detailed derivations, numerical protocols and diagnostic evidence of the previously released v2.0.0 benchmark (doi:10.5281/zenodo.23112065), whose files now have restricted access, and adds the fixed-order family, truncation proof and tilted-axis validation of the accompanying article. The current public supporting package, Zenodo v3.0.0 (doi:10.5281/zenodo.23249084), preserves the original benchmark archive and includes the new numerical reproducers and proof sources. Original archive files are unchanged. The main article separates the new claims from the previously deposited results. Ratios require nonzero signals. Except where explicitly stated, rates and stationary seeds are fixed before taking the weak-modulation coefficient and then its high-frequency limit.

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1 Full-periodic residual bound and resonant crossover

This section supplements the preserved benchmark with the new full-periodic residual certificate and resonant joint-limit analysis. The new checks distinguish intended mathematical parameters from their serialized numerical representations. The earlier chain-only error bound and operational numerical floors retain their original scopes.

A residual bound for the full periodic response. Let $\gamma_1 > 0$, $\Gamma_2 = \gamma_1/2 + \gamma_\phi$, and $\kappa = \min(\Gamma_2, \gamma_1) > 0$. For any differentiable periodic approximation $\tilde{\mathbf{r}}$, define $\mathbf{d} = \dot{\tilde{\mathbf{r}}} - M(t)\tilde{\mathbf{r}} - \mathbf{c}$. With $\|f\|_{T,2}^2 = T^{-1} \int_0^T \|f(t)\|_2^2 dt$, the exact periodic solution satisfies

$$\|\tilde{\mathbf{r}} - \mathbf{r}_*\|_{T,2} \leq \frac{\|\mathbf{d}\|_{T,2}}{\kappa}, \quad |\tilde{p}_h - p_h| \leq \frac{\|\mathbf{d}\|_{T,2}}{2\kappa}.$$

Consequently the error of $A_h = |w_h|$ is at most $\|\mathbf{d}\|_{T,2}/\kappa$. Indeed the Hamiltonian part of M is skew-Hermitian and $(M + M^\dagger)/2 = -\text{diag}(\Gamma_2, \Gamma_2, \gamma_1)$. For $\mathbf{e} = \tilde{\mathbf{r}} - \mathbf{r}_*$, periodicity gives $0 = \text{Re}\langle \mathbf{e}, \dot{\mathbf{e}} \rangle_T \leq -\kappa \|\mathbf{e}\|_{T,2}^2 + \|\mathbf{e}\|_{T,2} \|\mathbf{d}\|_{T,2}$. The Fourier estimate follows by Cauchy–Schwarz. The proof also allows complex approximations; conjugate symmetry is not required for the bound.

For $\tilde{\mathbf{r}} = \sum_{n=-N}^N \mathbf{x}_n e^{invt}$, extend $\mathbf{x}_n = 0$ outside the truncation. For sine phase zero,

$$\mathbf{d}_n = (in\nu I - M_0)\mathbf{x}_n - \frac{\lambda}{2i}B(\mathbf{x}_{n-1} - \mathbf{x}_{n+1}) - \mathbf{c}\delta_{n0}, \quad |n| \leq N,$$

$$\mathbf{d}_{N+1} = -\frac{\lambda}{2i}B\mathbf{x}_N, \quad \mathbf{d}_{-N-1} = +\frac{\lambda}{2i}B\mathbf{x}_{-N}, \quad \|\mathbf{d}\|_{T,2}^2 = \sum_{n=-N-1}^{N+1} \|\mathbf{d}_n\|_2^2.$$

These exterior terms bound the full truncation error, not merely the algebraic residual of the finite linear system. A floating point reduction is only an estimate; a rigorous numerical certificate requires an outward enclosure of every quantity used. This bound does not apply when $\kappa = 0$.

Intended parameters and serialized inputs. Use a common phase coordinate $\tau = \nu_* t$ for the intended model, and let D be the residual norm evaluated for serialized inputs. With $R = \|\tilde{\mathbf{r}}\|_{2\pi,2}$ and $R' = \|\partial_\tau \tilde{\mathbf{r}}\|_{2\pi,2}$,

$$D_* \leq D + |\nu_* - \nu|R' + \sup_{\tau} \|M_*(\tau) - M(\tau)\|_2 R + \|\mathbf{c}_* - \mathbf{c}\|_2.$$

The error bound uses D_*/κ_* of the intended model. Parseval gives $R^2 = \sum \|\mathbf{x}_n\|_2^2$ and $R'^2 = \sum n^2 \|\mathbf{x}_n\|_2^2$. For an axis-angle discrepancy $\delta\theta$ alone, $\|B_{\theta+\delta\theta} - B_\theta\|_2 = 4|\sin(\delta\theta/2)| \leq 2|\delta\theta|$; hence $D_* \leq D + 4|\lambda| |\sin(\delta\theta/2)| R$. A direct interval residual evaluated with the intended exact matrix is an alternative. Arbitrary precision agreement does not make a serialized parameter an exact symmetry point.

A resonant joint-limit crossover at minimal amplitude order. Fix $\Delta = 0$, $\Omega, \gamma_1 > 0$, $\gamma_\phi \geq 0$, and a common sine phase for both components, with $B_\theta = \cos\theta M_z + \sin\theta M_x$. Let $h = 2m$, and let $w_h^{[h]}$ denote only the coefficient at minimal amplitude order λ^h . Define

$$\chi = \frac{2\nu|w_0|\sin^2\theta}{\Omega v_0} = \frac{2\Gamma_2\nu\sin^2\theta}{\Omega^2}, \quad I_2 = \frac{\lambda^2\Omega v_0}{4\nu^3}, \quad I_4 = \frac{\lambda^4\Omega v_0}{96\nu^5}.$$

As $\nu \rightarrow \infty$ and $\theta \rightarrow 0$ with χ in any fixed bounded nonnegative interval,

$$w_2^{[2]}/I_2 = \chi + i + O(\nu^{-1}), \quad w_4^{[4]}/I_4 = -(2\chi + i) + O(\nu^{-1}).$$

Thus

$$R_{42}^{\min} \rightarrow \sqrt{\frac{1 + 4\chi^2}{1 + \chi^2}}, \quad C_{42}^{\min} = \frac{24\nu^2 w_4^{[4]}}{\lambda^2 w_2^{[2]}} \rightarrow -\frac{2\chi + i}{\chi + i}.$$

For the specified sine reference, $\text{Re } C = -(1 + 2\chi^2)/(1 + \chi^2)$, $\text{Im } C = \chi/(1 + \chi^2)$, and $(\text{Re } C + 3/2)^2 + (\text{Im } C)^2 = 1/4$. For a common modulation phase ϕ , use $e^{-2i\phi}C_{42}$. This curve assumes resonance, a single common-phase tilted axis, fixed nonzero stationary seed, and the stated joint minimal-order limit. It is not an identification theorem for arbitrary drive geometry or finite amplitude response.

Uniform remainder establishing that limit. Put $\mu \geq \|M_0\|_2$, $t = \mu/\nu < 1$, $G_h(t) = \prod_{j=1}^h (1 - t/j)^{-1}$ and $H_h = \sum_{j=1}^h 1/j$. For even h , the projected first-drift coefficient is an even smooth function of θ : $B_{-\theta} = -SB_\theta S$, $SM_0S = M_0$, $S\mathbf{r}_0 = \mathbf{r}_0$, $q^T S = q^T$, with $S = \text{diag}(-1, 1, 1)$. Its second derivative is bounded by $\|\mathbf{r}_0\|_2 \mu H_h h^2/h!$, since every B , B'

and B'' has norm 2 and the product contains h such factors. Writing $W_{h,0}(\theta)$ for the exact zero-drift term and $W_{h,1}(0)$ for the longitudinal first-drift term, one obtains

$$|w_h^{[h]} - W_{h,0}(\theta) - W_{h,1}(0)| \leq \frac{|\lambda|^h \|\mathbf{r}_0\|_2}{h! \nu^h} \left[G_h(t) - 1 - H_h t + \frac{\mu H_h h^2 \theta^2}{2\nu} \right].$$

The first term bounds every chain word containing at least two drifts; the second bounds the change of the first-drift coefficient. For $0 \leq \chi \leq C$, $\theta^2 \leq (\pi^2/4) \sin^2 \theta \leq \pi^2 C \Omega^2 / (8\Gamma_2 \nu)$. After dividing by $|\lambda|^h \Omega \nu_0 / (h h! \nu^{h+1})$ the bound is $O(\nu^{-1})$, uniformly over this interval. Constants are not uniform as Γ_2 or $\Omega \nu_0$ approaches zero.

For a relative excess tolerance $0 < \varepsilon < 1$, the limiting curve satisfies $R_{42}^{\min} \leq 1 + \varepsilon$ precisely when

$$\chi \leq \sqrt{\frac{(1 + \varepsilon)^2 - 1}{4 - (1 + \varepsilon)^2}}, \quad \sin^2 \theta \leq \frac{\Omega^2}{2\Gamma_2 \nu} \sqrt{\frac{(1 + \varepsilon)^2 - 1}{4 - (1 + \varepsilon)^2}}.$$

This is an asymptotic minimal-order alignment criterion; a finite-frequency or finite-amplitude guarantee requires its remainder and full residual or other amplitude-order error bounds.

Propagation to a complex harmonic ratio. Let E be a certified upper bound for every $|\tilde{w}_h - w_h|$, for example $E = D_*/\kappa_*$. If $|\tilde{w}_2| > E$, then

$$\left| \frac{24\nu^2 w_4}{\lambda^2 w_2} - \frac{24\nu^2 \tilde{w}_4}{\lambda^2 \tilde{w}_2} \right| \leq \frac{24\nu^2 E(|\tilde{w}_2| + |\tilde{w}_4|)}{|\lambda|^2 |\tilde{w}_2| (|\tilde{w}_2| - E)}.$$

The same upper bound applies to the error of the magnitude ratio. If the denominator enclosure contains zero, this argument certifies no ratio. It also does not establish uniqueness of a physical inference from that ratio. Parameters in the prefactor must be those of the intended model.

New validation membership and distinction from retained evidence. The source grids are `joint_points.csv` (12 points) and `master_points.csv` (9 controls), frozen before any corresponding outputs. At $\Delta = 0$, $\Omega = \gamma_1 = 1$, the new crossover inputs fix $\chi = 0.1, 0.5, 2$. Nine points use $\gamma_\phi = 0$ and frequencies 80,320,1280; three use $\gamma_\phi = 0.5$ at frequency 1280. Their angles are generated at high precision from the stated χ , not a binary64 inverse sine. The full response at finite $\lambda = 0.1$ is compared with the minimal-order joint-limit prediction, not treated as that limit itself. The maximum endpoint ratio error is 0.0837%; all twelve rows and all lower-frequency deviations are retained. The full complex coefficients and phase reference are recorded independently of the amplitude ratio.

The direct density-matrix control constructs the commutator and Lindblad jump super-operators on a 2×2 matrix and propagates its linear 4×4 density-matrix representation, imposing trace one when solving the periodic fixed point; its Fourier observable is the excited-state diagonal element. The nine frozen controls and two density-solver settings test weak positive relaxation, strong dephasing, signed detuning, common phase, finite modulation and the two sides of the state-cancellation condition. Twenty-seven $h = 1, 2, 4$ checks pass against the Bloch implementations. The algorithms share mpmath arithmetic and local Taylor integration, so representation independence is not claimed as total software or arithmetic independence. The initial exact-real-zero decoding exception and the corrected, separately frozen attempt are preserved, rather than omitted.

The 43 existing HB polynomials yield 440 required outward harmonic enclosures at two interval precisions and for serialized/canonical parameter targets. Every required bound passes; the maximum complex-population error upper bound relative to the stored coefficient magnitude is below 7.71×10^{-41} . It is not a relative bound with an unknown true coefficient in its denominator. If this stored-scale bound is $\rho < 1$, the true-relative bound is at most $\rho/(1 - \rho)$. Exact exterior harmonics $\pm(N + 1)$ are included. Intended pure-transverse and state-zero angles use canonical matrices $(\sin \theta, \cos \theta) = (1, 0)$ and $(4/\sqrt{41}, 5/\sqrt{41})$, while the exact serialized-decimal targets are recorded separately. Their approximately 80-digit angle representations are not silently identified with exact mathematical limits. All original coefficients are preserved.

The same a posteriori procedure is evaluated separately on the 21 density-control and joint-crossover HB polynomials. Its 168 additional harmonic enclosures pass, with a true-relative complex-population error bound below 7.12×10^{-43} . Quotient-error propagation gives 128 ratio enclosures across the 43 earlier and 21 additional physical inputs, with serialized and intended parameter targets treated separately. Every denominator enclosure excludes zero; the largest relative ratio-interval width is below 1.55×10^{-40} . These posterior error bounds assess the computed full solutions. They are not fitted tolerances or evidence that an asymptotic prediction is exact at a finite point.

Common phase and edge domains. The joint limit requires common-phase longitudinal/transverse components. With independent phases the positive-frequency coupling matrix is generally complex and need not obey the real-axis polynomial identity. For example, $B_+ = (M_z + iM_x)/\sqrt{2}$ satisfies $B_+^3 = 0$ although $q^T B_+^2 r_0 \neq 0$ at resonance. The nominal fourth-order leading coefficient then vanishes, so the same leading ratio need not approach one or two. This is outside the chosen fixed-real-axis model. It is a scope counterexample, not a prediction for arbitrary phase errors. The constants in the joint-limit remainder are not uniform as Γ_2 or Ωv_0 tends to zero. The full-periodic residual certificate requires $\kappa > 0$ and becomes less sharp as relaxation decreases. Pure-dephasing nulls retain their separate proof.

2 Stationary domain and relation to the retained benchmark

The model uses $\Omega > 0$, $\nu > 0$, nonnegative dissipation rates and excludes $\gamma_1 = \gamma_\phi = 0$. For $\gamma_1 > 0$ the homogeneous Bloch differences contract uniformly: the symmetric part is $\text{diag}(-\Gamma_2, -\Gamma_2, -\gamma_1)$, independently of the Hamiltonian modulation. There is consequently a unique attracting periodic state. For $\gamma_1 = 0$, $\gamma_\phi > 0$, the pure-dephasing null proof below supplies the attracting zero vector with the nonzero transverse drive. The fully undamped problem is outside this stationary benchmark.

The symbols $w_h^{[h]}$ and its high-frequency coefficients denote only the minimal modulation-order contribution. Comparisons between these asymptotic expressions and the full finite-amplitude response are numerical statements on declared grids; the full-periodic residual bound above is a separate analytic result. The norm and projected-amplitude diagnostics below do not control all omitted modulation orders. The historical ensemble is a supplied set of fixed tuples, without a recoverable historical random sampler.

3 Fixed-order family, certified chain truncation and axis tilt

Damping domain and limiting procedure. For non-null response we take $\Omega, \nu > 0$, $\gamma_1 > 0$ and $\gamma_\phi \geq 0$, with fixed M_0 , dissipation and population readout. Then $\Gamma_2 = \gamma_1/2 + \gamma_\phi > 0$ and the symmetric part of the homogeneous Bloch generator is $-\text{diag}(\Gamma_2, \Gamma_2, \gamma_1)$. Two solutions therefore approach one another at least as $\exp[-\min(\Gamma_2, \gamma_1)t]$, independently of the antisymmetric Hamiltonian modulation. The periodic state is unique and attracting; the stationary-state denominator is positive. Write $w_h(\lambda, \nu) = \lambda^h \chi_h(\nu) + O(\lambda^{h+2})$ near $\lambda = 0$. The expressions below concern χ_h , the lowest modulation order reaching sector h , and its high-frequency expansion at fixed M_0 . In particular, the ratio limits mean

$$\lim_{\nu \rightarrow \infty} \lim_{\lambda \rightarrow 0^+} \frac{24\nu^2 A_{4\nu}}{\lambda^2 A_{2\nu}} = \begin{cases} 1, & M_z, \\ 2, & M_x. \end{cases}$$

The denominators and stationary seeds must be nonzero. Numerical comparison with the full periodic response is a separate test of higher modulation orders. A scan at fixed phase depth or fixed λ/ν is a different limiting procedure and cannot be substituted into this finite-order statement.

Even-harmonic coefficient family at minimal modulation order. For each fixed $h = 2m \geq 2$, let $w_h^{[h]}$ be the population Bloch Fourier coefficient contributed by the order- λ^h all-positive resolvent chain. Then

$$w_{2m}^{[2m],z} = -i(-1)^m \frac{\lambda^{2m} \Omega v_0}{(2m)(2m)!\nu^{2m+1}} + O(\lambda^{2m} \nu^{-2m-2}), \quad (1)$$

$$w_{2m}^{[2m],x} = (-1)^m \frac{\lambda^{2m} w_0}{(2m)!\nu^{2m}} + O(\lambda^{2m} \nu^{-2m-1}). \quad (2)$$

These are the first terms of finite-order susceptibilities, not an assertion about summing every modulation order. Their population peak amplitudes are $|w_{2m}|$, because $p_{2m} = w_{2m}/2$ and the conjugate Fourier pair contributes $2|p_{2m}|$. To prove the longitudinal expression, $\mathbf{q}^T M_z = 0$ annihilates every leading product. At the next inverse-frequency order, all drift insertions except the outermost resolvent insertion are still annihilated; the surviving contraction is $\mathbf{q}^T M_0 M_z^{2m} \mathbf{r}_0 = (-4)^m \Omega v_0$. The all-positive denominator product is $(i\nu)^{2m} (2m)!$, and the outermost drift supplies another factor $(i2m\nu)^{-1}$. For the transverse model, $\mathbf{q}^T M_x^{2m} \mathbf{r}_0 = (-4)^m w_0$ already survives. Thus, for even h , the successive-amplitude constants at this order are

$$\frac{\nu^2 A_{(h+2)\nu}^{z,[h+2]}}{\lambda^2 A_{h\nu}^{z,[h]}} \longrightarrow \frac{h}{(h+1)(h+2)^2}, \quad \frac{\nu^2 A_{(h+2)\nu}^{x,[h+2]}}{\lambda^2 A_{h\nu}^{x,[h]}} \longrightarrow \frac{1}{(h+1)(h+2)}.$$

Their geometry quotient is $h/(h+2)$. At $h = 2$ the same factor 24 produces limits 1 and 2, respectively. These identities are elementary consequences of the stated chain and geometry; no priority or universal-system claim is made.

An absolute resolvent-chain remainder bound. Use the induced Euclidean norm and choose $\mu \geq \|M_0\|_2$ with $t = \mu/\nu < 1$. A Frobenius norm is one explicit conservative

choice. Set $b = \|M_1\|_2$ and

$$G_h(t) = \prod_{j=1}^h (1 - t/j)^{-1}, \quad K_h = \frac{|\lambda|^h \|q^T M_0\|_2 \|r_0\|_2 b^h}{2^h h! h \nu^{h+1}}.$$

If $q^T M_1 = 0$, the leading projected order is zero and the chain admits

$$w_h^{[h]} = \lambda^h \sum_{s \geq 1} C_{h,s} \nu^{-h-s}.$$

Define

$$W_h^{[S]} = \lambda^h \sum_{s=1}^S C_{h,s} \nu^{-h-s}.$$

Then

$$|w_h^{[h]} - W_h^{[1]}| \leq K_h [G_h(t) - 1],$$

$$|w_h^{[h]} - W_h^{[3]}| \leq K_h \left[G_h(t) - 1 - H_h t - \frac{H_h^2 + H_h^{(2)}}{2} t^2 \right],$$

where $H_h = \sum_{j=1}^h j^{-1}$ and $H_h^{(2)} = \sum_{j=1}^h j^{-2}$. Indeed, the resolvent identity and $q^T M_1 = 0$ give $q^T R_h M_1 = (ih\nu)^{-1} q^T M_0 R_h M_1$. The extra outer drift is exact. Expanding the remaining resolvents gives the majorant G_h ; its nonnegative coefficients at orders 0, 1, 2 are 1, H_h , $(H_h^2 + H_h^{(2)})/2$. Absolute convergence and submultiplicativity bound the remaining words by the displayed tails. The same bounds control the difference of the associated peak amplitudes by the reverse triangle inequality. This is a bound on the retained modulation-order chain, not on higher powers of λ , numerical roundoff, or detector noise. A relative bound requires division by a nonzero retained coefficient and becomes loose near a suppressed seed. It does not convert the finite amplitude-change diagnostic η_h into a certificate. In particular, $q^T M_1^2 r_0 = 0$ alone is weaker than $q^T M_1 = 0$ and does not justify this enhanced projected bound.

Sensitivity to a tilted modulation axis. Hold M_0 , the dissipator and readout fixed, and set $M_\theta = \sin \theta M_x + \cos \theta M_z$, $0 \leq \theta \leq \pi/2$. Then

$$q^T M_\theta^2 r_0 = 4 \sin \theta \cos \theta u_0 - 4 \sin^2 \theta w_0, \quad M_\theta^4 = -4 M_\theta^2.$$

For a nonzero projected seed the nominal ν^{-2} and ν^{-4} terms survive, and the same normalized R_{42} tends to 2 at minimal modulation order. At resonance $u_0 = 0$, $w_0 < 0$, so every fixed $\theta > 0$ has such a seed, whereas the exact longitudinal axis has $R_{42} \rightarrow 1$. Hence the $\theta \rightarrow 0$ and high-frequency limits do not commute. For small tilt at resonance, the competing second-harmonic magnitudes are approximately

$$\frac{\lambda^2 \sin^2 \theta |w_0|}{2\nu^2} \quad \text{and} \quad \frac{\lambda^2 \Omega |v_0|}{4\nu^3}.$$

They are comparable at $\nu \sin^2 \theta \simeq \Omega |v_0|/(2|w_0|)$; for $\gamma_1 = \Omega$, $\gamma_\phi = 0$, this is $\nu \sin^2 \theta \simeq \Omega$. It is a small-angle crossover estimate, not a certified finite-frequency boundary. At detuning, a nonzero tilted axis can instead have a state-dependent leading cancellation when $\tan \theta = u_0/w_0 = \Delta\Omega/(\Delta^2 + \Gamma_2^2)$. The next weighted projected sums must then be calculated; the structural longitudinal result and its ratio cannot be transferred solely from the vanishing leading seed.

Unital and degenerate edges. For $\gamma_1 = 0$, $\gamma_\phi > 0$ and $\Omega > 0$, the source is zero. For both axes and their tilts, $d\|\mathbf{r}\|^2/dt = -2\gamma_\phi(u^2 + v^2)$. A periodic homogeneous solution with constant norm must have $u = v = 0$. Its remaining w is constant, and the time-dependent transverse field $(\Omega + 2\lambda \sin \theta \sin \nu t)w$ must vanish identically. Because this field is not identically zero, $w = 0$. For attraction, suppose the homogeneous monodromy had an eigenvector with unit-modulus eigenvalue, allowing complex components. Its norm is equal at the beginning and end of a period, while the same dissipation identity uses $|u|^2 + |v|^2$. It therefore has $u = v = 0$ throughout the period; the nonzero static transverse component forces $w = 0$, contradicting a nonzero eigenvector. Thus all monodromy eigenvalues have modulus below one, and the maximally mixed state is the unique attracting periodic state; every population harmonic vanishes, and the ratios are undefined. This proof does not rely on uniform instantaneous strict contraction. With no dissipation there need not be an attracting periodic state; the stationary fractions become 0/0. With $\Omega = 0$ and exactly longitudinal modulation, pure dephasing conserves w and does not select the maximally mixed state. These degenerate cases are explicitly excluded from the non-null susceptibility/ratio statements.

4 Initial extension validation and portable reproduction

The initial post-rejection campaign uses the frozen parameter table `points.csv` in the portable reproducer. Its acceptance criteria and input hashes are recorded in `protocol.json` and `FREEZE.json`. These were frozen before the full physical solves. Units are $\tau = \Omega t$ and all angular rates divided by Ω , with no assumed experimental platform. Forty points, 104 complex harmonic rows and a separate three-point/six-row state-dependent-zero control are retained, including every numerical or asymptotic diagnostic outcome. The separate control was frozen before its own execution and did not change the 40-point primary registry.

The independent time method uses 60 digits/degree 52/48 segments and 80 digits/degree 72/64 segments. The independent harmonic-balance method uses 60 digits/ $N = 20$ and 80 digits/ $N = 28$. Both use `mpmath 1.3.0`, so independence concerns transcription and solution algorithm, not the arithmetic library. The predeclared relative cross-method and refinement gates are 10^{-20} , closure 10^{-35} , conjugacy 10^{-40} , sampled minimum density eigenvalue -10^{-35} and signal/floor at least 10^4 . Floors are ten times the maximum of refinement change, time operational tail estimate, harmonic-balance algebraic residual bound and a precision-scale minimum; they are not complete truncation or roundoff certificates. The exact minimal-order reference uses a separate positive resolvent chain and is compared separately with both the full solution and the leading asymptotic coefficient.

The primary set uses two seeds, both endpoint geometries, three frequencies 40, 80, 160 and $\lambda = 0.1$, with $\Omega = \gamma_1 = 1$. Seed two is $\Delta = 1$, $\gamma_\phi = 0.5$. The 16 minimal-order leading endpoint checks at 160 require complex error below 6% and amplitude error below 1%; four geometry ratios require deviation below 2%. Those endpoints and tolerances were fixed before results. All pass. The remaining six low-frequency, six fixed- $\lambda/\nu = 0.2$ and 16 tilted-axis points characterize validity/crossover and carry no mandatory asymptotic-success gate. None is removed. The main article reports the actual finite-grid deviations, rather than presenting every computed point as asymptotic confirmation.

The minimal reproducer requires only seven scientific input files and `mpmath 1.3.0`, with no historical repository path or stored response data. Python 3.13.7 is the tested interpreter.

From its directory run `python reproduce.py --output fresh-results`; the output directory must not exist. The full 40-point workflow runs both methods/settings; `--point-id P001` is explicitly a one-point smoke check and cannot reproduce the full-grid claims. The separately frozen control has its own equivalent command inside `seed_zero_control/`. Output tables retain full complex coefficients, operational floors, method/refinement errors, convergence, physicality, seed values and all rejection reasons. Source hashes and execution/environment metadata accompany each run.

The S2 helper preserves the original scientific producer files byte-for-byte and stages their historical directory hierarchy automatically. Its `--mode full` sequence includes all 683 scalar and arbitrary-precision checks, 16 time anchors, precision analysis, 36 unital controls and figure regeneration. The helper's `--mode smoke` rebuilds the complete registries, then solves four preselected parameter-diverse anchors with both harmonic-balance implementations and independent time propagation. This new smoke run passes 4/4 and the input/anchor registries match the archived registries byte-for-byte. The complete full workflow is available. A subsequent full reexecution on 8 October 2026 repeated all 683 inputs, the 16 independent time anchors and all 36 null controls with unchanged parameter memberships and gates; the canonical numerical tables reproduced the archived values exactly. The original isolated full rebuild is also preserved as evidence.

The published v2.0.0 Zenodo record was checked anonymously on 7 October 2026. Its 24,447,596-byte download has a SHA256 matching the original release manifest, matches the published MD5 and is byte-identical to the original local release. Exact checksums are retained in the public-access verification record. Outer and nested archives pass CRC checks and contain the original code and parameter tables. These checks established public access to the earlier benchmark on that date. Its record now retains public metadata with restricted file access. On 9 October 2026, Zenodo v3.0.0 (doi:10.5281/zenodo.23249084) was published and independently verified by anonymous download: its archive is byte-identical to the sealed release and passes outer and nested CRC checks. It preserves the original archive unchanged and includes the numerical data, portable reproducers and mathematical proof/check sources supplied in Online Resources 2 and 3. The current public DOI identifies this updated supporting package; the manuscript and this technical-companion PDF/TeX are excluded from the deposit.

4.1 State-dependent cancellation counterexample

Take $\Delta = \Omega = \gamma_1 = 1$, $\gamma_\phi = 0$ and $\tan \theta = 4/5$. The unmodulated stationary state is $\mathbf{r}_0 = (-4/7, 2/7, -5/7)^T$. It gives $s_\theta = 0$ although $\mathbf{q}^T M_\theta \neq 0$. The first surviving weighted drift sums yield

$$w_2^{[2]} = i \frac{113\lambda^2}{574\nu^3} + \mathcal{O}(\lambda^2\nu^{-4}), \quad w_4^{[4]} = -i \frac{305\lambda^4}{13776\nu^5} + \mathcal{O}(\lambda^4\nu^{-6}). \quad (3)$$

Thus $\lim_{\nu \rightarrow \infty} \lim_{\lambda \rightarrow 0^+} R_{42} = 305/113 \simeq 2.699115$, despite the same exponent pair as the longitudinal model. The exponent pair or leading seed zero alone does not identify a structural modulation axis. This exact counterexample was checked by independent rational matrix calculations and a separately frozen three-point full-solver campaign.

In this example the drift words can be checked directly without numerical integration:

$$\mathbf{q}^T (M_0 M_\theta^2 + 2M_\theta M_0 M_\theta) \mathbf{r}_0 = -\frac{904}{287}, \quad \mathbf{q}^T \mathcal{K}_4 \mathbf{r}_0 = \frac{2440}{287},$$

where the fourth-harmonic weighted insertion $\mathcal{K}_4$ is defined in the retained derivation below. Rational checks independent of the first derivation confirm the contractions and the

305/113 ratio. The three full periodic solutions agree at relative complex discrepancy below 3.80×10^{-65} , with no numerical failures. This is a counterexample to identifying structural geometry from the leading scalar zero, rather than a new physical symmetry.

5 Retained benchmark formulation and evidence

6 Model and affine formulation

6.1 Markovian two-level dynamics

We use angular frequencies and rates throughout, with $\Omega > 0$, $\nu > 0$, $\gamma_1 \geq 0$ and $\gamma_\phi \geq 0$. The sign of a modulation amplitude can be absorbed into its phase. Physical periodic-state claims below assume the stated attracting solution exists; the pure-dephasing-only case is treated separately.

We consider a two-level system with density operator ρ governed by a Markovian Gorini–Kossakowski–Sudarshan–Lindblad equation [1, 2, 3],

$$\dot{\rho} = -i[H(t), \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right), \quad (4)$$

where $\hbar = 1$. The Bloch representation is

$$\rho = \frac{1}{2} (\mathbb{I} + u\sigma_x + v\sigma_y + w\sigma_z), \quad \mathbf{r} = (u, v, w)^T. \quad (5)$$

The excited-state population is

$$P_e(t) = \frac{1 + w(t)}{2}. \quad (6)$$

Spontaneous emission and pure dephasing are represented by

$$L_1 = \sqrt{\gamma_1} \sigma_-, \quad L_2 = \sqrt{\gamma_\phi/2} \sigma_z, \quad (7)$$

with transverse damping

$$\Gamma_2 = \frac{\gamma_1}{2} + \gamma_\phi. \quad (8)$$

For longitudinal modulation and static detuning,

$$H_z(t) = \frac{\Delta}{2} \sigma_z + \frac{\Omega}{2} \sigma_x + \lambda \sin(\nu t) \sigma_z, \quad (9)$$

the Bloch equations are

$$\dot{u} = -\Gamma_2 u - [\Delta + 2\lambda \sin(\nu t)]v, \quad (10a)$$

$$\dot{v} = [\Delta + 2\lambda \sin(\nu t)]u - \Gamma_2 v - \Omega w, \quad (10b)$$

$$\dot{w} = \Omega v - \gamma_1(w + 1). \quad (10c)$$

They have the affine form

$$\dot{\mathbf{r}} = [M_0 + \lambda \sin(\nu t) M_1] \mathbf{r} + \mathbf{c}, \quad (11)$$

with

$$M_0 = \begin{pmatrix} -\Gamma_2 & -\Delta & 0 \\ \Delta & -\Gamma_2 & -\Omega \\ 0 & \Omega & -\gamma_1 \end{pmatrix}, \quad (12)$$

$$M_z = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ -\gamma_1 \end{pmatrix}.$$

For the transverse amplitude-modulation control model,

$$H_x(t) = \frac{\Delta}{2}\sigma_z + \left[\frac{\Omega}{2} + \lambda \sin(\nu t) \right] \sigma_x, \quad (13)$$

the time-independent matrix M_0 and source $\mathbf{c}$ are unchanged, whereas

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}. \quad (14)$$

Most explicit transverse-control results below are given at resonance and with spontaneous emission only, so that the two modulation geometries share the same unmodulated stationary state.

For comparison with phase-modulated atomic-candle work, a phase-following rotating frame maps a transverse carrier with phase depth m to a detuning modulation of amplitude $m\nu = 2\lambda$ [4]. Thus a fixed- m frequency scan differs from the fixed- λ scan studied here; coherence-sign and modulation-phase conventions do not change the population amplitudes.

6.2 Stationary state and observable normalization

Here $\mathbf{r}_0$ denotes the unmodulated stationary state used in the lowest-order response chain. The full driven zero-Fourier sector can differ from it; the full solvers include that change. The unmodulated stationary state satisfies

$$M_0 \mathbf{r}_0 + \mathbf{c} = 0. \quad (15)$$

For the unified longitudinal model,

$$u_0 = -\frac{\Delta}{\Gamma_2} v_0, \quad (16a)$$

$$v_0 = \frac{\gamma_1 \Gamma_2 \Omega}{\Gamma_2 \Omega^2 + \gamma_1 (\Delta^2 + \Gamma_2^2)}, \quad (16b)$$

$$w_0 = -\frac{\gamma_1 (\Delta^2 + \Gamma_2^2)}{\Gamma_2 \Omega^2 + \gamma_1 (\Delta^2 + \Gamma_2^2)}. \quad (16c)$$

At resonance with spontaneous emission only,

$$u_0 = 0, \quad v_0 = \frac{2\Omega\gamma}{\gamma^2 + 2\Omega^2}, \quad w_0 = -\frac{\gamma^2}{\gamma^2 + 2\Omega^2}, \quad (17)$$

where $\gamma \equiv \gamma_1$.

For a real periodic Bloch vector,

$$\mathbf{r}(t) = \sum_{n=-\infty}^{\infty} \mathbf{r}_n e^{in\nu t}, \quad \mathbf{r}_{-n} = \mathbf{r}_n^*. \quad (18)$$

A linear Bloch observable $O = O_c + \mathbf{q}^T \mathbf{r}$ has complex Fourier coefficient $\mathcal{O}_n = \mathbf{q}^T \mathbf{r}_n$ and real sinusoidal amplitude $2|\mathcal{O}_n|$. For the Bloch population coordinate w , $\mathbf{q}^T = (0, 0, 1)$. The measured excited-state population is $P_e = (1 + w)/2$, so its nonzero complex coefficient is $p_n = w_n/2$. The factor $1/2$ therefore cancels the factor of two from the conjugate Fourier pair: the $n > 0$ contribution to $P_e(t)$ is $\text{Re}[w_n \exp(in\nu t)]$, with sinusoidal amplitude $|w_n|$. In contrast, the corresponding amplitude in $w(t)$ itself is $2|w_n|$. Hence

$$A_{n\nu} = |w_n|. \quad (19)$$

The stationary-state shift $\mathbf{x} = \mathbf{r} - \mathbf{r}_0$ puts the affine equation into the standard single-input, single-output bilinear form $\dot{\mathbf{x}} = A\mathbf{x} + N\mathbf{x}u + bu$, with $A = M_0$, $N = M_1$, $b = M_1\mathbf{r}_0$, $u = \lambda \sin(\nu t)$, and output $\mathbf{q}^T \mathbf{x}$ [5, 6]. Thus the positive-harmonic resolvent chain is an established generalized frequency-response construction, evaluated here for a particular quantum observable and stationary state.

7 Observable-projected high-frequency coefficients

7.1 Fourier hierarchy and resolvent solution

Inserting the Fourier series of Eq. (18) into Eq. (11) yields

$$(in\nu I - M_0)\mathbf{r}_n = \frac{\lambda}{2i} M_1(\mathbf{r}_{n-1} - \mathbf{r}_{n+1}) + \mathbf{c} \delta_{n0}. \quad (20)$$

For $n \neq 0$ we define

$$R_n = (in\nu I - M_0)^{-1}. \quad (21)$$

At the lowest modulation order capable of reaching sector $n > 0$, the positive-harmonic chain reads

$$\mathbf{r}_n^{(n)} = \left(\frac{\lambda}{2i}\right)^n R_n M_1 R_{n-1} M_1 \cdots R_1 M_1 \mathbf{r}_0. \quad (22)$$

The modulation-induced correction to the zero mode contributes to $\mathbf{r}_n$ only two orders later, at λ^{n+2} .

For fixed n and $\|M_0\|/\nu \ll 1$,

$$R_n = \frac{1}{in\nu} \left[I + \frac{M_0}{in\nu} + \mathcal{O}(\nu^{-2}) \right]. \quad (23)$$

Two approximations are being used here and should not be conflated. The ratio λ/ν is an expansion scale for coupling nonzero Fourier sectors, whereas expansion of each resolvent requires $\|M_0\|/\nu \ll 1$. Conditioning of the zero sector and the stationary state can also affect truncation in modulation order, so a small λ/ν alone is not an error certificate. We compare the order- λ^h chain separately with the full periodic solution.

At its lowest modulation order, the all-positive chain for harmonic h is the standard single-tone bilinear response [6]. Expanding its resolvents yields weighted sums of the inverse-power matrix words familiar from bilinear realization theory [5]. The first nonzero *summed projected coefficient* determines the roll-off of this perturbative harmonic; individual nonzero words can still cancel within a common order. Related Markov-parameter/roll-off results are established for linear time-varying systems under regularity and stability assumptions [7, 8]. The latter are adjacent systems-theory antecedents, rather than a direct nonlinear-harmonic or Lindblad result.

7.2 Second-harmonic coefficient

At order λ^2 ,

$$\mathbf{r}_2^{(2)} = \left(\frac{\lambda}{2i}\right)^2 R_2 M_1 R_1 M_1 \mathbf{r}_0. \quad (24)$$

Expanding the two resolvents gives

$$\begin{aligned} \mathbf{q}^T \mathbf{r}_2^{(2)} &= \frac{\lambda^2}{8\nu^2} \mathbf{q}^T M_1^2 \mathbf{r}_0 \\ &\quad - \frac{i\lambda^2}{16\nu^3} \mathbf{q}^T (M_0 M_1^2 + 2M_1 M_0 M_1) \mathbf{r}_0 \\ &\quad + \mathcal{O}(\lambda^2 \nu^{-4}, \lambda^4). \end{aligned} \quad (25)$$

Equation (25) separates three possibilities:

1. If $\mathbf{q}^T M_1^2 \mathbf{r}_0 \neq 0$, the response generically begins at $\lambda^2 \nu^{-2}$.
2. If $\mathbf{q}^T M_1^2 \mathbf{r}_0 = 0$ but the coefficient on the second line is nonzero, the response begins at $\lambda^2 \nu^{-3}$.
3. If both coefficients vanish, the leading exponent is determined by higher orders in the resolvent expansion.

We call the cancellation *structural* when $\mathbf{q}^T M_1^2 = 0$ independently of $\mathbf{r}_0$, and *state dependent* when only the contraction with the particular stationary state vanishes.

For longitudinal modulation,

$$M_z^2 = \text{diag}(-4, -4, 0), \quad \mathbf{q}^T M_z^2 = 0, \quad \mathbf{q}^T M_z = 0. \quad (26)$$

The ν^{-2} term and the $M_z M_0 M_z$ contribution vanish structurally. Since

$$\mathbf{q}^T M_0 M_z^2 \mathbf{r}_0 = -4\Omega v_0, \quad (27)$$

we obtain

$$w_2^{(z)} = \frac{i\lambda^2 \Omega v_0}{4\nu^3} + \mathcal{O}(\lambda^2 \nu^{-4}, \lambda^4), \quad (28)$$

and therefore

$$\boxed{A_{2\nu}^{(z)} \simeq \frac{\lambda^2 \Omega |v_0|}{4\nu^3}.} \quad (29)$$

At resonance with spontaneous emission,

$$\boxed{A_{2\nu}^{(z)} \simeq \frac{\lambda^2 \Omega^2 \gamma}{2\nu^3 (\gamma^2 + 2\Omega^2)}.} \quad (30)$$

For transverse amplitude modulation,

$$\mathbf{q}^T M_x^2 \mathbf{r}_0 = -4w_0, \quad (31)$$

so the generic term survives:

$$\boxed{A_{2\nu}^{(x)} \simeq \frac{\lambda^2 |w_0|}{2\nu^2} = \frac{\lambda^2 \gamma^2}{2\nu^2 (\gamma^2 + 2\Omega^2)}.} \quad (32)$$

For $\gamma = \Omega$,

$$\frac{A_{2\nu}^{(x)}}{A_{2\nu}^{(z)}} \simeq \frac{\nu}{\Omega}. \quad (33)$$

7.3 Fourth-harmonic corollary

The fourth Fourier sector first appears at order λ^4 , for which the positive-harmonic chain is

$$\mathbf{r}_4^{(4)} = \left(\frac{\lambda}{2i}\right)^4 R_4 M_1 R_3 M_1 R_2 M_1 R_1 M_1 \mathbf{r}_0. \quad (34)$$

Its first two high-frequency orders are

$$\begin{aligned} \mathbf{q}^T \mathbf{r}_4^{(4)} &= \frac{\lambda^4}{384\nu^4} \mathbf{q}^T M_1^4 \mathbf{r}_0 \\ &\quad - \frac{i\lambda^4}{384\nu^5} \mathbf{q}^T \mathcal{K}_4 \mathbf{r}_0 + \mathcal{O}(\lambda^4 \nu^{-6}, \lambda^6), \end{aligned} \quad (35)$$

where

$$\mathcal{K}_4 = \frac{1}{4} M_0 M_1^4 + \frac{1}{3} M_1 M_0 M_1^3 + \frac{1}{2} M_1^2 M_0 M_1^2 + M_1^3 M_0 M_1. \quad (36)$$

A nonzero $\mathbf{q}^T M_1^4 \mathbf{r}_0$ therefore gives the generic $\lambda^4 \nu^{-4}$ response, whereas a vanishing leading projection shifts the response to $\lambda^4 \nu^{-5}$ if $\mathbf{q}^T \mathcal{K}_4 \mathbf{r}_0 \neq 0$.

For longitudinal modulation,

$$M_z^4 = \text{diag}(16, 16, 0), \quad \mathbf{q}^T M_z^4 = 0. \quad (37)$$

Only the first term in Eq. (36) survives population projection, and

$$\mathbf{q}^T M_0 M_z^4 \mathbf{r}_0 = 16\Omega v_0. \quad (38)$$

Therefore

$$w_4^{(z)} = -\frac{i\lambda^4 \Omega v_0}{96\nu^5} + \mathcal{O}(\lambda^4 \nu^{-6}, \lambda^6), \quad (39)$$

and

$$\boxed{A_{4\nu}^{(z)} \simeq \frac{\lambda^4 \Omega |v_0|}{96\nu^5}.} \quad (40)$$

At resonance with spontaneous emission,

$$\boxed{A_{4\nu}^{(z)} \simeq \frac{\lambda^4 \Omega^2 \gamma}{48\nu^5 (\gamma^2 + 2\Omega^2)}.} \quad (41)$$

For transverse amplitude modulation,

$$\mathbf{q}^T M_x^4 \mathbf{r}_0 = 16w_0 \neq 0, \quad (42)$$

so

$$\boxed{A_{4\nu}^{(x)} \simeq \frac{\lambda^4 |w_0|}{24\nu^4} = \frac{\lambda^4 \gamma^2}{24\nu^4 (\gamma^2 + 2\Omega^2)}.} \quad (43)$$

Table 1 collects the four response classes.

Table 1: Observable-level high-frequency classes for resonant population readout. “Structural zero” means that the leading projected power vanishes for every stationary state

Modulation	Harmonic	Leading projection	Class	Leading population amplitude
Longitudinal M_z	2ν	$\mathbf{q}^T M_z^2 = 0$	structural ν^{-3}	$\lambda^2 \Omega v_0 / (4\nu^3)$
Transverse M_x	2ν	$\mathbf{q}^T M_x^2 \mathbf{r}_0 = -4w_0$	generic ν^{-2}	$\lambda^2 w_0 / (2\nu^2)$
Longitudinal M_z	4ν	$\mathbf{q}^T M_z^4 = 0$	structural ν^{-5}	$\lambda^4 \Omega v_0 / (96\nu^5)$
Transverse M_x	4ν	$\mathbf{q}^T M_x^4 \mathbf{r}_0 = 16w_0$	generic ν^{-4}	$\lambda^4 w_0 / (24\nu^4)$

8 Detuning, dissipation, and domain of validity

8.1 Unified stationary-coherence dependence

Static detuning and additional pure dephasing alter M_0 but leave M_z unchanged. Consequently, the structural identities $\mathbf{q}^T M_z^2 = \mathbf{q}^T M_z^4 = 0$ remain exact. Detuning and dissipation enter the leading longitudinal harmonics through the stationary coherence in Eq. (16):

$$A_{2\nu}^{(z)} \simeq \frac{\lambda^2 \gamma_1 \Gamma_2 \Omega^2}{4\nu^3 [\Gamma_2 \Omega^2 + \gamma_1 (\Delta^2 + \Gamma_2^2)]}. \quad (44)$$

$$A_{4\nu}^{(z)} \simeq \frac{\lambda^4 \gamma_1 \Gamma_2 \Omega^2}{96\nu^5 [\Gamma_2 \Omega^2 + \gamma_1 (\Delta^2 + \Gamma_2^2)]}. \quad (45)$$

Their common seed cancels in the ratio

$$R_{42} = \frac{24\nu^2 A_{4\nu}}{\lambda^2 A_{2\nu}} \longrightarrow 1. \quad (46)$$

Equation (46) is a leading-order statement. Its finite-frequency departure from unity measures the unequal higher-order corrections received by the two harmonics.

The frequency at which Eqs. (44) and (45) become accurate depends on Δ and Γ_2 . The norm used here is the induced Euclidean norm, $\|M_0\|_2 = \sigma_{\max}(M_0)$; each sector has ratio $\sigma_{\max}(M_0)/(n\nu)$, with the largest at $n = 1$. The relevant operator condition is

$$\epsilon_{\text{op}} = \frac{\|M_0\|_2}{\nu} \ll 1, \quad (47)$$

which requires progressively larger ν as detuning or transverse damping is increased.

8.2 Pure dephasing gives exact zero

For $\gamma_1 = 0$, $\gamma_\phi > 0$, and $\Omega \neq 0$, the affine source vanishes and the Bloch equations are unital. The maximally mixed state $\mathbf{r} = 0$ is an exact solution for arbitrary λ , ν , and Δ . Moreover,

$$\frac{d}{dt}(u^2 + v^2 + w^2) = -2\gamma_\phi(u^2 + v^2) \leq 0. \quad (48)$$

The largest invariant set with $u = v = 0$ also requires $w = 0$ because $\dot{v} = -\Omega w$. Hence the periodic attractor is the maximally mixed state, and

$$A_{n\nu} = 0 \quad \text{for every } n. \quad (49)$$

The small nonzero values produced by time-domain quadrature at machine precision are therefore numerical floor, not physical response.

8.3 Projected amplitude-change diagnostic

A small ϵ_{op} controls the resolvent in operator norm, but it does not ensure a uniformly small *relative* error in every observable. When $|v_0|$ is strongly suppressed, subleading projected coefficients can compete with the leading one. We write the order- λ^h coefficient of harmonic h as

$$w_h^{(h)} = \lambda^h \sum_{s=0}^{\infty} \frac{\mathcal{C}_{h,s}}{\nu^{h+s}}, \quad (50)$$

where the coefficients are generated by all weak compositions of the additional resolvent order; an explicit formula is given in Appendix D. For the longitudinal population channel, $\mathcal{C}_{2,0} = \mathcal{C}_{4,0} = 0$, and the leading term is $s = 1$.

For $|\mathcal{C}_{h,1}| > 0$, define

$$A_h^{(0)} = \lambda^h \frac{|\mathcal{C}_{h,1}|}{\nu^{h+1}}, \quad (51)$$

$$A_h^{[2]} = \lambda^h \left| \frac{\mathcal{C}_{h,1}}{\nu^{h+1}} + \frac{\mathcal{C}_{h,2}}{\nu^{h+2}} + \frac{\mathcal{C}_{h,3}}{\nu^{h+3}} \right|, \quad (52)$$

and

$$\boxed{\eta_h = \left| \frac{A_h^{[2]}}{A_h^{(0)}} - 1 \right|, \quad \eta_* = \max(\eta_2, \eta_4).} \quad (53)$$

The operator ratio ϵ_{op} controls a Neumann expansion of each resolvent when it is below unity. The amplitude-change diagnostic η_h records the change produced by retaining two additional inverse-frequency orders at fixed modulation order. It does not bound the omitted tail, establish convergence, or control higher modulation orders. A small value can coexist with cancellations in the retained terms; comparison with the full periodic solution remains necessary.

For classification of the numerical data in this work, the strict asymptotic core is defined by

$$\epsilon_{\text{op}} \leq 0.05, \quad \lambda/\nu \leq 0.04, \quad \eta_* \leq 0.02, \quad (54)$$

with dimensionless harmonic-balance residual at most 10^{-12} . Truncation, conjugacy, physicality and coefficient-precision checks are reported separately from the three asymptotic thresholds. These numerical thresholds define the data set used here; they are not proposed as universal bounds.

9 Numerical methods

9.1 Independent periodic-state solutions

The time-domain implementation integrates the real Bloch equations independently of the Fourier implementation. Over $T = 2\pi/\nu$, the affine fundamental map is $\mathbf{r}(T) = A\mathbf{r}(0) + \mathbf{b}$ and the periodic state solves $(I - A)\mathbf{r}_* = \mathbf{b}$. We test the spectral radius of the homogeneous 3×3 block A , not the unavoidable unit eigenvalue of the augmented affine map. Closure, fixed-point residual and sampled density-matrix positivity are retained. Nonzero coefficients use one-period Fourier analysis with $\exp(-in\nu t)$ and subtraction of a constant population to reduce cancellation; that subtraction leaves every nonzero harmonic unchanged.

Binary64 diagnostic integrations use DOP853 at relative/absolute tolerances $10^{-11}/10^{-13}$ with maximum step $T/128$, and $10^{-12}/10^{-14}$ with $T/256$, with selected $(10^{-13}, 10^{-15}, T/512)$

checks. Adaptive real/imaginary quadratures use 64 subintervals with absolute and relative tolerances 10^{-16} and 10^{-12} . Centered discrete Fourier transforms (DFTs) use 32768 samples, and 48 fixed endpoint/reference points also use 16384 and 65536 samples. No transient cutoff or data-dependent Fourier window is used.

The primary fourth-harmonic time solutions use piecewise arbitrary-precision Taylor propagation in $\theta = \nu t$. Coarse and fine settings are respectively 50 digits/degree 32/24 segments and 70 digits/degree 44/32 segments, for both the monodromy map and direct Fourier integrals. Refinement changes, final Taylor terms, propagated error estimates and conditioning of $(I - A)$ supply operational coefficient-floor estimates. These estimates are not proven remainder or complete roundoff bounds.

An independently transcribed two-sided harmonic-balance system truncates Eq. (20) to $|n| \leq N$. Binary64 diagnostics use $N = 8, 10, 12, 14, 16$; primary arbitrary-precision solutions use 50 digits/ $N = 16$ and 70 digits/ $N = 20$. With $\gamma_1, \Gamma_2 > 0$, the Hermitian part of its matrix is the damping diagonal. Thus $\sigma_{\min}(H) \geq \min(\gamma_1, \Gamma_2)$ bounds the algebraic solution error by the residual divided by this lower bound. It does not bound all floating-point or truncation errors. Conjugacy, reconstructed positivity and coefficient refinement are checked separately. The two full solvers share physical input tuples and output convention, but no equation-transcription, integration, extraction or floor implementation.

9.2 Frozen primary grids and diagnostic retention

The dedicated campaign has 270 physical points and 2682 registered solver, Fourier-extraction and perturbative-reference configurations. For each modulation model, the primary fourth-harmonic set contains 37 distinct points (74 in total), all receiving both full arbitrary-precision methods and an independent order- λ^4 resolvent reference. Baseline $\Omega/2\pi = \gamma_1/2\pi = 0.5$ MHz, $\Delta = \gamma_\phi = 0$. The frequency fit uses all six values $\nu/2\pi$ equal to 15, 20, 25, 30, 40 and 50 MHz at $\lambda/2\pi = 0.5$ MHz. Strength fits use all six $\lambda/2\pi$ equal to 0.1, 0.15, 0.2, 0.3, 0.4 and 0.5 MHz at $\nu/2\pi = 20, 40, 80$ MHz. The joint fit uses the complete 6×5 strength/frequency grid with frequencies 15,20,30,40,50 MHz. No failed member or alternate endpoint can replace a required primary point.

Control conditions are $\lambda/\nu \leq 0.04$ and $\epsilon_{\text{op}} \leq 0.05$. Each primary population coefficient must exceed its independently estimated floor by at least 10^4 , satisfy periodic-closure, density-matrix physicality and convergence checks, and agree across methods to the predeclared relative complex tolerance 10^{-10} . The primary floor is ten times the largest same-method refinement change, residual bound or propagated error estimate, with a precision-scale minimum; method disagreement is not put into either floor. Multipliers 5, 10 and 30 and root-sum-square combinations are retained as sensitivity analyses. Binary64 floor-limited points remain diagnostics. Eight fixed low-signal anchors additionally use 80/120 digits and separate full time, harmonic-balance and perturbative solutions.

Ordinary least-squares log fits report deterministic finite-domain exponents, descriptive 95% residual-scatter intervals and residuals. They are not stochastic measurement uncertainty. Predeclared class tests require exponent deviations at most 0.1, $R^2 \geq 0.995$ and maximum log residual at most 0.1, with exact primary membership. Whether a descriptive interval includes the infinite-frequency target is reported separately: finite-frequency bias can exclude that target even for a correctly converging model. The joint residual degrees of freedom are $N_{\text{points}} - 3$.

9.3 Parameter-diverse ensemble and reproducibility

The ensemble rebuild freezes only parameters from four supplied input tables: detuning, mixed-bath, combined-grid and historical randomized draws. Their 695 source rows yield 683 unique points; every duplicate source-row membership is retained. The historical randomized seed/sampler is undocumented, so the supplied draws are fixed inputs rather than a newly regenerated sampling process. The exact parameter tuples and their source-row memberships are retained and replayable. All ensemble percentiles, correlations and class fractions describe these particular fixed tuples; they do not establish representativeness of a parameter distribution, sampling uncertainty or statistical generalization. No deposited response amplitude enters a new solve or classification.

An independently transcribed dimensionless scalar harmonic-balance producer evaluates every point at $N = 8, 10, 12, 16$, computes the projected coefficients through $s = 3$ in Eq. (50), and writes the complete complex-response master, thresholds, class statistics and figure inputs. All 683 points also receive the independent block solver at 50 digits with $N = 16$ and 70 digits with $N = 20$. Sixteen parameter-selected anchors, fixed before their solution, receive a direct 70-digit time-domain solution. The 314-member strict mask is evaluated from unrounded inputs with the three thresholds of Eq. (54), residual and additional solver-quality checks; observed leading errors are not used to select it. Empirical percentiles use linear/type-7 interpolation and relative amplitude errors use the stated theoretical prediction in the denominator.

A deposited v2.0.0 isolated-environment rebuild with the six deposited exact package versions regenerates the complete 683-point master and statistics byte-for-byte. All deposited raw configurations, failed implementation attempts, exclusions, source hashes, environments, commands and figure-row identities are retained. This establishes the v2.0.0 executable producer; it does not restore the historical producer’s undocumented provenance.

9.4 Use of computational assistance

Codex assisted with implementation, source comparison, editorial proposals and audit coordination. Protocols, hypotheses, settings and primary memberships were frozen before the corresponding computations; synthetic analytic tests, independent equation transcriptions, refinement, physicality and clean regeneration check its computational outputs. Agent agreement is not scientific validation. The author retains final responsibility for critical review and selection of the results and for verification of the cited original literature.

10 Results

10.1 Second-harmonic longitudinal–transverse comparison

Figure 1 compares the two modulation geometries for

$$\Omega/2\pi = \gamma/2\pi = 0.5 \text{ MHz}, \quad \lambda/2\pi = 0.5 \text{ MHz}. \quad (55)$$

Figure 1 overlays full time-domain and harmonic-balance amplitudes across the fixed frequency grid. The predicted leading coefficients provide the compensated limits below. At $\nu/2\pi = 40 \text{ MHz}$,

$$A_{2\nu}^{(z)} = 3.25519 \times 10^{-7}, \quad A_{2\nu}^{(x)} = 2.60562 \times 10^{-5}, \quad (56)$$

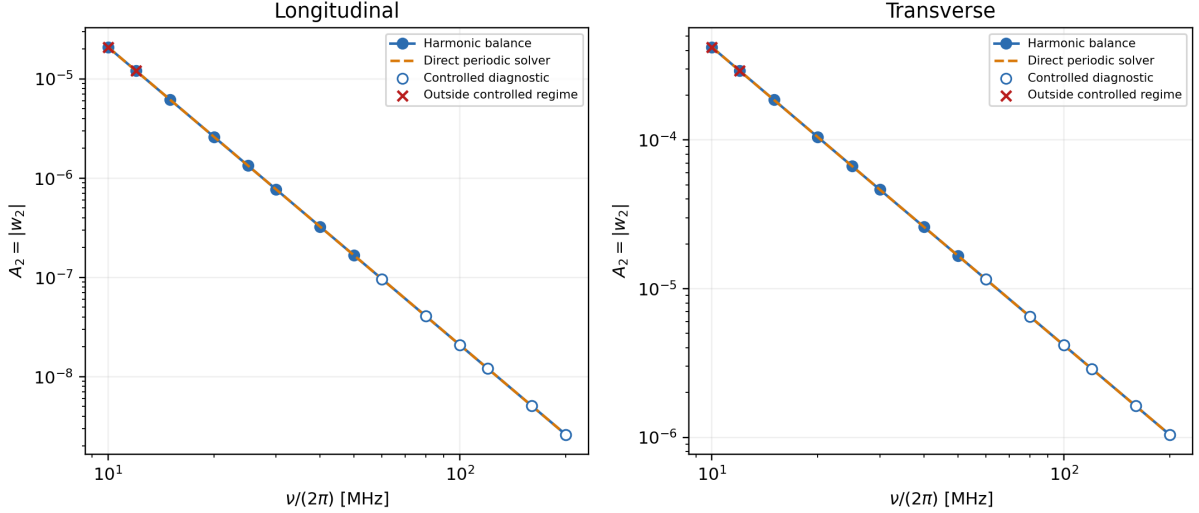


Fig. 1 Full second-harmonic response at the baseline parameters in Eq. (55). Left: longitudinal modulation. Right: matched transverse modulation. Symbols show harmonic balance and dashed curves the independently integrated periodic solution. The fixed frequencies range from 10 to 200 MHz; low-frequency points outside the primary control regime remain displayed as diagnostics. The analytic powers and compensated limits are given in the text

with ratio $A_{2\nu}^{(x)}/A_{2\nu}^{(z)} = 80.0452$, compared with the asymptotic prediction $\nu/\Omega = 80$.

The compensated longitudinal quantity

$$\frac{A_{2\nu}^{(z)} \nu^3}{\lambda^2 \Omega} \longrightarrow \frac{\Omega \gamma}{2(\gamma^2 + 2\Omega^2)} \quad (57)$$

and its transverse counterpart

$$\frac{A_{2\nu}^{(x)} \nu^2}{\lambda^2} \longrightarrow \frac{\gamma^2}{2(\gamma^2 + 2\Omega^2)} \quad (58)$$

share the value $1/6$ when $\gamma = \Omega$.

10.2 Fourth-harmonic benchmark

Figure 2 compares the full independent arbitrary-precision solutions with the leading asymptotic amplitudes on complete frozen grids. All 74 primary points pass the predeclared numerical and comparison gates, and all registered configurations complete. A compact summary of the harmonic-balance fits is The joint model is $\log A_{4\nu} = c + p \log \lambda + q \log \nu$. The longitudinal frequency fit has $R^2 = 0.9999999998$ and maximum log residual 4.16×10^{-5} . Its descriptive 95% interval is $[-5.000283, -5.000095]$: it excludes -5 because the calculation is at finite frequency, rather than because of numerical disagreement. Exact-target interval containment is therefore not an asymptotic class test. No post-selected correction regression is quoted.

Across the primary points, the maximum full-solution versus order- λ^4 complex discrepancy is 1.49×10^{-4} for longitudinal and 5.37×10^{-4} for transverse modulation. The maximum full-amplitude versus leading-amplitude discrepancies are respectively 0.0263% and 0.921%.

Table 2: Finite-domain fourth-harmonic log-fit exponents. Both independently transcribed methods give the displayed values. Frequency fits contain six points and joint fits 30 per model; each strength fit contains six points. These are finite-domain slopes, not exact asymptotic exponents

Fit	Longitudinal	Transverse
Frequency at $\lambda/2\pi = 0.5$ MHz	−5.000189	−4.006594
Strength at $\nu/2\pi = 20$ MHz	4.000047	4.000161
Strength at $\nu/2\pi = 40$ MHz	4.000012	4.000041
Strength at $\nu/2\pi = 80$ MHz	4.000003	4.000010
Joint strength exponent p	4.000034	4.000116
Joint frequency exponent q	−5.000126	−4.006501

These amplitude errors must be distinguished from complex coefficient errors: the order- λ^4 coefficient differs from its leading complex term by as much as 18.6% in the transverse set, substantially through finite-frequency phase corrections. The data support the scoped leading powers and amplitude accuracy, not uniformly negligible phase corrections.

The two independent primary solutions agree at relative complex discrepancy below 10^{-20} ; all eight predeclared low-signal full-solver anchors also meet their 10^{-20} comparison and refinement targets. Such numerical agreement is not a proof of a complete error bound. The compensated longitudinal amplitude obeys

$$C_4^{(z)} = \frac{A_{4\nu}^{(z)} \nu^5}{\lambda^4 \Omega} \longrightarrow \frac{x}{48(x^2 + 2)}, \quad x = \gamma_1/\Omega, \quad (59)$$

with target $1/144$ at $x = 1$. This quantity is dimensionless: the frequency powers in $\nu^5/(\lambda^4 \Omega)$ cancel. At resonance without additional pure dephasing, $v_0 = 2x/(x^2 + 2)$, so the displayed limit follows for any $\Omega > 0$; it does not require setting $\Omega = 1$. Angular frequencies or their values divided by 2π give the same compensated ratio when used consistently. The supplementary dissipation scan retains uncontrolled large-damping points as diagnostics.

10.3 Fundamental response, symmetry and signed detuning

The transverse model also has a resolved fundamental. At the leading retained order,

$$p_1^{(x)} \simeq -\frac{\lambda v_0}{2\nu}, \quad Q_1 \equiv \frac{p_1^{(x)}}{\lambda/\nu} \longrightarrow -\frac{v_0}{2} = -\frac{1}{3} \quad (60)$$

for the resonant $\gamma_1 = \Omega$ benchmark. Both independently integrated and harmonic-balance data give finite-domain strength and frequency slopes 0.99975 and -0.99961 . On the complete predetermined 20–80 MHz primary frequency set, the maximum relative complex error of Q_1 from $-1/3$ is 1.252%, decreasing to 0.313% at 80 MHz. Figure 3 shows the real and imaginary parts, so the phase correction is visible rather than hidden by an amplitude fit.

The resonant longitudinal odd-harmonic null has a separate exact symmetry. Let $S = \text{diag}(-1, 1, 1)$. At $\Delta = 0$, $SM_0S = M_0$, $SM_zS = -M_z$ and $S\mathbf{c} = \mathbf{c}$. Thus $S\mathbf{r}(t+T/2)$ obeys the same equation as $\mathbf{r}(t)$. Uniqueness of the periodic solution implies $\mathbf{r}(t+T/2) = S\mathbf{r}(t)$; since S fixes w , all odd population harmonics vanish exactly. For $\gamma_1, \Gamma_2 > 0$, uniqueness follows from strict contraction of the difference between two Bloch solutions. This symmetry

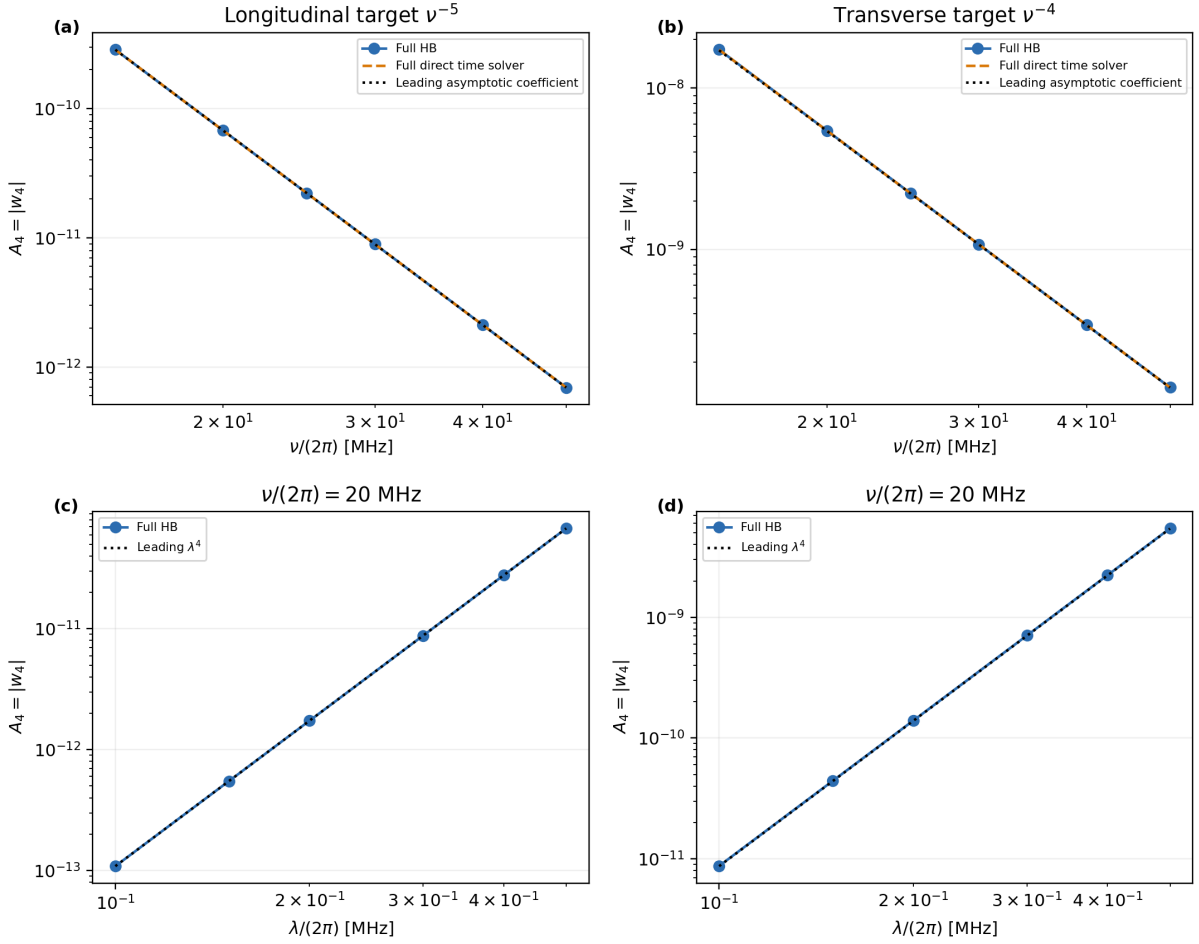


Fig. 2 Fourth-harmonic full-solution benchmark for the matched models. Top: the six fixed primary frequencies at $\lambda/2\pi = 0.5$ MHz, with both full arbitrary-precision methods and the leading coefficient. Bottom: the six fixed modulation strengths at $\nu/2\pi = 20$ MHz. The common baseline is Eq. (55), with zero detuning/pure dephasing. Complete memberships, floors, independent order- λ^4 references and finite-frequency phase/amplitude corrections are retained in the supplement

removes a harmonic family; the projected high-frequency calculation instead identifies which coefficient survives inside a symmetry-allowed even sector.

At signed detuning, $SM_0(\Delta)S = M_0(-\Delta)$ yields

$$p_n(-\Delta) = (-1)^n p_n(\Delta). \quad (61)$$

Parity alone does not prove a nonzero odd response. The deposited paired scans at $\Delta/\Omega = \pm 0.25, \pm 0.5, \pm 1, \pm 2$, frequencies 20 and 40 MHz and baseline modulation resolve a nonzero fundamental in both methods, retain its signed parity, and show a resolved departure from half-period population symmetry. All resonant $n = 1, 3, 5$ coefficients remain below the respective operational null floors. Separate phase controls at $\phi = 0, \pi/3, \pi/2, \pi$ satisfy $p_n(\phi) = \exp(in\phi)p_n(0)$ within the preregistered tolerances. The transverse model is not subject to the longitudinal odd-sector null.

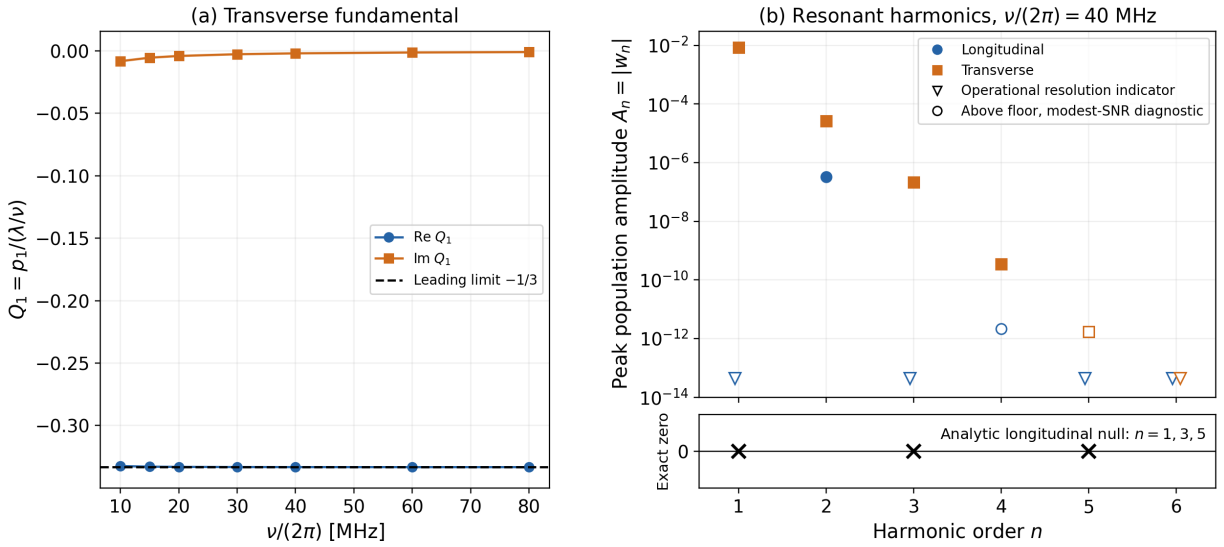


Fig. 3 Fundamental and symmetry controls. Left: complex Q_1 in the transverse model at $\lambda/2\pi = 0.5$ MHz, with baseline $\Omega = \gamma_1$, zero detuning and no pure dephasing. Right: binary64 harmonic diagnostics for both geometries at $\nu/2\pi = 40$ MHz. Downward triangles mark operational peak-resolution indicators $2 \max(F_A, F_B)$; they are neither measured nonzero responses nor certified bounds. Hollow circles and squares denote above-floor, modest signal-to-floor diagnostics; filled symbols pass the registered nonzero check. The separate linear strip displays the analytic longitudinal identities $A_{1\nu} = A_{3\nu} = A_{5\nu} = 0$, following from half-period symmetry and uniqueness. Coincident resolution markers are shifted by ± 0.045 in harmonic order for visibility. All complex coefficients, resolution indicators and display offsets are retained with source-row identities

10.4 Unified seed collapse and cross-harmonic asymptotic ratio

The detuning, mixed-bath, combined-grid, and randomized data are consolidated in Fig. 4. The leading predictions are

$$\frac{A_{2\nu}\nu^3}{\lambda^2\Omega} \longrightarrow \frac{|v_0|}{4}, \quad \frac{A_{4\nu}\nu^5}{\lambda^4\Omega} \longrightarrow \frac{|v_0|}{96}. \quad (62)$$

Ratios R_{42} and η_h are evaluated only for nonzero denominators; the exact null model is excluded from these ratios. The same points are classified using ϵ_{op} and η_* . In the strict core, the 95th-percentile relative errors of the leading formulae are

$$\begin{aligned}\delta A_{2\nu} &= 0.721\%, \quad \delta A_{4\nu} = 1.458\%, \\ |R_{42} - 1| &= 0.724\%.\end{aligned}\tag{63}$$

The respective maximum errors are 0.965%, 1.915%, and 0.961%, so the observed leading errors are below 2% throughout the 314-member core. This error level was not used to select the core.

Including the next two resolvent orders reduces the strict-core 95th-percentile errors to

$$\delta A_{2\nu}^{[2]} = 0.04198\%, \quad \delta A_{4\nu}^{[2]} = 0.03499\%.\tag{64}$$

The logarithmic Spearman correlations between η_h and the actual leading relative errors are 0.989 and 0.983 for $h = 2$ and 4 ; these describe this fixed numerical ensemble. By contrast, there are points with $\epsilon_{\text{op}} \simeq 0.023$ but strongly suppressed v_0 for which leading relative errors exceed unity. The projected amplitude-change diagnostic flags retained-order sensitivity near a small seed, while full-solution comparison establishes the actual errors.

For the combined grid at $\nu/2\pi = 40$ MHz, median errors are 0.585% for $A_{2\nu}$, 0.805% for $A_{4\nu}$, and 0.252% for R_{42} . The supplied randomized parameter draws include crossover and small-seed regions; its 95th-percentile R_{42} error is correspondingly much larger and is not represented as asymptotic-core performance.

10.5 Exact pure-dephasing null result

The 36 deposited pure-dephasing control combinations specified in Appendix C, with $\gamma_1 = 0$, return

$$A_{2\nu} = A_{4\nu} = 0\tag{65}$$

with zero linear-system residual in the zero-source harmonic-balance calculation. The independent homogeneous monodromy check also supports an attracting solution throughout this finite grid. This is the numerical counterpart of Eq. (49). A machine-floor trace would misrepresent this result as a small physical signal. The correct statement is the exact analytic null result.

A Explicit perturbative coefficients

A.1 Longitudinal second harmonic at resonance

For spontaneous emission at resonance, the Fourier hierarchy is

$$(i\nu + \gamma/2)u_n = i\lambda(v_{n-1} - v_{n+1}),\tag{66a}$$

$$(i\nu + \gamma/2)v_n + \Omega w_n = -i\lambda(u_{n-1} - u_{n+1}),\tag{66b}$$

$$(i\nu + \gamma)w_n - \Omega v_n = -\gamma\delta_{n0}.\tag{66c}$$

At first order,

$$u_1 = \frac{i\lambda v_0}{i\nu + \gamma/2}, \quad v_1 = w_1 = 0.\tag{67}$$

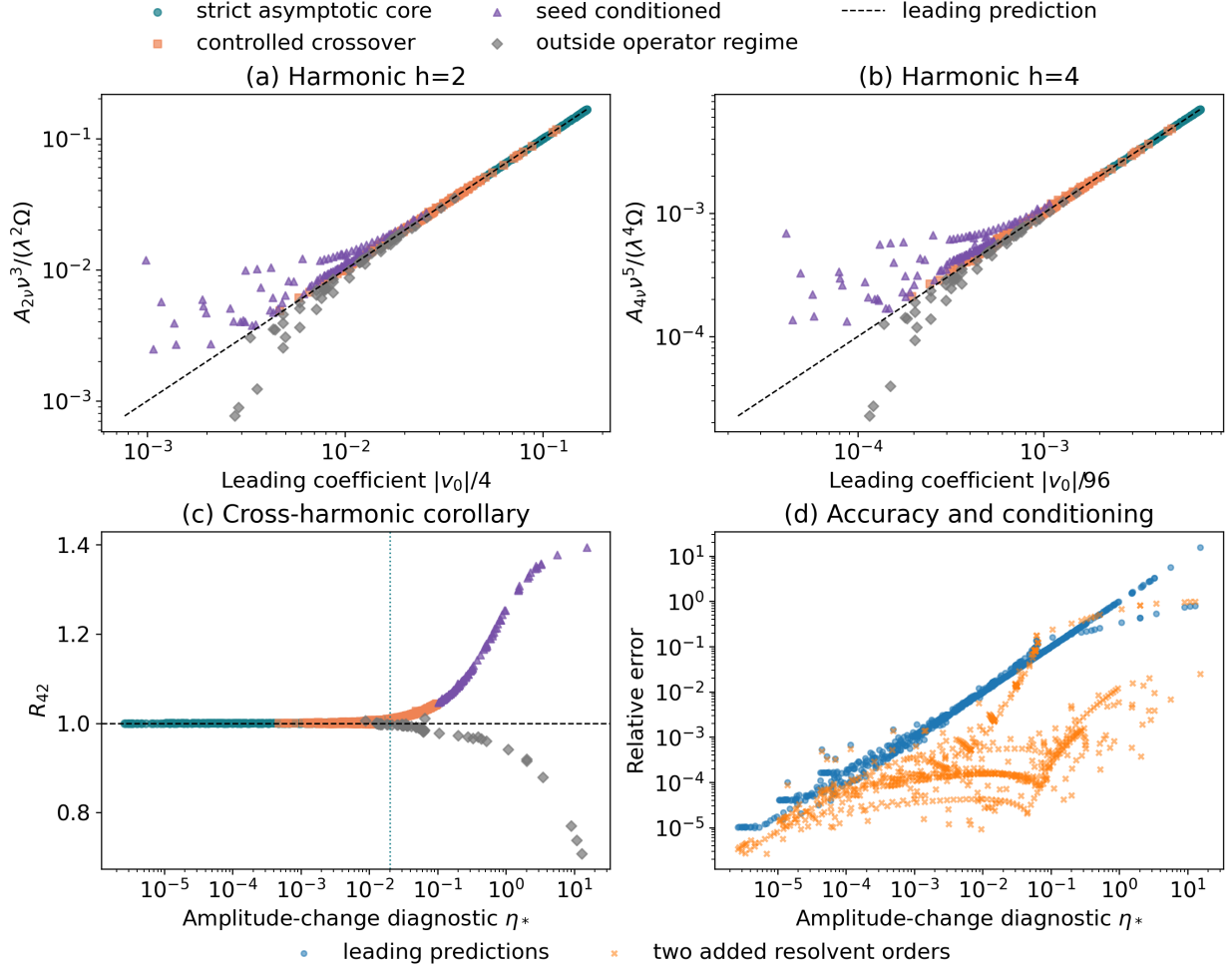


Fig. 4 Unified stationary-seed collapse and diagnostic-based domain of validity. Top: compensated second- and fourth-harmonic amplitudes against the common theoretical seed. Bottom left: cross-harmonic asymptotic ratio R_{42} versus the projected amplitude-change diagnostic η_* . Bottom right: actual leading error and the error after retaining two additional resolvent orders. The thresholds classify this data set and are not proposed as universal constants

At second order,

$$w_2^{(2)} = \frac{\Omega \lambda^2 v_0}{(i\nu + \gamma/2)[(2i\nu + \gamma/2)(2i\nu + \gamma) + \Omega^2]}. \quad (68)$$

No expansion in ν^{-1} has been made in Eq. (68); its only truncation is at order λ^2 .

A.2 Longitudinal fourth harmonic at resonance

Define

$$D_n = (i\nu + \gamma/2)(i\nu + \gamma) + \Omega^2. \quad (69)$$

Iterating the positive-harmonic chain to $n = 4$ gives

$$w_4^{(4)} = \frac{\lambda^4 \Omega v_0 (2i\nu + \gamma)}{(i\nu + \gamma/2) D_2 (3i\nu + \gamma/2) D_4}. \quad (70)$$

Expanding Eq. (70) at high frequency yields Eq. (39).

A.3 Transverse second harmonic at resonance

For transverse amplitude modulation define

$$a_n = in\nu + \gamma/2, \quad b_n = in\nu + \gamma, \quad D_n = a_n b_n + \Omega^2. \quad (71)$$

The first-harmonic coefficients are

$$v_1 = \frac{i\lambda(b_1 w_0 + \Omega v_0)}{D_1}, \quad (72)$$

$$w_1 = \frac{i\lambda(\Omega w_0 - a_1 v_0)}{D_1}. \quad (73)$$

The population coefficient exact to order λ^2 is

$$w_2^{(x)} = \frac{\lambda^2[(a_2 b_1 - \Omega^2)w_0 + \Omega(a_1 + a_2)v_0]}{D_1 D_2}. \quad (74)$$

Its high-frequency limit is Eq. (32).

B Numerical implementation and evidence limits

The frozen protocol, explicit configuration registry and write-once raw results define the deposited v2.0.0 numerical evidence. Each solution records the complex coefficients p_n , $n = 1, \dots, 6$, periodic or algebraic residuals, physicality and refinement. Phase is undefined below the adopted operational coefficient floor. The monodromy fixed system uses the homogeneous Bloch block, and every retained failure is distinguishable from a successful zero response.

For binary64 the floor minimum is $100\epsilon_{64}S$, with $S \geq 1$ the population scale. At the longitudinal baseline, even the leading $|p_4|$ at 15 MHz is approximately 1.43×10^{-10} , below the 10^4 -times minimum-floor requirement 2.22×10^{-10} . Consequently these binary64 values cannot establish the primary frequency fit under that conservative rule. They remain diagnostics; arbitrary-precision full solutions supply the primary evidence without changing the predetermined endpoints. The floor is an operational reproducibility criterion, not detector noise or a rigorous rounding certificate.

C Unified detuning–dissipation diagnostics

The stationary closed form is checked against an independent direct linear solve at every one of the 683 points. All pass the 10^{-12} dimensionless residual/difference criterion. All 683 binary64 amplitudes agree with the independent 70-digit harmonic-balance coefficients at relative complex discrepancy below 1.3×10^{-13} for $h = 2, 4$. All 16 preselected independent time-domain anchors pass relative 10^{-16} comparison and 10^{-30} closure checks. These checks cover all 683 points for harmonic-balance precision and 16 points for an independent solution method; they must not be interpreted as 683 independently integrated time series.

The 289-point combined grid spans $0 \leq \Delta/\Omega \leq 4$ and $0 \leq \gamma_\phi/\gamma_1 \leq 30$, with $\nu/2\pi = 40$ MHz and $\lambda/2\pi = \Omega/2\pi = \gamma_1/2\pi = 0.5$ MHz. Figure 6 retains every point. The nonuniform dephasing grid includes zero, displayed using a symmetric-log scale. Finite-frequency and small-seed corrections account for departures from the leading ratio; the colors do not define a new symmetry class.

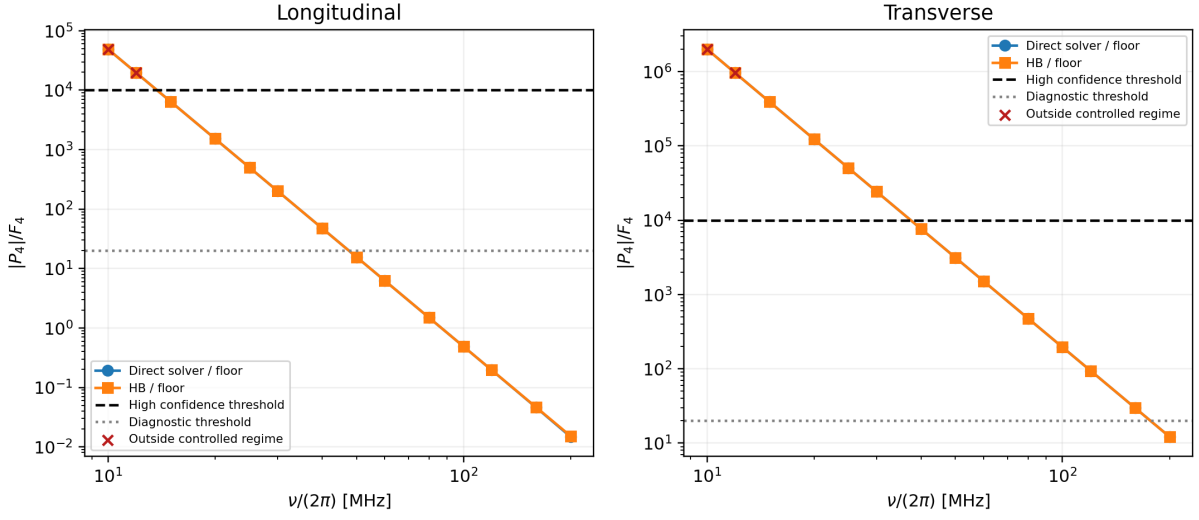


Fig. 5 Binary64 fourth-harmonic coefficient divided by its estimated floor for the complete fixed diagnostic frequency grid. The dashed 10^4 and dotted 20 levels are predeclared high-confidence and diagnostic thresholds. Low-signal points are retained and do not replace primary members. Arbitrary-precision evidence is reported separately

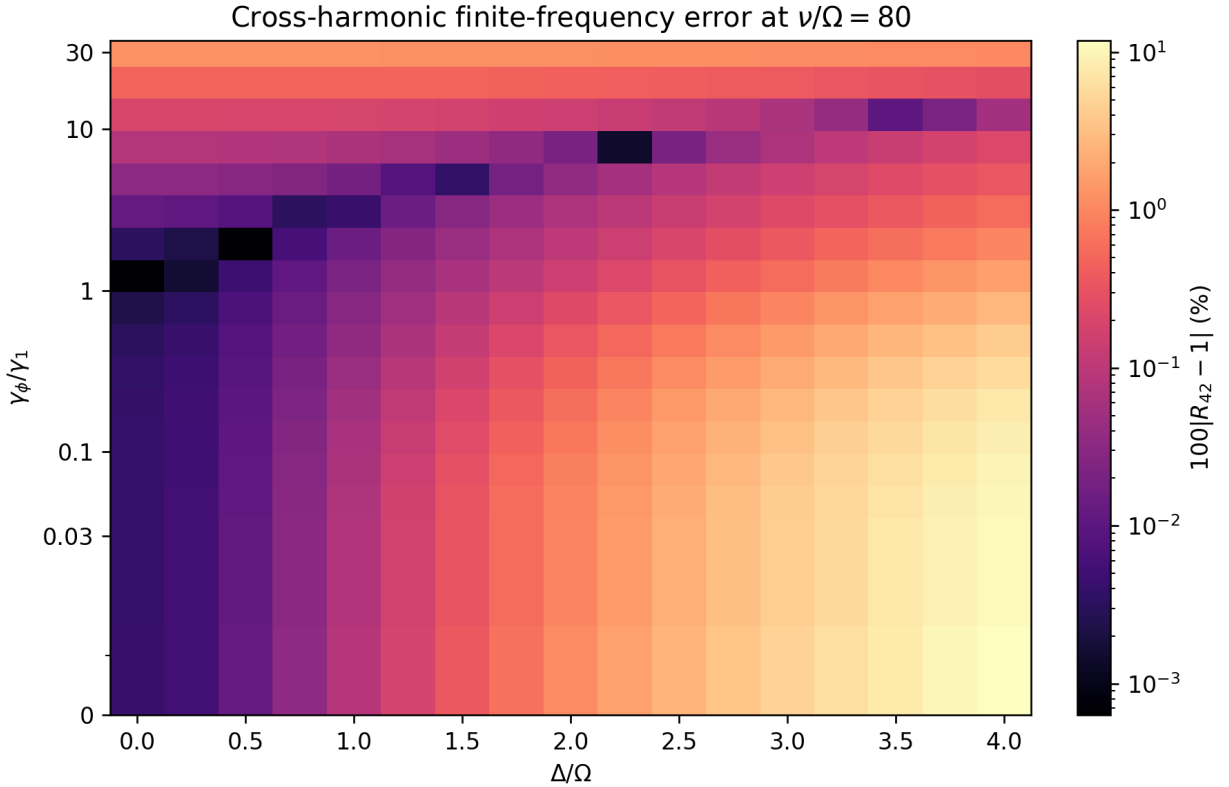


Fig. 6 Observed $100|R_{42} - 1|$ (percent) over the full 289-point combined grid, using the deposited v2.0.0 solved coefficients. Error colors are logarithmic; the dephasing axis includes zero. Parameters and the ensemble selection are specified in the text

For the exact pure-dephasing null model, 36 deposited control tuples combine $\gamma_\phi/\Omega = 0.1, 1, 10$, $\Delta/\Omega = 0, 1$, $\nu/\Omega = 20, 80$ and $\lambda/\Omega = 0, 0.5, 1$. Zero-source scalar harmonic balance at $N = 16$ returns zero coefficients throughout, with minimum matrix singular value 0.0497; an independent homogeneous time integration gives maximum monodromy spectral radius $0.9961 < 1$. These are finite-grid consistency checks; the exact null statement follows from the analytic invariant-set proof. They do not reconstruct the undocumented dissipation setting of the historical 60-point illustration.

D Projected coefficient generator

The coefficient generator below is a specialization of the established bilinear transfer-function expansion [5, 6]. In the retained v2.0.0 benchmark, its arbitrary- h form was a formal generator and model-specific evidence was restricted to the second and fourth harmonics. The new fixed-order family and validation are stated separately above.

Starting from Eq. (22), expand each resolvent as

$$R_n = \sum_{k=0}^{\infty} \frac{M_0^k}{(in\nu)^{k+1}}. \quad (75)$$

For harmonic h , let $(k_1, \dots, k_h)$ be a weak composition of s . Then

$$\begin{aligned} \mathcal{C}_{h,s} = & \left(\frac{1}{2i} \right)^h \sum_{\substack{k_1 + \dots + k_h = s \\ k_j \geq 0}} \left[\prod_{n=1}^h \frac{1}{(in)^{k_n+1}} \right] \\ & \times \mathbf{q}^T M_0^{k_h} M_1 M_0^{k_{h-1}} M_1 \dots M_0^{k_1} M_1 \mathbf{r}_0. \end{aligned} \quad (76)$$

Equation (76) determines the coefficients entering η_h directly from the matrices and the stationary state; no fit is used. For the 683-point master set, the logarithmic Spearman correlations between η_h and the true leading relative error are 0.989 and 0.983 for $h = 2$ and 4. In the strict core, the next-two-order approximation reduces the 95th-percentile errors below 5×10^{-4} .

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