

Model-Reference Compensator Synthesis for Uncertain Linear Systems Using Data-Driven Identified Plant Model

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Abstract

A model reference tracking controller that uses only input-output plant information is proposed for linear systems subjected to bounded unknown inputs. A design model for the compensator synthesis is obtained using a data-driven identification process. Using a data-driven identified plant model, an unknown input observer (UIO) is synthesized to estimate the state of the plant in the presence of uncertainties, which in this application are unknown inputs to the plant. A tracking controller is constructed to force the plant output to track a desired trajectory. A compensator is designed using linear matrix inequalities (LMI). The proposed compensator is tested on a vehicle model that performs a lane change maneuver. Bounds on the prescribed performance level for the state estimation and the tracking errors are given.

Keywords: Data-driven system identification, robust compensator synthesis, unknown input observer (UIO), model reference control.

1 Introduction

1.1 Motivation and literature overview

Data-driven modeling and control have been successfully applied to high-dimensional nonlinear systems such as autonomous robots, self-driving cars, turbulence, and neuroscience [1]. This approach requires collecting data from a system when it is not in operational mode to develop a mathematical model and identify the system dynamics, which are essential for controller design. Currently, there are numerous system identification methods available; see, for example, [2–6]. In many practical applications, an identified model is frequently used for control design under the assumption that it is “sufficiently close” to the true system without detailed quantitative analysis against the actual system. The primary reason for this is time and cost. While it may be important to determine that a model is accurate in the design of controllers, it requires the collection of large amounts of data, which is impractical and time consuming in many real-world applications. In response to these challenges, several approaches have recently been proposed to make system identification more practical. For example, a

data-driven system identification design algorithm was proposed in [6, 7] that combines limited measurements of the real system and data from a proxy system. The proxy system is not a perfect replica of the real system, but attempts to closely approximate true system behavior and can reduce the cost and increase the efficiency of the controller design. An example of a proxy system for vehicle dynamics could be another vehicle of the same type produced by the same manufacturer [7].

In [8], a parameterization of linear feedback systems was formulated using data-dependent LMI was formulated. This approach was used to solve the stabilization of state and output feedback and the linear quadratic regulation problem. In [9], a data-driven direct open-loop control method was derived using behavioral system theory. In [10], a stabilizing controller design was proposed for an actual system by collecting data from a proxy system. Stabilizing feedback controller designs for unknown dynamical systems based on noisy input-state data were proposed in [11, 12]. A data-driven tuning method for the controller parameters of a two-degree-of-freedom control system using initial experimental data was presented in [13]. In [14, 15], a data-driven predictive control method was presented using input/output data from the unknown system. In [16], a method was presented to solve a variety of linear quadratic control problems for linear systems using a dataset of input, state, and output trajectories. Data-driven linear quadratic regulator for linear time-invariant systems subject to process noise was proposed in [17].

Unknown input observer (UIO) based approaches have been applied to fault detection, state estimation, and controller design [18, 19]. In [20], a data-driven UIO formulation was proposed in which the UIO was designed to estimate states using the collected data. In [21], a finite-time integral controller was proposed for a class of non-linear systems perturbed by various types of uncertainties. In [22], an intermediate observer based on the plant's fuzzy model was proposed to simultaneously estimate the states and fault in interval form for discrete-time systems. In [23], two methods were implemented to detect actuator attacks on distributed energy resources of electrical power distribution systems. In [24], necessary and sufficient conditions were proposed for the design of a data-driven UIO to estimate the states of discrete-time linear time-invariant systems. In [25], a data-driven approach was proposed to design a reduced-order unknown input observer.

Incorporating UIO-based controller design into the online system compensator design process offers numerous advantages, especially in cases where identifying the complete state of the plant proves challenging or costly. This enables controller designers to achieve accurate state estimation and robust control performance without the need for large sensor arrays or conducting extensive data collection. The UIO-based controller design facilitates the estimation of unknown or unmeasured input to the system using only a subset of measured variables, minimizing the dependence on sensors and making the online system compensator design more feasible.

In this paper, we propose a model reference tracking controller-UIO compensator for uncertain linear systems. The compensator uses only the plant's input and output. We begin our compensator design by identifying the plant parameter matrices. We collect data from both the actual and proxy systems to obtain the identified system dynamics to be used in the compensator design. We perform our compensator design

in discrete time and apply it to the continuous-time vehicle model to achieve a lane change maneuver. We demonstrate the l_∞ -stability of our proposed model reference tracking and state estimation errors. The compensator design is given in terms of LMI.

1.2 Paper organization

The paper is organized as follows. We first discuss our paper's contributions and the notation used. Then in the following section, we describe in detail the objective of the paper of designing a tracking unknown input observer-based compensator using the system identified parameters. A block diagram of the proposed tracking controller-UIO compensator is shown in Figure 1. In Section 3, we describe the system identification method used in this paper to identify the parameters of the system. In Section 4, we propose a model-reference tracking controller design for a linear discrete-time system when the system matrices are not available to the designer. We also present a theorem concerning l_∞ -stability of the state tracking error dynamics. In Section 5, we present the unknown input observer architecture. In Section 6, we analyze the stability of the state estimation error dynamics. In Section 7, we present our proposed model reference tracking controller-UIO compensator. Then, the unknown input estimator is presented in Section 8. The error bounds are analyzed in Section 9. A numerical example illustrating our results is given in Section 10. Finally, conclusions can be found in Section 11.

1.3 Paper contributions

The contributions of our paper are as follows.

- We proposed a model reference tracking controller that uses only input-output plant information, rather than the plant's state. A design model for the compensator synthesis is obtained using a data-driven identification process.
- We proposed an unknown input observer (UIO) to estimate the plant's states. The proposed UIO is synthesized using a data-driven identified plant model.
- We derived bounds on the prescribed performance levels of the state estimation and tracking errors.
- We formulated the compensator design in terms of LMI.
- We proposed an estimator of the unknown input using identified system parameters.

1.4 Notation

The p -norm of a vector $x = [x_1 \cdots x_n]^\top \in \mathbb{R}^n$, denoted $\|x\|_p$, is defined as $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$. For notational simplicity, we denote the standard vector 2-norm, $\|x\|_2$, by $\|x\|$. Unless otherwise specified, we use the standard matrix spectral norm defined by $\|X\| = \sqrt{\lambda_{\max}(X^\top X)} = \sigma_{\max}(X)$, where $\lambda_{\max}$ and $\sigma_{\max}$ denote the largest eigenvalue and the largest singular value, respectively.

2 Problem Statement

We consider a class of discrete-time (DT) linear time-invariant (LTI) dynamical systems modeled as

$$\begin{aligned} x[k+1] &= Ax[k] + B(u_1[k] + u_2[k]) + w[k], & k \geq 0, \\ y[k] &= Cx[k], \end{aligned} \quad (1)$$

where $x[k] \in \mathbb{R}^n$, $u_1[k], u_2[k] \in \mathbb{R}^m$, $w[k] \in \mathbb{R}^n$, and $y[k] \in \mathbb{R}^r$ are the state, the control input, actuator disturbance, plant disturbance, and output, respectively. We assume that the elements of the plant disturbance vector $w[k]$ are independent and identically distributed normal random variables (i.i.d.), that is, $w[k] \sim \mathcal{N}(0, \sigma_w^2 I_n)$, where σ_w^2 is the variance. We make the following assumptions.

Assumption 1 The pair (A, B) is stabilizable.

Assumption 2 The pair (A, C) is detectable.

Assumption 3 The state $x[k]$ of the plant modeled by (1) and the actuator disturbance $u_2[k]$ are bounded.

We assume that the system matrices A and B are unknown to the designer, but a proxy system of the same order is available to the designer. The dynamics of the proxy system is modeled by

$$\tilde{x}[k+1] = \tilde{A}\tilde{x}[k] + \tilde{B}\tilde{u}_1[k] + \tilde{w}[k], \quad k \geq 0, \quad (2)$$

where $\tilde{u}_1[k] \in \mathbb{R}^m$ is the input to the proxy plant and $\tilde{w}[k] \in \mathbb{R}^n$ is the disturbance. Let $\Delta\tilde{A} = \tilde{A} - A$ and $\Delta\tilde{B} = \tilde{B} - B$ be the differences in the system matrices between the actual and the proxy systems. We assume that $\|\Delta\tilde{A}\| \leq \varepsilon_A$ and $\|\Delta\tilde{B}\| \leq \varepsilon_B$, where ε_A and ε_B are known to the designer. We also assume that the disturbance $\tilde{w}[k]$ has normal i.i.d. elements with $\tilde{w}[k] \sim \mathcal{N}(0, \sigma_w^2 I_n)$.

In the scenario under consideration, we have limited access to the actual system, but unlimited access to the proxy system. The system identification process will use a weighted average of the input-output of the actual system and the proxy system. For every usage of the actual system to obtain the measurement $(x[k], u_1[k])$, N_p measurements $(\tilde{x}[k], \tilde{u}_1[k])$ will be obtained from the proxy system. If $N_p = 0$, only measurements from the actual system are used, and if N_p is large, then the measurements from the proxy system will dominate in the identification process. The details of the identification process and the calculation of the identification error will be given below.

We denote the identified system by

$$\hat{x}[k+1] = \hat{A}\hat{x}[k] + \hat{B}\hat{u}_1[k], \quad (3)$$

where the system identification errors in the matrices are given by $(\Delta A, \Delta B)$, where $\Delta A = A - \hat{A}$ and $\Delta B = B - \hat{B}$.

Our objective is to design a tracking unknown input observer-based compensator using the identified system pair $(\hat{A}, \hat{B})$. The UIO estimates the plant's state that is then used to synthesize the compensator shown in Figure 1.

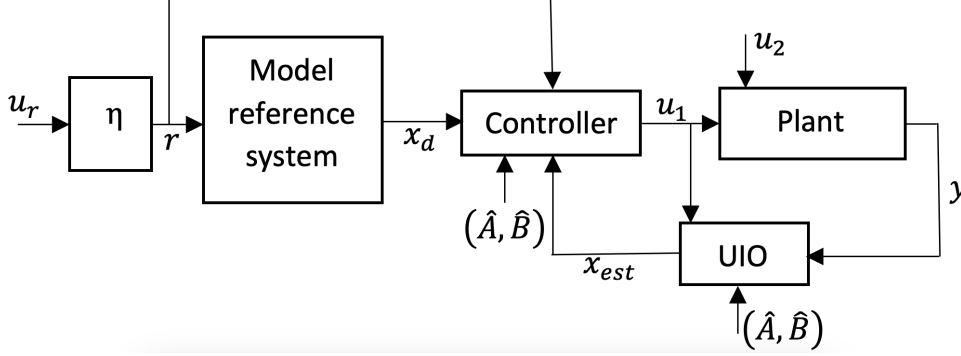


Fig. 1: Proposed tracking controller-UIO compensator for system modeled by (1).

Note that in Figure 1, η is a scaling factor for the reference input u_r which will be discussed in Section 10.

3 System Identification

In this section, we describe a system identification method that we use to identify the parameters of the plant given by (1). We then discuss the bounds of the system identification errors ΔA , and ΔB that will be used to design the compensator.

3.1 The Role of the Proxy System

In this section, we motivate the idea behind the identification procedure before giving the full technical details in the next section.

In the case where the actual system is not available to the designer and the design is completed using only the proxy system and its parameters, the initial error bound cannot be improved. However, if the designer has access to the actual system, even if for just one measurement of the actual system input and the corresponding output, the error can be reduced. We give a simple geometric explanation of this idea.

Suppose that we have system $x[k+1] = Ax[k] + Bu[k]$ with A, B unknown and a proxy system $\tilde{x}[k+1] = \tilde{A}\tilde{x}[k] + \tilde{B}\tilde{u}[k]$ with $\tilde{A}, \tilde{B}$ known. Suppose that the actual system can only be used once for a state and input pair x_1, u_1 , which are randomly chosen. Choose (A_1, B_1) such that $A_1x_0 + B_1u_0 = Ax_0 + Bu_0$, or equivalently $(A - A_1)x_0 + (B - B_1)u_0 = 0$. There are infinitely many such pairs (A_1, B_1) but we will see that all choices lead to the same result. Let $\mathcal{S}_1 = \{(K, L) : Kx_1 + Lu_1 = 0\}$ be the hyperplane in the space of matrices through the origin that is perpendicular to

(x_1, u_1) and let $\mathcal{P}_1 = (A_1, B_1) + \mathcal{S}_1$ be the hyperplane shifted to contain (A_1, B_1) . Then $(A, B) \in \mathcal{P}_1$. Let $(\hat{A}, \hat{B})$ be the projection of $(\tilde{A}, \tilde{B})$ onto $\mathcal{P}_1$. By the Pythagorean Theorem, $\|(\hat{A}, \hat{B}) - (A, B)\| \leq \|(\tilde{A}, \tilde{B}) - (A, B)\|$. This shows that having access to the actual system even once can improve the performance of the design process over using the proxy system alone.

If the actual system can be used J times, then we iteratively repeat the above procedure and obtain a nested sequence of hyperplanes $\mathcal{P}_k = \bigcap_{j=1}^k \{(A_j, B_j) + \mathcal{S}_j\}$ each containing (A, B) :

$$(A, B) \in \mathcal{P}_J \subset \cdots \subset \mathcal{P}_2 \subset \mathcal{P}_1.$$

For each k , let $(\hat{A}_k, \hat{B}_k)$ be the projection of $(\tilde{A}, \tilde{B})$ onto $\mathcal{P}_k$. Since the hyperplanes are nested, we have

$$\|(\hat{A}_J, \hat{B}_J) - (A, B)\| \leq \cdots \leq \|(\hat{A}_1, \hat{B}_1) - (A, B)\| \leq \|(\tilde{A}, \tilde{B}) - (A, B)\|.$$

We see that the approximation improves with increased access to the actual system. We next present the system identification procedure.

3.2 System identification procedure

In this paper, we use an identification method first presented in [6]. The method uses a randomly generated test control signal, denoted $u_1^t[k]$, as input to the actual and proxy systems. The state-control pairs of the actual and proxy plants, $(x[k], u_1[k])$ and $(\tilde{x}[k], \tilde{u}_1[k])$, respectively, are collected by the sample collector.

We begin by generating the test control signal $u_1^t[k]$ as i.i.d. Gaussian with $u_1^t[k] \sim \mathcal{N}(0, \sigma_u^2 I_m)$. In the system identification procedure, we let $u_1[k] = \tilde{u}_1[k] = u_1^t[k]$. We assume that the number of independent tests performed on the actual system (1) is N_a . In each test, the actual system restarts from a random initial state $x[0] \sim \mathcal{N}(0, \sigma_x^2 I_n)$. We assume that all tests are performed on a time interval of duration T . Let $u_1^i[k]$ and $x^i[k]$ be the samples collected from the i -th test. Then the samples collected from all N_a tests in the time interval $[0, T-1]$ are,

$$\left\{ (x^i[k], u_1^i[k]) : i = 1, \dots, N_a; k = 0, \dots, T-1 \right\}.$$

Let $z^i[k] = [x^i[k]^\top u_1^i[k]^\top]^\top$. Then for the i -th test of length T , we have $Z^i = [z^i[0] \cdots z^i[T-1]]$, where Z^i is the sample collection matrix of the i -th test. Let $X^i = [x^i[1] \cdots x^i[T]]$, $\Xi = [A \ B]$, and

$$W^i = [w^i[0] \cdots w^i[T-1]].$$

Then for the i -th test on the actual system given by (1), we have

$$X^i = \Xi Z^i + W^i, \tag{4}$$

where $X^i \in \mathbb{R}^{n \times T}$, $\Xi \in \mathbb{R}^{n \times (n+m)}$, $Z^i \in \mathbb{R}^{(n+m) \times T}$, and $W^i \in \mathbb{R}^{n \times T}$. Next, we collect data from the N_a tests on the actual system. Let $X = [X^1 \dots X^{N_a}]$, $Z = [Z^1 \dots Z^{N_a}]$, and $W = [W^1 \dots W^{N_a}]$. The data collected for the actual system can be represented as

$$X = \Xi Z + W, \quad (5)$$

where $X \in \mathbb{R}^{n \times N_a T}$, $Z \in \mathbb{R}^{(n+m) \times N_a T}$, and $W \in \mathbb{R}^{n \times N_a T}$.

The same procedure is then performed for the proxy system (2) except we conduct N_p independent tests on the time interval $[0, T-1]$. For each test, the test control input and the initial state of the proxy system are generated, as described above. Let $\tilde{u}_1^i[k]$ and $\tilde{x}^i[k]$ be the samples collected from the i -th test. For the proxy system, we have

$$\left\{ (\tilde{x}^i[k], \tilde{u}_1^i[k]) : i = 1, \dots, N_p; k = 0, \dots, T-1 \right\}.$$

Let $\tilde{z}^i[k] = [\tilde{x}^i[k]^\top \tilde{u}_1^i[k]^\top]^\top$. Then for the i -th test of length T , we have $\tilde{Z}^i = [\tilde{z}^i[0] \dots \tilde{z}^i[T-1]]$. We next rewrite (2) to obtain

$$\tilde{x}[k+1] = (A + \Delta \tilde{A}) \tilde{x}[k] + (B + \Delta \tilde{B}) \tilde{u}_1[k] + \tilde{w}[k]. \quad (6)$$

Let $\tilde{X}^i = [\tilde{x}^i[1] \dots \tilde{x}^i[T]]$, $\tilde{W}^i = [\tilde{w}^i[0] \dots \tilde{w}^i[T-1]]$, $\Delta_\Xi = [\Delta \tilde{A} \ \Delta \tilde{B}]$, and denote $\Delta^i = [\Delta_\Xi \dots \Delta_\Xi] \tilde{Z}^i$. Then for the i -th test on the proxy system given by (2), we have

$$\tilde{X}^i = \Xi \tilde{Z}^i + \tilde{W}^i + \Delta^i. \quad (7)$$

Let $\tilde{X} = [\tilde{X}^1 \dots \tilde{X}^{N_p}]$, $\tilde{Z} = [\tilde{Z}^1 \dots \tilde{Z}^{N_p}]$, $\tilde{W} = [\tilde{W}^1 \dots \tilde{W}^{N_p}]$, and $\Delta = [\Delta^1 \dots \Delta^{N_p}]$. The data collected using the proxy system can be represented as

$$\tilde{X} = \Xi \tilde{Z} + \tilde{W} + \Delta, \quad (8)$$

where $\tilde{X} \in \mathbb{R}^{n \times N_p T}$, $\tilde{Z} \in \mathbb{R}^{(n+m) \times N_p T}$, $\tilde{W} \in \mathbb{R}^{n \times N_p T}$, and $\Delta \in \mathbb{R}^{n \times N_p T}$. Let $X := [X \ \tilde{X}]$, $Z := [Z \ \tilde{Z}]$, and $W := [W \ \tilde{W}]$, where the symbol $:=$ denotes the arithmetic assignment, that is, the left hand side of the equation becomes the right hand side. Combining (5) and (8) gives

$$X = \Xi Z + W + [O_{n \times N_a T} \ \Delta], \quad (9)$$

where $O_{n \times N_a T}$ denotes the zero matrix with n rows and $N_a T$ columns.

To proceed, let $\tilde{v}_k \in \{\tilde{v}_k \in \mathbb{R}, \tilde{v}_k \geq 0\}$ be the design parameter that specifies the relative weight assigned to the samples generated from the proxy system at sample k . Let $\tilde{V} = \text{diag}\{\tilde{v}_0, \dots, \tilde{v}_{T-1}\}$ and $\tilde{V} = \text{diag}\{\tilde{V}, \dots, \tilde{V}\}$, where $\tilde{V} \in \mathbb{R}^{N_p T \times N_p T}$. Let $\hat{\Xi} = [\hat{A} \ \hat{B}]$. The identified system matrices $\hat{A}$ and $\hat{B}$ are obtained by solving the following weighted least squares (WLS) optimization problem:

$$[\hat{A} \ \hat{B}] = \arg \min_{\hat{\Xi}} \left\| (X - \hat{\Xi} Z) \begin{bmatrix} I_{N_a T} \\ \tilde{V} \end{bmatrix}^{\frac{1}{2}} \right\|_F^2, \quad (10)$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a given matrix.

4 Model Reference Tracking Controller Design

In this section, we propose a model-reference tracking controller design for a linear discrete-time system when the system matrices are not available to the designer. We use the system-identified pair $(\hat{A}, \hat{B})$ to design the controller.

We first represent the system model (1) in terms of the identified matrices as

$$x[k+1] = \hat{A}x[k] + \hat{B}(u_1[k] + u_2[k]) + \Delta Ax[k] + \Delta B(u_1[k] + u_2[k]). \quad (11)$$

The model reference generator signal is obtained from the system,

$$x_d[k+1] = A_d x_d[k] + B_d r[k], \quad (12)$$

where $A_d = \hat{A} - \hat{B}K_d$, $B_d = \hat{B}$, and K_d is selected in such a way that A_d is Schur.

We define the tracking error as $e_t[k] = x[k] - x_d[k]$. We note that, in general, ΔA and ΔB cannot be excessively large if we are to succeed in designing a controller using an identified model. To proceed with the controller design, we assume that ΔA and ΔB satisfy the matching conditions

$$\Delta A = \hat{B}G_A, \quad \Delta B = \hat{B}G_B, \quad (13)$$

where $G_A \in \mathbb{R}^{m \times n}$ and $G_B \in \mathbb{R}^{m \times m}$. The matching conditions are assumed to facilitate controller design analysis but are not strictly necessary as we will see in Section 10 that the controller can still function satisfactorily even if the matching conditions are not met.

Substituting (11), (13), and (12) into the tracking error dynamics gives

$$e_t[k+1] = A_d e_t[k] + \hat{B}((K_d + G_A)x[k] + (I_m + G_B)(u_1[k] + u_2[k]) - r[k]). \quad (14)$$

Let $E[k] = (I_m + G_B)u_2[k] + (K_d + G_A)x[k] - r[k]$. Then, we represent (14) as

$$e_t[k+1] = A_d e_t[k] + \hat{B}((I_m + G_B)u_1[k] + E[k]). \quad (15)$$

In the following, we use ℓ_∞ -stability with performance level γ as defined in [26, 27].

Definition 1 The error system, $e[k+1] = f(k, e[k], v[k])$, is globally uniformly ℓ_∞ -stable with performance level γ if the following conditions are satisfied:

1. The undisturbed system (that is, when $v[k] = 0$ for all $k \geq 0$) is globally uniformly exponentially stable with respect to the origin.
2. For zero initial condition, $e[k_0] = 0$, and every bounded unknown input $v[k]$, we have $\|e[k]\| \leq \gamma \|v[k]\|_\infty$ for all $k \geq k_0$.
3. For every initial condition, $e[k_0] = e_0$, and every bounded unknown input $v[k]$, we have $\limsup_{k \rightarrow \infty} \|e[k]\| \leq \gamma \|v[k]\|_\infty$.

We now present a lemma adapted from [26] that we use in the proof of the theorem that gives the design condition of the proposed compensator in LMI format.

Lemma 1 Suppose that for the error dynamics, $e[k+1] = f(k, e[k], v[k])$, there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, $V(0) = 0$, and scalars $\alpha \in (0, 1)$, $\beta_1, \beta_2 > 0$ and $\mu_1 \geq 0$ such that

$$\beta_1 \|e[k]\|^2 \leq V(e[k]) \leq \beta_2 \|e[k]\|^2, \quad (16)$$

and

$$V(e[k+1]) - V(e[k]) \leq -\alpha(V(e[k]) - \mu_1 \|v[k]\|^2), \quad (17)$$

for all $k \geq 0$. Then the error system is globally uniformly ℓ_∞ -stable with performance level $\gamma = \sqrt{\mu_1/\beta_1}$ with respect to the output disturbance $v[k]$.

Now, we present the following theorem:

Theorem 1 If there exist a matrix $P_d = P_d^\top \succ 0$ and a scalar $\rho \in (0, 1)$ such that the matrix inequality

$$A_d^\top P_d A_d - \rho P_d \prec 0 \quad (18)$$

is satisfied, and if the control input has the form

$$u_1[k] = \begin{cases} -(I_m + G_B)^{-1} R^{-1} \frac{\hat{B}^\top P_d A_d e_t[k]}{\|\hat{B}^\top P_d A_d e_t[k]\|_{R^{-1}}} \psi[k] & \text{if } e_t[k] \notin \text{Ker}(\hat{B}^\top P_d A_d) \\ 0 & \text{if } e_t[k] \in \text{Ker}(\hat{B}^\top P_d A_d) \end{cases} \quad (19)$$

where $R = \hat{B}^\top P_d \hat{B}$, $\psi[k] = \lambda_{\max}^{\frac{1}{2}}(R) \zeta[k]$, $\zeta[k] = \|(I_m + G_B)\| \|u_2[k]\| + (\|K_d\| + \|G_A\|) \|x[k]\| + \|r[k]\|$ and

$$\|\hat{B}^\top P_d A_d e_t[k]\|_{R^{-1}} = (e_t[k]^\top A_d^\top P_d \hat{B} R^{-1} \hat{B}^\top P_d A_d e_t[k])^{1/2},$$

then the state tracking error dynamics in (14) are l_∞ -stable with performance level

$$\gamma = \sqrt{\frac{4}{\lambda_{\min}(P_d)(1-\rho)}}.$$

Proof The control law given by (19) is a modification of the controller proposed in [28]. Let $V_t[k] = e_t[k]^\top P_d e_t[k]$ be a Lyapunov function candidate for the state tracking error system given by (15). Then, proceeding as in [28], we obtain

$$\Delta V_t[k] \leq e_t[k]^\top (A_d^\top P_d A_d - P_d) e_t[k] + 4\psi^2[k].$$

By Assumption 3, the function $\psi[k]$ is bounded. It follows from (18) that $A_d^\top P_d A_d \prec \rho P_d$. Combining this with the above, we obtain

$$\begin{aligned} \Delta V_t[k] &\leq -(1-\rho) e_t[k]^\top P_d e_t[k] + 4\psi^2[k] \\ &\leq -(1-\rho) (V_t[k] - \frac{4}{(1-\rho)} \psi^2[k]). \end{aligned}$$

Therefore, condition (17) in Lemma 1 holds with $\mu_1 = \frac{4}{(1-\rho)}$. Next, since $P_d = P_d^\top \succ 0$ then $P_d \succ \lambda_{\min}(P_d)I$. Premultiplying this by $e_t[k]^\top$ and postmultiplying it by $e_t[k]$ gives $e_t[k]^\top P_d e_t[k] > \lambda_{\min}(P_d) e_t[k]^\top e_t[k]$. Rearranging, we obtain $\|e_t[k]\|^2 \leq \frac{1}{\lambda_{\min}(P_d)} V_t(e_t[k])$ which yields the right hand side of condition (16) in Lemma 1 with $\beta_1 = \lambda_{\min}(P_d)$. By Lemma 1, the tracking error dynamics (14) are l_∞ -stable with performance level $\gamma = \sqrt{\frac{4}{\lambda_{\min}(P_d)(1-\rho)}}$. $\square$

We now present a method of solving matrix inequality (18) in Theorem 1 using linear matrix inequality (LMI). Let $N = P_d^{-1}$ and $Q = K_d N$. Then, for a given ρ , solving matrix inequality (18) is equivalent to solving the following LMI for N and Q ,

$$\begin{bmatrix} -N & \hat{A}N - \hat{B}Q \\ N\hat{A}^\top - Q^\top \hat{B}^\top & -\rho N \end{bmatrix} \prec 0. \quad (20)$$

Since $P_d = P_d^\top \succ 0$, taking the Schur complement of (20) then pre- and post-multiply it by P_d yields (18).

Remark 1 As mentioned in the beginning of the proof of Theorem 1, the architecture of the control law (19) was inspired by the controller proposed in [28], which was analyzed using the notion of uniform ultimate boundedness. Our analysis of the performance of our controller is from the point of view of l_∞ -stability. In addition, we provided an LMI to synthesize our proposed controller.

Remark 2 The control law given by (19) is a switched controller about a naturally defined surface

$$\sigma(e_t[k]) = R^{-1/2} \hat{B}^\top P_d A_d e_t[k]. \quad (21)$$

This type of controllers were discussed in [29]. The conclusion of that paper is that the gain of the controller (19) is to be turned off or reduced when the closed-loop system trajectory is in the neighborhood of the switching surface. The reason for the controller's gain reduction is that the closed-loop system trajectory will chatter around the switching surface with a chatter amplitude at least as large as the gain of the switched controller.

In the following section, we propose a design method for an unknown input observer used in our compensator synthesis.

5 Unknown Input Observer Architecture

For the observer design, we use the system model given by

$$x[k+1] = \hat{A}x[k] + \hat{B}(u_1[k] + u_2[k]) \quad (22)$$

Following the UIO design procedure from [27], we let $z[k] = (I - HC)x[k]$. Then, the UIO architecture that we use has the form

$$\left. \begin{aligned} z[k+1] &= (I - HC)(\hat{A}z[k] + \hat{A}Hy[k]) + L(y[k] - Cz[k] - CHy[k]) \\ x_{est}[k] &= z[k] + Hy[k], \end{aligned} \right\} \quad (23)$$

where L and H are the observer design matrices. The observer design matrix H is selected so that

$$(I - HC)\hat{B} = O, \quad (24)$$

where O is a zero matrix of appropriate dimensions. The calculation procedure of the observer gain matrix L is given in the following section.

6 Stability Analysis of the State Estimation Error

In this section, we analyze the stability of the state estimation error dynamics and give the conditions to find the UIO gain matrix L in terms of linear matrix inequalities.

6.1 State Estimation Error Dynamics

To proceed, we define the state estimation error dynamics as $e[k+1] = x[k+1] - x_{est}[k+1]$. Then, substituting (11) and (23) into the error dynamics, we obtain

$$\begin{aligned} e[k+1] &= \hat{A}x[k] + \hat{B}(u_1[k] + u_2[k]) + \Delta Ax[k] + \Delta B(u_1[k] + u_2[k]) \\ &\quad - z[k+1] - Hy[k+1] \\ &= \hat{A}x[k] + \hat{B}(u_1[k] + u_2[k]) + \Delta Ax[k] + \Delta B(u_1[k] + u_2[k]) \\ &\quad - (I - HC)(\hat{A}z[k] + \hat{A}Hy[k]) - L(y[k] - Cz[k] - CHy[k]) \\ &\quad - HC(\hat{A}x[k] + \hat{B}(u_1[k] + u_2[k]) + \Delta Ax[k] + \Delta B(u_1[k] + u_2[k])). \end{aligned}$$

Let $\bar{A} = (I - HC)\hat{A}$, taking into account $(I - HC)\hat{B} = O$, and performing simple manipulations, we obtain

$$e[k+1] = (\bar{A} - LC)e[k] + (I - HC)(\Delta Ax[k] + \Delta B(u_1[k] + u_2[k])). \quad (25)$$

In the following subsection, we propose a design procedure for a compensator comprising the feedback controller and the UIO. In Section 7, we design the combined model reference tracking controller-UIO compensator.

6.2 Feedback Controller-UIO Compensator

Substituting $u_1[k] = -Kx_{est}[k]$ into (25) gives

$$e[k+1] = (\bar{A} - LC)e[k] + (I - HC)(\Delta Ax[k] + \Delta B(-Kx_{est}[k] + u_2[k])).$$

Then substitute into the above $x_{est}[k] = x[k] - e[k]$ to obtain

$$\begin{aligned} e[k+1] &= (\bar{A} - LC)e[k] + (I - HC)\Delta B(Ke[k] + u_2[k]) \\ &\quad + (I - HC)(\Delta A - \Delta BK)x[k]. \end{aligned}$$

Let $\Phi(e[k]) = \Delta BKe[k]$, $\mathcal{J}_1[k] = (\Delta A - \Delta BK)x[k] + \Delta Bu_2[k]$, $\mathcal{A} = (\bar{A} - LC)$ and $\tilde{H} = I - HC$, then,

$$e[k+1] = \mathcal{A}e[k] + \tilde{H}\Phi[k] + \tilde{H}\mathcal{J}_1[k]. \quad (26)$$

To proceed, let M be a $\mathbb{R}^{2n \times 2n}$ matrix. We define the quadratic form, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$,

$$Q_M(a, b) = \begin{bmatrix} a^\top & b^\top \end{bmatrix} M \begin{bmatrix} a \\ b \end{bmatrix}. \quad (27)$$

Definition 2 ([26]) A symmetric $2n \times 2n$ matrix M is an incremental multiplier matrix (δMM) for Φ if for every $x, \delta \in \mathbb{R}^n$,

$$Q_M(\delta, \Phi(x + \delta) - \Phi(x)) \geq 0.$$

The usefulness of this concept will be demonstrated in Theorem 2. We have the following theorem.

Theorem 2 Suppose the unknown sequence of inputs $\{u_2[k]\}_{k=0}^\infty$ is uniformly bounded as a function of k , and M is an δMM for $\Phi[k]$. If there exist matrices $\tilde{P} = \tilde{P}^\top \succ 0$, L , and scalars $\kappa > 0$ and $\alpha \in (0, 1)$ such that

$$\left. \begin{aligned} \Gamma + \kappa \tilde{M} &\preceq 0 \\ \tilde{P} &\succ \frac{1}{1-\alpha} \kappa I_n \end{aligned} \right\} \quad (28)$$

where

$$\begin{aligned} \Gamma &= \begin{bmatrix} \mathcal{A}^\top \tilde{P} \mathcal{A} - (1-\alpha)\tilde{P} & * & * \\ \tilde{H}^\top \tilde{P} \mathcal{A} & \tilde{H}^\top \tilde{P} \tilde{H} & * \\ \tilde{H}^\top \tilde{P} \mathcal{A} & \tilde{H}^\top \tilde{P} \tilde{H} & \tilde{H}^\top \tilde{P} \tilde{H} - \alpha I \end{bmatrix}, \\ \tilde{M} &= \begin{bmatrix} M & O_{2n \times n} \\ O_{n \times 2n} & O_{n \times n} \end{bmatrix}, \end{aligned}$$

then the state observation error dynamics are ℓ_∞ -stable with performance level $\gamma = \frac{1}{\sqrt{\lambda_{\min}(\tilde{P})}}$.

Proof We verify the conditions of Lemma 1. Since $\tilde{P} = \tilde{P}^\top \succ 0$, conditions (16) and (17) are satisfied with $\beta_1 = \lambda_{\min}(\tilde{P})$ and $\beta_2 = \lambda_{\max}(\tilde{P})$.

Let $V[k] = e[k]^\top \tilde{P} e[k]$ be a Lyapunov function candidate for the estimation error dynamics given by (26). We evaluate the first forward difference, $\Delta V[k] = V[k+1] - V[k]$, on the trajectories of (26) to obtain

$$\Delta V[k] = e[k]^\top \left(\mathcal{A}^\top \tilde{P} \mathcal{A} - \tilde{P} \right) e[k] + 2e[k]^\top \mathcal{A}^\top \tilde{P} \tilde{H} \mathcal{J}_1[k] + 2e[k]^\top \mathcal{A}^\top \tilde{P} \tilde{H} \Phi[k]$$

$$+ \mathcal{J}_1[k]^\top \tilde{H}^\top \tilde{P} \tilde{H} \mathcal{J}_1[k] + 2\mathcal{J}_1[k]^\top \tilde{H}^\top \tilde{P} \tilde{H} \Phi[k] + \Phi[k]^\top \tilde{H}^\top \tilde{P} \tilde{H} \Phi[k].$$

Let $\Delta q = [e[k]^\top \ \Phi[k]^\top \ \mathcal{J}_1[k]^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (28) by $\Delta q^\top$ and Δq , respectively, we obtain

$$\Delta V[k] + \alpha(V[k] - \|\mathcal{J}_1[k]\|^2) + \kappa \Delta q[k]^\top \tilde{M} \Delta q[k] \preceq 0.$$

Substituting $\tilde{M}$ into the above gives

$$\Delta V[k] + \alpha(V[k] - \|\mathcal{J}_1[k]\|^2) + \kappa [e[k]^\top \ \Phi[k]^\top] M \begin{bmatrix} e[k] \\ \Phi[k] \end{bmatrix} \prec 0.$$

Since M is an δMM for $\Phi[k]$, we have $\Delta V[k] + \alpha(V[k] - \|\mathcal{J}_1[k]\|^2) \preceq 0$. Therefore, condition (17) in Lemma 1 holds with $\mu = 1$ and the observer error given by (26) satisfies

$$\limsup_{k \rightarrow \infty} \|e[k]\| \leq \gamma \limsup_{k \rightarrow \infty} \|\mathcal{J}_1[k]\|_\infty. \quad (29)$$

Therefore, the state error dynamics (26) are ℓ_∞ -stable with performance level $\gamma = \frac{1}{\sqrt{\lambda_{\min}(P)}}$. $\square$

We now present a method to solve matrix inequality (28) in Theorem 2 using an LMI. Let $Z = \tilde{P}L$. Then, solving the matrix inequality (28) is equivalent to solving the following LMI for $\tilde{P}$ and Z ,

$$\begin{bmatrix} -\tilde{P} & * \\ \Omega_{31} & \Omega_{33} \end{bmatrix} \preceq 0, \quad (30)$$

where

$$\Omega_{31} = \begin{bmatrix} \bar{A}^\top \tilde{P} - C^\top Z^\top \\ \tilde{H}^\top \tilde{P} \\ \tilde{H}^\top \tilde{P} \end{bmatrix},$$

$$\Omega_{33} = \begin{bmatrix} -(1-\alpha)\tilde{P} + \kappa\beta^2 I_n & O_{n \times n} & O_{n \times n} \\ O_{n \times n} & -\kappa I_n & O_{n \times n} \\ O_{n \times n} & O_{n \times n} & -\alpha I_n \end{bmatrix}.$$

Since $\tilde{P} = \tilde{P}^\top \succ 0$, taking the Schur complement of (30) gives,

$$\Omega_{33} + \Omega_{31} \tilde{P}^{-1} \Omega_{31}^\top \preceq 0,$$

which, in turn, produces the matrix inequality (28).

7 Combined Model Reference Tracking Controller-UIO Compensator

We now use $x_{est}[k]$ instead of the true state $x[k]$ in the controller part (19) of the proposed tracking controller-UIO compensator. Then, the state estimation error dynamics (25) takes the form

$$e[k+1] = (\bar{A} - LC)e[k] + (I - HC)(\Delta A x[k] + \Delta B u_2[k])$$

$$+(I - HC)\Delta B \frac{-R^{-1}\hat{B}^\top P_d A_d(x_{est}[k] - x_d[k])}{\|\hat{B}^\top P_d A_d(x_{est}[k] - x_d[k])\|_{R^{-1}}} \psi[k].$$

Let

$$\mathcal{J}_2[k] = \Delta A x[k] + \Delta B \left(\frac{-R^{-1}\hat{B}^\top P_d A_d(x_{est}[k] - x_d[k])}{\|\hat{B}^\top P_d A_d(x_{est}[k] - x_d[k])\|_{R^{-1}}} \psi[k] + u_2[k] \right), \quad (31)$$

$\mathcal{A} = (\bar{A} - LC)$, and $\tilde{H} = I - HC$, then,

$$e[k+1] = \mathcal{A}e[k] + \tilde{H}\mathcal{J}_2[k]. \quad (32)$$

We now state and prove a theorem which gives the conditions for the existence of the proposed compensator.

Theorem 3 Suppose the sequence of unknown inputs, $\{u_2[k]\}_{k=0}^\infty$, is uniformly bounded as a function of k and that $\mathcal{J}_2[k]$ is also bounded. If there exist matrices $P = P^\top \succ 0$, L , and a scalar $\alpha \in (0, 1)$ such that

$$\begin{bmatrix} \mathcal{A}^\top P \mathcal{A} - (1 - \alpha)P & * \\ \tilde{H}^\top P \mathcal{A} & \tilde{H}^\top P \tilde{H} - \alpha I \end{bmatrix} \preceq 0 \quad (33)$$

then the state observation error dynamics (32) are ℓ_∞ -stable with performance level $\gamma = \frac{1}{\sqrt{\lambda_{\min}(P)}}$.

Proof We verify the conditions of Lemma 1. Since $P = P^\top \succ 0$, conditions (16) and (17) are satisfied with $\beta_1 = \lambda_{\min}(P)$ and $\beta_2 = \lambda_{\max}(P)$.

Let $V[k] = e[k]^\top P e[k]$ be a Lyapunov function candidate for the estimation error dynamics given by (32). We evaluate the difference $\Delta V[k] = V[k+1] - V[k]$ on the trajectories of (32) to obtain

$$\Delta V[k] = e[k]^\top (\mathcal{A}^\top P \mathcal{A} - P) e[k] + 2e[k]^\top \mathcal{A}^\top P \tilde{H} \mathcal{J}_2[k] + \mathcal{J}_2[k]^\top \tilde{H}^\top P \tilde{H} \mathcal{J}_2[k].$$

Let $\Delta q = [e[k]^\top \mathcal{J}_2[k]^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (28) by $\Delta q^\top$ and Δq , respectively, we obtain $\Delta V[k] + \alpha(V[k] - \|\mathcal{J}_2[k]\|^2) \leq 0$. Therefore, condition (17) in Lemma 1 holds with $\mu = 1$ and the observation error given by (32) satisfies

$$\limsup_{k \rightarrow \infty} \|e[k]\| \leq \gamma \limsup_{k \rightarrow \infty} \|\mathcal{J}_2[k]\|_\infty. \quad (34)$$

Therefore, the state estimation error dynamics (32) are ℓ_∞ -stable with performance level $\gamma = \frac{1}{\sqrt{\lambda_{\min}(P)}}$. $\square$

We present a method to solve matrix inequality (33) in Theorem 3 using an LMI. Let $Z = PL$. Then solving the matrix inequality (33) is equivalent to solving the following LMI for P and Z ,

$$\begin{bmatrix} -P & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \preceq 0, \quad (35)$$

where $\Omega_{21}^\top = [P\bar{A} - ZC \ P\tilde{H}]$ and $\Omega_{22} = \begin{bmatrix} -(1-\alpha)P & O_{n \times r} \\ O_{r \times n} & -\alpha I \end{bmatrix}$. Since $P = P^\top \succ 0$, taking the Schur complement of (35) gives $\Omega_{22} + \Omega_{21}P^{-1}\Omega_{21}^\top \preceq 0$, which, in turn, yields the matrix inequality (33).

8 Unknown Input Estimation

Unknown inputs can model malicious attacks on the plant's actuators or modeling uncertainties. Unknown inputs can also model actuators' faults. Thus, being able to estimate unknown inputs will enable us to detect malicious attacks or actuators' faults.

In this section, we propose an unknown input estimator using the plant state estimate. This state estimate is generated by the observer constructed using the identified plant parameters.

If there exist matrices G_A and G_B such that $\Delta A = \hat{B}G_A$, $\Delta B = \hat{B}G_B$, and $(I_m + G_B)$ is invertible, then we rewrite (11) as

$$x[k+1] = \hat{A}x[k] + \hat{B}(G_A x[k] + (I_m + G_B)(u_1[k] + u_2[k])).$$

Following the unknown input reconstruction design in [27], we obtain the unknown input estimator as

$$\hat{u}_2[k-1] = (I_m + G_B)^{-1} \hat{B}^\dagger x_{est}[k] - (I_m + G_B)^{-1} (\hat{B}^\dagger \hat{A} + G_A) x_{est}[k-1] - u_1[k-1].$$

The above unknown input estimator reconstructs the unknown input with one sampling period time-delay.

9 Analysis of the Error Bounds

In this section, we give bounds on the tracking and state estimation error dynamics when the identified system pair $(\hat{A}, \hat{B})$ has known identification error bounds on $(\Delta A, \Delta B)$. We show how the identification error bounds affect the tracking and the state estimation.

9.1 Tracking error bound as a function of identification error bound

Recall from Theorem 1 that $R = \hat{B}^\top P_d \hat{B}$, $\psi[k] = \lambda_{\max}^{\frac{1}{2}}(R) \zeta[k]$, $\zeta[k] = \|u_2[k]\| + (\|K_d\| + \|G\|)\|x[k]\| + \|r[k]\|$. We next substitute $G = \hat{B}^\dagger \Delta A$, $\hat{B} = B + \Delta B$ into $\psi[k]$, to obtain

$$\psi[k] = \lambda_{\max}^{\frac{1}{2}}(R) (\|u_2[k]\| + (\|K_d\| + \|\hat{B}^\dagger \Delta A\|)\|x[k]\| + \|r[k]\|)$$

Suppose that $\max\{\|\Delta A\|, \|\Delta B\|\} \leq \xi$, then

$$\psi[k] \leq \lambda_{\max}^{\frac{1}{2}}(R) (\|u_2[k]\| + (\|K_d\| + \xi \|\hat{B}^\dagger\|)\|x[k]\| + \|r[k]\|) \stackrel{\text{def}}{=} \beta[k]. \quad (36)$$

Note that by Assumption 3, $\beta[k]$ is bounded. From Theorem 1, we have

$$\limsup_{k \rightarrow \infty} \|e_t[k]\| \leq \gamma \limsup_{k \rightarrow \infty} \psi[k] \leq \gamma \limsup_{k \rightarrow \infty} \beta[k].$$

9.2 State estimation error bound as a function of identification error bounds

From Theorem 3

$$\limsup_{k \rightarrow \infty} \|e[k]\| \leq \gamma \limsup_{k \rightarrow \infty} \|\mathcal{J}_2[k]\|_\infty,$$

where $\mathcal{J}_2[k]$ is given by (31). Therefore

$$\begin{aligned} \|\mathcal{J}_2[k]\| &\leq \|\Delta A\| \|x[k]\| + \|\Delta B\| \frac{\|R^{-1}\| \|\hat{B}^\top P_d A_d (x_{est}[k] - x_d[k])\|}{\|\hat{B}^\top P_d A_d (x_{est}[k] - x_d[k])\|_{R^{-1}}} \psi[k] \\ &\quad + \|\Delta B\| \|u_2[k]\|. \end{aligned}$$

Then performing some manipulations, we obtain

$$\begin{aligned} \|\mathcal{J}_2[k]\| &\leq \xi \|x[k]\| + \xi \frac{\lambda_{\max}(R^{-1}) \|\hat{B}^\top P_d A_d (x_{est}[k] - x_d[k])\|}{\lambda_{\min}^{1/2}(R^{-1}) \|\hat{B}^\top P_d A_d (x_{est}[k] - x_d[k])\|} \psi[k] + \xi \|u_2[k]\| \\ &\leq \xi \left(\|x[k]\| + \frac{\lambda_{\max}(R^{-1})}{\lambda_{\min}^{1/2}(R^{-1})} \psi[k] + \|u_2[k]\| \right). \end{aligned}$$

Substituting (36) into the above gives

$$\|\mathcal{J}_2[k]\| \leq \xi \left(\|x[k]\| + \frac{\lambda_{\max}(R^{-1})}{\lambda_{\min}^{1/2}(R^{-1})} \beta[k] + \|u_2[k]\| \right). \quad (37)$$

From (34), (36), and (37), we see that the state estimation error bound depends on the bounds of the plant state, disturbances, and reference signal. Note that a linear combination of these bounds in (37) is multiplied by the identification error bound ξ . If ξ and the bound on the reference signal are small, then this will produce a reasonably small state estimation error. We illustrate this in the example in Section 10. On the other hand, if our objective is to design a stabilizing compensator, then the state converges to zero and the reference signal is zero, and therefore the state estimation error bound will depend only on the bound of the disturbance.

10 Example

In this section, we illustrate the design of our proposed compensator on the continuous-time vehicle bicycle model given in [30, Subsection 1.7.2]. Our objective is to design a compensator for a lane-changing maneuver when the actual system model is not available to the designer. We follow the Society of Automotive Engineers (SAE) standard coordinate system.

The state of the vehicle dynamical model is

$$x = [\dot{y} \ \theta \ \omega_z \ Y]^\top = [x_1 \ x_2 \ x_3 \ x_4]^\top,$$

where $\dot{y}$ is the vehicle lateral velocity to the right side of the vehicle, θ is the yaw angle, ω_z is the yaw angular velocity, and Y is the global coordinate of the global coordinate system. In our simulation, we used the longitudinal velocity of the vehicle $v_x = 17.3$ m/sec. We first discretized the system model with sampling period $T_s = 0.02$ sec. The discretized state-space system matrices are:

$$A = \begin{bmatrix} 0.8972 & 0 & -0.3137 & 0 \\ 0 & 1 & 0.0192 & 0 \\ 0.0005 & 0 & 0.9219 & 0 \\ -0.0190 & -0.3460 & -0.0001 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.7952 \\ 0.0056 \\ 0.5534 \\ -0.0091 \end{bmatrix}.$$

The output matrix is $C = [0 \ 0 \ 0 \ 1]$. We assume that we do not have access to the system matrices. Instead, we use a replica of the model (proxy model) to perform an offline experiment to identify the system matrices $(\hat{A}, \hat{B})$. The proxy system was implemented in MATLAB by adding random errors to the actual system parameter matrices. The resulting proxy system matrices are:

$$\tilde{A} = \begin{bmatrix} 0.9026 & 0.0032 & -0.2779 & 0.0073 \\ 0.01834 & 0.9869 & 0.0469 & -0.0006 \\ -0.0221 & -0.0043 & 0.9084 & 0.0071 \\ -0.0103 & -0.3426 & 0.0302 & 0.9979 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0.7939 \\ 0.0205 \\ 0.5675 \\ 0.0051 \end{bmatrix}.$$

In our analysis of the proposed tracking algorithm given in Theorem 1, we assumed the matching conditions on ΔA and ΔB given by (13). In our simulations, however, the matching conditions are not necessarily satisfied in order to show the robustness of the proposed control algorithm.

We used the length of the time interval of the identification experiment $T = 8$, the number of tests on the actual system $N_a = 20$, and the number of tests on the proxy system $N_p = 2000$. Applying the identification system procedure from Subsection 3.2, we obtained the following identification model matrices:

$$\hat{A} = \begin{bmatrix} 0.8939 & -0.0031 & -0.3046 & -0.0045 \\ -0.0037 & 1.0224 & 0.0235 & -0.0083 \\ -0.0156 & 0.0012 & 0.9173 & -0.0062 \\ -0.0065 & -0.3449 & -0.0059 & 1.0185 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0.8046 \\ 0.0084 \\ 0.5752 \\ -0.0059 \end{bmatrix}.$$

From the identification experiment, we obtain the following bounds on the system identification matrices: $\|\Delta A\| = 0.03062$ and $\|\Delta B\| = 0.02412$.

We consider the lane-changing maneuver corresponding to the reference signal $r[k] = 4 \times 1[k - 75]$, where $1[k]$ is the unit step function. The reference input scaling factor $\eta = (C(I_4 - A_d)^{-1} \hat{B})^{-1} = -8.1887$ as shown in Figure 1. To solve the LMI system given by (20), we let $\rho = 0.85$, and we obtain

$$K_d = [-0.1517 \ 23.1357 \ 2.3805 \ -4.1121],$$

and

$$P_d = \begin{bmatrix} 0.0377 & -0.4852 & -0.0618 & 0.0783 \\ -0.4852 & 56.3596 & 3.1013 & -10.3157 \\ -0.0618 & 3.1013 & 0.3640 & -0.5378 \\ 0.0783 & -10.3157 & -0.5378 & 1.9720 \end{bmatrix}.$$

Next, we design our UIO using $\alpha = 0.085$ and solve (24). We solve the LMI given by (35) to obtain the UIO gain matrices,

$$H = \begin{bmatrix} -137.3658 \\ -1.4390 \\ -98.2047 \\ 1.0000 \end{bmatrix}, \quad L = \begin{bmatrix} 139.9087 \\ 1.4574 \\ 100.0196 \\ 0 \end{bmatrix}.$$

We proceed to implement the switched controller (19) by setting the controller's gain such that if $\|\hat{B}^\top P_d A_d e_t[k]\| \geq 1.5$, then $\psi[k] = \lambda_{\max}^{\frac{1}{2}}(R)\zeta[k]$, else we set $\psi[k] = 0.046$ — see Remark 2 for a discussion on the tuning the controller's gain. That is,

$$u_1[k] = \begin{cases} -\frac{(R(I_m + G_B))^{-1} \hat{B}^\top P_d A_d e_t[k]}{\|\hat{B}^\top P_d A_d e_t[k]\|_{R^{-1}}} \lambda_{\max}^{\frac{1}{2}}(R)\zeta[k] & \text{if } \|\hat{B}^\top P_d A_d e_t[k]\| \geq 1.5 \\ -0.046 \frac{(R(I_m + G_B))^{-1} \hat{B}^\top P_d A_d e_t[k]}{\|\hat{B}^\top P_d A_d e_t[k]\|_{R^{-1}}} & \text{if } \|\hat{B}^\top P_d A_d e_t[k]\| < 1.5 \end{cases}$$

In our simulation, we use the initial state vector, $x[0] = [0.1 \ 0 \ -0.1 \ 0.2]^\top$, and zero initial state for the UIO. We first simulate the case where $u_2[k] = 0$. The simulation results for this case are shown in Figures 2, 3, 4, 5, 6, and 7. In Figure 2, we show plots of the vehicle states and the corresponding states of the model reference system. In Figure 3, we depict the tracking errors between the vehicle states and the corresponding states of the model reference system. We see that the vehicle follows the desired trajectories. In Figure 4, we compare the plots of the vehicle states against their estimates generated by the UIO. In Figure 5, we show plots of state estimation errors. The state estimation errors of other states of the vehicle can be attributed to the difference between the identified model, which was used as a design model, and the true model. In Figure 6, we present the plot of the vehicle path in the global coordinates. As can be seen, the lane change maneuver was performed as desired. The control input behavior in Figure 7 is explained in Remark 2.

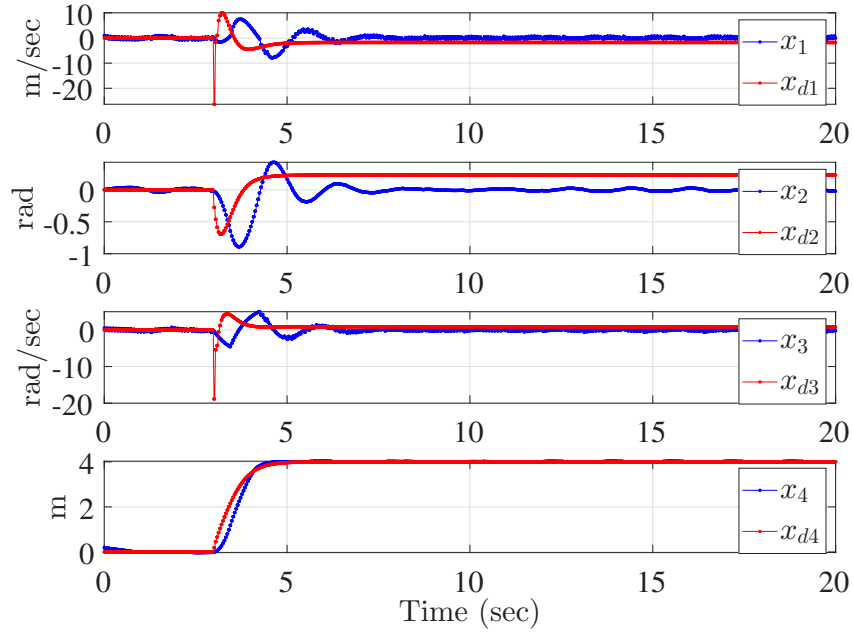


Fig. 2: State and reference signals for $u_2 = 0$.

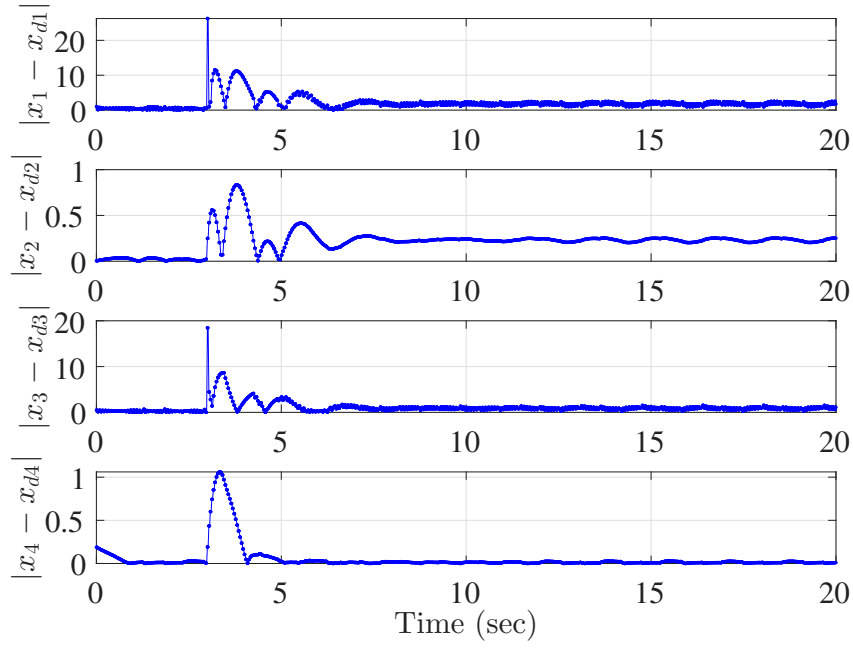


Fig. 3: Tracking error plots for $u_2 = 0$.

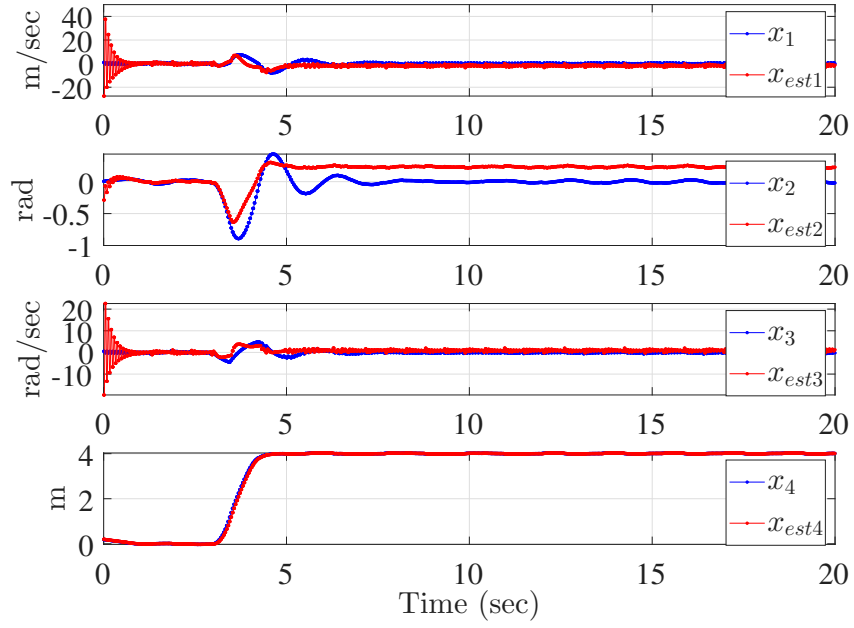


Fig. 4: State and its estimate for $u_2 = 0$.

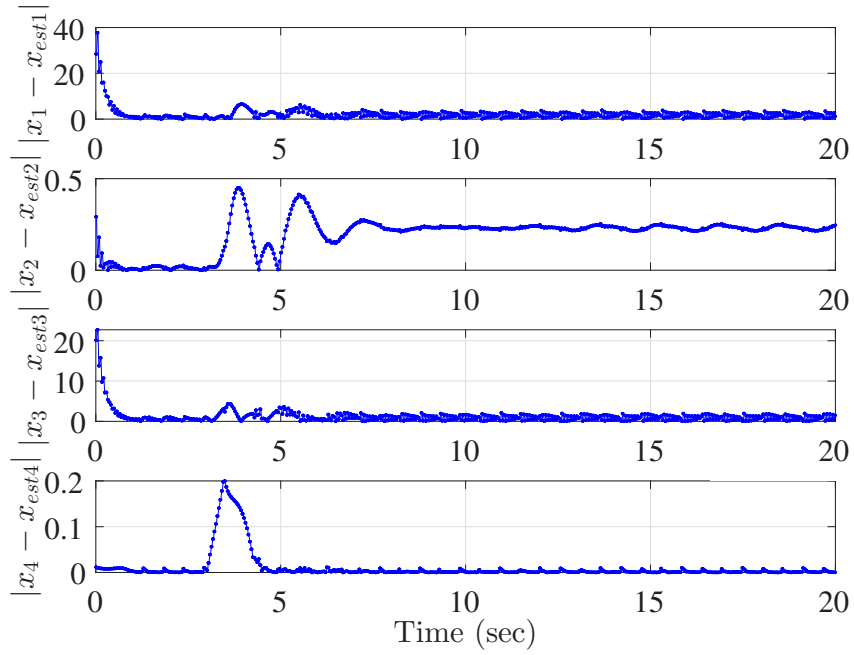


Fig. 5: State estimation error plots for $u_2 = 0$.

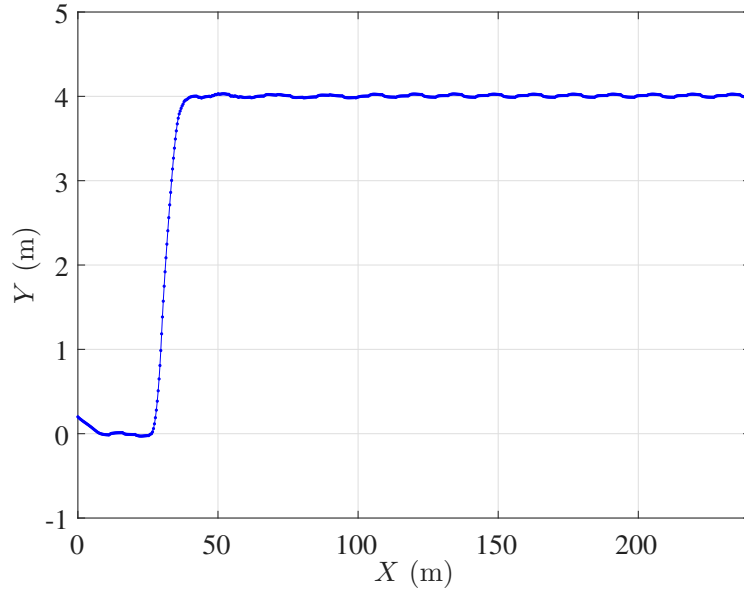


Fig. 6: Vehicle path in the global coordinates for $u_2 = 0$.

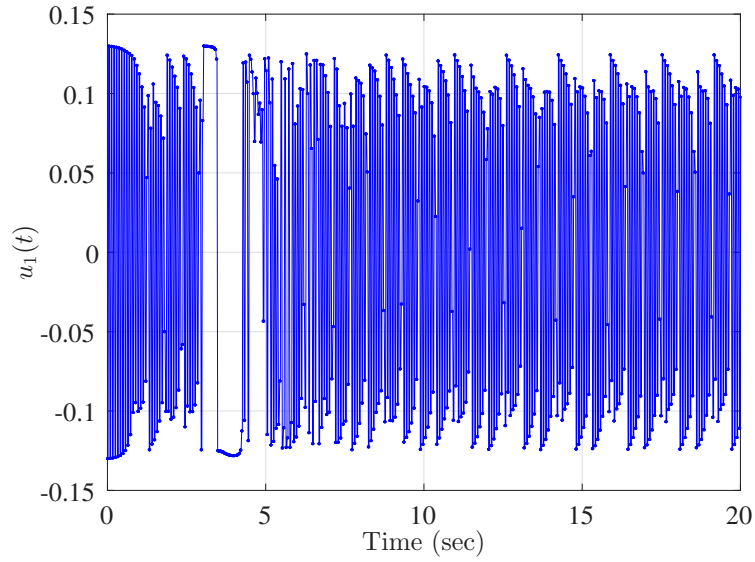


Fig. 7: The control input $u_1(t)$ when $u_2 = 0$.

Next, we consider a scenario where an unknown external input affects vehicle dynamics, such that if $k < 200$, $u_2[k] = -0.05$, else $u_2[k] = 0.15$. The simulation results

of the state and the tracking signals, the tracking errors, the state and its estimates, the state estimation errors, the vehicle path in global coordinates, the control input, and the unknown input reconstruction are shown in Figures 8, 9, 10, 11, 12, 13, and 14, respectively. In Figure 8 and 9, we see the tracking errors caused by the difference between the identified and true models. In Figure 10, we compare the plots of the vehicle states with their estimates generated by the UIO in the presence of unknown input. In Figure 11, the state estimation errors of some states of the vehicle can be attributed to the difference between the identified model, which was used as a design model, and the true model. In Figure 12, we see that the lane change maneuver was performed as desired. The control input behavior in Figure 13 is explained in Remark 2. Finally, in Figure 14, we show the reconstructed unknown input and its estimation error. Our proposed compensator tracks well reference signals in the lane-changing maneuver while estimating the vehicle state and unknown input in the presence of system identification errors.

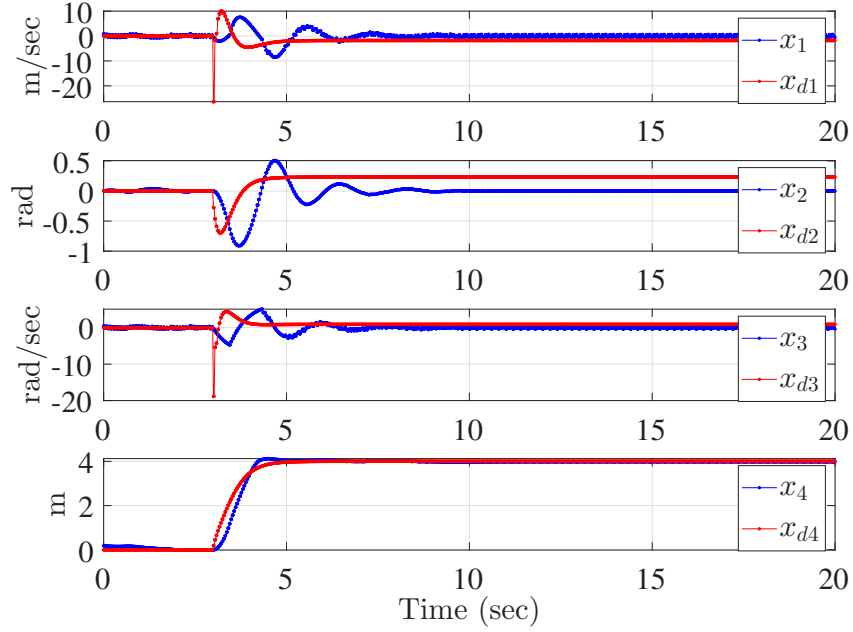


Fig. 8: State and reference signals for nonzero unknown input.

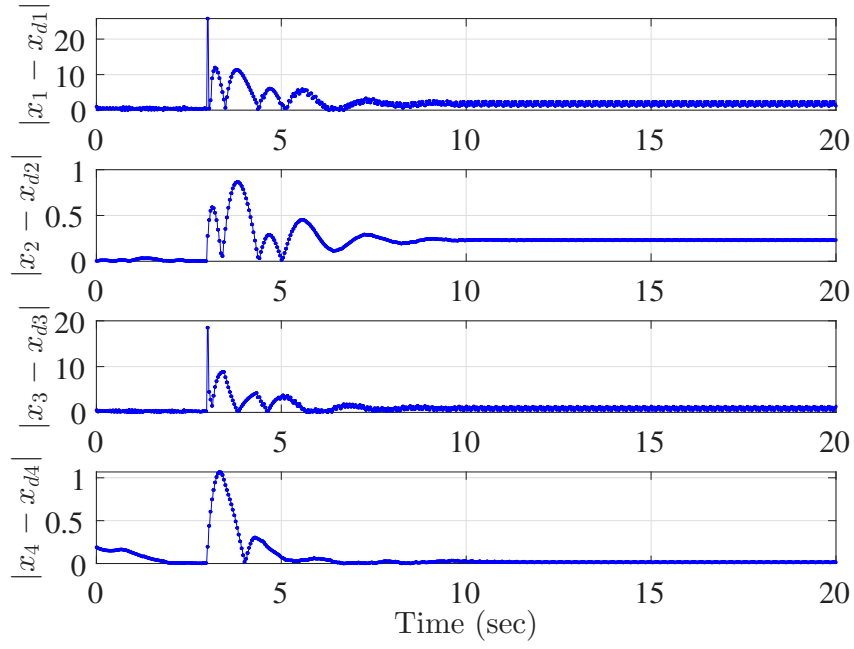


Fig. 9: Tracking error plots for nonzero unknown input.

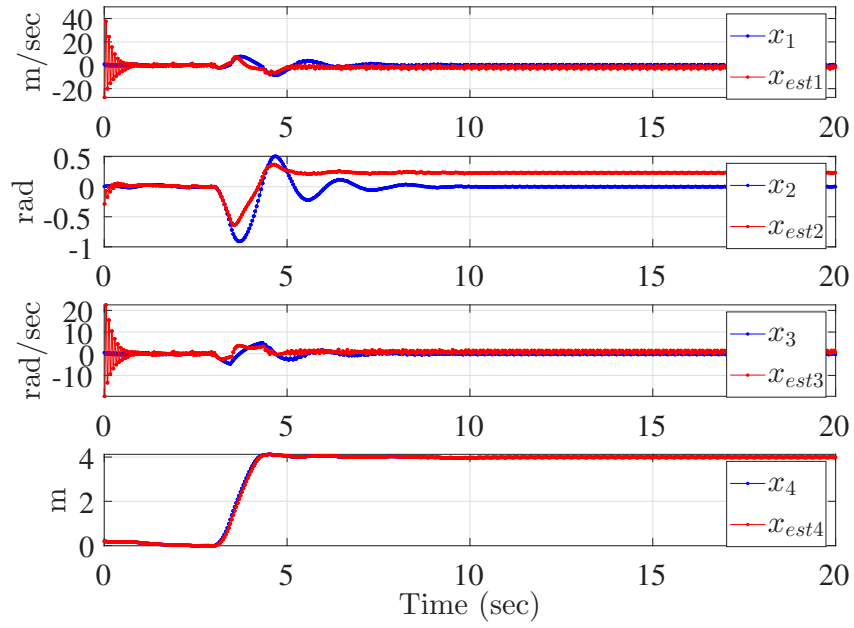


Fig. 10: State and its estimate for nonzero unknown input.

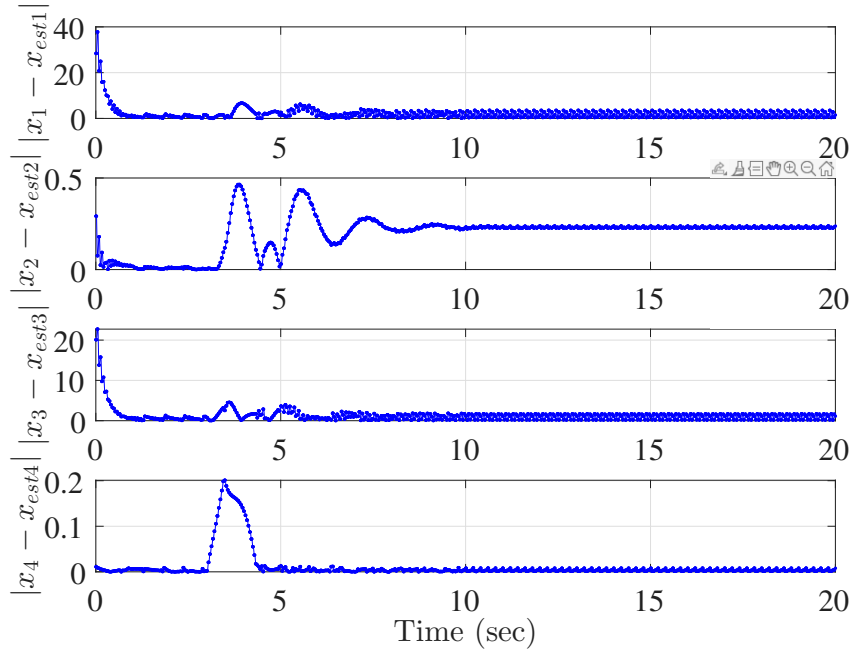


Fig. 11: State estimation error plots for nonzero unknown input.

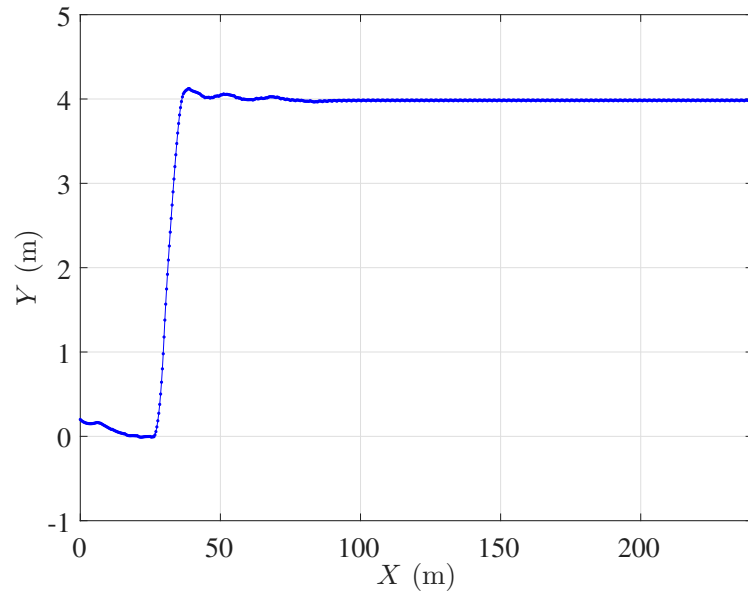


Fig. 12: Vehicle path in the global coordinates for nonzero unknown input.

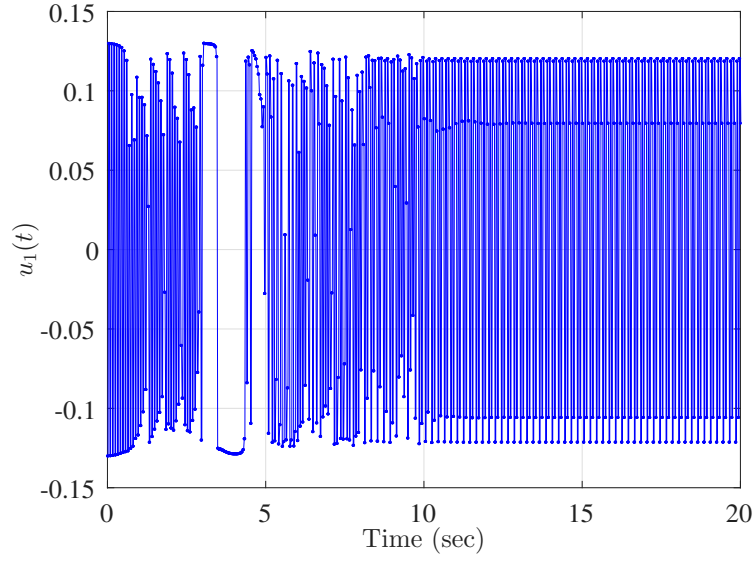


Fig. 13: The control input $u_1(t)$ for nonzero unknown input.

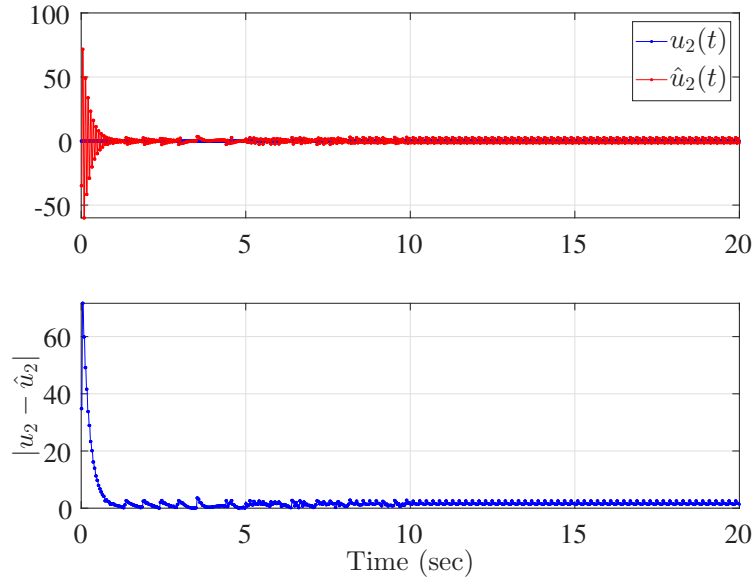


Fig. 14: The unknown input, its estimate, and its estimation error for nonzero unknown input.

11 Conclusions

In this paper, an unknown-input-observer (UIO) based model reference tracking controller compensator was proposed for linear systems subjected to bounded disturbances. The proposed compensator uses only the input-output information of the plant. The tracking error and the state estimation error were proven to be l_∞ -stable with a prescribed performance level. The design of the proposed compensator was given in terms of linear matrix inequalities. The UIO-based tracking controller compensator was then applied to a vehicle model to perform a lane change maneuver, and two different cases were simulated. The first case was without the presence of an unknown input, and the second case was in the presence of an unknown input that affects vehicle dynamics. In both cases, our proposed compensator performed well in the lane change maneuver.

12 Data Availability Statement

The authors confirm that the data supporting the findings of this study are available within the article.

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