

Supplementary Information for: How Recurrent Neural Information Flow Can Generate a Setpoint

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This document supplements the main text for *Biological Cybernetics*. Sections, tables and equations numbered with the prefix S belong to this document; all other numbers refer to the main text. The notation is that of the main text (Table 1), and only the symbols that do not appear there are defined below. Every number reported here was read from the stored result files by the scripts `paper_numbers.py` and `supp_tables.py`, which also write the data tables of Sections S3–S9; no number was entered by hand except those of the original reference network series (Sections S10 and S11), which are copied from the saved records of that series.

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S1 Model parameters and the exact construction

The parameters listed in Tables S1 and S2 are choices of the study design. The form of the model and the Lorentzian reduction derive from Montbrió et al. (2015), and the theta-neuron form from Izhikevich (2000). It is important to note that none of these numerical choices is presented as a set of physiological parameters measured experimentally. Under the normalization of Eq. (6), the coupling magnitudes, the excitabilities and the rescaled variables R_X , V_X and S_X are dimensionless; on the other hand, the firing rates r_X and the synaptic variables s_X are in s^{-1} . The reference network is used in every simulation, whereas the exact construction is used in the proofs; the two parameter sets are distinct.

Table S1 Parameters of the reference network. The values are fixed a priori and are distinct from those of the exact construction used in the proofs (Table S2)

Parameter	Value	Meaning and qualification
N_E, N_I	8000, 2000	Sizes of the excitatory and inhibitory populations in every spiking network (4000/1000 and 16,000/4000 in the size comparison of the original series).
τ	20 ms	Membrane time constant.
τ_s	10 ms	Time constant of the exponential synaptic filter.
Δ	1	Half-width of the Lorentzian distribution of excitabilities.
$\bar{\eta}_E, \bar{\eta}_I$	1, -1	Centers of the distributions of excitability; they have no interpretation as desired rates.
J_{EE}	20	Magnitude of the excitatory-to-excitatory coupling.
J_{EI}	15	Magnitude of the inhibitory-to-excitatory coupling, which enters the current with a negative sign.
J_{IE}	10	Magnitude of the excitatory-to-inhibitory coupling.
J_{II}	3	Magnitude of the inhibitory-to-inhibitory coupling, which enters the current with a negative sign.
Level	41.253/36.066 Hz	Excitatory/inhibitory equilibrium rates of the population model at $u_E = 0$ (Proposition 2).
Spiking step	0.2 ms	The synaptic input is held during the step. In the original series the phase was advanced by fourth-order Runge–Kutta substeps (phase increment of at most 0.6 rad); in the revised experiments it is propagated exactly within the step (Section 2.9).
Quantile placement	$(j + 1/2)/N_X$	Deterministic midpoint quantiles of the Lorentzian distribution, Eq. (5), in the all-to-all networks.
Sparse network	$K = 200$	Fixed in-degree from each population, weights J_{XY}/K , excitabilities drawn at random from the Lorentzian distribution; five realizations (seeds 11–15), each with its own connectivity and excitabilities.
Reference signal	36.03 Hz	Each spiking network’s own time-averaged inhibitory signal, used as $S_I^*/(\pi\tau)$ by the interventions (population value 36.07 Hz). In the sparse networks, each excitatory neuron uses its own time-averaged inhibitory input.
Autonomous runs	$u_E = u_I = 0$	The tonic excitability remains present.

Table S1 (continued)

Parameter	Value	Meaning and qualification
Population solvers	—	Dormand–Prince (RK45 or LSODA where stated; relative tolerance 10^{-8} to 10^{-9}), split at input discontinuities; fixed-step fourth-order Runge–Kutta (RK4) with steps of 0.02 to 0.25 ms where channels or delays are present; Euler–Maruyama with $\Delta t = 0.025$ ms for the noisy channel (Table S3).
Root search	Eq. (9)	One positive root at every drive (Proposition 2); the level at $u_E = 0$ is refined to a residual below 10^{-14} .
Random seeds	—	Original series: 20261001 (phases and pulses), 20261101 (probes of the information assay), 20261002 (Latin hypercube). Revised experiments: 0–4 (all-to-all networks), 11–15 (sparse networks), 1 (initial perturbations in the population model), 0 (channel noise in the population model), 100+network seed (channel noise in spiking networks), 2026 (random networks).

Table S2 Exact construction. The coefficients are chosen for algebraic transparency; the derived values follow from the proofs in Appendix A and are not outputs of simulations, except where stated

Quantity	Exact value	Status
Δ	1	Chosen parameter.
$\bar{\eta}_E, \bar{\eta}_I$	$-1/4, -1/4$	Chosen parameters.
τ_s/τ	$1/2$	Chosen ratio.
J_{EE}, J_{EI}	$3\pi, 2\pi$	Chosen couplings.
J_{IE}, J_{II}	$2\pi, \pi$	Chosen couplings.
R_E^*, R_I^*	1, 1	Derived; unique positive equilibrium (Proposition 1).
V_E^*, V_I^*	$-1/2, -1/2$	Derived centers of the voltage distributions.
S_E^*, S_I^*	1, 1	Derived synaptic states.
r_E^*, r_I^*	$1/(\pi\tau)$	Derived rates; 15.915 Hz for $\tau = 20$ ms, an illustration of the time scaling and not a fitted prediction.
Eigenvalues at $g = 1$	$-1/\tau, (-3 \pm i\sqrt{15})/(2\tau)$	Derived; each with algebraic multiplicity two (Proposition 3).
Clamped excitatory polynomial	$\lambda^3 + 4\lambda^2 + 9\lambda - 2$	Derived, with λ in units of $1/\tau$; its positive root, 0.20298, gives the unstable eigenvalue $\lambda_u = 0.20298/\tau$ (Proposition 5).
Minimum channel capacity	$h = \lambda_u/\ln 2$	$0.2928/\tau$ bits/s, that is, 14.6 bits/s for $\tau = 20$ ms (Theorem 1).
Feedback window	$7/16 < g < 1187/512$	Derived (Proposition 4); at $g_H = 1187/512$ the critical pair is $\pm 1.2247 i/\tau$, and the first Lyapunov coefficient is -0.235 (computed; Table S8).
Critical delay at $g = 1$	$d_c = 1.5675\tau$	Derived (Proposition 6); 31.3 ms for $\tau = 20$ ms.

Table S2 (continued)

Quantity	Exact value	Status
Sensitivity to sustained input	$(14/9, 8/9)$	Derived values of $(dR_E^*/du_E, dR_I^*/du_E)$; amplification by 35/9 relative to an uncoupled population (Proposition 8).
Bode integral	$\int_0^\infty \ln S_g(i\omega) d\omega = \pi\lambda_u$	Derived for every g in the window, with ω and λ_u in units of $1/\tau$ (Theorem 3).
Population size	Infinite population limit	The simulations with 10,000 neurons use, instead, the reference network of Table S1.

S2 Numerical protocols

The protocols of Table S3 are, likewise, choices of the study design. The experiments form two groups. The *original reference network series* comprises the brief pulses, the constant and replayed signals, the comparisons of integration step, of network size and of initial states, the conditional information assay and, in a later attraction study, the equilibrium scan, the sampling of initial states, the local certificates and the maintained input; these programs are kept unchanged with the code. The *revised experiments* comprise the linear and bifurcation analyses, the spiking gain experiments, the efficient coder, the static codes, the one-bit code in spiking networks, the messengers, the noisy channel, the sinusoidal drive and the random networks; they use the exact spiking integrator. In this sense, each row of the table states what was run, over which window the verdict was taken, and what the result can and cannot show. No duration, amplitude or tolerance is attributed to a physiological measurement.

The rule that decides whether the level was held is stated in Section 2.5, and, with the exception of the deterministic population runs without drive, it is the same in every experiment: (i) the mean of r_E over the final window lies within 5% of the level, and (ii) its standard deviation (s.d.) is at most ten times the s.d. that the imposed input alone produces in the linearized intact loop, with a floor of 0.54 Hz (the finite-size s.d. of the intact spiking network). For deterministic population runs without drive, we chose a different criterion: r_E must stay within 5% of the level throughout the final window. It implies (i), but it is not equivalent to (i)–(ii): deterministic oscillations around the level can occur in this model, and this criterion accepts them only when their excursions stay within 5% of the level. The last column of Table S3 gives the final window of each experiment, and Table S5 shows how the spiking verdicts change under looser tolerances.

Table S3 Numerical protocols, final windows and limits of interpretation. All experiments use the reference network of Table S1, except the random networks. Population starts carry independent Gaussian perturbations of relative size 2% on every coordinate (seed 1) unless stated

Experiment	Implementation	Final window; scope
<i>Original reference network series (spiking: original Runge–Kutta phase integrator)</i>		
Finite starts	Three uniform draws of the phases (seeds 20261001 + k , $k = 0, 1, 2$); initial $(s_E, s_I) = (0, 0)$, (80, 5) and (5, 80) Hz; runs of 2.5 s.	Means over 1.5–2.5 s. A sample of initial states.
Population starts	(0.1, −2, 0.1, −2, 0, 0), (4, 1, 2, 1, 4, 2) and (1, −1, 3, −1, 1, 3) in the state order of Eq. (6), integrated for 2.5 s.	Three selected full states.

Table S3 (continued)

Experiment	Implementation	Final window; scope
Population sampling	The three fixed starts and 64 Latin hypercube points (Section S10) at each of $u_E = -3, 0$ and 3 , integrated for 8 s.	67 starts per input; the sampling is not exhaustive.
Equilibrium scan	u_E from -4 to 4 in steps of 0.1 (81 inputs); one positive root recorded at each input and classified by the eigenvalues of the full Jacobian.	A root search, not a global count; uniqueness is proved separately (Section S6).
Brief pulses	Six seconds, with $u_E = +3$ during 2–2.2 s and -3 during 4–4.2 s.	Means over 1–2 s (baseline) and over 3–4 and 5–6 s (recovery).
Maintained input	Twelve seconds per branch: no input for 2 s, $u_E = -3$ or $+3$ for 6 s, and no input for 4 s; the baseline is shared by the two branches.	Means over 1–2, 6–8 and 10–12 s; 22 unique simulated seconds.
Constant and replayed signals	Warm-up of 2 s, then the unperturbed inhibitory input is recorded for 4 s. The live signal is compared with the constant recorded mean and with the record circularly shifted by one third of its length; perturbation of $u_E = +0.5$ during 0.5–0.55 s.	Means over 2–4 s. The replay preserves the mean and the variance, not every statistic of the live signal.
Step comparison	The pulse protocol repeated with a step of 0.1 ms.	Means over 5–6 s; not a formal order of convergence.
Size comparison	$(N_E, N_I) = (4000, 1000)$ and $(16000, 4000)$; autonomous runs of 2.5 s, compared with the default initialization.	Means over 1.5–2.5 s; no ensemble of random wirings.
Information assay	20 s with independent Gaussian probe currents to the E and I populations (s.d. 0.2, each value held for 10 ms; seed 20261101), identical across the live, constant and replayed conditions; the replacement signals derive from the live record; the first 2 s are discarded.	A separate regime of observation (Section 2.8).
Information predictors	Spike counts in 10-ms bins; three history bins of the source and of the target; the current value and three lags of both probe currents.	Linear-Gaussian proxy with finite history, not a general nonlinear transfer entropy.
Source-shift null	99 circular shifts of the source, each by at least $\max(50, 0.1 n_{\text{bin}})$ bins, with n_{bin} the number of bins of the record; rank statistic with the plus-one correction.	Exploratory; neither a confidence interval nor a correction for multiplicity.
Estimator sanity process	5000 samples of a bivariate autoregressive process in which the source drives the target (coefficients 0.7 for the source, 0.4 for the target and 0.9 for the coupling) with independent Gaussian innovations.	Checks the direction found by the estimator; not neural evidence.
Numerical flags	The recovery, the step difference and the spread of the excitatory rate across initial states are each compared with 5%; no automatic retuning.	Tolerances of the analysis, not desired rates.
<i>Population model: linear analysis and bifurcations</i>		
Equilibrium branch	u_E from -4 to 4 in steps of 0.02 (401 inputs), through Eq. (9); full Jacobian for stability and Jacobian with clamped inhibition for h .	Linear quantities along the branch.

Table S3 (continued)

Experiment	Implementation	Final window; scope
Feedback gain	Reference network: leading eigenvalue of $M(g)$ for 1201 values of $g \in [0, 1.2]$, with g_c refined by Brent's method (SciPy developers n.d.a); upper boundary g_H by Brent's method on $[1, 20]$. Exact construction: 3001 values of $g \in [0, 1.5]$, in addition to the Routh–Hurwitz proof.	Linear stability.
Feedback delay	Linearized delay equation on an 81×81 grid of $g \in [0, 1.6]$ and $d \in [0, 20]$ ms (Euler method, step 0.02 ms, 1.5 s); growth rate from the maximum norm over 1–1.5 s relative to 0.5–1 s; at $g = 1$, the first unstable delay of the grid.	Linear stability; the random networks use the exact quasi-polynomial (below).
Lower gain boundary	Equilibria with $S_I^{(E)} = S_I^* + g(S_I - S_I^*)$ for ten values of g from 0 to 0.6 (reference network) and six from 0.40 to 0.47 (exact construction), each from 344 starting points (Powell's hybrid method, tolerance 10^{-13}), classified by the leading real part of the Jacobian.	Table S8(e).
Clamped inhibition	R_E changed by +1% or –1% at the level, then 1 s with $S_I^{(E)} = S_I^*$ (RK45, relative tolerance 10^{-9}).	The escape is finite; the instability is local.
Hopf points	Input Hopf points by Brent's method on the leading real part along the branch; gain Hopf points as above; first Lyapunov coefficient by Kuznetsov's formula with the exact bilinear term (Kuznetsov 2004).	Table S8(a).
Beyond the boundaries	Gain $g = g_H(1 + \varepsilon)$, $\varepsilon \in \{0.5, 1, 2, 4\}\%$, from the level with R_E raised by 0.1% (LSODA, relative tolerance 10^{-9} ; 3000τ for the exact construction, 60 s for the reference network). Delay $d \in \{12, 13, 13.6, 14, 15, 16, 20\}$ ms at $g = 1$ (RK4 with a history buffer, step 0.02 ms, 30 s). Inside the oscillatory window, u_E at a distance $\varepsilon \in \{0.0025, 0.005, 0.01, 0.02, 0.04\}$ from each Hopf point, from the equilibrium with R_E raised by 1% (RK45, 150 s).	Amplitude: half the peak-to-peak range over the last 20% of the run (gain), the last 4 s (delay) or the last 10 s (window); slope from the three smallest ε .
Interval cover	mpmath 1.3.0 interval arithmetic (mpmath developers n.d.), 30 significant digits (103-bit mantissa), outward rounding; bisection until the enclosure of $u'_E(q)$ is positive, then nine-decimal endpoints and a second certification from them (Section S6).	Table S7.
<i>Spiking networks, revised experiments (exact integrator; rates in 10-ms bins unless stated)</i>		
Maintained input	All-to-all (seed 0): baseline 2 s, $u_E = \pm 3$ for 6 s, recovery 4 s; population overlay in 50-ms bins. Sparse (seeds 11–15): warm-up 1 s, then the same schedule.	Means over 1–2, 6–8 and 10–12 s.
Feedback gain, all-to-all	Warm-up of 1 s in two halves, the second of which gives the network's own reference signal; pulse $u_E = \pm 0.5$ for 50 ms, 0.2 s after the gain is applied; runs of 3 s. Seed 0: $g \in \{0, 0.2, 0.3, 0.35, 0.4, 0.45, 0.5, 0.6, 0.8, 1\}$; seeds 1–4: $g \in \{0.3, 0.4, 0.45, 0.5, 0.6, 0.8, 1\}$.	Last second. Condition (ii) only for seed 0 (traces stored); means only for seeds 1–4.

Table S3 (continued)

Experiment	Implementation	Final window; scope
Feedback gain, sparse	Seeds 11–15; warm-up of 1 s, after which each excitatory neuron’s reference is its own inhibitory input averaged over the next second; pulse $u_E = \pm 0.5$ for 50 ms, 0.2 s after the gain is applied; runs of 3 s; $g \in \{0, 0.3, 0.6, 1\}$.	Last second. Condition (i) in every run; condition (ii) only for seed 11, whose traces were stored, where it changed no verdict; means only for seeds 12–15.
One-bit code	Seeds 0–2, gain 1, $\beta = 0.02$ relative to the network’s own reference; $\mathcal{R} \in \{50, 100, 125, 151.5, 172.4, 200, 417, 1000, 2500\}$ bits/s, each realized with an integer number of 0.2-ms steps per sample; pulse $u_E = \pm 0.5$ for 50 ms, 0.2 s after the channel is switched on; runs of 4 s.	Last 2 s; mean within 5% of the network’s own baseline and s.d. at most 5.4 Hz.
Messengers	$N_{\text{msg}} \in \{2000, 300, 100, 30, 10, 5, 3, 2, 1\}$, drawn among the inhibitory neurons with stationary rates between 5 and 150 Hz, their spikes rescaled so that the mean signal equals that of the population (exact single-neuron rates); 5 s after a warm-up of 1 s. For $N_{\text{msg}} \leq 10$, four further subsets (4 s each).	Last 2 s; mean within 5% of the run with 2000 messengers and s.d. at most 5.4 Hz.
Noisy channel	Seeds 0–2, warm-up 1 s; 60 s for each $\sigma \in \{0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 1\}$; noise seed 100+network seed, regenerated exactly for the cross-spectral estimate. Spectra: Welch’s method, Hann window, non-overlapping segments of 2^{13} steps (1.64 s), $f_{\text{max}} = 100$ Hz (50 to 1000 Hz in Table S6).	Spectra and rates after the first 2 s of the record ($t > 3$ s from the start of the warm-up); mean within 5% of the noiseless baseline; s.d. at most $10 [0.54^2 + (20.6 \sigma)^2]^{1/2}$ Hz.
Sinusoidal drive	Seeds 0–4, warm-up 1 s; $U = 0.5$ and $f \in \{1, 2, 5, 10, 20, 40\}$ Hz at $g = 1$ and $g = 0.6$; $U = 2$ at 2 and 10 Hz with $g = 1$ (seed 0); two further runs of seed 0, $U = 2$ at 2 Hz with $g = 0.6$ and $U = 0.5$ at 5 Hz with $g = 0.3$ (means 20.3 and 3.2 Hz, so that the level was lost in both), which enter only the check of condition (ii); runs of 4 s; 5-ms bins.	Last 3 s; amplitude from the Fourier coefficient at f ; condition (i) in every run; condition (ii) only for seed 0, whose traces were stored (10-ms bins, whole cycles of the last 3 s), where it changed no verdict in any of its 16 runs; seeds 1–4 stored only the mean, the amplitude and the RMS deviation of each run.
<i>Population model: channels, noise and drive</i>		
Efficient coder	29 settings of (T_s, n_q) (Table S4); 24 starts for each $a_0 \in \{0.02, 0.2, 2\}\%$; initial cell of half-width 4.5 s.d. of the unstable modal coordinate of the perturbation; allowance of 10^{-3} of the initial cell per sample; perturbation, cell and allowance proportional to a_0 , with the same 24 Gaussian draws (seed 1) at every a_0 ; RK4, step 0.2 ms, 40 s.	Last 10% (4 s); within 5% throughout.

Table S3 (continued)

Experiment	Implementation	Final window; scope
Static codes	Mid-rise and mid-tread quantizers with $n_b \in \{1, 2, 3, 4, 6, 8\}$ and $T_s \in \{0.5, 1, 2, 4, 8, 16, 32, 64\}$ ms (96 settings); one-bit code with $\beta \in \{0.005, 0.01, 0.02, 0.05, 0.1\}$ and 16 values of T_s spaced logarithmically from 2 to 64 ms (80 settings); 24 starts; RK4, step 0.05 ms, 3 s.	Last second; within 5% throughout. Finite-time comparison with the necessary condition of Theorem 1, without noise.
Noisy channel	Eq. (10) at $g = 1$; Euler–Maruyama, $\Delta t = 0.025$ ms, 80 s; eight trials for each $\sigma \in \{0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, 1, 1.5\}$, with the same eight noise realizations (seed 0) at every σ ; $\hat{I}_G$ and C computed for each trial; every value reported in the main text and plotted in Fig. 5a is the mean (with its s.e., where given) of the eight single-trial values at the given σ , whatever the verdict of each trial (all eight trials held for $\sigma \leq 0.3$, 2 of 8 at $\sigma = 0.5$ and none for $\sigma \geq 0.7$); an estimate from the spectrum pooled over the held trials only is also stored by the script, but it is not reported. Linear check: the linearized loop at $\sigma = 0.01$ with the same realizations and with three further sets of eight (seeds 1–3).	After the first 4 s; mean within 5%; s.d. at most $10 \max(0.54, 22.6 \sigma)$ Hz. Comparison with Theorem 4, which concerns the linearized network.
Estimator check	Linearized reference network loop with the channel; Euler–Maruyama, step 0.1 ms; eight trials of 100 s (first 2 s discarded) for $g \in \{0.5, 1, 1.5\}$ and $\sigma \in \{0.01, 0.1, 1\}$; segments of 2^{14} steps (1.64 s).	The exact answer is $h = 28.91$ bits/s; validates Eq. (11).
Sinusoidal drive	$U \in \{0.25, 1, 2\}$, $f \in \{0.5, 1, 2, 5, 10, 20, 40, 80\}$ Hz, $g \in \{0.5, 0.7, 1, 1.5\}$; from the level; RK45 (relative tolerance 10^{-8} , maximum step $\min(1 \text{ ms}, 0.02/f)$) for $\max(3 \text{ s}, (n_c + 4)/f)$, with $n_c = \max(4, \lceil 1.5f \rceil)$ cycles. Linear theory: quadrature of $\ln S_g $ for the reference network Jacobian.	Whole cycles within the last four periods; rule for periodic drive.
Drive through channels	$f = 5$ Hz, $U = 0.5$, $g = 1$; 12 starts; RK4, step 0.05 ms, 3 s; one-bit code with $\beta \in \{0.01, 0.02\}$ at 11 rates from 15.6 to 500 bits/s; 8-bit mid-tread code at 7 rates from 250 to 16,000 bits/s.	Last 2 s; RMS deviation from the level; rule for periodic drive.
Drive with delay	$f = 5$ Hz, $U = 0.5$, $g = 1$, $d \in \{0, 2, 5, 8, 11, 13, 14, 16\}$ ms; RK4 with a history buffer, step 0.05 ms, 3 s.	Whole cycles within the second half of the run.
<i>Random networks (population model; Section 2.7)</i>		
Draw and selection	641 parameter sets drawn uniformly (seed 2026) on the ranges of Section 2.7; positive equilibria located with Eq. (9) on a logarithmic grid of 40,000 values of q , refined by Brent’s method.	74 retained, of which 68 inhibition-stabilized.
Linear quantities	h from the unstable eigenvalues of the excitatory block with clamped inhibition; g_c by Brent’s method on the leading eigenvalue in $[0, 1]$; d_c at $g = 1$ from the quasi-polynomial, with the crossing frequencies located on a logarithmic grid from 0.1 to 10^5 s^{-1} and refined; dr_1^*/du_I by central differences ($u_I = \pm 10^{-4}$).	Exact up to the tolerances of the root searches.

Table S3 (continued)

Experiment	Implementation	Final window; scope
One-bit code and zero-information control	$\beta = 0.02$; all 25 rates of a logarithmic grid from 5 to 5000 bits/s, each realized with an integer number of 0.05-ms steps per sample ($68 \times 25 = 1700$ tests), and the control with $S_1^{(E)} = S_1^*$; 12 starts; RK4, step 0.05 ms, $\max(2 \text{ s}, 10/\lambda_u)$.	Last quarter; within 5% throughout, from all 12 starts; $\mathcal{R}_{\text{hold}}$ as in Section 2.7.
Efficient coder	$n_q = 2$ and $T_s = \ln 2 / [(\mathcal{R}/h) \text{Re } \lambda]$ for each unstable mode λ , with $\mathcal{R}/h \in \{0.8, 0.95, 1.05, 1.2, 1.5\}$; 12 starts for each $a_0 \in \{0.02, 2\}\%$; RK4, step 0.25 ms, $\max(4 \text{ s}, 250 T_s)$, with T_s at $0.95 h$.	Last 10%; within 5% throughout. Reported for the 63 networks whose clamped instability is excitatory only.
Correlations	Spearman's ρ with 95% intervals from 10,000 bootstrap resamples of the networks (Efron and Tibshirani 1993); partial correlation from the residuals of the ranks after linear regression on the ranks of d_c .	Descriptive statistics over one family of models.

S3 Efficient coder in the reference network

Table S4 lists every setting of the efficient coder in the reference network. At a fixed sampling period, only the number of cells n_q changes from one row to the next, and with it the number of bits sent per sample; the timing, the mean of the received signal and the gain do not change. With the smallest perturbations ($a_0 = 0.02\%$), the outcome is all or nothing: each setting held the level from all 24 starts or from none. Across all tested settings, the largest rate at which the level was lost was $0.988 h$ (at $T_s = 35$ ms) and the smallest rate at which it was held was $1.017 h$ (at $T_s = 34$ ms). This bracket is not established separately at each sampling period: at 100 ms the bracket is $0.971 h$ – $1.038 h$, at 150 ms $0.979 h$ – $1.028 h$ and at 200 ms $0.976 h$ – $1.022 h$; it is wider at 40, 50 and 66 ms; and at 20, 30, 34 and 35 ms a single setting was tested. The outcomes at the sampling periods from 20 to 200 ms are therefore consistent with a common threshold near h , but they do not locate it at each period. With larger perturbations, some starts were lost above h , but not because the initial error fell outside the initial cell. Since the perturbation, the half-width of the initial cell and the allowance are all proportional to a_0 , and the same 24 Gaussian draws are used at every a_0 (Table S3), the initial error occupies the same position relative to the initial cell at every a_0 ; it lay inside the cell in every start, because at $a_0 = 0.02\%$ no sample of any setting above h , the first sample included, found the error outside the cell. For linear dynamics, the position of the error relative to the propagated cell would not depend on a_0 at the later samples either, so that a dependence on a_0 of whether the error remains inside the cell can arise only from the nonlinear terms of the population equations. To leading order, these terms grow relative to the cell in proportion to a_0 , whereas the cell is propagated with the linearized clamped dynamics and the allowance remains a fixed fraction of the initial cell. The stored runs show that the losses above h coincide with such escapes: at $a_0 = 0.2\%$ and 2% , the error left the propagated cell at some later sample in every setting above h that lost a start, and in none of the settings above h that held all 24 starts. We did not examine which nonlinear terms are responsible, nor whether a larger allowance would prevent these losses. No setting below h held any start at any a_0 ; this is an observed finite-time outcome, consistent with the necessary condition of Theorem 1, which concerns every initial state in a neighborhood of the level and all future time, and which therefore does not require that each sampled start be lost within a finite run.

Table S4 Efficient coder in the population model of the reference network. Each row is one combination of the sampling period T_s and the number of cells n_q . Since the clamped reference network Jacobian has a single real unstable eigenvalue ($\ell = 1$), the bit rate is $\mathcal{R} = \log_2 n_q/T_s$, with T_s realized as an integer number of 0.2-ms integration steps; $\mathcal{R}/h$ uses $h = 28.912$ bits/s, the rate of Eq. (18) for that eigenvalue. The last three columns give the number of starts, out of 24, from which the level was held (r_E within 5% of the level throughout the last 4 s of a 40-s run) for initial perturbations of relative size a_0

T_s (ms)	n_q	$\mathcal{R}$ (bits/s)	$\mathcal{R}/h$	$a_0 = 0.02\%$	$a_0 = 0.2\%$	$a_0 = 2\%$
20	2	50.0	1.729	24	24	23
30	2	33.3	1.153	24	23	14
34	2	29.4	1.017	24	14	0
35	2	28.6	0.988	0	0	0
40	2	25.0	0.865	0	0	0
40	3	39.6	1.370	24	23	19
50	2	20.0	0.692	0	0	0
50	3	31.7	1.096	24	23	15
50	4	40.0	1.383	24	24	22
66	3	24.0	0.831	0	0	0
66	4	30.3	1.048	24	22	10
66	5	35.2	1.217	24	23	21
100	5	23.2	0.803	0	0	0
100	6	25.8	0.894	0	0	0
100	7	28.1	0.971	0	0	0
100	8	30.0	1.038	24	23	7
100	9	31.7	1.096	24	21	16
100	12	35.8	1.240	24	23	19
100	16	40.0	1.383	24	24	24
150	16	26.7	0.922	0	0	0
150	19	28.3	0.979	0	0	0
150	22	29.7	1.028	24	22	7
150	32	33.3	1.153	24	24	22
200	32	25.0	0.865	0	0	0
200	45	27.5	0.950	0	0	0
200	50	28.2	0.976	0	0	0
200	60	29.5	1.022	24	23	11
200	64	30.0	1.038	24	23	15
200	90	32.5	1.123	24	24	24

S4 Sensitivity of the verdicts to the rule for a held level

The 5% band of the rule is an analysis tolerance, and it is therefore important to show to what extent the conclusions depend on it. Table S5 gives the verdict of every stored spiking run under mean tolerances of 5, 10 and 20%, with and without the s.d. condition. On the one hand, loosening the tolerance changes few verdicts, and only where the network moved to another state close to the level or where a slow drift crossed the band: the noisy channel at $\sigma = 0.5, 0.7$ and 1, where the mean fell by 4.6–9.5% (7–9.5% at $\sigma \geq 0.7$), which we attribute to the curvature of the rate equations, so that 1 of the 9 runs met the 5% rule and all 9 met a tolerance of 10%, including the s.d. condition (under channel noise of this size, the verdict therefore records a displacement of the operating level whose classification depends on the tolerance, and it does not by itself demonstrate a loss of

attraction); the gain experiments at $g = 0.3$ – 0.35 , where the positive pulse left the network 8–13% above the level; and the single messenger, whose low state is far from the level but whose subsets with oscillations have means close to it. On the other hand, the s.d. condition is what distinguishes a held level from a self-sustained oscillation around it: with the one-bit code at 50–175 bits/s and with 1–10 messengers, several runs met the mean condition while the excitatory rate oscillated with an s.d. of tens of hertz, and the s.d. condition classifies these runs as lost.

Table S5 Verdicts of every stored spiking experiment under the rule of Section 2.5 and under looser tolerances. For each condition, n is the number of runs (one pulse sign in one network for the gain experiments and the one-bit code; one subset for the messengers; one network for the noisy channel), and the next columns give the number of runs held when the mean of r_E over the final window lies within 5, 10 or 20% of the level, without and with condition (ii), the s.d. bound (s.d. in 10-ms bins at most $10 [0.54^2 + (20.6 \sigma)^2]^{1/2}$ Hz, that is, 5.4 Hz without channel noise). The primary verdict is the column “5%, s.d.”. The level is each network’s own noiseless baseline (for the all-to-all networks of seeds 1–4, whose gain runs stored no baseline, the common value of 41.35 Hz, their baselines in the other experiments being 41.35–41.36 Hz; for the messengers, the run with all 2000 messengers). The gain experiments of four of the five all-to-all networks (seeds 1–4, which share the rows $g = 0.3$ – 1 with seed 0) and of four of the five sparse networks (seeds 12–15, which share every sparse row with seed 11) stored mean rates only; since each row counts all of its runs, the s.d. condition is not available (—) in any row that contains these networks, and the all-to-all rows with $n = 2$ contain only the network of seed 0. Traces were stored for the all-to-all network of seed 0 (20 runs) and for the sparse network of seed 11 (8 runs); in these runs, adding condition (ii) changed no verdict (Section 2.5). One-bit rates are the realized rates

Experiment	Condition	n	Mean within 5%		Mean within 10%		Mean within 20%	
			alone	s.d.	alone	s.d.	alone	s.d.
Spiking, noisy channel	$\sigma = 0.05$	3	3	3	3	3	3	3
	$\sigma = 0.1$	3	3	3	3	3	3	3
	$\sigma = 0.2$	3	3	3	3	3	3	3
	$\sigma = 0.3$	3	3	3	3	3	3	3
	$\sigma = 0.5$	3	1	1	3	3	3	3
	$\sigma = 0.7$	3	0	0	3	3	3	3
	$\sigma = 1$	3	0	0	3	3	3	3
Spiking, one-bit code	$\mathcal{R} = 50$	6	0	0	0	0	3	0
	$\mathcal{R} = 100$	6	2	0	4	0	6	0
	$\mathcal{R} = 125$	6	4	3	6	3	6	3
	$\mathcal{R} = 151.5$	6	6	5	6	5	6	5
	$\mathcal{R} = 172.4$	6	5	5	5	5	6	5
	$\mathcal{R} = 200$	6	6	6	6	6	6	6
	$\mathcal{R} = 417$	6	6	6	6	6	6	6
	$\mathcal{R} = 1000$	6	6	6	6	6	6	6
	$\mathcal{R} = 2500$	6	6	6	6	6	6	6
Spiking, messengers	$N_{\text{msg}} = 1$	5	0	0	1	0	4	0
	$N_{\text{msg}} = 2$	5	2	2	5	2	5	2
	$N_{\text{msg}} = 3$	5	3	0	5	0	5	0
	$N_{\text{msg}} = 5$	5	5	0	5	0	5	0
	$N_{\text{msg}} = 10$	5	5	0	5	0	5	0
	$N_{\text{msg}} = 30$	1	1	1	1	1	1	1
	$N_{\text{msg}} = 100$	1	1	1	1	1	1	1
	$N_{\text{msg}} = 300$	1	1	1	1	1	1	1

Table S5 (continued)

Experiment	Condition	n	Mean within 5%		Mean within 10%		Mean within 20%	
			alone	s.d.	alone	s.d.	alone	s.d.
Spiking gain, all-to-all	$g = 0$	2	0	0	0	0	0	0
	$g = 0.2$	2	0	0	0	0	0	0
	$g = 0.3$	10	0	—	0	—	5	—
	$g = 0.35$	2	0	0	1	1	1	1
	$g = 0.4$	10	5	—	5	—	5	—
	$g = 0.45$	10	5	—	5	—	5	—
	$g = 0.5$	10	5	—	5	—	5	—
	$g = 0.6$	10	8	—	8	—	8	—
	$g = 0.8$	10	10	—	10	—	10	—
	$g = 1$	10	10	—	10	—	10	—
Spiking gain, sparse	$g = 0$	10	0	—	0	—	0	—
	$g = 0.3$	10	0	—	1	—	5	—
	$g = 0.6$	10	8	—	8	—	8	—
	$g = 1$	10	10	—	10	—	10	—

S5 Upper frequency of the spiking information estimates

The Gaussian spectral estimate, Eq. (11), integrates the logarithm of the spectrum of the received signal up to $f_{\max}$. For the linearized reference network loop, the band up to 100 Hz contains 99.92% of the theoretical integral, so that the choice of $f_{\max}$ is immaterial there. In a finite spiking network, however, the received signal also carries the intrinsic fluctuations of the 2000 inhibitory neurons, which are broadband, and $\hat{I}_G$ then keeps growing with $f_{\max}$ (Table S6a); this is most visible with weak channel noise, where the intrinsic fluctuations dominate, a situation that Corollary 2 describes for the linearized network with independent additive noise, but not for this nonlinear spiking estimate. The empirical cross-spectral integral, computed from the ratio of Eq. (12), discards the contributions to the received signal that are uncorrelated with the injected noise, and it is nearly independent of $f_{\max}$ (Table S6b). Therefore, the value of $\hat{I}_G$ at weak noise is not a measurement of a single rate. The cross-spectral integral, in turn, is consistent with the population prediction h at weak noise; however, since in a nonlinear network this ratio describes the best linear approximation of the loop for the given excitation (Schoukens et al. 2005; Pintelon and Schoukens 2012), the agreement is evidence of consistency with h , not an independently validated measurement of the instability rate of the finite network (Section 2.6). It is important to note, in addition, that its mean over the three networks changed little between $\sigma = 0.05$ and 0.2 and depended clearly on σ for larger noise, where the response is no longer linear (Table S6b); at $\sigma = 1$, where no run met the 5% rule, it was again close to h , which shows that agreement with h at a single amplitude does not, by itself, validate the estimate.

Table S6 Dependence of the spiking information estimates on the upper frequency $f_{\max}$. All-to-all spiking network with the noisy channel of Eq. (10) at $g = 1$; three networks (seeds 0–2), records of 58 s. (a) Gaussian spectral estimate $\hat{I}_G$, Eq. (11). (b) Empirical cross-spectral integral $\int_0^{f_{\max}} \log_2 |\hat{S}(f)|^2 df$, from the ratio of Eq. (12); for the linearized loop, the corresponding integral of $\log_2 |S_g|^2$ equals $h = 28.9$ bits/s, whereas in the spiking network the ratio is a best linear approximation that depends on σ , and the integral is a test of consistency with h (Section 2.6). The rows “mean $\pm$ s.d.” give the mean and the s.d. ($n - 1$ in the denominator) over the three networks at $f_{\max} = 100$ Hz, the band used in the main text. For $\sigma \leq 0.2$, the cross-spectral integral at 1000 Hz differed from that at 50 Hz by at most 2.5 bits/s, and its largest spread over the five values of $f_{\max}$ was 3.3 bits/s (seed 0, $\sigma = 0.05$)

σ (Hz s ^{1/2})	Seed	$f_{\max}$ (Hz)				
		50	100	200	500	1000
<i>(a) Gaussian spectral estimate $\hat{I}_{\text{G}}$, bits/s</i>						
0.05	0	58.9	76.7	92.6	106.0	109.5
	1	57.2	73.9	87.6	92.7	105.4
	2	60.4	78.9	92.5	102.4	106.1
	mean $\pm$ s.d.	76.5 $\pm$ 2.5				
0.1	0	40.1	47.5	53.8	59.1	61.5
	1	35.8	41.0	44.8	43.7	54.0
	2	40.1	46.6	50.1	54.1	55.9
	mean $\pm$ s.d.	45.0 $\pm$ 3.5				
0.2	0	38.1	43.1	47.1	50.5	52.1
	1	33.5	35.2	37.0	34.9	45.1
	2	36.0	38.7	39.5	42.7	43.8
	mean $\pm$ s.d.	39.0 $\pm$ 4.0				
0.3	0	46.9	51.1	54.5	57.8	59.2
	1	42.1	42.8	44.0	41.8	52.3
	2	42.6	45.3	45.7	48.2	49.1
	mean $\pm$ s.d.	46.4 $\pm$ 4.3				
0.5	0	58.3	62.4	65.8	69.0	70.3
	1	56.0	57.2	58.2	55.9	66.1
	2	57.2	59.6	60.1	62.6	63.6
	mean $\pm$ s.d.	59.7 $\pm$ 2.6				
0.7	0	57.6	61.8	65.3	68.5	69.9
	1	54.2	55.2	56.2	54.0	64.2
	2	55.9	58.5	58.8	61.4	62.3
	mean $\pm$ s.d.	58.5 $\pm$ 3.3				
1	0	50.0	54.3	57.7	60.8	62.2
	1	46.7	48.0	49.0	46.7	56.9
	2	48.5	51.3	51.6	54.2	55.1
	mean $\pm$ s.d.	51.2 $\pm$ 3.1				
<i>(b) Cross-spectral integral $\int_0^{f_{\max}} \log_2 \hat{S}(f) ^2 df$, bits/s</i>						
0.05	0	29.5	26.5	27.3	29.8	29.1
	1	27.4	27.9	28.2	27.7	27.5
	2	31.7	33.3	34.5	34.2	34.1
	mean $\pm$ s.d.	29.2 $\pm$ 3.6				
0.1	0	28.5	27.9	27.9	28.0	28.4
	1	26.6	27.7	27.6	26.9	26.3
	2	28.8	29.9	30.1	29.6	29.8
	mean $\pm$ s.d.	28.5 $\pm$ 1.2				

Table S6 (continued)

σ (Hz s ^{1/2})	Seed	$f_{\max}$ (Hz)				
		50	100	200	500	1000
0.2	0	31.1	31.5	31.6	31.5	31.6
	1	29.5	30.5	30.8	30.6	30.5
	2	28.8	29.2	29.1	29.3	29.3
	mean $\pm$ s.d.	30.4 $\pm$ 1.2				
0.3	0	34.8	34.9	34.8	34.8	34.8
	1	34.5	34.9	35.0	35.0	35.3
	2	31.6	32.3	32.2	32.1	32.0
	mean $\pm$ s.d.	34.0 $\pm$ 1.5				
0.5	0	38.1	37.9	37.9	37.9	37.8
	1	38.5	39.0	39.1	39.1	39.1
	2	37.9	38.1	38.2	38.2	38.3
	mean $\pm$ s.d.	38.3 $\pm$ 0.6				
0.7	0	35.2	34.6	34.7	34.8	34.8
	1	35.0	34.9	35.0	35.0	35.1
	2	35.2	35.2	35.2	35.2	35.2
	mean $\pm$ s.d.	34.9 $\pm$ 0.3				
1	0	29.4	28.3	28.4	28.4	28.3
	1	30.5	30.3	30.4	30.4	30.4
	2	29.6	29.3	29.3	29.3	29.3
	mean $\pm$ s.d.	29.3 $\pm$ 1.0				

S6 Interval cover in the proof of Proposition 2

The proof of Proposition 2 (Appendix A.2) reduces the uniqueness of the reference network level, at every drive, to the strict monotonicity of the function $u_E(q)$ of Eq. (9) on $q > q_0$, where q_0 is the inhibitory rate at which $R_E(q)$ vanishes. The derivative $u'_E(q)$, Eq. (A4), is positive by an analytic argument wherever $R_E(q) \leq [\pi\Delta^2/(2J_{EE})]^{1/3} \approx 0.428$ or $R_E(q) \geq J_{EE}/(2\pi)$; the latter holds for $q \geq q_b$, where q_b is defined by $R_E(q_b) = J_{EE}/(2\pi)$. It remains to certify the sign of u'_E on the compact interval of q on which $0.428 < R_E(q) < J_{EE}/(2\pi)$, which lies inside $(q_0, q_b]$, with $q_0 = 0.402159$ and $q_b = 2.588056$.

The certificate covers the interval $[q_{\text{first}}, q_{\text{last}}] = [0.402159316, 2.588055847]$, whose endpoints are exact nine-decimal numbers: q_{first} is $q_0(1 + 10^{-12})$, with $q_0 = 0.4021593154\dots$, rounded up at the ninth decimal, and q_{last} is $q_b(1 + 10^{-7})$, with $q_b = 2.5880555874\dots$, rounded up at the ninth decimal, so that q_{last} lies about 2.6×10^{-7} above q_b . Since q_0 and q_b come from a numerical root search, the certificate does not rely on them: the end checks below establish, in interval arithmetic, that the cover reaches the required values of R_E at both ends. The 17 subintervals are listed in Table S7; their endpoints are also exact nine-decimal numbers, and adjacent subintervals share an endpoint, so that they cover $[q_{\text{first}}, q_{\text{last}}]$ without gaps. Each endpoint is read from its decimal string and converted to an enclosing interval, so that no rounding of the endpoints can open a gap. On each subinterval Q , the program evaluates $R_E(Q)$, $R'_E(Q)$ and then Eq. (A4) in the interval arithmetic of mpmath 1.3.0 (mpmath developers n.d.), with 30 significant digits (a 103-bit mantissa) and outward rounding: every lower endpoint is rounded toward $-\infty$ and every upper endpoint toward $+\infty$, and π enters as mpmath's interval enclosure of π . The result is an interval guaranteed to contain $u'_E(q)$ for every $q \in Q$. The subintervals were found by bisection; their endpoints were

then rounded to nine decimals, and the whole cover was certified again from these decimal endpoints. The table reports this second certification. Every certified lower bound is positive; the smallest is 0.109662236..., on subinterval 3. Two end checks close the argument. At the left end, the upper endpoint of the enclosure of $R_E(q_{\text{first}})$ is $1.79 \times 10^{-9} < 0.428$, so the cover starts inside the interval next to q_0 on which the analytic argument gives $u'_E > 0$. At the right end, the lower endpoint of the enclosure of $R_E(q_{\text{last}})$ is 3.18309936..., larger than the upper endpoint of the enclosure of $J_{EE}/(2\pi)$, 3.18309886..., so the cover reaches beyond q_b . Together with the analytic argument, the cover shows that $u'_E(q) > 0$ for every $q > q_0$, and hence that $u_E(q) = u$ has exactly one solution for every drive u . The program is `prop2_cover.py`, and every enclosure, printed with 40 digits, is stored in `prop2_cover.json`.

Table S7 Interval-arithmetic cover used in the proof of Proposition 2. Each row is one subinterval $[q_{\text{lo}}, q_{\text{hi}}]$ of the inhibitory equilibrium rate q ; the endpoints are exact decimals, and adjacent rows share an endpoint. The next two columns give the interval enclosure of $R_E(q)$ (Eq. (9)) on the subinterval, rounded outward at the sixth decimal; the last column gives the certified lower bound of $u'_E(q)$ (Eq. (A4)), rounded down at the fifth decimal, so that every displayed bound is itself a lower bound. Arithmetic: mpmath 1.3.0 intervals, 30 significant digits (103-bit mantissa), outward rounding. Every bound is positive. The cover spans $[q_{\text{first}}, q_{\text{last}}] = [0.402159316, 2.588055847]$

Subinterval	q_{lo}	q_{hi}	$\min R_E$	$\max R_E$	Lower bound of $u'_E(q)$
1	0.402159316	0.538777849	1.78×10^{-9}	0.296424	24.75287
2	0.538777849	0.675396382	0.296423	0.487910	2.23592
3	0.675396382	0.812014915	0.487909	0.645797	0.10966
4	0.812014915	0.880324182	0.645796	0.720376	0.49004
5	0.880324182	0.948633448	0.720375	0.794188	0.35138
6	0.948633448	1.016942715	0.794187	0.868193	0.26731
7	1.016942715	1.085251981	0.868192	0.943058	0.21159
8	1.085251981	1.153561248	0.943057	1.019260	0.17480
9	1.153561248	1.221870514	1.019259	1.097144	0.15401
10	1.221870514	1.290179781	1.097143	1.176968	0.14891
11	1.290179781	1.358489048	1.176967	1.258927	0.16018
12	1.358489048	1.426798314	1.258926	1.343170	0.18885
13	1.426798314	1.495107581	1.343169	1.429812	0.23606
14	1.495107581	1.631726114	1.429811	1.610638	0.12698
15	1.631726114	1.768344647	1.610637	1.801936	0.34686
16	1.768344647	2.041581714	1.801935	2.217224	0.39797
17	2.041581714	2.588055847	2.217223	3.183100	1.04834

S7 Stability boundaries: transcritical and Hopf bifurcations

The level loses its stability in three ways in the population model: through a real eigenvalue at the gain threshold g_c (Section 3.2); through a complex pair at the upper gain boundary g_H and at the critical delay d_c ; and through a complex pair along the equilibrium branch of the reference network, for $-1.36324 < u_E < -1.08416$ (Section 3.10). Table S8 characterizes the boundaries of all three kinds. At the boundary of the first kind, a simple real eigenvalue crosses zero while the level remains an equilibrium for every g . These two facts do not, by themselves, establish a transcritical bifurcation, and we therefore computed its nondegeneracy coefficients (`transcritical_check.py`, `transcritical_check.json`). With v_c and w_c the right and left null vectors of the Jacobian at g_c , normalized so that the R_E component of v_c is 1 and $w_c \cdot v_c = 1$, the reduced equation on the center

manifold is $\zeta = \Theta_1(g - g_c)\zeta + \Theta_2\zeta^2 + \dots$, with $\Theta_1 = w_c \cdot (\partial_g D_x F)v_c$ and $\Theta_2 = \frac{1}{2}w_c \cdot D_x^2 F(v_c, v_c)$, and the bifurcation is transcritical if, in addition to the simple zero eigenvalue with no other eigenvalue on the imaginary axis and to $\partial_g F = 0$ along the level, $\Theta_1 \neq 0$ and $\Theta_2 \neq 0$ (Kuznetsov 2004). In the exact construction, in rational arithmetic, the characteristic polynomial at $g_c = 7/16$ is $\lambda(\lambda^2 + 4\lambda + 9)(\lambda^3 + 4\lambda^2 + 9\lambda + 12)$, so that the zero eigenvalue is simple and the other eigenvalues have real parts of at most $-0.8278/\tau$; $v_c = (1, \frac{1}{2}, \frac{4}{7}, \frac{2}{7}, 1, \frac{4}{7})$ in the coordinates $(R_E, V_E, R_I, V_I, S_E, S_I)$, $\Theta_1 = -16/27$ and $\Theta_2 = -5/42$, in units of $1/\tau$. In the reference network, $g_c = 0.41971100$, the other eigenvalues have real parts of at most -19.55 s^{-1} , $\Theta_1 = -61.41 \text{ s}^{-1}$ (equal to the derivative of the critical eigenvalue with respect to g , computed independently from the spectrum) and $\Theta_2 = -19.37 \text{ s}^{-1}$. Both bifurcations are therefore transcritical: the level exchanges stability with a second branch of equilibria, whose slope at g_c is $-\Theta_1/\Theta_2$ in R_E , that is, $-224/45$ in the exact construction and -3.171 in the reference network, where it corresponds to -50.46 Hz of $r_E = R_E/(\pi\tau)$ per unit of gain; the computed reference network branch agrees with this slope within 0.5% at $g = 0.41$ and 0.43 , and the computed slope of the exact branch converges to $-224/45$ as $g \rightarrow g_c$. In the reference network, the branch is stable and lies above the level for $g < g_c$, and for $g > g_c$ it lies below the level and has exactly one eigenvalue of positive real part (Table S8e). For $g < g_c$, the level has exactly one unstable eigenvalue, and small perturbations of opposite signs led to this branch and to a low state near 3 Hz. For $g > g_c$, the stable manifold of the unstable equilibrium has codimension one and is a natural candidate for part of the boundary of the basin of the level; we did not compute this manifold, and the basin is not characterized by a single firing rate. At $g = 0.43$, starts that differed from the level only in R_E returned to the level unless r_E was lowered below 31.4 Hz, whereas starts that differed only in S_E went to the low state as soon as S_E was lowered below the value that corresponds to 40.8 Hz, against 40.7 Hz for the unstable equilibrium; at $g = 0.5$, the two values were 12.2 and 37.8 Hz, against 37.0 Hz (bisection over 30-s runs, with one change of outcome on a grid of 96 starting values in each case). The exact construction behaves in the same way at $g = 7/16$: the branch lies above the level for $g < 7/16$ ($R_E = 1.127, 1.069$ and 1.033 at $g = 0.40, 0.42$ and 0.43 , stable) and below it for $g > 7/16$ ($R_E = 0.953$ at $g = 0.445$, with one unstable eigenvalue), where it meets a lower stable state and disappears between $g = 0.445$ and 0.45 ; from then on the level is the only positive equilibrium found. The four Hopf points of Table S8a, namely the two input-driven points of the reference network and the upper gain boundaries of the reference network and of the exact construction, have negative first Lyapunov coefficients, so that these four bifurcations are supercritical: just beyond the boundary, a small stable limit cycle replaces the level. Accordingly, the amplitude of the oscillation grows as the square root of the distance beyond the gain boundary (the amplitude at $\varepsilon = 4\%$ is close to twice that at $\varepsilon = 1\%$, in both models) and inside the oscillatory window of the branch (log-log slopes of 0.50 and 0.47). For the Hopf point at the critical delay, no first Lyapunov coefficient was computed: the delayed system is infinite-dimensional, and its normal form requires a center-manifold reduction for delay equations (Hale and Verduyn Lunel 1993) that we did not carry out. There we report only what the simulations show: the amplitude was zero at 12 and 13 ms and grew from 6.5 Hz at 13.6 ms to an oscillation between 3.5 and 87 Hz at 20 ms, which is consistent with a supercritical bifurcation but does not establish it. Here ε denotes the relative distance of the gain beyond g_H , or the distance of the input from the Hopf point, as stated in each panel.

Table S8 Stability boundaries of the population model. (a) Hopf points at which the first Lyapunov coefficient was computed (the Hopf point at the critical delay is not included): location, first Lyapunov coefficient (Kuznetsov 2004) (negative: supercritical), angular frequency ω and frequency f of the critical pair, and the absolute real part of that pair at the computed point (s^{-1} for the reference network, $1/\tau$ for the exact construction), which measures the accuracy of the location. (b) Oscillations just beyond the upper gain boundary, $g = g_H(1 + \varepsilon)$: amplitude (half the peak-to-peak range) of R_E in units of the level for the exact construction, and of r_E for the reference network, with its range. (c) Reference network at $g = 1$ with delayed inhibitory feedback (critical delay 13.3 ms): amplitude, mean and range of r_E over the last 4 s of 30 s. (d) Amplitude of the limit cycle of the reference network inside the oscillatory window, at a distance ε in u_E from each input Hopf point, and log-log slope over the three smallest ε (1/2 for a supercritical Hopf bifurcation). (e) Equilibria of the reference network near the lower gain boundary (a transcritical bifurcation, by the nondegeneracy coefficients given in the text): excitatory rate of the low state, of the branch that crosses the level at g_c , and the leading real parts of the Jacobian (s^{-1}) at the level and on the branch (the low state is stable throughout); positive real part, unstable

<i>(a) Hopf points</i>					
Boundary	Location	First Lyapunov coeff.	ω	f (Hz)	$ \text{Re } \lambda $ at the point
Reference network, input (lower)	$u_E = -1.36324$	-0.253	28.71 s^{-1}	4.57	7.8×10^{-9}
Reference network, input (upper)	$u_E = -1.08416$	-0.246	28.28 s^{-1}	4.50	1.4×10^{-8}
Exact construction, gain	$g_H = 1187/512$	-0.235	$1.2247/\tau$	—	3.9×10^{-11}
Reference network, gain	$g_H = 2.303$	-0.053	147.46 s^{-1}	23.47	1.2×10^{-7}
<i>(b) Beyond the upper gain boundary, $g = g_H(1 + \varepsilon)$</i>					
ε	g (exact)	Amplitude (exact; level = 1)	g (reference network)	Amplitude (Hz)	Range of r_E (Hz)
0.5%	2.3300	0.119	2.3142	5.5	35.9–46.9
1.0%	2.3415	0.168	2.3257	7.7	33.8–49.3
2.0%	2.3647	0.237	2.3488	10.9	31.0–52.8
4.0%	2.4111	0.334	2.3948	15.4	27.4–58.1
<i>(c) Reference network at $g = 1$ with delayed feedback</i>					
d (ms)	Amplitude (Hz)	Mean (Hz)	Minimum (Hz)	Maximum (Hz)	
12.0	0.0	41.3	41.3	41.3	
13.0	0.0	41.3	41.3	41.3	
13.6	6.5	40.8	34.8	47.9	
14.0	10.0	40.3	31.6	51.5	
15.0	14.5	39.5	27.5	56.5	
16.0	17.2	38.9	25.3	59.7	
20.0	41.6	31.2	3.5	86.7	

Table S8 (continued)

(d) Reference network inside the oscillatory window, u_E at a distance ε from each input Hopf point

ε	Amplitude, lower point (Hz)	Amplitude, up- per point (Hz)
0.0025	1.81	2.91
0.0050	2.56	4.05
0.0100	3.63	5.55
0.0200	5.16	7.43
0.0400	7.22	9.51
log-log slope	0.50	0.47

(e) Reference network near the lower gain boundary, $g_c = 0.420$ (transcritical; see the text)

g	Low state (Hz)	Level: max Re λ	Branch (Hz)	Branch: max Re λ	Branch vs. level
0.00	2.69	20.04	59.80	-9.96	+45.0%
0.20	3.02	11.64	51.50	-9.93	+24.8%
0.30	3.24	6.75	47.02	-6.16	+14.0%
0.35	3.38	4.06	44.67	-3.85	+8.3%
0.40	3.53	1.19	42.24	-1.17	+2.4%
0.41	3.57	0.59	41.74	-0.59	+1.2%
0.43	3.64	-0.64	40.73	0.64	-1.3%
0.45	3.71	-1.91	39.70	1.95	-3.8%
0.50	3.92	-5.28	37.04	5.63	-10.2%
0.60	4.50	-13.17	31.21	15.01	-24.3%

S8 Sections of the basin of the level

The local certificates of Section S10 prove attraction in a small neighborhood of the level, and the 201 starts of Table S10 sample a box of states at random. To see how far the basin extends, we computed two-dimensional sections of it in the population model of the reference network (Fig. S1; `basin_slices.py`, `fig_basin.py`), at the intact gain $g = 1$ and at two gains just above the threshold, $g = 0.5$ and $g = 0.43$ ($g_c = 0.4197$). In the first section, both populations start as if they were stationary at the rates on the axes, that is, $R_X = \pi\tau r_X$, $V_X = -\Delta/(2R_X)$ and $S_X = R_X$ for $X = E, I$. In the second, the start differs from the level only in R_E and in the excitatory synaptic variable S_E , so that the horizontal and vertical lines through the level are the one-coordinate displacements of Section 3.2. Each section is a grid of 151×151 starts, with rates from 0.5 to 120 Hz (about three times the level). Each start was integrated for 3 s (150τ) by the classical fourth-order Runge–Kutta method with a step of 0.1 ms, and its outcome was classified as held by the deterministic rule of Section 2.5 (excitatory rate within 5% of the level throughout the last 0.5 s) or as the low state (excitatory rate below 10 Hz at the end). For each section, 60 starts drawn at random were integrated again with an adaptive LSODA solver (relative tolerance 10^{-10}), and all 360 classifications agreed.

With intact feedback, every one of the $2 \times 22,801$ starts returned to the level, including those at 0.5 and at 120 Hz in either population (Fig. S1a,d,g). At $g = 0.5$, 61.0% of the starts in the first section and 67.8% in the second returned, and all the others went to the low state (3.9 Hz). At $g = 0.43$, where the level and the nearby unstable equilibrium change only at a rate of about 0.64 s^{-1} , both sections were integrated for 40 s (RK4 step 0.2 ms; classification over the last

1 s; `basin_long_check.py`): 59.9% and 65.4% of the starts returned, and 9,143 and 7,886 starts, respectively, went to the low state (3.6 Hz). In the second section, two of the 22,801 integrations produced nonfinite states and were classified as numerical failures, with unresolved dynamical outcomes. Figure S1 uses these 40-s records for $g = 0.43$; the 360 LSODA comparisons above concern the original 3-s integrations. At these two gains the boundary passes close to the level. In the second section it lies just below the level along S_E (on the grid line nearest to the level, at 37.9 Hz for $g = 0.5$ and 41.1 Hz for $g = 0.43$, in agreement, within the grid spacing of 0.8 Hz, with the values of 37.8 and 40.8 Hz of Section 3.2), whereas r_E alone can be lowered to 12.2 or 31.4 Hz before the level is lost. A small loss of excitatory synaptic activity is therefore enough to lose the level near g_c , which agrees with the asymmetry of the pulse experiments of Section 3.2. These sections are two two-dimensional sets of initial conditions in a six-dimensional state space (the first is a curved surface, because $V_X = -\Delta/(2R_X)$ varies nonlinearly with the rate, and the second is a plane in which the other four coordinates are fixed at the level), and they do not prove that the basin at $g = 1$ contains every positive state. They show that, over the sampled range, the attraction of the level at intact gain is not only local, and that the basin shrinks as the gain approaches the threshold g_c .

To examine the six coordinates simultaneously, we also integrated, at each of 19 gains from 0.43 to 2.25, the same 10,000 starts of a Latin hypercube (seed 20261006) in the sampling box of Eq. (S1), which is the box of the 201 random starts of Table S10 (`basin_box.py`; Table S9 lists the gains and the settings of each one). The integrator is the fourth-order Runge–Kutta routine of the sections, with a step of $5 \mu\text{s}$ during the first 50 ms, to resolve the fast transient of the starts with large $|V_X|$, and of 0.2 ms afterwards. The integration time was chosen so that the slowest mode of the level decays at least by the factor e^{-25} , that is, $T_g = \min\{40, \max[5, 25/|\text{Re } \lambda_1(g)|]\}$ s, where $\lambda_1(g)$ is the eigenvalue of the level with the largest real part; at the gains 0.78, 0.79 and 0.80, which are near the saddle-node described below, $T_g = 20$ s. Each outcome was classified by the deterministic rule of Section 2.5, applied to the last 1 s (instead of the last 0.5 s used for the sections), and a start was assigned to the low state when r_E was below 10 Hz at the end; every start fell in one of the two classes. For each gain, 30 starts drawn at random were integrated again with LSODA (relative tolerance 10^{-10}), and all 570 classifications agreed. For each of the 11 gains from 0.80 to 2.25, which include the intact network, every start returned to the level. For the gains from 0.43 to 0.79, between 53.4% and 64.2% of the starts returned, and the others went to the low state, whose excitatory rate increases with the gain from 3.6 to 8.9 Hz. The low state disappears at $g = 0.796767$, where it meets, in a saddle-node bifurcation, the unstable equilibrium that branches off the level at g_c ($r_E = 11.10$ Hz). We located this point by solving, with Newton’s method, the equilibrium equations reduced to (R_E, R_I) by $V_X = -\Delta/(2R_X)$ and $S_X = R_X$, together with the condition that their Jacobian matrix be singular (`saddle_node.py`). At this point, one eigenvalue of the full Jacobian matrix is zero and the others have real parts of -18.9 s^{-1} or less; at a gain 0.001 below it, the two equilibria exist (at 10.19 and 12.12 Hz), and at a gain 0.001 above it, neither equilibrium was found in the numerical search (240 solver starts, retaining positive equilibria with $r_E < 30$ Hz). This last search is a numerical check, not a proof that no other equilibrium exists.

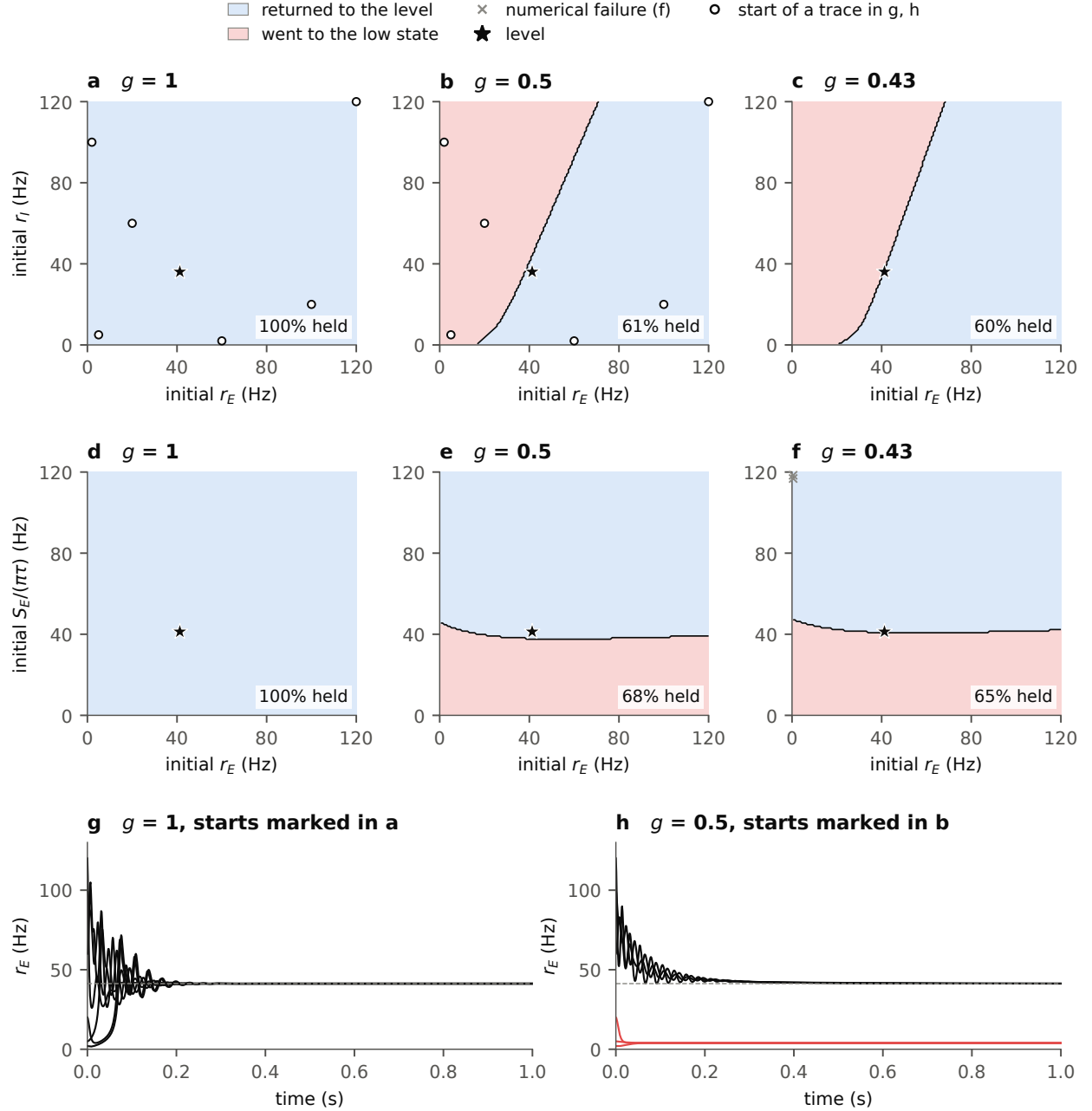


Fig. S1 Sections of the basin of the level in the population model of the reference network. (a–c) Both populations start as if stationary at the initial rates r_E and r_I ; (d–f) the start differs from the level only in $r_E = R_E/(\pi\tau)$ and in the excitatory synaptic variable, expressed as the rate $S_E/(\pi\tau)$. Columns: feedback gain $g = 1$ (intact network), 0.5 and 0.43 ($g_c = 0.4197$). For $g = 0.43$, the maps use the 40-s integrations and the last 1 s for classification; the other maps use 3-s integrations and the last 0.5 s. Gray crosses mark the two numerical failures, whose outcomes are unresolved. Blue, starts that returned to the level (rule of Section 2.5); pink, starts that went to the low state; black line, boundary between them; star, the level; percentages, fraction of the 151×151 starts that returned. (g, h) Excitatory rate from the six starts marked by circles in (a) and (b); black, returned to the level; red, went to the low state; dashed, the level

Table S9 Six-dimensional sample of the basin of the level (population model of the reference network; `basin_box.py`, `data/basin_box.json`). At every gain g , the same 10,000 starts of a Latin hypercube (seed 20261006) in the box of Eq. (S1) were integrated for the time T_g , by the fourth-order Runge–Kutta method with a step of $5\ \mu\text{s}$ during the first 50 ms and of 0.2 ms afterwards. $|\text{Re } \lambda_1|$, decay rate of the slowest mode of the level, which sets T_g (Section S8). Returned, r_E within 5% of the level throughout the last 1 s; low state, $r_E < 10\ \text{Hz}$ at the end; no start fell outside these two classes. Low-state r_E , median excitatory rate at the end of the starts that went to the low state. LSODA, agreement of the classification of 30 starts integrated again with LSODA (relative tolerance 10^{-10}). The low state disappears at the saddle-node $g = 0.796767$

g	$ \text{Re } \lambda_1 $ (s^{-1})	T_g (s)	Returned (%)	Low state (%)	Low-state r_E (Hz)	LSODA
0.43	0.64	39.2	53.39	46.61	3.64	30/30
0.45	1.91	13.1	54.10	45.90	3.71	30/30
0.50	5.28	5.0	55.63	44.37	3.92	30/30
0.60	13.17	5.0	58.50	41.50	4.50	30/30
0.70	23.46	5.0	60.78	39.22	5.54	30/30
0.75	24.05	5.0	62.30	37.70	6.61	30/30
0.78	24.39	20.0	63.60	36.40	7.96	30/30
0.79	24.50	20.0	64.23	35.77	8.93	30/30
0.80	24.61	20.0	100.00	0.00	–	30/30
0.82	24.82	5.0	100.00	0.00	–	30/30
0.85	25.13	5.0	100.00	0.00	–	30/30
0.90	25.60	5.0	100.00	0.00	–	30/30
1.00	26.42	5.0	100.00	0.00	–	30/30
1.25	27.68	5.0	100.00	0.00	–	30/30
1.50	27.63	5.0	100.00	0.00	–	30/30
1.75	25.18	5.0	100.00	0.00	–	30/30
2.00	16.10	5.0	100.00	0.00	–	30/30
2.20	5.06	5.0	100.00	0.00	–	30/30
2.25	2.52	9.9	100.00	0.00	–	30/30

S9 Summary of the numerical results

Table S10 collects the numerical results of the study with their scope. The rows of the original reference network series are means over time windows from one realization of 10,000 neurons, and the population values are independent equilibrium calculations; unless the table states that the finite baseline is used, the percent differences use the population value as denominator. The rows of the revised experiments are written by `supp_tables.py` from the stored data. The results of the population model with a noisy channel are given in Section 3.6 and are not repeated here.

Table S10 Numerical results and their scope. Rates are excitatory/inhibitory population means in Hz unless stated; “held” follows the rule of Section 2.5

Outcome	Result	Scope
<i>Original reference network series and attraction study</i>		
Exact construction	The characteristic polynomials, the Hurwitz minors, the critical delay and the sensitivities were verified symbolically (<code>proofs_check.py</code>).	Analytical results.
Population local certificates	The three certificates passed at $u_E = -3, 0$ and 3 (Section S10).	Local attraction only.
Population sampled starts	67 of 67 starts approached the equilibrium at each of $u_E = -3, 0$ and 3 , that is, 201 of 201.	Evidence about the basin from sampling, not a global proof.
Finite baseline	$E = 41.359625$ Hz, $I = 36.026500$ Hz; the population values are $E = 41.253103$ Hz, $I = 36.066472$ Hz.	Relative differences of $+0.2582\%$ and -0.1108% .
Finite maintained input -3	$E = 4.746625$ Hz, $I = 8.495500$ Hz; the population values are $E = 4.789684$ Hz, $I = 8.618902$ Hz.	Relative differences of -0.8990% and -1.4318% .
Finite maintained input $+3$	$E = 60.309438$ Hz, $I = 45.838500$ Hz; the population values are $E = 60.297570$ Hz, $I = 45.942981$ Hz.	Relative differences of $+0.0197\%$ and -0.2274% .
Finite recovery after -3	$E = 41.355750$ Hz, $I = 36.026750$ Hz.	Changes of -0.00937% and $+0.00069\%$ relative to the finite baseline.
Finite recovery after $+3$	$E = 41.350938$ Hz, $I = 36.019500$ Hz.	Changes of -0.02100% and -0.01943% relative to the finite baseline.
Brief pulses, step, size and initial states	After pulses of ± 3 , the means recovered to within 0.017% of baseline; halving the step changed them by at most 0.0015% ; networks of 5000 and 20,000 neurons by at most 0.12% ; the excitatory means of the three initializations differed by at most 0.016% (inhibitory, 0.033%).	Comparisons in one realization (Table S12).

Table S10 (continued)

Outcome	Result	Scope
Constant and re-played signals	Live communication 41.36/36.03 Hz; constant mean 59.58/45.50 Hz; circular replay 59.58/45.49 Hz.	Both state-independent signals lost the level in these finite runs, an observed outcome consistent with Corollary 1, which is stated for the population model.
<i>Revised experiments</i>		
Reference network level and linearization	Level 41.253/36.066 Hz; leading real part of the intact spectrum -26.42 s^{-1} ; with clamped inhibition, $\lambda_u = 20.04 \text{ s}^{-1}$, that is, $h = 28.912 \text{ bits/s}$. Stable for $g_c < g < g_H$, with $g_c = 0.420$ and $g_H = 2.303$; critical delay at $g = 1$, 13.3 ms.	Population model; linear stability.
Clamped inhibition	With $S_1^{(E)} = S_1^*$, perturbations of R_E by +1% and -1% led, after 1 s, to 59.8/45.7 Hz and to 2.7/7.5 Hz.	With clamped inhibition, the level has exactly one unstable eigenvalue; perturbations of opposite signs led to two different states.
Population equilibrium branch	One positive equilibrium at every input (Proposition 2; Table S7); unstable only for $-1.36324 < u_E < -1.08416$. Both Hopf points are supercritical (first Lyapunov coefficients -0.253 and -0.246 ; 4.57 and 4.50 Hz); the amplitude of the limit cycle grows with log-log slopes of 0.50 and 0.47 (Table S8).	Uniqueness is proved; supercriticality is computed.
Sensitivity to sustained input (reference network)	9.98 Hz per unit of u_E , against 3.05 Hz for an uncoupled population at the same rate (factor 3.3).	Proposition 8; a settling point.
Maintained input, all-to-all, exact integrator	Baseline 41.358/36.021 Hz; $u_E = -3$: 4.746/8.496 Hz; $u_E = +3$: 60.308/45.841 Hz; recovery 41.357/36.030 and 41.357/36.026 Hz.	Reproduces the original series with the exact integrator.
Maintained input, sparse	Five networks (seeds 11–15), excitatory rates. Baselines 41.50, 41.18, 42.52, 41.91 and 42.02 Hz; $u_E = -3$: 4.80, 6.15, 4.98, 6.35 and 5.52 Hz; $u_E = +3$: 60.52, 60.56, 61.02, 61.03 and 60.75 Hz. After removal of the drive, every network returned to within 0.018 Hz of its own baseline.	Random connectivity and excitabilities ($K = 200$).
Spiking gain, all-to-all	Five networks (seeds 0–4), pulses of ± 0.5 for 50 ms. $g \leq 0.35$: level lost in every run (positive pulse at $g = 0.2$ – 0.35 to 44.3–51.2 Hz; negative pulse, and both pulses at $g = 0$, to the low state, 2.6–3.3 Hz). $g = 0.4$ – 0.5 : after the positive pulse, within 5% of the level (40.5–42.5 Hz), negative pulse to the low state (3.5–3.9 Hz) in every network. $g = 0.6$: held in 8 of 10 runs (negative pulse to 4.4 Hz in the other two). $g = 0.8$ and 1: every run at 41.34–41.37 Hz.	Population threshold $g_c = 0.420$; Table S5.

Table S10 (continued)

Outcome	Result	Scope
Spiking gain, sparse	Five networks (seeds 11–15; 10 runs per gain). $g = 0$ and 0.3: level lost in every run (positive pulse at $g = 0.3$ to 45.6–48.4 Hz); $g = 0.6$: held in 8 of 10 runs (negative pulse to 5.9 and 6.1 Hz in the other 2); $g = 1$: held in 10 of 10.	Random connectivity and excitabilities; verdicts by condition (i); condition (ii), available for seed 11 only (traces stored), changed none of its verdicts.
Efficient coder, reference network	29 settings ($T_s = 20$ –200 ms, $n_q = 2$ –90, $\mathcal{R}/h = 0.69$ –1.73), 24 starts each (Table S4). $a_0 = 0.02\%$: all 17 settings with $\mathcal{R} \geq 1.017h$ held every start, and all 12 settings with $\mathcal{R} \leq 0.988h$ failed from every start. The lowest setting at which all 24 starts were held was at $1.12h$ with $a_0 = 0.2\%$ and at $1.12h$ with $a_0 = 2\%$, although some settings above it still lost starts (1 of 24 at $a_0 = 0.2\%$; 1 to 10 at $a_0 = 2\%$); no setting below h held any start at any a_0 .	The bound is approached in finite simulations.
Static codes, population model	176 settings (mid-rise and mid-tread quantizers with $n_b \in \{1, 2, 3, 4, 6, 8\}$ and $T_s = 0.5$ –64 ms; one-bit code with five values of β and 16 values of T_s from 2 to 64 ms), 24 starts each. None below h held any start; the lowest rate that held any start was 62.5 bits/s ($\beta = 0.01$, 1 of 24). One-bit code with $\beta = 0.02$: 24, 23, 11, 7 and 0 of 24 starts held at 157.5, 125, 99, 79 and 62.5 bits/s. Quantizers: held from 250 bits/s with 8 bits and from 750 bits/s with 6 bits; with 4 bits or fewer, at no rate up to 8000 bits/s.	Codes that do not track the dynamics need much more than h .
One-bit code, spiking	Three networks, two pulse signs ($\beta = 0.02$, $g = 1$). Held in 0 of 6 runs at 50 and 0 at 100 bits/s (s.d. 30–36 Hz), 3 at 125, 5 at 151.5 and 5 at 172.4, and 6 of 6 at 200, 417, 1000 and 2500 bits/s; s.d. 1.0–1.3 Hz at 200 and 0.5–0.6 Hz at 2500 bits/s.	Information manipulated at fixed gain and fixed mean.
Messengers	S.d. of r_E (10-ms bins): 0.54 (2000 messengers), 1.5 (300), 2.2 (100), 4.0 (30), 7.4 (10), 9.0 (5) and 11.9 Hz (3), with means of 41.0–43.5 Hz (41.0 Hz with 2000). Held down to 30; lost with 10, 5 and 3 in all five subsets; held in 2 of 5 subsets with 2; lost with 1 in all five (low state, 2.8 Hz, or s.d. 20.8–21.1 Hz).	A noise manipulation at fixed mean.
Sinusoidal drive, linear theory	$\int_0^\infty \ln S_g d\omega = 62.96 \text{ s}^{-1}$ for $g = 0.5, 0.7, 1$ and 1.5, equal to $\pi\lambda_u = 62.96 \text{ s}^{-1}$.	Theorem 3.
Sinusoidal drive, population model	$U = 1$: amplitude 11.7–12.2 Hz at 0.5–5 Hz and 7.3 Hz at 20 Hz for $g = 1$; 5.4–5.5 Hz at 0.5–2 Hz and 11.0 Hz at 20 Hz for $g = 1.5$; level lost at $g = 0.7$ for drives of 10 Hz or slower (means 6.6–38.6 Hz) and at $g = 0.5$ for every drive slower than 40 Hz (means 3.9–4.0 Hz).	Redistribution of the response across frequencies.

Table S10 (continued)

Outcome	Result	Scope
Sinusoidal drive, spiking	Five networks, $U = 0.5$, $g = 1$: amplitudes 5.09, 5.11, 5.14, 4.52, 3.52, 1.53 Hz at 1, 2, 5, 10, 20 and 40 Hz (s.d. across networks at most 0.04 Hz), against 5.00, 5.01, 5.03, 4.50, 3.58, 1.54 Hz predicted. $g = 0.6$: low state (4.5 Hz) at 1 and 2 Hz; level lost at 5 Hz (means 36.5, 20.8, 37.2, 37.0 and 36.7 Hz); held at 10, 20 and 40 Hz with amplitudes 3.77, 2.74, 1.82 Hz, against 3.75, 2.77, 1.83 Hz predicted.	Predictions corrected for the 5-ms bins.
Sinusoidal drive through channels and with delay	5 Hz, $U = 0.5$, RMS deviation from the level: 3.72 Hz with ideal feedback; one-bit code ($\beta = 0.02$) 3.91, 4.01, 4.15 and 4.75 Hz at 500, 354, 250 and 177 bits/s (8 of 12 starts held at 177), lost from every start at 125 bits/s and below (34–40 Hz); 8-bit mid-tread code 3.78, 3.90, 4.69 and 6.64 Hz at 16,000, 4,000, 1,000 and 500 bits/s, lost at 250 bits/s. Delay: amplitude 5.2, 5.8, 6.9 and 8.8 Hz at 0, 2, 5 and 8 ms; lost from 11 ms on.	Regulation error grows as information becomes scarce.
Noisy channel, population model	Reported in Section 3.6.	Compared with Theorem 4, which concerns the linearized network.
Estimator check, linearized reference network loop	Pooled $\hat{I}_G$ of 28.1–29.5 bits/s (mean 28.86) over the nine combinations of $g \in \{0.5, 1, 1.5\}$ and $\sigma \in \{0.01, 0.1, 1\}$; single 100-s trials 25.1–32.5 bits/s (s.d. 1.8). Capacity from the sent power: 73–75 ($g = 0.5$), 77–81 ($g = 1$) and 160–162 bits/s ($g = 1.5$).	Exact answer $h = 28.91$ bits/s; validates Eq. (11).
Noisy channel, spiking	Three networks. Held in all runs for $\sigma \leq 0.3$; under the 5% rule, in 1 of 3 at $\sigma = 0.5$ and in none at $\sigma \geq 0.7$ (mean 7.0–9.5% below baseline), whereas 9 of 9 runs at $\sigma = 0.5$ –1 met a tolerance of 10%, including the s.d. condition (Table S5): a displacement of the operating level whose classification depends on the tolerance, not by itself a loss of attraction. S.d. of r_E between 19.2σ and 23.5σ . $\hat{I}_G$ (100 Hz; mean $\pm$ s.d. over networks): 76.5 ± 2.5 , 45.0 ± 3.5 , 39.0 ± 4.0 , 46.4 ± 4.3 , 59.7 ± 2.6 , 58.5 ± 3.3 , 51.2 ± 3.1 bits/s for $\sigma = 0.05, 0.1, 0.2, 0.3, 0.5, 0.7$ and 1. Cross-spectral integral (100 Hz): 29.4 ± 2.1 bits/s for $\sigma \leq 0.2$ (nine runs, 26.5–33.3; means over networks 29.2, 28.5 and 30.4 bits/s at $\sigma = 0.05, 0.1$ and 0.2), 32–39 bits/s for $\sigma = 0.3$ –0.7 and 28.3–30.3 bits/s at $\sigma = 1$ (no run met the 5% rule).	Observed $\hat{I}_G \geq h$ in every run, an empirical comparison in the nonlinear spiking network, not a consequence of Corollary 2 (which concerns the linearized network with independent additive noise); cross-spectral integral consistent with h at weak noise, not an independent measurement of the instability rate (Section 2.6); Table S6.
Random networks, selection	641 draws; 74 passed the first selection, of which 68 were inhibition-stabilized and analyzed (6 excluded). r_E 6.7–63 Hz (median 26; 13 below 10 Hz); h 0.23–101 bits/s (median 24.9); g_c 0.004–0.96 (median 0.39); d_c 0.17–60 ms (median 8.0; 10 below 3 ms).	Population model; Section 3.7.

Table S10 (continued)

Outcome	Result	Scope
Efficient coder, random networks	63 networks (the other 5 have an unstable inhibitory block), 756 starts per setting. $a_0 = 0.02\%$: none held at $0.8h$ or $0.95h$; 97.5% at $1.05h$ (all starts in 58 networks), 99.2% at $1.2h$, 100% at $1.5h$; the incomplete cases at $1.05h$ were networks with $h < 6.2$ bits/s. $a_0 = 2\%$: none below h ; 45, 74 and 87% at 1.05 , 1.2 and $1.5h$.	Tightness across parameters.
One-bit code below h , random networks	In the 63 networks with h above 5 bits/s, every grid rate below h was tested (397 tests, 12 starts each); none held any start. Highest rate below h : 75–99.8% of h (median 87%; above 90% in 20 networks).	Finite-time outcomes consistent with the necessary condition of Theorem 1 across parameters.
Zero-information control, random networks	Received signal fixed at S_I^* : level lost in all 68 networks; in 3 networks, with h of 0.23, 0.39 and 1.31 bits/s, 4 to 6 of 12 starts ended within 5% of the level.	Another stable state of the clamped network lies within 3% of the level.
One-bit code above h , random networks	Every holding rate was at least $2.7h$. $\mathcal{R}_{\text{hold}}$ defined in 60 networks, with $\mathcal{R}_{\text{hold}}/h$ from 2.7 to 506 (median 8.2); holding monotone in the rate in 57 of the 61 networks held at some rate. 7 networks were not held at 5000 bits/s (h of 43–101 bits/s; 6 with $d_c < 1.6$ ms, the other with $g_c = 0.96$).	Inefficiency of a code not matched to each network.
Rank correlations, random networks	Spearman ρ (95% bootstrap interval): h and g_c , 0.91 (0.84 to 0.94; 68 networks); h and d_c , -0.71 (-0.82 to -0.56 ; 68 networks); h and $\mathcal{R}_{\text{hold}}$, 0.76 (0.62 to 0.85; 60 networks); d_c and $\mathcal{R}_{\text{hold}}$, -0.83 (-0.93 to -0.67 ; 60 networks). Partial correlation of h and $\mathcal{R}_{\text{hold}}$ given d_c : 0.57.	10,000 resamples of the networks.
Paradoxical response, random networks	$dr_I^*/du_I < 0$ in 68 of 68 networks; ρ between h and dr_I^*/du_I : -0.91 (-0.95 to -0.84 ; 68 networks).	Inhibition-stabilized networks.
Basin sections	At $g = 1$, all $2 \times 22,801$ starts with rates from 0.5 to 120 Hz returned to the level; at $g = 0.5$ and 0.43 , 59.9–67.8% did; the other starts went to the low state, except for two unresolved numerical failures at $g = 0.43$ (Section S8).	Two two-dimensional sets of initial conditions in a six-dimensional state space; evidence from sampling, not a global proof.

S10 Population sampling and local certificates

The settings of the scan and of the sampling are choices of the study, and not fitted physiological parameters. The Latin hypercube sampler follows the SciPy implementation ([SciPy developers n.d.b](#)). The sampling box is

$$[0.05, 6] \times [-4, 4] \times [0.05, 6] \times [-4, 4] \times [0, 6] \times [0, 6], \quad (\text{S1})$$

where the six intervals give, in the state order of Eq. (6), the sampled ranges of R_E , V_E , R_I , V_I , S_E and S_I , and $\times$ denotes the Cartesian product. The 64 sampled points are generated with seed 20261002 and reused at each of the three tested inputs, whereas the three additional fixed starts are

listed in Table S3. The output of the population model is sampled every 5 ms, and the adaptive integration uses a relative tolerance of 10^{-8} , an absolute tolerance of 10^{-10} and a maximum step of 1 ms. Finally, the guard $\max_j |x_j| = 10^6$, applied to the coordinates x_j of the state, detects numerical failure; it is not a biological boundary.

The certificate uses a standard local Lyapunov argument (Khalil 2002), with inequalities that we derive for Eq. (7). Let x_0 be a numerical candidate for the root, let $B = x_0 + [-\epsilon, \epsilon]^6$ be a box of coordinate radius ϵ around it, let C be a proposed inverse Jacobian, and let $G = I - CJ(B)$. Interval arithmetic then verifies that

$$\max_i \sum_j |G_{ij}| < 1, \quad x_0 - CF(x_0) + G(B - x_0) \subset \text{int } B. \quad (\text{S2})$$

In these two conditions, G_{ij} are the entries of G , with the indices i and j running over the six state coordinates; I is the six-dimensional identity matrix; $J(B)$ is the interval enclosure of the Jacobian matrix of F over the box B ; F is the vector field of Eq. (7) at the tested input; and $\text{int } B$ is the interior of B . In this section, C denotes the proposed inverse Jacobian and not a channel capacity, and B denotes the root box and not the coupling block of the Appendix. The absolute values and the inclusion denote outward interval bounds. When both conditions hold, the map $T(x) = x - CF(x)$ maps B into itself and is a contraction there. Its unique fixed point is a root, because the norm bound makes $CJ(x)$, and hence C , invertible, so that $CF(x) = 0$ implies $F(x) = 0$. It is important to note that this establishes uniqueness inside B only. The candidate radii of the box are 10^{-8} , 10^{-7} and 10^{-6} ; all three reported certificates passed with $\epsilon = 10^{-8}$. In addition, the rate and synaptic coordinates of each box must be positive.

Next, a floating-point solver proposes a symmetric matrix P satisfying $J(x_0)^\top P + PJ(x_0) = -I$ (Virtanen et al. 2020). Interval $LDL^\top$ pivot checks then verify, independently, that

$$P \succ p_{\min} I \succ 0, \quad -(J_*^\top P + PJ_*) \succ q_{\min} I \succ 0 \quad (\text{S3})$$

for every Jacobian J_* in the root box, including the Jacobian at the exact enclosed root x_* . In these inequalities, P plays the role of a Lyapunov matrix (in this section, P denotes this matrix, and not the polynomial of Eq. (15) or a signal power); $p_{\min}$ is a certified positive lower bound on the eigenvalues of P ; $q_{\min}$ is a certified positive lower bound on the eigenvalues of $-(J_*^\top P + PJ_*)$; and $\succ$ means that the difference between its left-hand and right-hand sides is positive definite. The implemented value of $q_{\min}$ is 0.25, and $p_{\max} = \sum_{ij} |P_{ij}|$, the sum of the absolute values of the entries P_{ij} of P , is an upper bound on the spectral norm of P . For the deviation $e = x - x_*$ of the state from the root, the exact quadratic remainder of the vector field is

$$Q(e) = \frac{1}{\tau} (2e_1 e_2, e_2^2 - e_1^2, 2e_3 e_4, e_4^2 - e_3^2, 0, 0)^\top, \quad (\text{S4})$$

that is, the part of $F(x_* + e)$ that remains after the subtraction of its linear approximation at x_* , where $e_1, \dots, e_4$ are the components of e along R_E , V_E , R_I and V_I , and the two zero components correspond to the synaptic coordinates. Since $(2ab)^2 + (b^2 - a^2)^2 = (a^2 + b^2)^2$ for any real numbers a and b , applying this identity to the pairs (e_1, e_2) and (e_3, e_4) shows that $\|Q(e)\|_2 \leq \|e\|_2^2 / \tau$. Therefore, the quadratic function $\mathcal{V}(e) = e^\top P e$ obeys

$$\dot{\mathcal{V}} \leq -q_{\min} \|e\|_2^2 + \frac{2p_{\max}}{\tau} \|e\|_2^3 \leq -\frac{q_{\min}}{2} \|e\|_2^2 \quad \text{when } \|e\|_2 \leq \rho_c, \quad 4p_{\max} \rho_c < q_{\min} \tau, \quad (\text{S5})$$

where $\dot{\mathcal{V}}$ is the derivative of $\mathcal{V}$ along the solutions of Eq. (7) and ρ_c is the radius of the ball of deviations on which the inequalities hold. The first inequality follows from the bounds $q_{\min}$ and $p_{\max}$

and from the bound on $\|Q(e)\|_2$; the second, from $\|e\|_2 \leq \rho_c$ together with $4p_{\max}\rho_c < q_{\min}\tau$. The code also checks the positivity of the domain for this ball and selects a level α_c with $0 < \alpha_c < p_{\min}\rho_c^2$. In this manner, the ellipsoid $e^T P e \leq \alpha_c$ is contained in the ball, because $P \succ p_{\min}I$; it is, moreover, forward invariant, and every trajectory in it converges to x_* (Khalil 2002). We emphasize that the certified set of attraction is the ellipsoid, and not the whole ball. The saved matrices define each ellipsoid explicitly.

Table S11 Bounds of the local certificates in dimensionless state coordinates. Each root box has coordinate radius 10^{-8} , and $q_{\min} = 0.25$

u_E	$p_{\min}$	$p_{\max}$	ρ_c	α_c
-3	8.61604×10^{-4}	0.211812	0.00295074	3.75094×10^{-9}
0	2.29885×10^{-3}	0.802235	0.000779074	6.97649×10^{-10}
3	2.49882×10^{-3}	0.562061	0.00111198	1.54489×10^{-9}

The table rounds the displayed values, whereas the saved records of the certificates retain the bounds used for the verification. The calculations use 50 decimal digits in `mpmath.iv`, exact representations of the proposed binary64 centers and matrices, decimal model constants and an interval for π ; the exported endpoints are widened by one binary64 step. It should be noted that the interval implementation is documented as experimental ([mpmath developers n.d.](#)); these are, therefore, reproducible certificates conditional on that implementation, and not an independently formalized proof. Furthermore, they neither enclose integrated trajectories nor certify global boundedness or global attraction.

S11 Original reference network series

All the results in this section come from the original reference network series with 10,000 neurons. Accordingly, they use the schedules of brief pulses and probes of Table S3, and not the maintained input of 6 s. The recorded means are descriptive statistics, and not independent replicate estimates. Figures S2–S5 show the records behind Tables S12 and S13: the brief pulses (Fig. S2), the constant and replayed signals (Fig. S3), the information assay (Fig. S4), and the local stability together with the comparisons of step, size and initial state (Fig. S5). They are drawn from the saved records of the run (`data/original_benchmark`, `fig_original_series.py`); no simulation was repeated for them.

Table S12 Mean rates in the original reference network series, in Hz per neuron. The pulse recovery uses 3–4 and 5–6 s; the control means use 2–4 s of their separate record; and the initialization and size means use 1.5–2.5 s

Condition	Excitatory	Inhibitory
Baseline	41.359625	36.026500
Recovery after positive pulse	41.358250	36.020500
Recovery after negative pulse	41.359000	36.029500
Unperturbed control source	41.353375	36.022500
Live control	41.355625	36.025250
Constant mean control	59.581438	45.501000
Circular replay control	59.577000	45.492250
Pulse protocol at half step (0.1 ms)	41.359625	36.029500

Table S12 (continued)

Condition	Excitatory	Inhibitory
Pulse protocol at the default step (0.2 ms)	41.359000	36.029500
Initialization 1	41.355125	36.015500
Initialization 2	41.356500	36.019000
Initialization 3	41.349750	36.027500
5000 neurons	41.405250	36.021000
20,000 neurons	41.321813	36.032250

All 11 saved flags passed, namely: the equilibrium residual, the agreement of the analytic Jacobian with finite differences, the stability of the live population, the instability under replacement of inhibition by its equilibrium value, the three population initial states, the finite pulse recovery, the step comparison, the spread over finite excitatory initializations, the preservation of the replay mean, the preservation of the replay variance and the direction found by the information estimator on the sanity process. It should be noted that these flags test their programmed criteria; in particular, the size comparison is descriptive and was not a separate flag.

Table S13 Proxies of conditional information, in bits per 10-ms bin, in the probe assay of the original series. The null statistics use 99 circular shifts of the source. The rank is $(1 + \#\{T_{\text{null}} \geq T_{\text{observed}}\})/100$, where T_{observed} is the estimate and T_{null} its value for a shifted source, and is exploratory

Condition	Direction	Estimate	Null median	Null 95th percentile	Rank
Live	$E \rightarrow I$	0.209660	0.000930	0.003110	0.01
Live	$I \rightarrow E$	0.202872	0.000953	0.002569	0.01
Constant	$E \rightarrow I$	0.000539	0.001071	0.003174	0.71
Constant	$I \rightarrow E$	0.000298	0.000903	0.003307	0.84
Replay	$E \rightarrow I$	0.007127	0.000790	0.002273	0.01
Replay	$I \rightarrow E$	0.002279	0.001106	0.003118	0.20

The same probe assay gave mean excitatory/inhibitory rates of 41.309299/35.974444 Hz with live communication, 2.626264/7.351639 Hz with constant replacement and 2.627563/7.352389 Hz with replay. These values belong to the probe assay and, for this reason, must not replace the separate means of the brief controls in Table S12. Finally, the estimator uses histories of rates rather than complete neural states, and the comparison with shifted sources is neither a causal test nor a test corrected for multiplicity (Vicente et al. 2011).

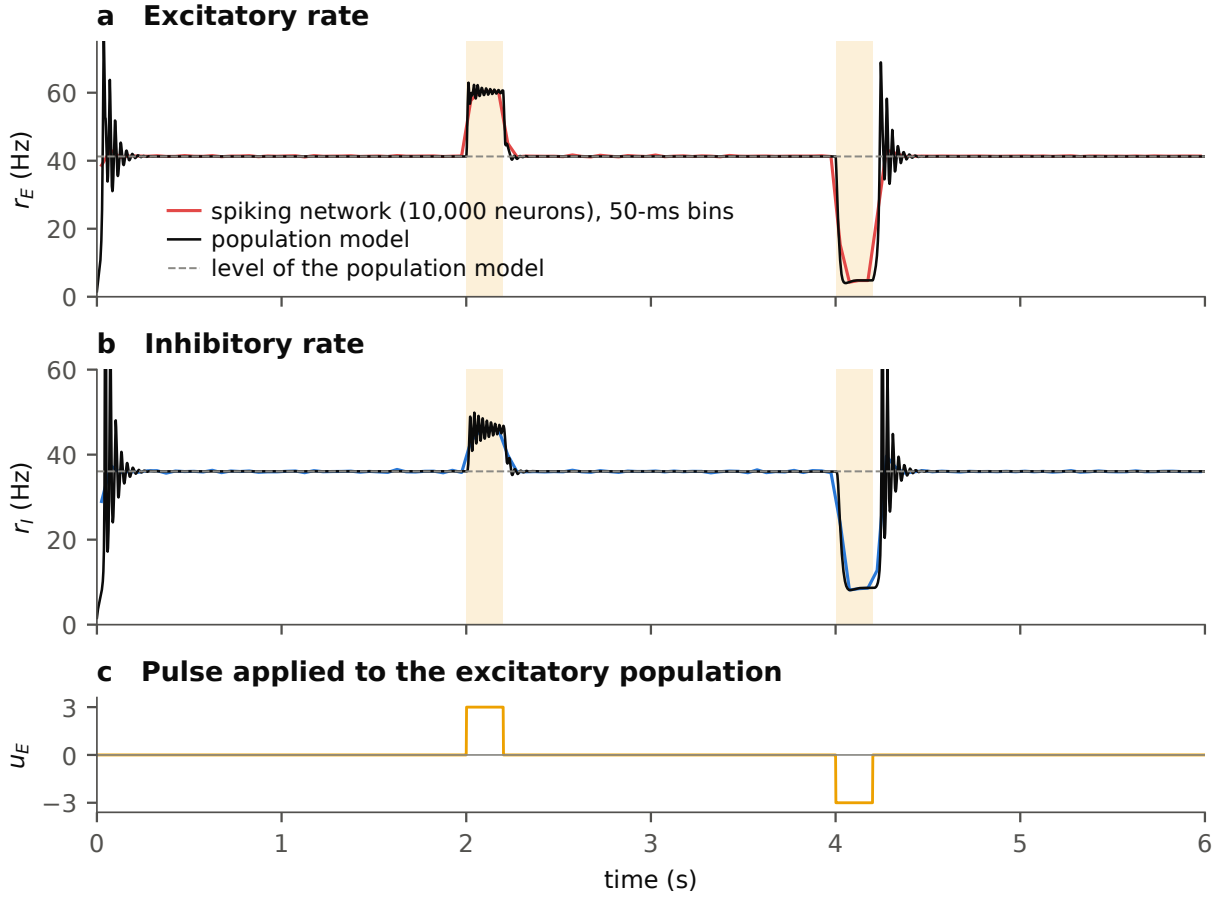


Fig. S2 Recovery after brief pulses (original reference network series). (a) Excitatory and (b) inhibitory rates of the all-to-all spiking network of 10,000 neurons, in 50-ms bins, and of the population model, which starts away from the level at $t = 0$ (1.6/1.6 Hz). Shaded, the pulses of $u_E = +3$ (2.0–2.2 s) and $u_E = -3$ (4.0–4.2 s), shown in (c). Dashed, the level of the population model (41.25/36.07 Hz). During the positive pulse the excitatory rate of both models rises to about 60 Hz, and during the negative pulse it falls to about 4.8 Hz; after each pulse both models return to the level. The recovery means are in Table S12

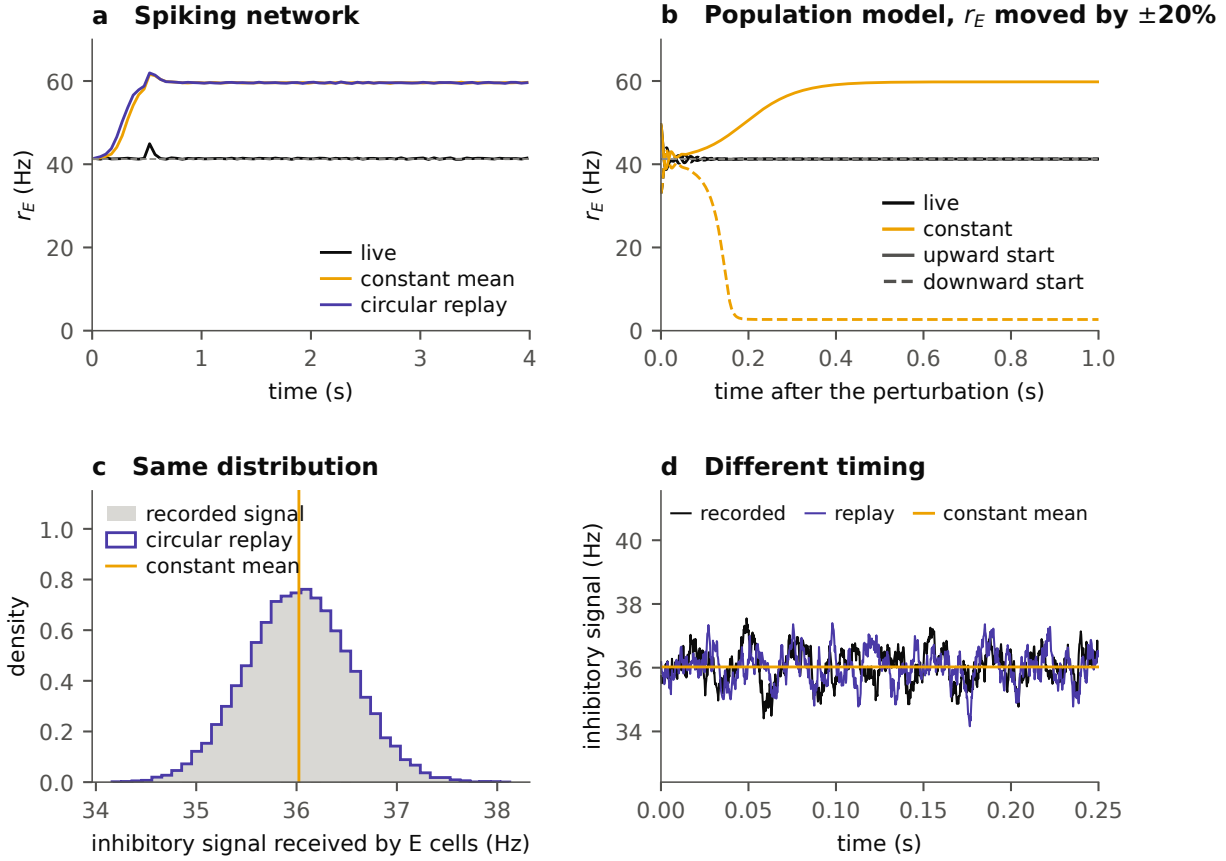


Fig. S3 Live, constant and replayed inhibitory signals (original reference network series). (a) Excitatory rate of the spiking network, in 50-ms bins, when the inhibitory signal received by E cells is the live one, its recorded mean, or a circular replay of the record; all three branches start from the same warmed state and receive the same brief excitatory perturbation. With the live signal the rate returns to the level (dashed); with either state-independent signal it moves to the upper state of the clamped network. The late means (2–4 s) are in Table S12. (b) Population model after the excitatory rate is moved 20% above or below the level, with intact feedback (live) or with the received signal clamped at its equilibrium value (constant). The clamped network leaves the level in the direction of the displacement, to 59.8 or 2.7 Hz; the intact network returns from both sides. (c) Distribution of the recorded inhibitory signal and of its circular replay, which is the same by construction, and the constant mean. (d) The first 0.25 s of the three signals: the replay keeps the statistics of the record but not its relation to the current state of the network

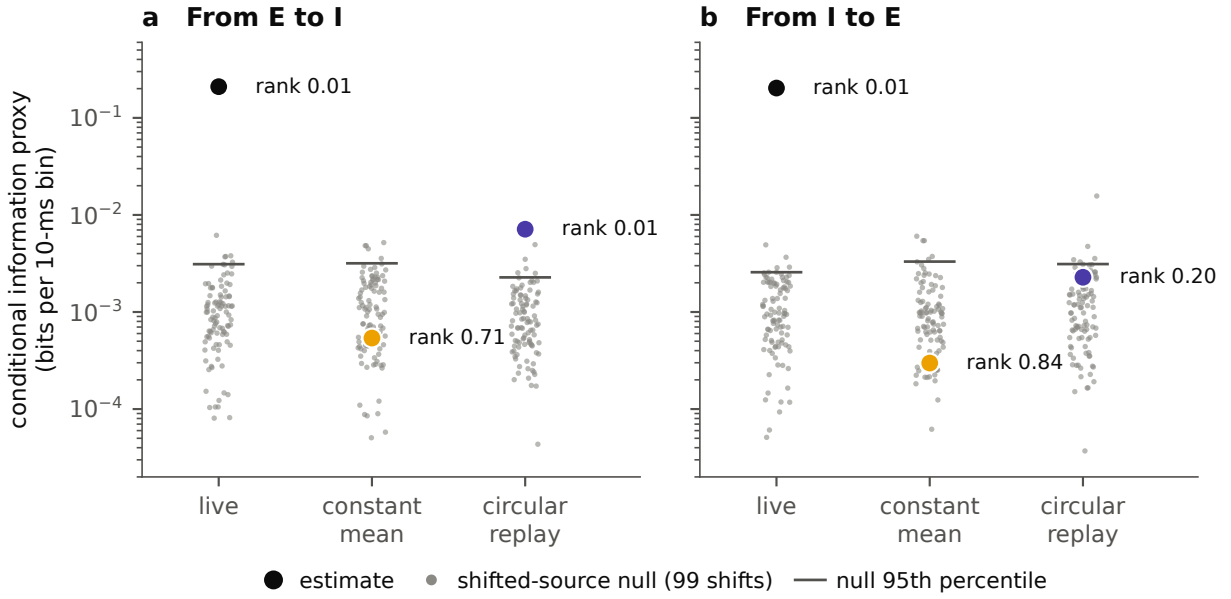


Fig. S4 Conditional information proxy in the probe assay (original reference network series), in bits per 10-ms bin on a logarithmic scale, (a) from E to I and (b) from I to E. Large dots, the estimates of Table S13; small gray dots, the 99 values for circularly shifted sources; horizontal bars, their 95th percentile; rank, as defined in Table S13. Live communication gives about 0.2 bits per bin in both directions, far above all 99 shifted-source values. In this assay, which used separate 20-s runs with random probe currents, both state-independent conditions ended in the low state, so that the three conditions differ in activity regime as well as in signal; the proxy is exploratory and is not a measurement of the channel capacity of Theorem 1

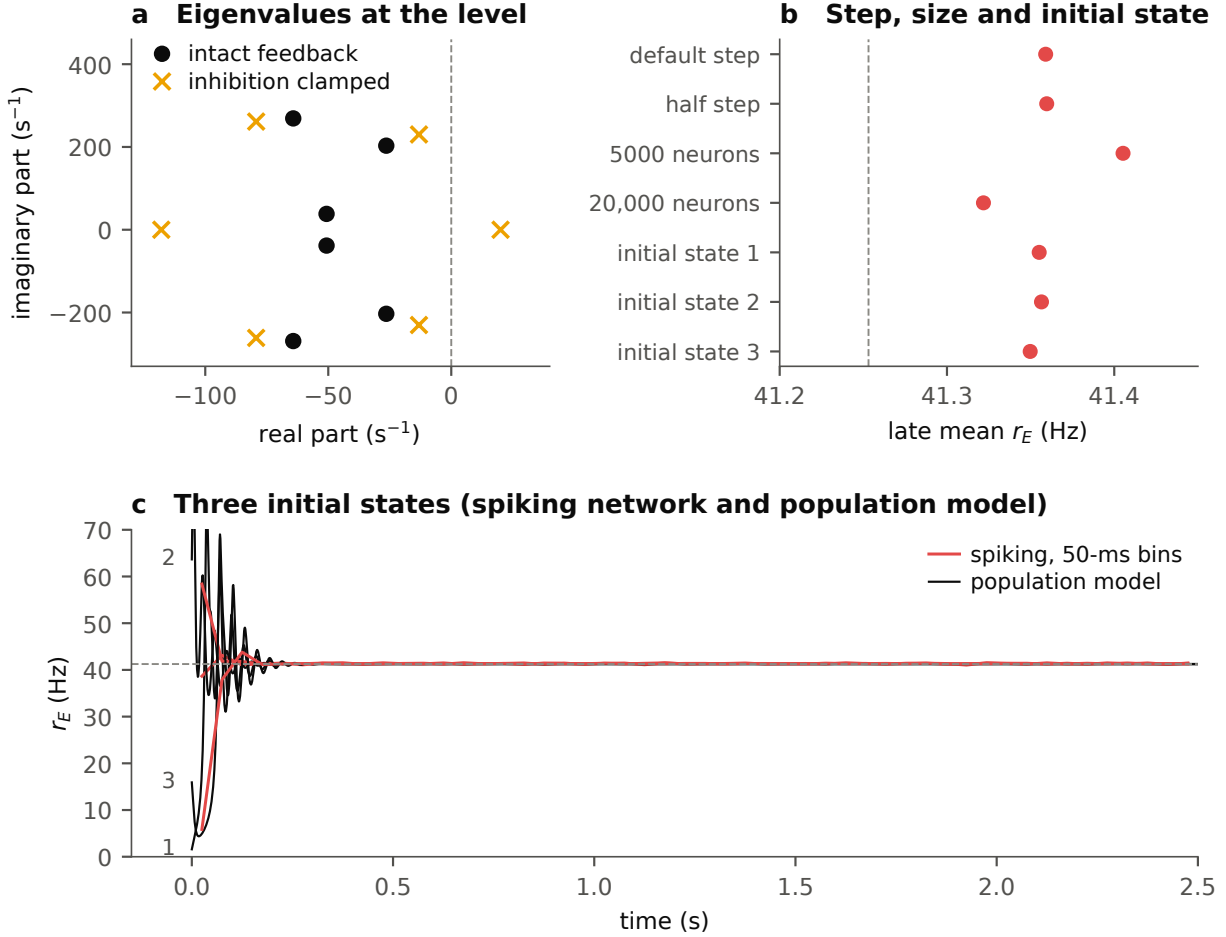


Fig. S5 Local stability and numerical comparisons (original reference network series). (a) Eigenvalues of the Jacobian of the population model at the level, with intact feedback and with the received inhibitory signal clamped at its equilibrium value. Clamping does not move the equilibrium, but one real eigenvalue becomes positive ($+20.0 s^{-1}$). (b) Late mean excitatory rate of the spiking network for the default step (0.2 ms) and half step, for networks of 5000 and 20,000 neurons, and for three initializations (means as defined in Table S12); dashed, the level of the population model. All values lie between 41.32 and 41.41 Hz. (c) Excitatory rate from three initial states: spiking network, in 50-ms bins, for the three initializations of Table S3, and population model from three initial states (numbers at the left, at their starting rates). All six runs stay within 5% of the level from 0.18 s on. These are comparisons in one realization each, not replicate statistics

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