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A CGMY–Hawkes Clock Model: Esscher Martingale Measures and Model-Preserving Minimal Entropy under Self-Exciting Jump Activity

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Abstract

We introduce the CGMY–Hawkes clock model, in which the log-price is a CGMY Lévy process evaluated at the integrated intensity of a Hawkes process. Jump activity therefore clusters in time while the jump-size law remains CGMY. We prove that the two-parametric Esscher transform, which shifts the CGMY tail parameters $(G, M) \mapsto (G + \theta_2, M - \theta_1)$, leaves the law of the Hawkes process unchanged: self-excitation is preserved under the change of measure. We close the model with a risk premium proportional to the intensity, so that the martingale condition holds pathwise, and we show that it reduces to one scalar equation that does not involve the Hawkes parameters. The relative entropy of the Esscher measure is computed in closed form and equals the expected integrated intensity times the Lévy entropy rate. Within the one-parameter Esscher class the martingale measure is unique, which defines a model-preserving minimal entropy martingale measure whose Radon–Nikodým density contains an explicit stochastic term driven by the fluctuations of the integrated intensity. In the two-parametric class the entropy minimiser is characterised by Lagrange conditions and, is not the one-parameter measure. Finally, European option prices are obtained from the exact affine Laplace transform of the integrated intensity; we show that, unlike the mean-field approximation, they depend on the excitation parameters and not only on the mean intensity.

Keywords: CGMY process, Hawkes process, time-changed Lévy process, Esscher transform, minimal entropy martingale measure, relative entropy, incomplete markets

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1 Introduction

Asset returns exhibit heavy tails, excess kurtosis and jump clustering: large moves arrive in bursts rather than in isolation [7, 9]. Lévy models such as the CGMY (KoBoL) process [7] capture the first two features, but their increments are stationary and independent, which excludes clustering by construction. Hawkes processes [17] are a natural remedy: the intensity increases after each event and decays back to a baseline. They are used in finance for contagion [1], high-frequency jumps [8] and order flow [18].

Stochastic clocks offer a tractable way to combine the two ingredients. Carr et al. [6] introduced time-changed Lévy processes $L(\Lambda_t)$ with $\Lambda_t = \int_0^t \lambda_s ds$ to model stochastic volatility. Here we take λ to be a Hawkes intensity. The resulting jump activity is self-exciting through the clock, and, because the clock is independent of the Lévy process, the model remains analytically tractable. To our knowledge this combination has not been studied.

Measure changes.

In incomplete markets the choice of a pricing measure is a modelling decision. The minimal entropy martingale measure (MEMM) [13–15, 19] has a dual interpretation in terms of exponential utility. For the CGMY model, Kim and Lee [20] computed the relative entropy in closed form and identified the model-preserving MEMM (MPMEMM); Boughamouira and Trabelsi [3] introduced the two-parametric Esscher transform, and Ben Abdelwahed and Trabelsi [2] linked it to utility indifference prices. For Hawkes-type models, measure changes typically alter the intensity [4]. In the clock model, in contrast, the change of measure acts only on the jump-size law (Theorem 4).

Contributions.

1. The CGMY–Hawkes clock model (Definition 2), with the exact affine transform of the integrated intensity (Lemma 3).
2. The two-parametric Esscher measures preserve both the CGMY family and the Hawkes law (Theorem 4); with an intensity-proportional risk premium the martingale condition is the scalar equation (13), free of Hawkes parameters (Proposition 5).
3. Closed-form relative entropy (Proposition 7).
4. Uniqueness in the one-parameter Esscher class and the density of the resulting MPMEMM, which carries a stochastic correction term (Theorem 8).
5. Characterisation of the two-parametric entropy minimiser, which is off the diagonal (Proposition 9).
6. Option pricing with the exact clock transform and a comparison with the mean-field approximation (Section 6).

Empirical calibration to market data is left for future work.

2 The CGMY process and its Esscher tilts

Let $C, G, M > 0$ and $Y \in (0, 2) \setminus \{1\}$ with $M > 1$. A CGMY process $L = (L_s)_{s \geq 0}$ is a Lévy process without Gaussian part whose Lévy measure is

$$\nu(dx) = C \left(\frac{e^{-Mx}}{x^{1+Y}} \mathbf{1}_{x>0} + \frac{e^{-G|x|}}{|x|^{1+Y}} \mathbf{1}_{x<0} \right) dx, \quad (1)$$

with the drift fixed so that, for $u \in [-G, M]$,

$$\mathbb{E}[e^{uL_s}] = e^{s\kappa(u)}, \quad \kappa(u) = C \Gamma(-Y) [(M-u)^Y - M^Y + (G+u)^Y - G^Y]. \quad (2)$$

By analytic continuation, (2) holds for complex u with $\operatorname{Re} u \in [-G, M]$ (this is the classical characteristic exponent of [7]). Two elementary facts are used repeatedly. First,

$$\Gamma(-Y)Y(Y-1) = \Gamma(2-Y) > 0 \quad (Y \in (0, 2) \setminus \{1\}), \quad (3)$$

since $\Gamma(2-Y) = (1-Y)(-Y)\Gamma(-Y)$. (Recall that $\Gamma(-Y) < 0$ for $Y \in (0, 1)$ and $\Gamma(-Y) > 0$ for $Y \in (1, 2)$.) Second, κ is strictly convex: by (3),

$$\kappa''(u) = C \Gamma(2-Y) [(M-u)^{Y-2} + (G+u)^{Y-2}] > 0. \quad (4)$$

Lemma 1 (Equivalence [20, Cor. 3]) *Consider two CGMY laws on path space, with parameters*

$$(C, G, M, Y, m) \quad \text{and} \quad (\tilde{C}, \tilde{G}, \tilde{M}, \tilde{Y}, \tilde{m}).$$

If they are equivalent on $\mathcal{F}_t$ for $t > 0$, then $\tilde{C} = C$ and $\tilde{Y} = Y$.

Two-parametric Esscher tilt of a Lévy process.

For $\theta = (\theta_1, \theta_2) \in \mathcal{D} := (-\infty, M) \times (-G, \infty)$ put $\psi(x) = \theta_1 x \mathbf{1}_{x>0} + \theta_2 x \mathbf{1}_{x<0}$ and let $\tilde{\mu}$ be the compensated jump measure of L . Define

$$Z_s^{L, \theta} = \exp \left(\int_0^s \int_{\mathbb{R}} \psi(x) \tilde{\mu}(dr, dx) - s \kappa_\psi \right), \quad \kappa_\psi := \int_{\mathbb{R}} (e^{\psi(x)} - 1 - \psi(x)) \nu(dx). \quad (5)$$

The integral κ_ψ is finite because $\theta_1 < M$, $\theta_2 > -G$ and x^2 is ν -integrable near the origin. If $\theta_1 = \theta_2 = \theta$, then $Z_s^{L, \theta} = e^{\theta L_s - s \kappa(\theta)}$, the classical Esscher density.

Lemma 2 *For $\theta \in \mathcal{D}$, $Z^{L, \theta}$ is a positive martingale with mean one. Under $\mathbb{Q}^{L, \theta} := Z_s^{L, \theta} \cdot \mathbb{P}$ on $\mathcal{F}_s^L$, L is a CGMY process with parameters $(C, G + \theta_2, M - \theta_1, Y)$ and*

$$\begin{aligned} \kappa_\theta(u) &:= \log \mathbb{E}_{\mathbb{Q}^{L, \theta}} [e^{uL_1}] \\ &= C \Gamma(-Y) [(M - \theta_1 - u)^Y - (M - \theta_1)^Y \\ &\quad + (G + \theta_2 + u)^Y - (G + \theta_2)^Y], \end{aligned} \quad (6)$$

for $\operatorname{Re} u \in [-(G + \theta_2), M - \theta_1]$. The entropy rate is

$$\mathbb{E}_{\mathbb{Q}^{L,\theta}} [\log Z_s^{L,\theta}] = s h(\theta), \quad h(\theta) = f_1(\theta_1) + f_2(\theta_2), \quad (7)$$

$$f_1(\theta_1) = C \Gamma(-Y) [-\theta_1 Y (M - \theta_1)^{Y-1} - (M - \theta_1)^Y + M^Y], \quad (8)$$

$$f_2(\theta_2) = C \Gamma(-Y) [\theta_2 Y (G + \theta_2)^{Y-1} - (G + \theta_2)^Y + G^Y]. \quad (9)$$

Moreover $f_1'(\theta_1) = C \Gamma(2 - Y) \theta_1 (M - \theta_1)^{Y-2}$ and $f_2'(\theta_2) = C \Gamma(2 - Y) \theta_2 (G + \theta_2)^{Y-2}$, so that

$$f_1''(\theta_1) = C \Gamma(2 - Y) (M - \theta_1)^{Y-3} (M + (1 - Y) \theta_1), \quad (10)$$

$$f_2''(\theta_2) = C \Gamma(2 - Y) (G + \theta_2)^{Y-3} (G + (Y - 1) \theta_2).$$

Proof $U_s = \int_0^s \psi \, d\tilde{\mu}$ is a Lévy process with $\mathbb{E} e^{U_1} = e^{\kappa_\psi} < \infty$, hence $Z^{L,\theta} = e^{U - s \kappa_\psi}$ is a martingale. By the Girsanov theorem for Lévy processes, the Lévy measure under $\mathbb{Q}^{L,\theta}$ is $e^\psi \nu$, i.e. (1) with $(G + \theta_2, M - \theta_1)$.

For (6), write $L_1 = \int x \, d\tilde{\mu} + b$ with $b = \mathbb{E} L_1$. Then $\mathbb{E}_{\mathbb{Q}} e^{u L_1} = \mathbb{E}[e^{u L_1} Z_1] = \exp\{ub + \int (e^{ux+\psi} - 1 - ux - \psi) d\nu - \kappa_\psi\}$, and subtracting the analogous expression for $\theta = 0$ gives

$$\kappa_\theta(u) - \kappa(u) = \int_{\mathbb{R}} (e^{\psi(x)} - 1)(e^{ux} - 1) \nu(dx).$$

On $x > 0$ the integrand is $e^{(\theta_1+u)x} - e^{\theta_1 x} - e^{ux} + 1$, whose linear terms cancel; hence the integral against $C e^{-Mx} x^{-1-Y}$ is $C \Gamma(-Y) [(M - \theta_1 - u)^Y - (M - \theta_1)^Y - (M - u)^Y + M^Y]$, and similarly for $x < 0$. Adding $\kappa(u)$ gives (6).

For the entropy rate, $\mathbb{E}_{\mathbb{Q}}[\log Z_s] = s \int (\psi e^\psi - e^\psi + 1) d\nu$, because the $\mathbb{Q}$ -compensator of μ is $e^\psi \nu \, ds$. Since $\partial_{\theta_1} \int_{x>0} (e^{\theta_1 x} - 1) d\nu = -C \Gamma(-Y) Y (M - \theta_1)^{Y-1}$, one obtains (8); (9) is symmetric. Differentiating (8) and using (3) gives f_1' , and then f_1'' ; the same for f_2 . $\square$

Remark 1 By (10), h is strictly convex on $\mathcal{D}_+ = \{M + (1 - Y)\theta_1 > 0, G + (Y - 1)\theta_2 > 0\} \cap \mathcal{D}$. For $Y \in (1, 2)$ this is all of $\mathcal{D}$; for $Y \in (0, 1)$ it excludes only the far tails ($\theta_1 \leq -M/(1 - Y)$ or $\theta_2 \geq G/(1 - Y)$). Also $h \geq 0$, with equality only at $\theta = 0$. For $\theta_1 = \theta_2 = \theta$ one checks $h(\theta, \theta) = \theta \kappa'(\theta) - \kappa(\theta)$.

3 The CGMY–Hawkes clock model

Definition 1 (Hawkes process) A Hawkes process N has intensity

$$\lambda_t = \mu + (\lambda_0 - \mu) e^{-\beta t} + \alpha \sum_{T_k \leq t^-} e^{-\beta(t-T_k)}, \quad (11)$$

that is, $d\lambda_t = \beta(\mu - \lambda_t)dt + \alpha dN_t$, with $\mu, \beta > 0$, $\alpha \geq 0$ and branching ratio $\rho = \alpha/\beta < 1$. Put $\bar{\lambda} = \mu/(1 - \rho)$ and $\Lambda_t = \int_0^t \lambda_s \, ds$.

Lemma 3 (i) $\mathbb{E}[\lambda_t] = \bar{\lambda} + (\lambda_0 - \bar{\lambda}) e^{-(\beta-\alpha)t}$: the intensity mean-reverts at rate $\beta - \alpha$ (not β). Hence $\mathbb{E}[\Lambda_T] = \bar{\lambda} T + (\lambda_0 - \bar{\lambda}) \frac{1 - e^{-(\beta-\alpha)T}}{\beta - \alpha}$, which equals $\bar{\lambda} T$ if $\lambda_0 = \bar{\lambda}$.

(ii) For $z \in \mathbb{C}$ with $\operatorname{Re} z \leq 0$,

$$\mathbb{E}[e^{z\Lambda_T} \mid \lambda_0] = \exp(A(T) + B(T)\lambda_0), \quad (12)$$

where (A, B) solves $A' = \beta\mu B$, $B' = -\beta B + z + e^{\alpha B} - 1$, $A(0) = B(0) = 0$.

Proof (i) Taking expectations in $d\lambda_t = \beta(\mu - \lambda_t)dt + \alpha dN_t$ with $dN_t = \lambda_t dt + d(\text{mart.})$ gives $\frac{d}{dt}\mathbb{E}\lambda_t = \beta\mu - (\beta - \alpha)\mathbb{E}\lambda_t$. (ii) (λ, Λ) is an affine jump-diffusion with generator $\mathcal{A}f = \beta(\mu - \lambda)\partial_\lambda f + \lambda\partial_\Lambda f + \lambda[f(\lambda + \alpha, \Lambda) - f(\lambda, \Lambda)]$; applying it to $f = \exp(z\Lambda + B\lambda + A)$ yields the stated ODEs [10–12]. For $\operatorname{Re} z \leq 0$, $|e^{z\Lambda_T}| \leq 1$, so the solution exists on $[0, T]$. $\square$

Definition 2 (CGMY–Hawkes clock model) Let L be a CGMY process (2) and N a Hawkes process with parameters (μ, α, β) , independent of L , on $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$X_t = L(\Lambda_t), \quad S_t = S_0 \exp(rt + \eta\Lambda_t + X_t - \kappa(1)\Lambda_t),$$

with risk-free rate $r \geq 0$ and a risk-premium parameter $\eta \in \mathbb{R}$ per unit of intensity. The filtration is $\mathcal{F}_t = \sigma(N_u, u \leq t) \vee \sigma(L_{s \wedge \Lambda_t}, s \geq 0)$.

Remark 2 (Interpretation) Jumps of X occur at the times when the clock Λ reaches the jump times of L ; the jump activity of X at time t is proportional to λ_t and clusters when λ is high. The Hawkes process N is *not* the jump counter of X (that counter has infinite intensity, since $\nu(\mathbb{R}) = \infty$): self-excitation acts on the clock, as in the time-change framework of Carr et al. [6]. For $\alpha = 0$ and $\lambda_0 = \mu$ one recovers $X = L(\mu \cdot)$. Unlike L , X has neither stationary nor independent increments. Given the path of N , Λ_T is a deterministic time, and

$$\mathbb{E}[e^{iuX_t} \mid \sigma(N)] = \exp(\kappa(iu)\Lambda_t), \quad \mathbb{E}[e^{iuX_t}] = \mathbb{E}[e^{\kappa(iu)\Lambda_t}],$$

the latter being given by Lemma 3(ii) with $z = \kappa(iu)$, $\operatorname{Re} z \leq 0$. This is the exact characteristic function; it is *not* $\exp(\lambda t \kappa(iu))$.

Remark 3 (Why a premium proportional to λ) Since $\mathbb{E}[e^{-rt}S_t \mid \sigma(N)] = e^{\eta\Lambda_t}$ under $\mathbb{P}$, η is the expected excess return per unit of clock time. A constant drift $\mu_S - r$ with a pathwise compensator $\Lambda_t \kappa(1)$ cannot be corrected by a measure change preserving the clock law: the discounted price could only be a local martingale if the stochastic term Λ_t were cancelled by a deterministic one. Replacing λ_s by $\bar{\lambda}$ in the compensator would make the martingale condition hold only on average, not pathwise.

4 Esscher martingale measures

For $\theta \in \mathcal{D}$ let $Z_t^\theta := Z_{\Lambda_t}^{L, \theta}$ and $\mathbb{Q}^\theta := Z_T^\theta \cdot \mathbb{P}$ on $\mathcal{F}_T$.

Theorem 4 (Preservation of the CGMY–Hawkes structure) Let $\theta \in \mathcal{D}$.

- (a) Z^θ is a strictly positive $(\mathbb{P}, \mathbb{F})$ -martingale with $\mathbb{E}Z_T^\theta = 1$; hence $\mathbb{Q}^\theta \sim \mathbb{P}$.
- (b) The law of N under $\mathbb{Q}^\theta$ coincides with its law under $\mathbb{P}$: N is a Hawkes process with the same (μ, α, β) and the same intensity λ , and $\mathbb{E}_{\mathbb{Q}^\theta}[\Lambda_t] = \mathbb{E}_{\mathbb{P}}[\Lambda_t]$.

- (c) Conditionally on N , $(L_s)_{s \leq \Lambda_T}$ is under $\mathbb{Q}^\theta$ a CGMY process with parameters $(C, G + \theta_2, M - \theta_1, Y)$ and drift normalised as in (2), (6). Hence under $\mathbb{Q}^\theta$, X is a CGMY–Hawkes process with parameters $(C, G + \theta_2, M - \theta_1, Y; \mu, \alpha, \beta)$.

Proof Let $\mathcal{G} = \sigma(N_u : u \leq T)$, independent of L , and let Λ_T be $\mathcal{G}$ -measurable. (a) For $t \leq T$ and $s = \Lambda_t$ deterministic given $\mathcal{G}$, Lemma 2 and independence give $\mathbb{E}[Z_t^\theta | \mathcal{G} \vee \mathcal{F}_{\Lambda_u}^L] = Z_{\Lambda_u}^{L, \theta}$ for $u \leq t$; the tower property yields the martingale property with respect to $\mathbb{F}$, and $\mathbb{E}[Z_T^\theta | \mathcal{G}] = 1$. Positivity is clear.

(b),(c) For bounded measurable F, H ,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^\theta} [F(N) H((L_s)_{s \leq \Lambda_T})] &= \mathbb{E} \left[F(N) \mathbb{E}[Z_s^{L, \theta} H(L)] \Big|_{s=\Lambda_T} \right] \\ &= \mathbb{E} \left[F(N) \mathbb{E}_{\mathbb{Q}^{L, \theta}} [H(L)] \Big|_{s=\Lambda_T} \right], \end{aligned}$$

where the first equality conditions on $\mathcal{G}$. Taking $H \equiv 1$ shows that N has the same law under $\mathbb{Q}^\theta$ as under $\mathbb{P}$; the general case, together with Lemma 2, gives (c). Note that Z^θ is continuous in the arrivals of N , so no change of the N -intensity can occur. $\square$

Remark 4 Because arrivals and sizes are decoupled by the clock, no normalisation of an “infinite-activity arrival intensity” is required: the invariance of the Hawkes law is a direct consequence of $\mathbb{E}[Z_T^\theta | \sigma(N)] = 1$. This is the structural reason why self-excitation is a property of the market, not of the risk premium.

Proposition 5 (Martingale condition) *Let $\theta \in \mathcal{D}_1 := (-\infty, M-1) \times (-G, \infty)$. Then $(e^{-rt} S_t)$ is a $\mathbb{Q}^\theta$ -martingale if and only if*

$$\begin{aligned} \eta &= \kappa(1) - \kappa_\theta(1), \\ \kappa_\theta(1) &= C\Gamma(-Y) [(M - \theta_1 - 1)^Y - (M - \theta_1)^Y \\ &\quad + (G + \theta_2 + 1)^Y - (G + \theta_2)^Y]. \end{aligned} \tag{13}$$

In particular the equation involves only (C, G, M, Y) , η and θ , and not (μ, α, β) . For $\theta_1 = \theta_2 = \theta$ it reads $\eta = \iota(\theta) := \kappa(1) + \kappa(\theta) - \kappa(\theta + 1)$.

Proof $e^{-rt} S_t / S_0 = \exp(L_{\Lambda_t} - \Lambda_t(\kappa(1) - \eta))$. Conditionally on N and under $\mathbb{Q}^{L, \theta}$, $\exp(L_s - s c)$ is a martingale in s iff $c = \kappa_\theta(1)$ by (6); since Λ is a strictly increasing continuous clock, this must hold for almost every path. Integrability of e^{L_1} under $\mathbb{Q}^\theta$ requires $M - \theta_1 - 1 > 0$. $\square$

Proposition 6 (Structure of the constraint) *Write $\kappa_\theta(1) = a(\theta_1) + b(\theta_2)$ with $a(\theta_1) = A(M - \theta_1 - 1) - A(M - \theta_1)$, $b(\theta_2) = A(G + \theta_2 + 1) - A(G + \theta_2)$ and $A(s) = C\Gamma(-Y)s^Y$. Then a is strictly increasing and convex, b is strictly increasing and concave, so that, for each η in the range of $\kappa(1) - \kappa_\theta(1)$, the set $\mathcal{C}_\eta = \{\theta \in \mathcal{D}_1 : (13)\}$ is the graph of a strictly decreasing continuous function $\theta_2 = \varphi(\theta_1)$. On the diagonal, ι is strictly decreasing on $(-G, M-1)$ with $\iota(0) = 0$ and*

$$\iota((M-1)^-) = \kappa(1) + \kappa(M-1) - \kappa(M) < 0 < \iota(-G^+) = \kappa(1) + \kappa(-G) - \kappa(1-G).$$

Proof $A''(s) = C\Gamma(2-Y)s^{Y-2} > 0$ is decreasing in s . Then $a'(\theta_1) = A'(s) - A'(s-1) > 0$ with $s = M - \theta_1$, and $a'' = -(A''(s) - A''(s-1)) > 0$; likewise $b' = A'(G + \theta_2 + 1) - A'(G + \theta_2) > 0$ and $b'' = A''(G + \theta_2 + 1) - A''(G + \theta_2) < 0$. On the diagonal $\iota'(\theta) = \kappa'(\theta) - \kappa'(\theta + 1) < 0$ by strict convexity of κ (4); $\iota(0) = 0$ is immediate. $\square$

5 Relative entropy and the model-preserving minimal entropy measure

Proposition 7 (Relative entropy) *For $\theta \in \mathcal{D}$,*

$$H(\mathbb{Q}^\theta \parallel \mathbb{P}) = \mathbb{E}_{\mathbb{P}}[\Lambda_T] h(\theta) = \left(\bar{\lambda}T + (\lambda_0 - \bar{\lambda}) \frac{1 - e^{-(\beta - \alpha)T}}{\beta - \alpha} \right) (f_1(\theta_1) + f_2(\theta_2)), \quad (14)$$

with f_1, f_2 in (8)–(9). In particular, the Hawkes dynamics enter only through $\mathbb{E}[\Lambda_T]$, i.e. through $\bar{\lambda}T$ in the stationary case.

Proof $\log Z_T^\theta = \log Z_{\Lambda_T}^{L, \theta}$. Conditionally on N , (7) gives $\mathbb{E}_{\mathbb{Q}}[\log Z_T^\theta \mid N] = \Lambda_T h(\theta)$, and by Theorem 4(b), $\mathbb{E}_{\mathbb{Q}}[\Lambda_T] = \mathbb{E}_{\mathbb{P}}[\Lambda_T]$. Lemma 3(i) yields the explicit form. $\square$

Definition 3 (One-parameter Esscher class) Ess_1 is the set of $\mathbb{Q}^{(\theta, \theta)}$, $\theta \in (-G, M - 1)$, which are martingale measures, i.e. satisfy $\iota(\theta) = \eta$.

Theorem 8 (Singleton and MPMEMM) *Assume $\eta \in (\iota((M - 1)^-), \iota(-G^+))$ (this interval contains 0 and thus all sufficiently small premia). Then:*

- (i) $\iota(\theta) = \eta$ has a unique root $\theta^* \in (-G, M - 1)$; hence $\text{Ess}_1 = \{\mathbb{Q}^{\theta^*}\}$ is a singleton. It is the model-preserving minimal entropy martingale measure of the class, $\theta^* < 0$ if $\eta > 0$, and $\theta^* = 0$ iff $\eta = 0$.
- (ii) θ^* depends on the Hawkes parameters neither through (μ, α, β) nor through $\bar{\lambda}$; it is determined by (C, G, M, Y) and η alone.
- (iii) The density is exactly

$$\left. \frac{d\mathbb{Q}^{\theta^*}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp(\theta^* X_t - \kappa(\theta^*) \Lambda_t) = \exp\left(\theta^* X_t - \bar{\lambda} \kappa(\theta^*) t - \kappa(\theta^*) \int_0^t (\lambda_s - \bar{\lambda}) ds\right). \quad (15)$$

The last term is a stochastic correction absent in the pure CGMY case $\alpha = 0$, $\lambda_0 = \mu$ (where $\Lambda_t = \bar{\lambda}t$). It captures the fluctuations of the clock around its mean.

- (iv) $H(\mathbb{Q}^{\theta^*} \parallel \mathbb{P}) = \mathbb{E}_{\mathbb{P}}[\Lambda_T] (\theta^* \kappa'(\theta^*) - \kappa(\theta^*))$.

Proof (i) By Proposition 6, ι is continuous and strictly decreasing on $(-G, M - 1)$, so the intermediate value theorem gives a unique root. Since $\iota(0) = 0$, $\theta^* = 0$ iff $\eta = 0$, and $\theta^* < 0$ for $\eta > 0$. The class is a singleton, so the infimum of H on it is attained there. (ii) is immediate from (13). (iii) Since $Z_s^{L, \theta} = e^{\theta L_s - s\kappa(\theta)}$ for $\theta_1 = \theta_2 = \theta$, set $s = \Lambda_t$ and note $\Lambda_t = \bar{\lambda}t + \int_0^t (\lambda_s - \bar{\lambda}) ds$; this is an identity, not an approximation. (iv) follows from Proposition 7 and Remark 1. $\square$

Remark 5 Theorem 8(ii) is the precise sense in which the MPMEMM parameter is insensitive to self-excitation. It does not imply that option prices are: those depend on the law of Λ_T through (12) (Section 6).

Proposition 9 (Two-parametric class) *On $\mathcal{C}_\eta \cap \mathcal{D}_+$ consider $\min h(\theta_1, \theta_2)$ subject to $a(\theta_1) + b(\theta_2) = \kappa(1) - \eta$. Any interior minimiser satisfies, with ξ the Lagrange multiplier,*

$$f'_1(\theta_1) = \xi a'(\theta_1), \quad f'_2(\theta_2) = \xi b'(\theta_2), \quad \text{i.e.} \quad \frac{f'_1(\theta_1)}{a'(\theta_1)} = \frac{f'_2(\theta_2)}{b'(\theta_2)}, \quad (16)$$

with $a'(\theta_1) = A'(M - \theta_1) - A'(M - \theta_1 - 1)$ and $b'(\theta_2) = A'(G + \theta_2 + 1) - A'(G + \theta_2)$. The two-parametric entropy minimiser $\mathbb{Q}^{\theta^\sharp}$ has the same structure as in Theorem 4 and the density $\exp(\int_0^{\Lambda_t} \psi d\tilde{\mu} - \kappa_\psi \Lambda_t)$. In general $\theta_1^\sharp \neq \theta_2^\sharp$, so that $H(\mathbb{Q}^{\theta^\sharp} \|\mathbb{P}) < H(\mathbb{Q}^{\theta^*} \|\mathbb{P})$ (Table 2).

Proof h is smooth and, on $\mathcal{D}_+$, strictly convex (Remark 1); the constraint curve is a smooth graph with non-vanishing gradient ($a', b' > 0$), so (16) is the first-order condition of the Lagrangian $h + \xi(a + b)$. The diagonal point θ^* lies on $\mathcal{C}_\eta$, so the minimum over the curve is at most $h(\theta^*, \theta^*)$. Uniqueness of the minimiser is *not* implied by convexity of h , since $\mathcal{C}_\eta$ is not a convex set (a convex, b concave); in all cases of Table 2 a grid search along $\mathcal{C}_\eta$ finds a single interior minimiser, strictly below the diagonal value. $\square$

Remark 6 Proposition 9 shows that the one-parameter measure $\mathbb{Q}^{\theta^*}$ is the model-preserving minimal entropy measure only within the one-parameter class; enlarging the class to independent tail tilts strictly lowers the entropy, by roughly 4% at the benchmark parameters.

6 Numerical illustrations

We use the benchmark CGMY parameters of [2]: $C = 1$, $G = M = \pi$, $Y \in \{0.75, 1.25\}$, with $r = 0.04$ and $\mu_S = 0.07$. The annualised variance of L is $CT(2 - Y)(M^{Y-2} + G^{Y-2}) \approx 0.434$ for $Y = 0.75$ and ≈ 1.038 for $Y = 1.25$, so absolute price levels below are high; only relative effects are of interest. To compare with a constant excess return $\mu_S - r$, we set $\eta = (\mu_S - r)/\bar{\lambda}$, which matches the expected excess return at the stationary mean intensity to first order. The root θ^* is computed by Brent's method, and the two-parametric minimiser by a fine grid along $\mathcal{C}_\eta$.

MPMEMM parameter and entropy.

Table 1 reports θ^* and the entropy rate $h(\theta^*)$ per unit of clock time; $H(\mathbb{Q}^{\theta^*} \|\mathbb{P}) = \bar{\lambda} T h(\theta^*)$ for stationary start. Since $\eta = (\mu_S - r)/\bar{\lambda}$, θ^* varies with $\bar{\lambda}$ through the premium; for a fixed η it is independent of all Hawkes parameters (Theorem 8(ii)). Note that $\theta^* < 0$ in both cases ($\mu_S > r$): the measure change fattens the left tail and thins the right tail.

One versus two parameters.

Table 2 compares the diagonal solution with the minimiser of Proposition 9 (here $G = M$).

Table 1 MPMEMM parameter θ^* (one-parameter class) for $C = 1$, $G = M = \pi$, $r = 0.04$, $\mu_S = 0.07$, $\eta = (\mu_S - r)/\bar{\lambda}$.

Y	$\bar{\lambda}$	η	θ^*	$G + \theta^*$	$M - \theta^*$	$h(\theta^*)$
0.75	0.5	0.060	-0.1329	3.0087	3.2745	3.834×10^{-3}
0.75	1.0	0.030	-0.0662	3.0754	3.2078	9.49×10^{-4}
0.75	2.0	0.015	-0.0330	3.1086	3.1746	2.36×10^{-4}
1.25	0.5	0.060	-0.0566	3.0850	3.1982	1.661×10^{-3}
1.25	1.0	0.030	-0.0283	3.1133	3.1698	4.15×10^{-4}
1.25	2.0	0.015	-0.0141	3.1275	3.1557	1.04×10^{-4}

Table 2 One-parameter (θ^*, θ^*) versus two-parametric entropy minimiser $(\theta_1^\#, \theta_2^\#)$; entropy rate h per unit of clock time.

Y	η	θ^*	$h(\theta^*, \theta^*)$	$(\theta_1^\#, \theta_2^\#)$	$h(\theta^\#)$
0.75	0.03	-0.0662	9.49×10^{-4}	$(-0.0767, -0.0512)$	9.12×10^{-4}
0.75	0.06	-0.1329	3.834×10^{-3}	$(-0.1547, -0.1035)$	3.685×10^{-3}
1.25	0.03	-0.0283	4.15×10^{-4}	$(-0.0313, -0.0244)$	4.08×10^{-4}
1.25	0.06	-0.0566	1.661×10^{-3}	$(-0.0626, -0.0491)$	1.637×10^{-3}

Option prices: exact transform versus mean field.

Let $\varphi(u) = \mathbb{E}_{\mathbb{Q}^\theta}[\exp(iu(\log S_T - \log S_0 - rT))]$. By Theorem 4 and the independence of L and N ,

$$\varphi(u) = \mathbb{E}[e^{\zeta(u)\Lambda_T}], \quad \zeta(u) = iu(\eta - \kappa(1)) + \kappa_\theta(iu), \quad (17)$$

which is evaluated exactly by Lemma 3(ii) because $\text{Re} \zeta(u) \leq 0$ for real u and $\text{Re} \zeta(u - i) \leq \zeta(-i) = 0$ (the latter is the martingale condition (13)). The *mean-field* approximation replaces Λ_T by $\bar{\lambda}T$ and gives $\varphi_{\text{MF}}(u) = \exp(\bar{\lambda}T \zeta(u))$; by Jensen's inequality it differs from the exact value in general. Call prices $e^{-rT} \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+] = S_0 P_1 - K e^{-rT} P_2$ are computed by the Gil-Pelaez inversion [16] with $P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} [e^{-iu \log(K/S_0 e^{rT})} \varphi(u)/(iu)] du$ and P_1 obtained with $\varphi(u - i)$ in place of $\varphi(u)$ (recall $\varphi(-i) = 1$). The ODE of Lemma 3(ii) is integrated by an adaptive Runge-Kutta scheme (complex arithmetic) and was checked against Monte Carlo simulation of Λ_T (40,000 paths; agreement to about three decimals).

Table 3 reports the ATM call ($S_0 = K = 100$, $T = 1$) for four Hawkes specifications with the *same* stationary mean intensity $\bar{\lambda} = 1$ (so the same $\eta = 0.03$ and the same θ^*), started at $\lambda_0 = \bar{\lambda}$.

Discussion.

For a fixed mean intensity the exact European price *decreases* with the branching ratio, by about 0.3%, 1.0% and 2.6% at $\rho = 0.4, 0.6, 0.9$ for $Y = 0.75$ (and similarly for $Y = 1.25$). The reason is that more excitation spreads the law of Λ_T around its

Table 3 ATM call price under $\mathbb{Q}^{\theta^*}$: exact transform versus mean field (the mean-field price equals the pure-CGMY price for every specification since $\bar{\lambda} = 1$). Parameters (μ, α, β) with $\rho = \alpha/\beta$.

Specification	(μ, α, β)	ρ	$Y = 0.75$		$Y = 1.25$	
			exact	mean field	exact	mean field
Pure CGMY	(1, 0, 1)	0	27.042	27.042	40.303	40.303
Low excitation	(0.6, 0.3, 0.75)	0.4	26.970	27.042	40.191	40.303
High excitation	(0.4, 0.6, 1)	0.6	26.768	27.042	39.874	40.303
Very high excitation	(0.1, 0.9, 1)	0.9	26.351	27.042	39.219	40.303

mean, and the transform in (17) is a convex function of Λ_T . Hence θ^* is insensitive to (α, β) (Theorem 8(ii)), but prices are not: the mean-field approximation only sees $\bar{\lambda}$. Path-dependent payoffs are expected to be even more sensitive.

7 Conclusion

We proposed a CGMY–Hawkes clock model whose Esscher measures can be handled by conditioning on the Hawkes path. The two-parametric Esscher transform preserves the CGMY family and the Hawkes law; with an intensity-proportional risk premium the martingale condition is a scalar equation without Hawkes parameters; the entropy is the expected integrated intensity times the Lévy entropy rate; the one-parameter MPMEMM is unique with an explicit density containing a stochastic correction term; and the two-parametric entropy minimiser is off the diagonal. Prices computed from the exact clock transform differ from the mean-field values.

Perspectives.

(i) Multivariate mutually exciting clocks for contagion [1]; (ii) rough (power-law) Hawkes kernels [5], for which the affine structure is lost; (iii) models in which the intensity is excited by the jumps of the price itself (self-excitation of jump sizes) [4, 8], where the change of measure also alters the intensity; (iv) calibration to option data with the exact transform and comparison with market prices; (v) exotic and barrier payoffs, for which the mean-field approximation is expected to fail most. Uniqueness of the two-parametric minimiser on $\mathcal{C}_\eta$ is left as an open question.

Declarations

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Conflict of interest

The author declares no conflict of interest.

Data availability

No external datasets were used. All numerical results can be reproduced from the formulas and parameter values stated in Section 6.

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