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Stanislaw Ambroszkiewicz

sambrosz@gmail.com

University of Siedlce

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Stanislaw Ambroszkiewicz

Institute of Computer Science, University of Siedlce, 3 Maja 54, 08-110
Siedlce, Poland.

Corresponding author(s). E-mail(s): sambrosz@gmail.com, ORCID
[0000-0002-8478-6703](https://orcid.org/0000-0002-8478-6703);

Abstract

We present a computational approach to asymptotic sequences of functions defined on partitions of the unit interval of the real numbers. These functions can be viewed as increasingly accurate approximations of certain functions defined on the unit interval. The term "asymptotic" means that these sequences yield, in the limit, continuous functions on the unit interval if these sequences satisfy some simple conditions. The approach corresponds to the intuitionistic view of Continuum (real numbers) proposed by Brouwer. The famous Brouwer Continuity Theorem is proved in our framework, on the basis of different principle than the Axiom of Continuity.

Keywords: computable functions on real numbers, interval arithmetic, asymptotic sequences of finite functions on intervals

1 Introduction

Representing real numbers in a computer always involves an approximation and a potential loss of significant digits. Performing arithmetic on very small or very large numbers can lead to errors that are not admissible in abstract Mathematics.

The more bits we use to represent a number, the greater the precision of the representation and the more memory we consume. For an excellent recent work on this subject see Braverman 2013 [1].

Real numbers (as defined in Mathematics) are represented in computers as double-precision (64-bit) binary floating-point numbers. It is a computer data type conforming to the IEEE 754 standard. The complete range of the positive normal floating-point numbers is from $2^{-1022} \approx 2 \times 10^{-308}$ to approximately $2^{1024} \approx 2 \times 10^{308}$.

The collection of such normal floating-point numbers is finite, and they are in fact rational numbers from the point of view of Mathematics.

Most floating-point number representations allow for the existence of two zeros, often denoted by -0 (negative zero) and $+0$ (positive zero).

Floating-point arithmetic operations, such as addition and division, approximate the corresponding real number arithmetic operations by rounding any result that is not a floating-point number itself to a nearby floating-point number.

Computers are discrete and finite machines, and they cannot cope with some of the continuous and infinite concepts of Mathematics. Since even the decimal fraction $\frac{1}{10}$ has no exact finite representation in binary notation, a computer can only approximate it. No computer can even read it in or print it out exactly, much less perform exact calculations with it.

To deal with these types of numerical errors, a technique called interval arithmetic was invented.

It is also called interval computation. It is used to mitigate rounding and measurement errors in mathematical computation by computing function bounds that can guarantee relatively reliable and mathematically correct results. Instead of working with an uncertain real-valued variable x , interval arithmetic works with an interval $[a, b]$ that defines the range of values that x can have. In other words, any value of the variable x is in the closed interval between a and b . A function f , when applied to x , gives an interval $[c, d]$ which includes all the possible values for $f(x)$ for all $x \in [a, b]$.

The foundations of interval arithmetic were set forth independently by three mathematicians: Mieczyslaw Warmus 1956 [2] and [3] in Poland, Teruo Sunaga [4] in Japan and Ramon E. Moore [5] and [6] in the United States.

Despite a simple idea underling the interval arithmetic, it is not easy nor simple, see [7]. New alternative versions of this arithmetic have been defined and developed for years to remove shortcomings and weaknesses of previously proposed versions.

The concept of a computable function on real numbers, as understood in mathematical analysis, plays a crucial role here. The four basic arithmetic operations (addition, subtraction, multiplication, and division) can be elegantly defined as computable operations within the framework of interval arithmetic. However, this is insufficient to capture the general concept of a computable function on real numbers that could be applied in practical computations.

Let us cite Braverman 2013 [1]: “*The study of algorithms dealing with real numbers and functions over the reals requires extending the reach of traditional computability theory but is a notable challenge for mathematicians and computer scientists.*”

There are two basic definitions of computable functions on the real numbers: Banach–Mazur’s definition [8] [9], and Grzegorzczuk’s definition [10].

Let X and Y be subsets of the real numbers $\mathbb{R}$. A function $f : X \rightarrow Y$ is *Banach–Mazur computable* if, for every computable sequence $(x_n)_{n \in \mathbb{N}}$ in X , the image sequence $(f(x_n))_{n \in \mathbb{N}}$ is computable in Y .

It is the weakest notion of computability in analysis, and is not appropriate for practical computer computations.

A remarkable property proved by Stanisław Mazur is that every Banach–Mazur computable function with a computably separable domain (e.g. the set of the real numbers) is continuous.

Let $(a; b)$ and $(c; d)$ be intervals of the real numbers $\mathbb{R}$. A function $f : (a; b) \rightarrow (c; d)$ is *Grzegorzczak computable* if there is a discrete algorithm A_f that, given a precision parameter n and access to arbitrarily good rational approximations of an arbitrary input $x \in (a; b)$, outputs a rational y_n such that $|y_n - f(x)| < 2^{-n}$.

Also here any computable function is continuous.

Let us notice, that for practical computations we do not need computable function as complete set theoretical function. It is sufficient to have its approximations for arbitrary small precision parameter.

Coming back to the idea of interval arithmetic, we can approximate the domain $(a; b)$ of function f . For a precision parameter $k \in \mathbb{N}$ of the domain, the corresponding approximation is the partitioning of $(a; b)$ into sub-intervals each of length $2^{-k}(b - a)$. Let this partition be denoted $\Pi(a; b)_k$.

We can also approximate the co-domain $(c; d)$ of the function. For a precision parameter $n \in \mathbb{N}$ of the co-domain, the corresponding co-domain approximation is the partitioning of $(c; d)$ into sub-intervals each of length $2^{-n}(d - c)$. Let the partition be denoted $\Pi(c; d)_n$.

Now we can rewrite the Grzegorzczak's definition as follows.

Function $f : (a; b) \rightarrow (c; d)$ is *computable* if there is an algorithm A_f that, given co-domain precision parameter n ; the algorithm outputs domain precision parameter k , such that for any element $I \in \Pi(a; b)_k$, there is element $O \in \Pi(c; d)_n$ such that

$$f(I) \subseteq O$$

where $f(I)$ is the image of set I , i.e., the set of all $f(x)$ such that $x \in I$.

So that for a given co-domain precision parameter n , the algorithm determines a mapping from $\Pi(a; b)_k$ into $\Pi(c; d)_n$. This mapping may be considered as $2^{-n}(d - c)$ – approximation of function f .

This leads to an effective and entirely practical construction of consecutive, increasingly accurate approximations of the function f .

Given the algorithm A_f , we no longer require a set-theoretic definition of the function f . This algorithm suffices to compute the value of the function f to any desired degree of precision.

Thus, the algorithm itself (or rather, the mappings it determines) can serve as a constructive definition of the function f .

This raises the question: what general and necessary conditions must these mappings satisfy to be considered a practically computable function of real numbers?

This article provides an answer to precisely this question – within a more general framework. We therefore present an approach to reliable real-number computations that extends the basic idea of interval arithmetic.

For more details see the complete version of our work on Continuum: [11], and also a mathematical approach to asymptotic, finite approximations of geometric spaces [12].

2 Continuum and the real numbers

A real number $x \in \mathbb{R}$ is *computable* if there is an algorithm that can compute x with any prescribed precision. For a given natural number $n \in \mathbb{N}$ the algorithm outputs a rational number x_n such that $|x_n - x| < 2^{-n}$.

However, the question remains: what is a real number?

It is a ubiquitous opinion among mathematicians that a real number is just a point in the line. If this rough definition is not enough, then a mathematician may provide a formal definition of the real numbers in the set theoretic and axiomatic fashion, i.e. via Cauchy sequences or Dedekind cuts, or as the collection of axioms characterizing exactly (up to isomorphism) the set of real numbers as the complete and totally ordered Archimedean field. Actually, the above notions of the real numbers are abstract and do not have a constructive grounding. Definition of Cauchy sequences, and equivalence classes of these sequences explicitly use the actual infinity. The same is for Dedekind cuts, where the set of rational numbers is used as actual infinity. Although there is no direct constructive grounding for these abstract notions, there are so called intuitions on which they are based.

In the sequel, we propose a rigorous approach to express these very intuition in a constructive way. It is based on the concept of the adjacency relation that seems to be a missing primitive concept in type theory. The approach corresponds to the intuitionistic view of Continuum proposed by Brouwer. The famous and controversial Brouwer Continuity Theorem is discussed on the basis of different principle than the Axiom of Continuity.

Since, in the classical view, the closed unit interval $[0, 1]$ – a subset of the set of real numbers $\mathbb{R}$ – serves as a mathematical example of a continuum, let us adopt this interpretation.

Let the n -th dimensional Continuum be interpreted as the n -th dimensional unit cube $[0; 1]^n$, that is, the Cartesian product of n -copies of the unit interval. Let us fix the dimension n and consider a unit cube. The cube may be divided into parts (smaller n -cubes) in many ways. However, the most uniform division is to divide it into 2^n the same parts, where n is the dimension of the cube. Each of the parts may be divided again into 2^n the same sub-parts (even smaller n -cubes), and so on, finitely many times. There is a natural relation of adjacency between the elementary cubes. Two cubes (parts) are adjacent if their intersection (as sets) is a cube of dimension $n - 1$.

Our purpose is to liberate (step by step) our reasoning from the Euclidean interpretation of Continuum as n -cubes. It seems that only the following properties are essential.

- Any part is *the same* (modulo size) as the original unit cube. It can be divided arbitrary many times into smaller parts. The same patten of partition, and the same adjacency pattern between the resulting parts is repeated in the consecutive divisions.
- Any part (except the border parts) has the same number of its adjacent parts and the same adjacency structure.

Hence, only the following aspects are important:

- the adjacency relation between parts resulting from consecutive divisions;
- fractal-like structure of the unit cube, i.e. its parts are similar (isomorphic) to the cube;
- the partition pattern is the same for all divisions;
- the pattern of adjacency between sub-parts, after consecutive divisions, is one and the same.

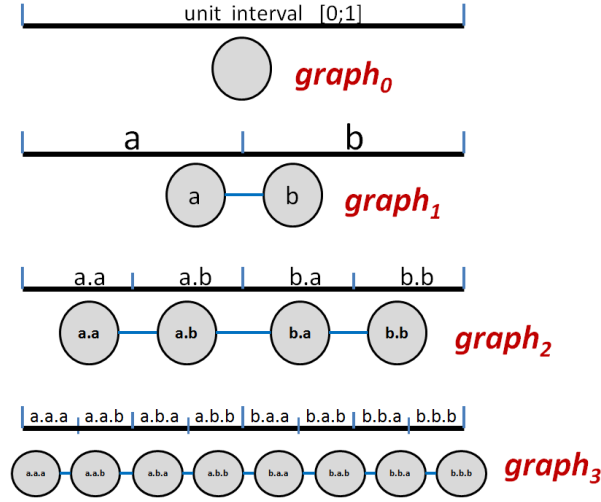


Fig. 1 Consecutive partitions of Continuum of dimension 1, i.e. the unit interval $[0;1]$, and the corresponding graphs as adjacency relations between parts

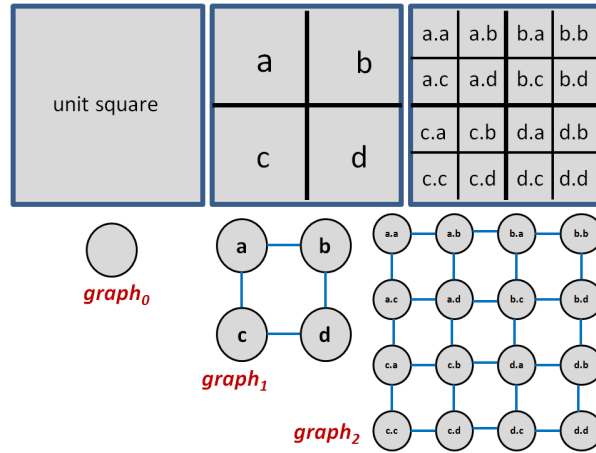


Fig. 2 Consecutive partitions of Continuum of dimension 2, i.e. the unit square $[0;1]^2$, and the corresponding graphs as adjacency relations between parts

In order to abstract completely from the intuitions of the unit cube, as well as from its interpretation as a subset of $\mathbb{R}^n$, only the above aspects should be taken into account in further investigations.

The adjacency relations between parts in consecutive divisions may be represented as graphs.

Let us take (for a while) the classic view and consider again the unit interval as Continuum. The interval may be represented as the $graph_0$, see Fig. 1, consisting only of a single node. The unit interval can be divided into two adjacent parts (two Continua). Each of the parts is the same (modulo size) as the original Continuum (the unit interval). This may be represented as the $graph_1$ consisting of two nodes, and one edge between them, corresponding to the adjacency of these two parts. We may continue the divisions, so that we get $graph_2$, $graph_3$, and so on, representing the parts as nodes, and adjacency as edges. Notice the pattern of inductive construction of the sequence of graphs.

Now, let the unit square be considered as the Continuum, see Fig. 2. The method of construction of the sequence of graphs resulting from consecutive division of the square is similar. That is, the nodes represent small squares (Continua), whereas the edges represent the adjacency between them. The pattern of inductive construction of the sequence of graphs is important.

Let us consider the single node of the $graph_0$ that represents an abstract Continuum. The $graph_1$ is constructed from the $graph_0$, by dividing the single node (Continuum) into 8 nodes (Continua), and determining adjacency between the nodes. The $graph_2$ is constructed from the $graph_1$ by dividing each of the 8 nodes again into 8 nodes with the same adjacency (edges) as in the $graph_1$, and then some additional adjacency (edges) are determined. Each of the additional edges in $graph_2$ corresponds to an edge in the $graph_1$. Notice the pattern of inductive construction of the graphs.

The sequence $(graph_0, graph_1, graph_2, \dots, graph_k, \dots)$, and the pattern of its inductive construction, is crucial to grasp the proper grounding of the notion of Continuum.

The patterns corresponding to the unit interval and the unit square are formally defined in Section 3.2.

If more edges are added during the construction of the graph sequences for the unit interval and the unit square, then this results in some interesting Continua like circle, torus, Möbius strip, Klein bottle, and many others interesting Riemannian manifolds.

Let us summarize the above examples.

Sequence $(graph_0, graph_1, graph_2, \dots, graph_k, \dots)$ of the adjacency graphs constructed inductively according to a *division and adjacency pattern*, can be seen as the grounding for Continuum. We are going to prove that.

3 Abstraction: a new primitive type

Introducing a new primitive type requires: primitive objects, object constructors and destructors, and primitive relations.

Let c_0 denote an abstract unit as a primitive object. It corresponds to the $graph_0$ from the above examples. The object c_0 can be divided into parts. Each

of the parts is again divided, and so on, inductively, to construct the sequence $(graph_0, graph_1, graph_2, \dots, graph_k, \dots)$.

Each division is done according to the same division and adjacency pattern denoted by $(\mathbb{C}_d, Adj_d)$ where $\mathbb{C}_d$ is a finite collection of elementary parts (denoted by $a_1, a_2, \dots, a_{l-1}, a_l$), Adj_d is an adjacency relation between these parts. The adjacency relation (as $graph_1$ from the examples) is connected, so that uniting all adjacent parts results again in the unit c_0 .

To denote the parts resulting from consecutive divisions, it is convenient to use the *dot* notation introduced in Abstract Syntax Notation One (ASN.1). The notation has been already used in the above examples. So that, the parts resulted from division of c_0 are denoted by $c_0.a_1, c_0.a_2, \dots, c_0.a_{l-1}, c_0.a_l$. Each of the parts $(c_0.a_i, \text{ for } i = 1, 2, \dots, l)$ is divided again, resulting in $c_0.a_i.a_1, c_0.a_i.a_2, \dots, c_0.a_i.a_{l-1}, c_0.a_i.a_l$. Analogously, it is done for the consecutive divisions.

The parts are objects of type (we are going to introduce) denoted by C . Actually, this is a simple example of Martin-Löf's W-type, that is, the type of lists (finite sequences) over C_d . However, contrary to W-types, for the type C also a specific primitive adjacency relation will be introduced.

Since a formal introduction of primitive operations and primitive relations is not necessary to follow and understand the proposed grounding of the Continuum, it will be omitted here. It is done elsewhere, see [13], where the importance for type theory is stressed and explained in detail.

Let C_k denote the collection of objects of length $k + 1$ in C .

3.1 Adjacency as a primitive relation

We are going to extend the adjacency relation Adj_d to all objects of the same length, i.e. to the objects in C_k for arbitrary natural number k . This very extended relation corresponds to the $graph_k$ from the examples in Section 2.

For a fixed x of type C , adjacency between $x.a_1, x.a_2, \dots, x.a_l$ (where $a_1, a_2, \dots, a_l$ are the all elements belonging to C_d) is determined by Adj_d , that is, $x.a_i$ and $x.a_j$ are defined as adjacent if $Adj_d(a_i, a_j)$ holds. So that, $(\mathbb{C}_d, Adj_d)$ determines the division pattern, and it is called *d-pattern*.

In order to extend the adjacency to all objects of C_k (then, also to C), a merging pattern (in short *m-pattern*) is necessary to determine if there is adjacency between $x.a_i$ and $y.a_j$, where x and y are different adjacent objects of the same length.

Important restriction on the m-pattern. For any x and y of the same length, if they are adjacent, then there are a_i and a_j such that $x.a_i$ and $y.a_j$ are also adjacent. The above restriction follows from our intuition that if two parts are adjacent then after their divisions, there must be at least two sub-parts (one sub-part from each division) that are also adjacent.

3.2 Euclidean patterns

For the unit interval $[0; 1]$, the d-pattern $(\mathbb{C}_d, Adj_d)$ consists of two elements (parts), denoted by a and b , that are adjacent. The consecutive three divisions are as follows, and they are also shown in Fig. 1:

1. $c_0.a$ and $c_0.b$
2. $c_0.a.a$, $c_0.a.b$, $c_0.b.a$, $c_0.b.b$
3. $c_0.a.a.a$, $c_0.a.a.b$, $c_0.a.b.a$, $c_0.a.b.b$, $c_0.b.a.a$, $c_0.b.a.b$, $c_0.b.b.a$, $c_0.b.b.b$

The lexicographical order can be defined on the type C , if it is supposed that a is preceding (lesser than) b .

Since c_0 is the common prefix of all objects of type C , let it be omitted. For the first partition, a and b are adjacent. For the second partition, also $a.b$ and $b.a$ are adjacent. According to the d-pattern, for any x of type C , objects $x.a$ and $x.b$ are adjacent. Also $x.a.b$ and $x.b.a$ are adjacent.

For the third partition, also $a.b.b$ and $b.a.a$ are adjacent.

For the fourth partition, also $a.b.b.b$ and $b.a.a.a$, $a.a.b.b$ and $a.b.a.a$, $b.a.b.b$ and $b.b.a.a$ are adjacent.

The m-pattern is defined as follows. For any x and y of type C of the same length, if $x.b$ and $y.a$ are adjacent, then

- if $x.b$ is less (in the lexicographical order) than $y.a$, then $x.b.b$ and $y.a.a$ are adjacent,
- if $y.a$ is less than $x.b$, then $y.a.b$ and $x.b.a$ are adjacent.

This completes the definition of the division and adjacency pattern for the unit interval.

For the unit cube of dimension 2 (i.e. the unit square), $\mathbb{C}_d$ consists of four elements denoted by a, b, c, d . Consecutive partitions are shown in Fig. 2. The adjacency relation Adj_d is between a and b , b and d , d and c , and between c and a . For any x the following pairs of objects are adjacent: $x.a$ and $x.b$, $x.b$ and $x.d$, $x.d$ and $x.c$, $x.c$ and $x.a$. The m-pattern for dimension 2 is based on our intuition of the 2-dimensional unit cube.

More adjacent pairs of parts after the second and third division are shown in Tab. 1 and Tab. 2.

$x.a.b$ and $x.b.a$	$x.a.d$ and $x.b.c$
$x.a.c$ and $x.c.a$	$x.a.d$ and $x.c.b$
$x.b.c$ and $x.d.a$	$x.b.d$ and $x.d.b$
$x.c.b$ and $x.d.a$	$x.c.d$ and $x.d.c$

Table 1 For any x of C the pairs in the table are adjacent

$x.a.b.b$ and $x.b.a.a$	$x.a.b.d$ and $x.b.a.c$	$x.a.d.b$ and $x.b.c.a$	$x.a.d.d$ and $x.b.c.c$
$x.a.d.c$ and $x.c.b.a$	$x.a.d.d$ and $x.c.b.b$	$x.a.c.c$ and $x.c.a.a$	$x.a.c.d$ and $x.c.a.b$
$x.b.c.c$ and $x.d.a.a$	$x.b.c.d$ and $x.d.a.b$	$x.b.d.c$ and $x.d.b.a$	$x.b.d.d$ and $x.d.b.b$
$x.c.b.b$ and $x.d.a.a$	$x.c.b.d$ and $x.d.a.c$	$x.c.d.b$ and $x.d.c.a$	$x.c.d.d$ and $x.d.c.c$

Table 2 Additional adjacent pairs after third division

a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	d	c	d	c	d

Fig. 3 Unit square after 4-th division. The parts are denoted only by single letters a , b , c and d . Their complete denotation is of the form $o_1.o_2.o_3.o_4$, where o_4 is either a or b or c or d , and o_1 , o_2 and o_3 can be decoded (as one of the letters a , b , c and d) from the position of the part in the unit square

Let the alphabetical order on a, b, c, d be extended to the lexicographical order on C_k for any k . The m-pattern for the unit cube of dimension 2 is defined inductively as follows. Let x and y be of the same length. There are four following possible cases for which the adjacency relation is defined inductively.

- If $x.a$ and $y.b$ are adjacent and
 - $x.a$ is less than $y.b$ (according to the lexicographical order), then $x.a.b$ and $y.b.a$ are adjacent, and $x.a.d$ and $y.b.c$ are adjacent.
 - $y.b$ is less than $x.a$, then $x.a.a$ and $y.b.b$ are adjacent, and $x.a.c$ and $y.b.d$ are adjacent.
- If $x.b$ and $y.d$ are adjacent and
 - $x.b$ is less than $y.d$, then $x.b.c$ and $y.d.a$ are adjacent, and $x.b.d$ and $y.d.b$ are adjacent.
 - $y.d$ is less than $x.b$, then $x.b.a$ and $y.d.c$ are adjacent, and $x.b.b$ and $y.d.d$ are adjacent.
- If $x.d$ and $y.c$ are adjacent and
 - $x.d$ is less than $y.c$, then $x.d.b$ and $y.c.a$ are adjacent, and $x.d.d$ and $y.c.c$ are adjacent.
 - $y.c$ is less than $x.d$, then $x.d.a$ and $y.c.b$ are adjacent, and $x.d.c$ and $y.c.d$ are adjacent.
- If $x.a$ and $y.c$ are adjacent and

- $x.a$ is less than $y.c$, then $x.a.c$ and $y.c.a$ are adjacent, and $x.a.d$ and $y.c.b$ are adjacent.
- $y.c$ is less than $x.a$, then $x.a.a$ and $y.c.c$ are adjacent, and $x.a.b$ and $y.c.d$ are adjacent.

The Fig. 3 may help to verify the above cases. This completes the definition of the Euclidean pattern for the unit square.

Note that the above m-pattern is not so simple, so that in order to grasp it we must refer to the pictures, i.e. to our intuitions.

In the very similar way, for any n dimensional cube, the appropriate d-pattern and m-pattern can be formally defined. Let it be called *the Euclidean pattern of dimension n* .

4 Generic form of the patterns

In its generic form, the division and adjacency pattern inductively determines the adjacency relation (graph) on the objects (nodes) of C_k for any k that corresponds to $graph_k$ from the examples in the Section 2. From now on, d-pattern may be arbitrary, and is determined by a collection $\mathbb{C}_d$ of elementary parts (primitive objects) $a_1, a_2, \dots, a_l$; and an arbitrary relation Adj_d (a connected graph) on these parts.

m-pattern that (along with d-pattern) determines the adjacency relation (denoted by Adj_{C_k}) for objects of C_k , for arbitrary k , also may be arbitrary. The only necessary condition is that the graph related to Adj_{C_k} must be connected. Note that Adj_{C_k} is symmetric, i.e. for any x and y in C_k , if $Adj_{C_k}(x; y)$, then $Adj_{C_k}(y; x)$. It is also convenient to assume that Adj_{C_k} is reflexive, i.e. any x is adjacent to itself, formally $Adj_{C_k}(x; x)$.

The relation Adj_{C_k} can be extended (by appropriate construction) to the type C , that is, to the relation Adj_C in the following way. For any x and y of different length (say n and k respectively, and $n < k$), relation $Adj_C(x; y)$ (as well as its symmetric version $Adj_C(y; x)$) holds if there is object $\bar{x}$ of length k such that x is a prefix (initial segment) of $\bar{x}$, and $Adj_{C_k}(\bar{x}; y)$ holds, i.e. $\bar{x}$ is of the same length as y and is adjacent to y .

Let us note that the above definitions of d-pattern and m-pattern are so general that the resulting adjacency relation, i.e. the sequence $(graph_0, graph_1, graph_2, \dots, graph_k, \dots)$, may be different than the adjacency relation for the unit cubes. The only necessary condition imposed on the patterns is that the corresponding adjacency relation on C_k , as $graph_k$, must be connected for any k . That is, the result of uniting all adjacent parts must be again the initial unit object c_0 .

Let d-pattern and m-pattern together be denoted by $\mathbb{P}$. For a fixed pattern $\mathbb{P}$, let C be denoted by $C_{\mathbb{P}}$ whereas Adj_C by $Adj_{\mathbb{P}}$. Analogously, $C_{\mathbb{Q}}$ denotes the primitive type for pattern $\mathbb{Q}$, and $Adj_{\mathbb{Q}}$ denotes its adjacency relation. Euclidean pattern of dimension n is denoted by $\mathbb{E}^n$.

Let us recall the intuitive meaning of Continuum: “... no part of which can be distinguished from neighboring parts except by arbitrary division”. That is, the self-similar fractal structure resulting from the adjacency patterns is not enough. The following conditions are essential.

- *Indiscernibility of the parts:* The parts from $\mathbb{C}_d$ can not be distinguished only on the basis of their neighborhood (adjacency) structures. That is, any part in $\mathbb{C}_d$ has exactly the same neighborhood structure.
- *Homogeneity of the parts:* For any x and y (except the border objects) in C_k , their neighborhoods in C_k are isomorphic with regard to the adjacency relation, i.e. any object from C_k (except the border objects) has exactly the same fixed adjacency structure in C_k .

The Euclidean patterns satisfy the Indiscernibility condition and Homogeneity condition. The natural question is if there are other patterns that satisfy these two conditions.

Let us summarize this section. For any d-pattern and any m-pattern (that together determine division and adjacency pattern $\mathbb{P}$), a primitive W-type can be introduced (denoted by $C_{\mathbb{P}}$) along with adjacency relation $Adj_{\mathbb{P}}$. Usually, the type corresponds to a fractal. However, pattern $\mathbb{P}$ of inductive construction of sequence $(graph_0^{\mathbb{P}}, graph_1^{\mathbb{P}}, graph_2^{\mathbb{P}}, \dots, graph_k^{\mathbb{P}}, \dots)$ may be arbitrary, and sometimes it does not have a fractal-like structure.

Let us note that a sequence of the form $(graph_0, graph_1, graph_2, \dots, graph_k, \dots)$ and corresponding adjacency relation may be constructed inductively not necessary using d-pattern and m-pattern. Hence, pattern $\mathbb{P}$ is defined as an inductive method for generating the sequence $(graph_k^{\mathbb{P}}, k = 0, 1, 2, \dots)$ and corresponding adjacency relation $Adj_{\mathbb{P}}$.

5 Abstraction from patterns to topological spaces

The primitive types augmented with adjacency relations correspond to important and interesting topological spaces as was argued in Section 2. It was claimed there that such types are the grounding for the spaces that themselves are abstract notions. Now, we are going to show how to abstract from a primitive type to a topological space.

Functions on the types (say from $C_{\mathbb{P}}$ into $C_{\mathbb{Q}}$), are important for the abstraction. The patterns $\mathbb{P}$ and $\mathbb{Q}$ are arbitrary.

A function $f : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$ is defined as *monotonic* if for any x and y such that x is prefix of y , then $f(x)$ is a prefix of $f(y)$, or $f(x)$ is equal to $f(y)$.

Let us consider sequences of the form $(x_k; k = 1, 2, \dots)$ of objects of type $C_{\mathbb{P}}$ such that x_k is of length k and x_k is a prefix of x_{k+1} . Monotonic function f is *strictly* monotonic if for any such sequence $(x_k; k = 1, 2, \dots)$ and for any n there is $j > n$ such that $f(x_n)$ is not $f(x_j)$.

A strictly monotonic function $f : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$ is defined as *continuous* if for any two adjacent objects x and y , the objects $f(y)$ and $f(x)$ are also adjacent. This continuity may be seen as more intuitive if x and y are of the same length. Actually, it is the same; it follows from the definition of adjacency relation for objects of different length (see Section 4), and from the strict monotonicity.

The strictly monotonic and continuous functions are important for the Brouwer's Continuum and the famous Brouwer's Continuity Theorem, as we will see in the next sections.

Let $\mathbb{C}_d$ correspond to pattern $\mathbb{P}$, and consider the type $C_{\mathbb{P}}$. Let us introduce the abstract notion of infinite sequences over $\mathbb{C}_d$ and denote the set of such infinite sequences by $C_{\mathbb{P}}^{\infty}$. For an infinite sequence denoted by u , let $u(k)$ denote its prefix of length k , i.e. the initial finite sequences of length k .

The set $C_{\mathbb{P}}^{\infty}$ may be considered as a topological space (Cantor space) with topology determined by the family of open sets U_x such that for any x of length n : $U_x = \{u : u(n) = x\}$. Note that the pattern $\mathbb{P}$ is not used in the definition.

We are going to introduce a different topological structure on the set $C_{\mathbb{P}}^{\infty}$ determined by the pattern. In order to do so the pattern $\mathbb{P}$ must satisfy the following two properties.

Property 1. *If x and y (of the same length) are adjacent, then there are two elements of $\mathbb{C}_d$ (say c_1 and c_2) such that $x.c_1$ and $y.c_2$ are adjacent.*

Property 2. *If x and y (of the same length) are not adjacent, then for any two elements c_1 and c_2 of $\mathbb{C}_d$, the objects $x.c_1$ and $y.c_2$ are not adjacent.*

Let a pattern that satisfies the above two conditions be called *regular*. For regular patterns, if x and y (of the same length k) are adjacent, then their prefixes of length $i < k$, i.e. $x(i)$ and $y(i)$, also are adjacent; they may be also the same.

For a regular pattern $\mathbb{P}$, the corresponding $(graph_k^{\mathbb{P}}, k = 0, 1, 2, \dots)$ is an inverse sequence and $C_{\mathbb{P}}^{\infty}$ is the well known inverse limit of the sequence, see [14], and [15] for a recent review of the subject.

Euclidean patterns are regular.

From now on, we consider only regular patterns. So that the term *pattern* means regular pattern. The reason for restricting patterns to the regular ones, is the following definition of extension of adjacency relation to the infinite sequences.

Two infinite sequences u and v are defined as *adjacent* if for any k , the prefixes $u(k)$ and $v(k)$ are adjacent, i.e. $Adj_{\mathbb{P}}(u(k); v(k))$ holds. Let this adjacency relation, defined on the infinite sequences, be denoted by $Adj_{\mathbb{P}}^{\infty}$. Note that any infinite sequence is adjacent to itself. Two different adjacent infinite sequences may have the same prefixes.

The transitive closure of $Adj_{\mathbb{P}}^{\infty}$ is an equivalence relation denoted by $\sim$. Let the quotient set be denoted by $C_{\mathbb{P}}^{\infty}/\sim$, whereas its elements, i.e. the equivalence classes be denoted by $[v]$ and $[u]$. The topological space $C_{\mathbb{P}}^{\infty}/\sim$ is the Hausdorff reflection of $C_{\mathbb{P}}^{\infty}$ relatively to the equivalence relation $\sim$.

Let us take the pattern $\mathbb{E}^1$, corresponding to the unit interval $[0; 1]$, as an example. Let us recall that $\mathbb{C}_d$ consists of a and b that are adjacent. $C_{\mathbb{E}^1}$ is the type of finite sequences over $\mathbb{C}_d$. Objects of this type are denoted, in the dot notation, as $c_0.c_1.c_2. \dots c_k$ where c_i (for $i = 1, 2, \dots, k$) are elements of $\mathbb{C}_d$. Since c_0 is the common prefix of all objects, it is omitted. For any x of length k , let the infinite sequences v and u be such that $x = v(k) = u(k)$, and the $(k+1)$ -th element of v is b , and the $(k+1)$ -th element of u is a , and for any n the $(k+1+n)$ -th element of v is a , and the $(k+1+n)$ -th element of u is b . That is, $v = x.b.a.a.a. \dots$ and $u = x.a.b.b.b. \dots$. Then, v and u are adjacent, and the equivalence class $[v]$ consists exactly of two elements v and u . For the rest of infinite sequences z , the equivalence classes $[z]$ are singletons, i.e. each one consists of one element only.

The transitive closure of the adjacency relation $Adj_{\mathbb{E}^1}^{\infty}$ does not extend the relation; it remains the same. However, it is not so for higher dimensional patterns, e.g. $\mathbb{E}^2$,

where some equivalence classes have 4 elements, others have 2 elements, and the rest classes are singletons.

Let $\mathbb{P}$ be again an arbitrary regular pattern. The relation $Adj_{\mathbb{P}}$ determines natural topology on the quotient set $C_{\mathbb{P}}^{\infty}/\sim$.

Usually, a topology on a set is defined by the family of open subsets that is closed under finite intersections and arbitrary unions. The set and the empty set belong to that family. Equivalently, the topology is determined by families of closed neighborhoods of the points of the set. For any equivalence class $[v]$ (i.e. a point in $C_{\mathbb{P}}^{\infty}/\sim$), the neighborhoods (indexed by k) are defined as the sets $U_k^{[v]}$ of equivalence classes $[u]$ such that there are $v' \in [v]$ and $u' \in [u]$ such that $u'(k)$ is adjacent to $v'(k)$, i.e. $Adj_{\mathbb{P}}(u'(k); v'(k))$ holds. Note that $u'(k)$ may be equal to $v'(k)$.

A sequence $(e_1, e_2, \dots, e_n, \dots)$ of equivalence classes of $\sim$ (elements of the set $C_{\mathbb{P}}^{\infty}/\sim$) converges to e , if for any i there is j such that for all $k > j$, $e_k \in U_i^e$.

Note that the topological space $C_{\mathbb{P}}^{\infty}/\sim$ is a Hausdorff compact space. It is an abstraction done from the type $C_{\mathbb{P}}$ and its adjacency relation $Adj_{\mathbb{P}}$. The abstraction uses actual infinity, i.e. infinite sequences.

For the pattern $\mathbb{E}^1$ corresponding to the unit interval $[0; 1]$, the space $C_{\mathbb{E}^1}^{\infty}/\sim$ is homeomorphic to that interval.

For another patterns $\mathbb{P}$ (e.g. $\mathbb{E}^2$ from Section 2), the space is homeomorphic to topological spaces: the unit square, Sierpinski sieve, etc. Hence, the patterns and the resulting types should be considered as the groundings for these topological spaces.

The patterns may be quite sophisticated so that the abstracted topological spaces may be homeomorphic to important spaces like torus, Möbius strip and Klein bottles of higher dimensions, Sierpinski sieve and carpet, and many others fractals and Riemannian manifolds.

The extension of a strictly monotonic function $f : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$ to the function f^{∞} from $C_{\mathbb{P}}^{\infty}$ into $C_{\mathbb{Q}}^{\infty}$ is natural. That is, $f^{\infty}(v)$ is defined as element z in $C_{\mathbb{Q}}^{\infty}$ such that for any i there is k such that $z(i)$ is a prefix of $f(v(k))$. The strict monotonicity of f is essential here. An equivalent definition of the extension is that $f^{\infty}(v)$ is the limit of the sequence $(f(v(n)), \text{ for } n = 1, 2, \dots)$. By the monotonicity, for any n , $f(v(n))$ is a prefix of $f(v(n+1))$ or equal to $f(v(n+1))$. By the strict monotonicity, the length of consecutive elements of the sequence is unbounded.

6 Brouwer's choice sequences

Let us consider again the type $C_{\mathbb{E}^1}$ where the pattern $\mathbb{E}^1$ corresponds to the unit interval $[0; 1]$, that is, a subset of the real numbers.

The finite sequences may be interpreted (according to Fig. 1) as the subintervals of the consecutive partitions of the unit interval $[0; 1]$. That is, a is $[0; \frac{1}{2}]$ and b is $[\frac{1}{2}; 1]$, $a.a$ is $[0; \frac{1}{4}]$, $a.b$ is $[\frac{1}{4}; \frac{2}{4}]$, $b.a$ is $[\frac{2}{4}; \frac{3}{4}]$, $b.b$ is $[\frac{3}{4}; 1]$, and so on. Then, the adjacency (according to pattern $\mathbb{E}^1$) between the sequences of length k is determined by the adjacency of the sub-intervals resulting from the k -th division. However, it is a bit misleading because it is based on our intuition, whereas the aim is to provide the concrete grounding for this intuition. The pattern of this adjacency (without referring to this intuition) is simple, and is the basis for inductive construction of the adjacency

relation for the objects of length k , and then to extend the relation to $Adj_{\mathbb{E}^1}$ on all objects of type $C_{\mathbb{E}^1}$. This has been already done in the Section 3.2.

Now let us consider type $C_{\mathbb{P}}$ where the regular pattern $\mathbb{P}$ is arbitrary. For any infinite sequence v , the sequence of finite sequences $(v(n), \text{ for } n = 1, 2, \dots)$ is defined as a *choice sequence* over C_d . Although from the formal point of view, a choice sequence is the same as the corresponding infinite sequence, according to Brouwer, it makes the difference, see the Intuitionism entry at Stanford Encyclopedia of Philosophy <https://plato.stanford.edu/entries/intuitionism/>. A choice sequence is a potential infinity, whereas an infinite sequence is an actual infinity. It means that in mathematical constructions, if we take a choice sequence, we use only its prefixes (finite sequences), whereas if we take infinite sequence, we use it as an actual infinity, as it was already done in the definition of topological space $C_{\mathbb{P}}^\infty/\sim$.

It is convenient to use the same symbols (e.g. u, v, z) to denote infinite sequences and choice sequences. It will be clear, from the context, what they denote.

Note that for the pattern $\mathbb{E}^1$ corresponding to the unit interval, the topological space $C_{\mathbb{E}^1}^\infty/\sim$ is homeomorphic to the unit interval $[0; 1]$ with the usual Euclidean topology. Then, a choice sequence $v(n)$, for $n = 1, 2, \dots$ may be interpreted as a sequence of nested intervals converging to a point in $[0; 1]$.

Note that the sequence of adjacency graphs $(graph_k^{\mathbb{P}}, \text{ for } k = 0, 1, 2, \dots)$ corresponds to Brouwer's notion of spread, whereas the set of nodes of $graph_k^{\mathbb{P}}$ to a Brouwer's fun. However, the adjacency determined by $graph_k^{\mathbb{P}}$ is more rich structure than a tree with branches of length n . The notion of choice sequences as well as spread and fun can be defined for any pattern $\mathbb{P}$. However, the adjacency relation is not used in these definitions.

The Axiom of Continuity is crucial for the Brouwer's approach to Continuum. The weak version of the axiom is as follows.

*For any predicate $A(v, n)$ on the choice sequences v and the natural numbers n :
if for any choice sequence v there is n such that $A(v, n)$,
then for any choice sequence v there are natural numbers m and n such that for any choice sequence u :
if $u(m) = v(m)$, then $A(u, n)$.*

It means that, although the choice sequences are potentially infinite, the truth of the predicate $A(v, n)$ is determined by an initial segment (prefix) of v . The strong axiom of continuity requires that this very initial segment can be effectively determined by a computable functional taking a choice sequence as an argument, and returning the natural number as the length of the desired prefix of the argument.

In our framework, the weak axiom may be rewritten as follows.

The postulate. *Any function F on choice sequences (i.e. from $C_{\mathbb{P}}^\infty$ into $C_{\mathbb{Q}}^\infty$) is the extension of a strictly monotonic function f of type $C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$.*

The extension (defined at the end of Section 5) implies that if $F(v) = f^\infty(v) = z$, then for any i there is k such that $z(i)$ is a prefix of $f(v(k))$. So that in order to compute an initial segment (prefix) of the value $F(v)$, the function F needs only an initial segment of its argument v . So that, the computations done by the function F are only potentially infinite.

In the classic Analysis, there are functions that contradict the postulate. As an example, let us define function F on $C_{\mathbb{E}^1}^\infty$ (homeomorphic to the unit interval $[0; 1]$) in the following way. If v corresponds to a rational number, then let $F(v)$ be the infinite constant sequence where each its element is a . Otherwise, let the infinite sequence $F(v)$ be also constant and consists of elements that all are equal to b . Another simple example is the function G such that for any v that corresponds to a real number less or equal $\frac{1}{2}$, the infinite sequence $G(v)$ is constant and consists of elements that are equal to a . Otherwise, $G(v)$ consists of elements that each one is b . For the choice sequence v such that $v(1) = b$, and k -th element of v is a for $k > 1$, the determination of the value $G(v)$ must take the whole infinite sequence v . Note that the functions F and G are not continuous.

The Postulate may be reformulated using predicates instead of functions in the following way.

Let f be the strictly monotonic function corresponding to F from the postulate. The predicate A_F is defined as follows.

$A_F(v, z, i) \equiv \exists_k z(i)$ is a prefix of $f(v(k))$.

The truth of the predicate $A_F(v, z, i)$ is determined by an initial segment (prefix) of v and a prefix of z . Hence, the predicate satisfies The Axiom of Continuity. The predicate $\forall_i A_F(v, z, i)$ is true if and only if $f^\infty(v) = F(v) = z$.

The Postulate is sufficient to prove the famous Brouwer Continuity Theorem. It is done in the Section 8.

7 Functions on types and their extensions to functions on topological spaces

Let us consider again a strictly monotonic function $f : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$, and its extension to $f^\infty : C_{\mathbb{P}}^\infty \rightarrow C_{\mathbb{Q}}^\infty$.

The transformation of f^∞ to the function $f^\infty/\sim$ from $C_{\mathbb{P}}^\infty/\sim$ into $C_{\mathbb{Q}}^\infty/\sim$ is not always possible.

If f is not continuous, then $f^\infty/\sim$ need not to be a function. That is, there may be two adjacent infinite sequences v and u (i.e. $v \sim u$) such that $f^\infty(v)$ is different than $f^\infty(u)$. Since $[v]$ is the same as $[u]$, the value $f^\infty/\sim([v])$ can not be defined.

For the type $C_{\mathbb{P}}$ and type $C_{\mathbb{Q}}$ where the both patterns $\mathbb{P}$ and $\mathbb{Q}$ are $\mathbb{E}^1$ (i.e Euclidean pattern of dimension 1 corresponding to the unit interval), the proof is simple. Suppose that f is strictly monotonic and not continuous. Then, there are sequences x and y (of the same length, because $\mathbb{E}^1$ is a regular pattern) that are adjacent, and $f(x)$ and $f(y)$ are not adjacent. So that, either $x.a$ and $y.b$ are adjacent, or $x.b$ and $y.a$ are adjacent. That is, either $x.a.a.a.a.a. \dots$ and $y.b.b.b.b.b. \dots$ are equivalent (adjacent), or $y.a.a.a.a.a. \dots$ and $x.b.b.b.b.b. \dots$ are equivalent. Let r_1 and r_2 be the infinite sequences from the above ones that are equivalent, i.e. they represent the same real number. By the assumption that $f(x)$ and $f(y)$ are not adjacent, and by the strict monotonicity of f and the regularity of pattern $\mathbb{E}^1$, $f^\infty(r_1)$ and $f^\infty(r_2)$ are not equivalent and correspond to different real numbers.

Note that in the proof presented above, only the regularity of the pattern $\mathbb{E}^1$ was essential. That is, the construction of r_1 and r_2 can be carried out for any regular

pattern. So that we may generalize the proof and get the following conclusion.

Theorem 1. For arbitrary *regular* patterns $\mathbb{P}$ and $\mathbb{Q}$, and strictly monotonic function $f : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$:

1. If $f^{\infty}/\sim$ is a function, then f is continuous.
2. If f is continuous, then $f^{\infty}/\sim$ is also continuous in the classical sense, i.e. as the continuity of a function from topological space $C_{\mathbb{P}}^{\infty}/\sim$ into topological space $C_{\mathbb{Q}}^{\infty}/\sim$.
3. It follows from (1) i (2) that if $f^{\infty}/\sim$ is a function, then $f^{\infty}/\sim$ is a continuous function.

The proof of (2) is simple and is as follows. If $f^{\infty}/\sim$ is not continuous, then there is sequence $(e_1, e_2, \dots, e_n, \dots)$ converging to e such that the sequence $(f^{\infty}/\sim(e_1), f^{\infty}/\sim(e_2), \dots, f^{\infty}/\sim(e_n), \dots)$ converges to $r \neq f^{\infty}/\sim(e)$. Let $[u] = e$, and $[v] = f^{\infty}/\sim(e)$, and $[u_n] = e_n$ for any n . Then, there are m and k such that for all $i > k$ and all $j > m$, the finite sequences $v(j)$ and $(f^{\infty}(u_i))(j)$ are not adjacent. Since f is strictly monotonic, there is l_j such that $v(j)$ is a prefix of $f(u(l_j))$, and there is n_j such that $(f^{\infty}(u_i))(j)$ is a prefix of $f(u_i(n_j))$. By the regularity of the pattern $\mathbb{Q}$, also $f(u(l_j))$ and $f(u_i(n_j))$ are not adjacent. Since the sequence $([u_i], i = 1, 2, \dots)$ converges to $[u]$ (here the regularity of $\mathbb{P}$ is essential), for sufficiently large i , the finite sequences $u(l_j)$ and $u_i(n_j)$ are adjacent. Hence, function f is not continuous, and the proof is completed.

On the other hand, for any continuous function $\bar{f}$ on the unit interval $[0; 1]$ (in the classic sense), there is a strictly monotonic and continuous function $f : C_{\mathbb{E}^1} \rightarrow C_{\mathbb{E}^1}$, such that $f^{\infty}/\sim$ and $\bar{f}$ are the same up to the homeomorphism between $C_{\mathbb{E}^1}^{\infty}/\sim$ and the unit interval. The required function f is defined as follows. For any object x of $C_{\mathbb{E}^1}$, let I_x denote the sub-interval of $[0; 1]$ corresponding to x . For any x , the let $f(x)$ be defined as y such that I_y is the smallest (with regard to inclusion) interval such that for any $r \in I_x$, $\bar{f}(r) \in I_y$. If such smallest interval does not exists, then by the continuity of $\bar{f}$, the function $\bar{f}$ is constant on I_x . Then, let $f(x)$ be defined as the first y of length greater than the length of x , such that for any $r \in I_x$, $\bar{f}(r) \in I_y$. The function f defined in this way is strictly monotonic and continuous. The classic continuity of $\bar{f}$ is essential here. The extension of f , i.e $f^{\infty}/\sim$ is the same (modulo the homomorphism) as $\bar{f}$.

In the general case, the above statement is also true. That is, for arbitrary *regular* patterns $\mathbb{P}$ and $\mathbb{Q}$, if $C_{\mathbb{P}}^{\infty}/\sim$ corresponds (and is homeomorphic) to topological space T_P , and $C_{\mathbb{Q}}^{\infty}/\sim$ corresponds (and is homeomorphic) to topological space T_Q , then for any continuous function $\bar{g} : T_P \rightarrow T_Q$ (in the classic sense), there is a strictly monotonic and continuous function $g : C_{\mathbb{P}} \rightarrow C_{\mathbb{Q}}$, such that $g^{\infty}/\sim$ and $\bar{g}$ are the same up to the homeomorphisms.

Proof is almost the same as for the Euclidean pattern $\mathbb{E}^1$. The only difference consists in definition of the set I_x . Now, it is defined as $[u] \in I_x$ if x is a prefix of u .

Note that the restriction of the function g to the domain of objects (of type $C_{\mathbb{P}}$) of length k may be seen as the k -th approximation of the continuous function $\bar{g}$.

The conclusion for this section is as follows. The continuity of strictly monotonic function f is necessary and sufficient condition for $f^{\infty}/\sim$ to be a function, and to be a continuous function.

8 The Brouwer Continuity Theorem

Let us recall that the unit interval $[0; 1]$ is homeomorphic to $C_{\mathbb{E}^1}^\infty/\sim$.

According to Brouwer, every real number (from the unit interval $[0; 1]$) is represented by a choice sequence that is an element of the set $C_{\mathbb{E}^1}^\infty$. However, there may be several different choice sequences (say v and u) that correspond to the same real number r . Then, the sequences are adjacent (equivalent), i.e. $v \sim u$, and determine one equivalence class (an element of the set $C_{\mathbb{E}^1}^\infty/\sim$) that may be identified with the real number r .

The Brouwer Continuity Theorem states that all functions on the unit interval are continuous, that is, any function that is defined everywhere on the unit interval of real numbers is also continuous. This is the famous “paradox” in intuitionistic Analysis. The original proof of the “paradox theorem” was based on the Brouwer’s axiom of continuity. The axiom has been presented in Section 6. For the proof see [16], and Stanford Encyclopedia of Philosophy <https://plato.stanford.edu/entries/intuitionism/>.

Let us recall our Postulate, however, restricted only to pattern $\mathbb{E}^1$ that corresponds to the unit interval $[0; 1]$.

Any function F on choice sequences (i.e. from $C_{\mathbb{E}^1}^\infty$ into $C_{\mathbb{E}^1}^\infty$) is the extension of a strictly monotonic function f of type $C_{\mathbb{E}^1} \rightarrow C_{\mathbb{E}^1}$. That is, F is one and the same as f^∞ .

It means that in order to compute an initial segment (prefix) of the value $F(v)$, the function F needs only an initial segment of its argument v .

Since the unit interval $[0; 1]$ is homeomorphic to $C_{\mathbb{E}^1}^\infty/\sim$, the Postulate implies that any function on the unit interval is of the form $f^\infty/\sim$. For $f^\infty/\sim$ to be a function, f must be continuous. From the Conclusion 1, the continuity of f implies the continuity of $f^\infty/\sim$ in the classic sense.

The Conclusion. The Postulate implies that any function on the unit interval $[0; 1]$ is continuous.

Note that (without assuming the Postulate) any classic continuous function on the unit interval is (up the homeomorphism) the extension, and then the transformation of a strictly monotonic continuous function on $C_{\mathbb{E}^1}$; see the previous Section. Hence, the Postulate is satisfied by the continuous functions on the unit interval. However, the Postulate is for *all* functions.

We may conclude that The Brouwer Continuity Theorem is not a paradox if the real numbers (i.e. the interval $[0; 1]$) are grounded as the type $C_{\mathbb{E}^1}$, and functions on the unit interval are grounded as strictly monotonic and continuous functions of type $C_{\mathbb{E}^1} \rightarrow C_{\mathbb{E}^1}$.

In the original work of Brouwer (also unpublished manuscripts, see the PhD thesis by Johannes [17] as a comprehensive source) on Continuum, there is no explicit adjacency relation. It seems that this very relation is the key to understand the Continuum and its grounding, as well as to explain the controversies of the intuitionistic Continuum with the classic Mathematics.

The Brouwer’s continuity theorem can be generalized to functions on the topological spaces constructed according to arbitrary regular patterns.

Note that $f^\infty/\sim$ is an abstract notion (its definition involves actual infinity) and its grounding (computational contents) is in the function f . It seems that the strict

monotonicity and continuity of f is much more simple and primitive than the classic notion of continuity of $f^\infty/\sim$. Moreover, function f that is not continuous (and/or not strictly monotonic) may be also of interest.

One may say that the proposed grounding is nothing but approximation of the Continuum, and a strictly monotonic and continuous function is an approximation of a continuous function on Continuum. If it is supposed that such Continuum *really does exist*, as well as the functions on that Continuum *do exist*, then indeed the proposed grounding provides only their approximations.

Strictly monotonic and continuous functions on the types $C_{\mathbb{Q}}$, (where $\mathbb{Q}$ is an arbitrary regular pattern) are of particular interest in finite element method (numerical method for solving problems of engineering and mathematical physics), as well as in computer graphics where 2-dimensional grid of pixels, and 3-dimensional grid of voxels, correspond to $graph_k^{\mathbb{Q}}$, where k determines the (2D and 3D) image resolution, finer if k is bigger.

Let us summarize the Section. The regular division and adjacency patterns, and the resulting types may be considered as a grounding for the Continuum – “*a concept in its own right and independent of number*”. This may be seen as a realization of the idea of Brouwer’s Continuum.

It seems that Professor Luitzen E. J. Brouwer was right and his intuitionism (when properly grasped and elaborated) constitutes the ultimate Foundations of Mathematics.

8.1 Conclusion

Coming back to the computable function from the Introduction, let us recall that $\mathbb{E}^1$ denote the 1-dimensional Euclidean pattern. The type $C_{\mathbb{E}^1}$ is the grounding of the topological space $C_{\mathbb{E}^1}^\infty/\sim$ that is homeomorphic to the 1-dimensional unit cube, i.e. $[0; 1] \subset \mathbb{R}$.

Any strictly monotonic and continuous function on the type $C_{\mathbb{E}^1}$ can be seen as computable counterpart (grounding) of the abstract function $f^\infty/\sim$ on the interval of real numbers $[0; 1] \subset \mathbb{R}$.

The final conclusion is as follows. Strictly monotonic and continuous functions on $C_{\mathbb{E}^1}$ can be used for practical reliable computations on the real numbers.

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