

The Synergy Framework: A Four-View, Domain-Self-Calibrated Instrument for Measuring the Structure of Complex Systems

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Method Article

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Software repository: <https://gitee.com/lwlzmck/symbiotic-complex-dynamics>

Abstract

Complex systems in different disciplines are described by variables with different units and reference scales, so a single numeric index computed on one system cannot in general be compared with the same index on another. In practice, analysts rely on one estimator at a time—an estimated Lyapunov exponent, a variance threshold, a pairwise correlation, or a higher-order information statistic—and each fails in a documented way when its operating assumptions are violated. This paper presents a *method*, not a new physical or mathematical law: a four-view measurement instrument that reports (V1) pairwise structural coupling via the non-diagonal energy fraction, effective rank and participation ratio, (V2) multibody organisation via O-information with weak and strong estimators, (V3) dynamical stability and causal asymmetry via the leading Jacobian eigenvalue, and (V4) perturbation recovery, together with an explicit input class (white-box A versus observed-only B), domain-self-calibrated significance tests, and premise gates. No single view is allowed to determine a conclusion. The method is calibrated analytically: for an m -ary nested hierarchy the participation ratio scales as $PR \sim N^\alpha$ with $\alpha = 2 \log_m \rho$, and numerical points fall on these predicted slopes. It is validated out of sample on ten known-answer systems and on a natural physiological domain (heart-rate variability, healthy $n = 54$ versus congestive heart failure $n = 29$), where it honestly records that V1 and effective rank separate the groups while O-information does not and the final classifier consequently does not—a single-point dependency reported as a known boundary rather than hidden. Cross-level correspondence is carried by V2, while V4 is insensitive to inter-layer meshing, a remaining limitation. The contribution is an instrument plus a catalogue of thirteen reproducible failure modes.

Keywords: complex systems; measurement; multiscale; O-information; participation ratio; critical slowing down; observability; failure modes.

1 Introduction

1.1 The measurement problem

Across physics, physiology, ecology and the social sciences, a complex system is observed as a multivariate time series. Two difficulties recur. First, variables carry domain-specific units and

reference scales; there is no obvious common quantity in which to compare, for example, an electrocardiogram, a fluid simulation and a food web. Second, standard single estimators each have a well-defined operating region and a failure region outside it. An estimated largest Lyapunov exponent is positive for some bounded, perfectly predictable driven systems; a fixed variance or autocorrelation threshold raises false alarms on noisy records; pairwise correlations are blind to genuinely multibody organisation; and higher-order information statistics depend on sample size and data density. This paper addresses the practical problem of deciding, for a given recording, which readings are trustworthy and which have silently left their operating region, without adopting the substantive theory of any discipline.

1.2 Related approaches and their limits

- **Lyapunov exponents and bifurcation theory** give the rigorous stability description when the vector field and Jacobian are known. From data, finite-time estimates are sensitive to noise, embedding and window choice, and a single positive estimate is not specific to deterministic chaos.
- **Critical slowing down and generic early-warning signals** use rising variance and autocorrelation as a bifurcation is approached; the signals are necessary but not sufficient.
- **Mutual information and O-information** quantify redundancy and synergy beyond pairwise terms; the estimators have known sample-size dependence, require a minimum record length, and behave differently on sparse, point-process-like data.
- **Effective rank and the participation ratio** summarise how many independent modes are active; they are uninformative when dominated by a single near-constant mode or by temporal rather than spatial structure.
- **Structural controllability and observability** address how many actuators or sensors a network needs; they are network-structural and do not by themselves give a dynamical reading.
- **Self-organised criticality, synergetics, and the edge of stability** describe organising mechanisms; the present work treats them as domain content to be cited.

1.3 Contributions

1. A four-view instrument with the rule that no single view determines a conclusion.
2. An explicit input-class protocol (white-box A versus observed-only B) fixing which views are reliable before interpretation.
3. Domain-self-calibrated significance and premise gates (surrogates, false-discovery-rate control, confidence intervals, effective-sample-size checks, and an autocorrelation prerequisite for participation-ratio readings).
4. An analytic calibration on a nested hierarchy, deriving the participation-ratio scaling exponent and confirming it numerically.
5. A validation package spanning known-answer synthetic systems and a natural physiological domain, including negative results, and a catalogue of thirteen failure modes.

The instrument only measures. What a measured value *means* is left to the discipline that owns the variables.

1.4 The organisational proposition that motivates the design

The instrument was designed around a simple organisational proposition, here called the **Synergy proposition** and stated in full as a conceptual statement in Appendix A: that coupled complex systems are organised as nested, self-similar levels; that each level is examined under the same views; and that the most viable organisation is neither a single enslaving order parameter nor excessive fragmentation, but a small number of resourced, independent yet coupled factors. This motivates why the instrument has four views and why it is applied at every level. It is *not* the empirical claim tested here; the tested claim is whether the instrument reads known-answer and natural systems correctly, and where it does not.

2 Methods

2.1 Design principles

1. **No conclusion from a single view.** A structural claim requires agreement across the views available and reliable for that input.
2. **Domain self-calibration.** Significance is assessed against nulls generated from the same recording.
3. **Observed-only inputs do not use the data-only stability view for classification;** it is marked `data_only` and excluded.
4. **Measurement and interpretation are separated;** the discipline supplies the meaning.
5. **The instrument states no physical or mathematical law.**

2.2 Input classes

Each analysis begins by declaring the input class (Figure 1).

- **A-class (white-box).** The recording is accompanied by a one-step function `step_fn`, a Jacobian `jacobian_fn`, and a step duration `dt`. V3 is reliable, V4 is available, and the region is decided jointly by V1, V2 and V3.
- **B-class (observed-only).** Only the time series is provided. Significance uses surrogate tests; reconstruction uses delay embedding and local linear / S-map forecasts. V3 is marked `data_only`; the region is determined by V1 and V2 with explicit confidence labels.

A missing time index is a blocker: the data-side stability view cannot be computed without a declared physical step duration.

2.3 View 1 — pairwise structural coupling

V1 reports three quantities.

- **Non-diagonal energy fraction ϕ .** $\phi = (\sum_{i \neq j} a_{ij}^2) / (\sum_{i,j} a_{ij}^2)$, a fraction on $[0, 1 - 1/n]$, whose scale is fixed by definition.

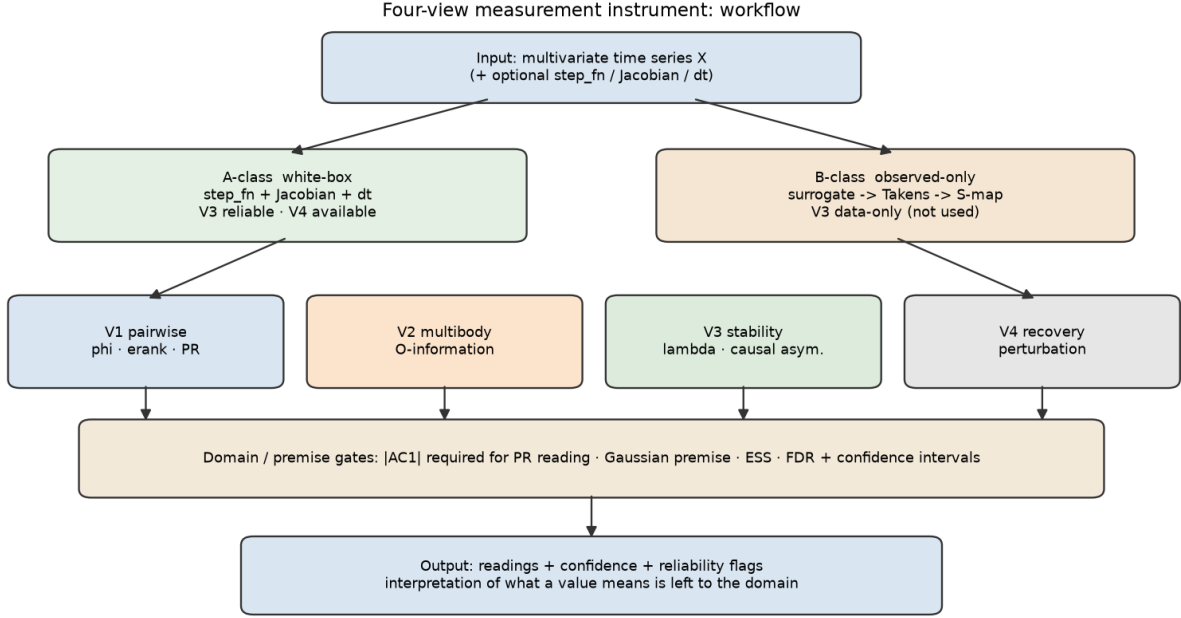


Figure 1: Instrument workflow: input is split into white-box A and observed-only B classes, passed through four views and premise gates, and emitted as readings with confidence and reliability flags; interpretation is left to the domain.

- **Effective rank.** $\text{erank} = \exp(-\sum_k p_k \log p_k)$, the entropy-based number of modes.
- **Participation ratio.** $\text{PR} = (\sum_k \lambda_k)^2 / (\sum_k \lambda_k^2)$.

Premise gate. A participation-ratio reading is interpreted only when the lag-1 autocorrelation satisfies $|\text{AC1}| \geq 0.4$; below this an apparent PR expansion is not specific to coordinated structure.

2.4 View 2 — multibody information

V2 reports the O-information, distinguishing redundancy (positive) from synergy (negative).

- **Weak estimator.** Discrete, triplet/bin-based O-information for observed-only data, requiring a minimum record length (approximately $T \geq 1715$ for the default binning), reported with a surrogate significance flag.
- **Strong estimator (--ogauss).** A Gaussian/copula closed form with a lower sample-length requirement, adopted only when its premise holds.

Premise and bias gates. A multivariate Gaussianity check at entry marks `gaussian_premise_violated` on failure. Near the length gate the weak estimator carries a measured bias (roughly -0.09 near $T = 1750$ while the truth is zero); the record carries `0_bias_estimate` and `0_bias_dominated`, and a bias-dominated reading is not interpreted as synergy. When more than half the frames are zero, the data are smoothed to a continuous signal before O-information is computed. The redundant and synergistic categories are evaluated against the median sign with a heterogeneity flag.

2.5 View 3 — stability and causal asymmetry

V3 reports the leading Jacobian eigenvalue $\lambda_{\max}$ on the frozen dynamics: $\lambda_{\max} < 0$ (or $|\lambda| < 1$ for a discrete map) indicates return to a stable state, and $\lambda_{\max} \rightarrow 0$ indicates critical slowing.

- **A-class versus B-class.** For A-class inputs the Jacobian is known and $\lambda_{\max}$ is reliable. For B-class inputs a least-squares autoregressive Jacobian is a linear autoregressive matrix rather than the one-step Jacobian; it is marked `data_only` and excluded.
- **Causal distinction rather than proof.** V3 measures forward/backward asymmetry and narrows a question to the smallest causally distinguishable units; it does not prove causation.

A relative causal-strength quantity `strength_rel` and an intrinsic time scale $\tau_{\text{eff}} = 1/\gamma_{\text{diag}}$ (the diagonal decay rate) are used. The eigenvalue is evaluated at the window-mean state (on the operating branch), not at noisy instantaneous points where fluctuations erase the slowing.

2.6 View 4 — perturbation recovery

V4 applies a standardised perturbation to the frozen dynamics and records whether and on what time scale the system returns, under a single registered criterion. V4 captures return to stability but is insensitive to whether inter-layer meshing has been established (Section 3.4), a named limitation.

2.7 Significance, multiplicity and statistical quality

- **Surrogates.** FFT phase randomisation and circular time shifting are both computed and the more conservative result used; at least 50 surrogates are used, with percentile $(k+1)/(n+1)$.
- **Multiple comparisons** use false-discovery-rate control; the family size is reported.
- **Uncertainty** is given by bootstrap confidence intervals, including for stability and relative strength.
- **Effective sample size** is reported per window; it seals confidence, and label suppression is applied only under declared domain-conditional rules.
- **Power:** a minimum detectable effect is reported.

2.8 Cross-level measurement

For a nested system the instrument is applied at each level and to the coupling between levels. A genuine cross-level transition is assessed against five criteria: (1) a new order parameter unreachable at the old level; (2) crossing the barrier with irreversibility via a hysteresis loop; (3) self-stabilisation; (4) the new state being alive rather than single-pole or full-rank; (5) genuine bidirectional coupling. The views are not assumed isomorphic across levels.

2.9 Data and computational protocol

All synthetic runs use fixed recorded seeds and a frozen calibration file; commits, seeds and pre-registration files appear in the data availability statement. Numerical work uses NumPy/SciPy; high-dimensional scans use a CPU/GPU blocked PyTorch implementation. The instrument ships a `--selftest` suite and per-version verification scripts.

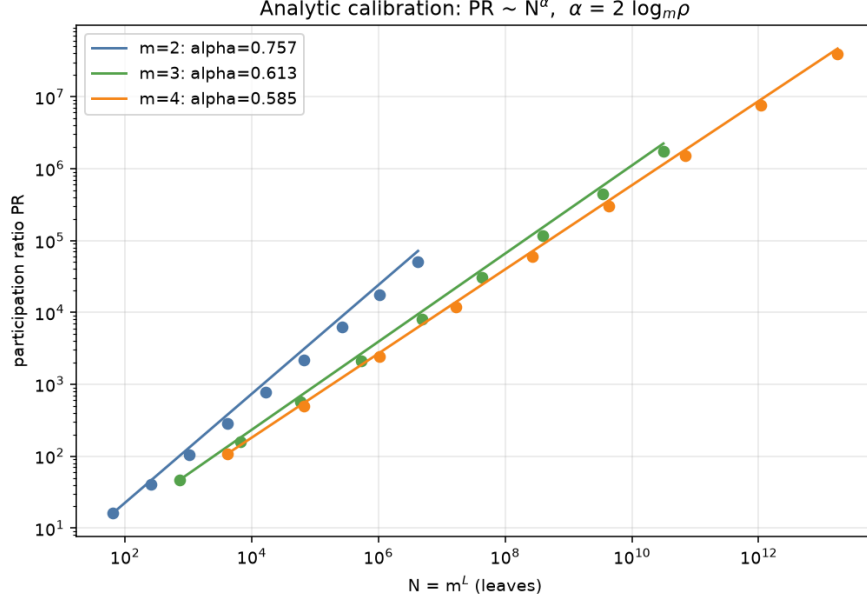


Figure 2: Analytic calibration: numerical participation ratios (points) follow the derived power laws $PR \sim N^\alpha$ (lines) for three branching factors.

3 Validation and results

3.1 Analytic calibration on a nested hierarchy

For an m -ary nested hierarchy, $PR \sim N^\alpha$ with $\alpha = 2 \log_m \rho$ (derived step by step in Appendix D). Figure 2 plots numerical PR against N for $m = 2, 3, 4$ ($\rho = 1.3, 1.4, 1.5$); points fall on the predicted slopes ($\alpha = 0.757, 0.613, 0.585$), with finite- L end corrections.

3.2 Known-answer systems (out of sample)

The instrument was run out of sample on ten known-answer systems (Figure 3), comparing linear fixed-point, nonlinear S-map and persistence forecasts by held-out error. It records single-view misreadings corrected by complementary views: forecast-optimal localisation differs from the Lyapunov point on the tent map; a bounded predictable 27-dimensional driven norm is not chaotic despite a single misleading estimate; and a noisy limit cycle is mis-stable under a single eigenvalue but separated by its spectrum.

3.3 Natural physiological domain: heart-rate variability

The instrument was applied to public de-identified heart-rate data (healthy $n = 54$; congestive heart failure $n = 29$); the result includes the failure (Figure 4).

- **V1 ϕ separates:** healthy median ≈ 0.75 versus CHF ≈ 0.36 .
- **Effective rank separates correctly:** $\text{erank}/n \approx 0.36$ healthy versus ≈ 0.80 CHF.
- **V2 O-information does not separate:** medians ≈ 0.30 and ≈ 0.31 .

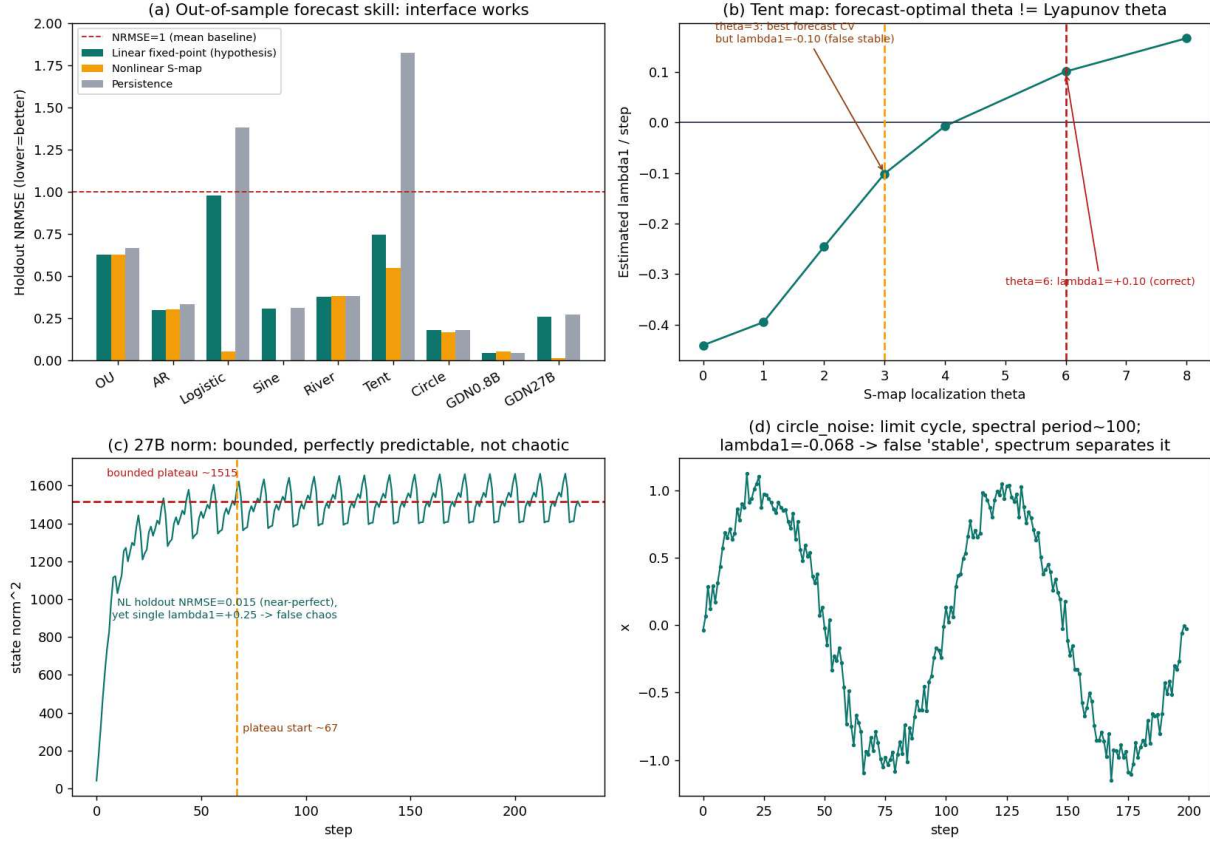


Figure 3: Out-of-sample validation across known-answer systems: (a) held-out forecast error; (b) tent-map forecast optimum versus Lyapunov point; (c) bounded predictable 27-dimensional norm; (d) noisy limit cycle separated by its spectrum.

- **The region classifier does not separate:** both groups fall predominantly in `redundant_factor` (51 healthy, 24 CHF), with 3–5 synergistic cases each.

This is not a passed benchmark; it exposes a single-point dependency: the region decision is dominated by O-information, and when O-information is insensitive, the separation captured by V1 and effective rank is discarded. Frozen classifier labels overlap similarly (normal 31 stable / 21 chaotic / 2 indeterminate; CHF 15 stable / 14 chaotic). Removing this dependency is designated future work and is not performed on this dataset, where post-seeing adjustment would be post-hoc.

3.4 Cross-level correspondence is carried by V2

On a two-level nested system, the per-view change when cross-layer blocks are restored (cut $\rightarrow$ coupled) is non-uniform (Figure 5). V2 is the principal carrier: Gaussian O-information rises from 4.09 to 7.21 (net standardised change 5.44), while V1 moves weakly (ϕ 0.496 $\rightarrow$ 0.579), V3 weakly (λ $-0.199 \rightarrow -0.138$), and V4 not at all. The views are not isomorphic across levels. Over the slaving scan, the live interval (erank peak ≈ 41 near $\gamma = 0.05$) is distinct from the enslavement interval ($\gamma \geq 0.50$, erank ≤ 21). A single unified system satisfies all five cross-level criteria, including a hysteresis loop (up-sweep jump 0.84, down-sweep jump 0.19) and a self-stabilised near-critical branch ($\lambda \approx -0.235$).

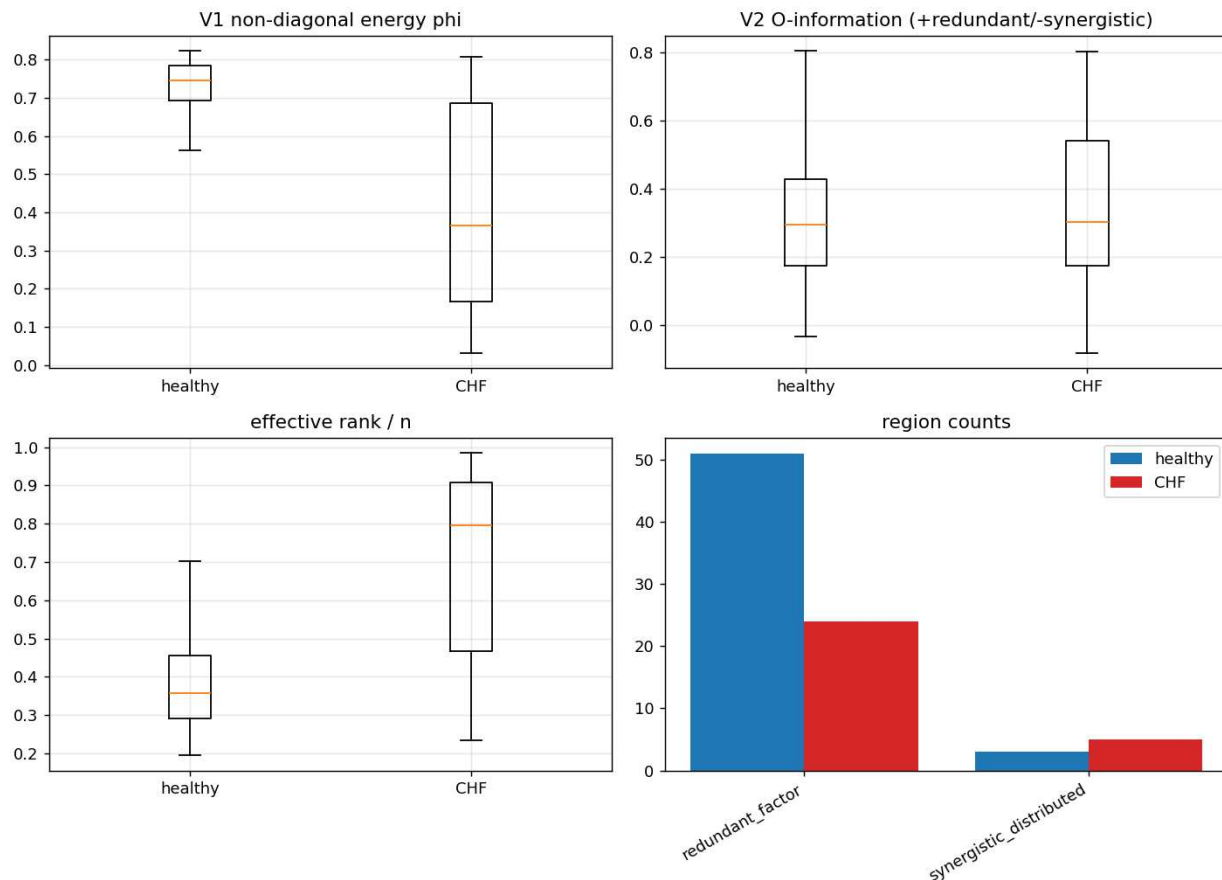


Figure 4: Natural-domain HRV: V1 ϕ and effective rank separate healthy from CHF; O-information does not; the O-dominated region classifier does not separate the groups. The failure is reported as a known boundary.

3.5 Failure-mode catalogue

Testing produced thirteen reproducible failure modes, each with a known-answer case and a catching gate (Appendix B). The catalogue is the instrument’s audit layer: it names, per reading, the specific way an interpretation can be invalid.

4 Discussion

4.1 What the method does and does not do

The instrument converts a multivariate recording into view-specific readings with declared input class, significance, uncertainty and premise flags. It establishes no universal law and assigns no meaning: a high reading may be healthy in one domain and pathological in another. Its most defensible claim is narrow—for a multivariate reading, it states the conditions under which it is interpretable, and the ways those conditions fail.

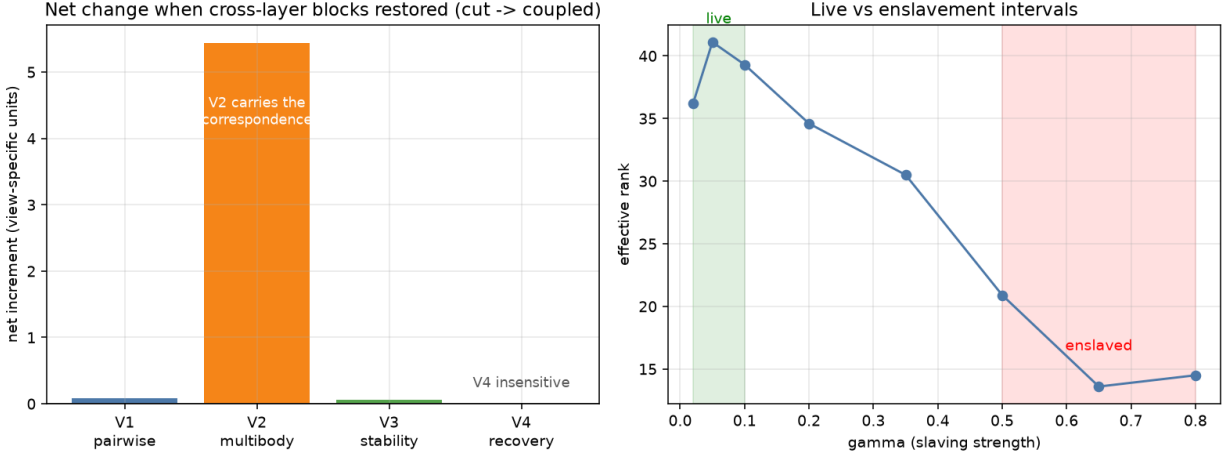


Figure 5: Cross-level correspondence. Left: the net change on restoring cross-layer blocks is carried by V2; V4 is insensitive. Right: live and enslavement intervals over the slaving scan.

4.2 Relation to existing methods

The method does not re-derive the edge of stability, self-organised criticality, synergetics, or graph deviation methods; these are cited as domain content. The added element is an audit layer deciding where an estimator’s operating region ends, analogous to uncertainty budgets and traceability in metrology.

4.3 Limitations

- **External domains.** Independent external validation is limited to HRV and held-out synthetic systems; multi-lead EEG, fMRI, seismic and cross-market data require access, approval or collaborators.
- **Pure-blind abrupt transitions.** When a transition is driven by an unobserved external forcing crossing a threshold, internal views carry no advance information; a fully unobservable forcing is a hard information-theoretic endpoint.
- **V4 and inter-layer meshing.** Recovery does not show whether inter-layer coupling is established.
- **General closed forms** for high-dimensional quasi-potentials and non-quasi-static regimes are beyond current analytic tools.

4.4 The arbitration layer

Readings meeting a trigger condition pass to an arbitration procedure returning `robust_signal`, `artifact_unreliable`, or `inconclusive`, explaining the result without altering the frozen reading. The layer is calibrated for long cleaned records and is itself a named failure mode for short forecast windows.

5 Conclusions

A four-view, domain-self-calibrated instrument can report the structure of a coupled complex system without adopting any discipline’s substantive theory, provided the input class is declared, significance is calibrated against same-recording nulls, and premise gates are enforced. An analytic calibration is confirmed numerically, known-answer systems are read with single-view misreadings made visible, and a natural physiological domain is reported honestly, including a single-point classification failure. The durable contribution is an executable statement of when a multivariate reading is interpretable and the thirteen ways it can silently fail.

Declarations

Use of large language models. A large language model (Doubao) assisted language drafting and formatting of the author’s own analysis, code and results. All technical content, figures, computations and conclusions were produced, verified and approved by the author, who takes full responsibility. The model is not an author.

Ethics and human-subjects data. The physiological validation uses public de-identified heart-rate data via PhysioNet; no new human-subject data were collected.

Data availability. Code, calibration, seeds, pre-registrations and results are at <https://gitee.com/lwlzmck/symbiotic-complex-dynamics>; the commit and seed list are in `papers/DATA_AVAILABILITY.md`. The companion dataset is at Zenodo (DOI 10.5281/zenodo.22979099) with companion paper (DOI 10.21203/rs.3.rs-11160904/v1).

Competing interests. The author declares no competing interests.

Author contributions. Wenlong Li is the sole author.

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A The Synergy proposition (conceptual statement)

The following organisational statement motivates the design and is conceptual, not an empirical result of this paper.

- **A1 Nested self-similarity.** A coupled complex system is organised as nested levels examined under the same views, related by scale changes.
- **A2 Cross-level correspondence.** A slow variable enslaves a group of fast variables and receives feedback; correspondence is carried per view and is not uniform.
- **A3 Viable organisation is neither pole** — neither a single enslaving order parameter nor excessive fragmentation, but a small number of resourced, independent yet coupled factors.
- **A4 Liveness versus stability** are distinct and measured separately.
- **A5 Infinite-nesting limit** expresses per-view readings through scaling exponents at a scale-invariant point; this is structural, not a proof of universality.

Older philosophical literature contains structural analogues of nested and complementary organisation; these are historical analogues only and carry no evidential weight.

B Failure-mode catalogue (summary)

ID	Failure mode	Gate that catches it
S1	Closure artefact creates apparent coupling	boundary/closure check
S2	Frame versus physical-window mismatch	declared time index
S3	Smoothing-boundary collapse manufactures coupling	kernel-mass renormalisation
S4	ESS seals confidence but not label	domain-conditional declaration
S5	Single view claims category when V2 unavailable	V1 structure hint only
S6	Smoothing collapse inflates coupling	boundary fix + regression
S7	Strong-estimator output dropped by record layer	record-layer pass-through
S8	Redundant judged before synergistic	median-sign + heterogeneity flag
S9	Near-singular closed-form O unusable	condition-number guard
S10	Data-side stability estimate wrong sign	scalar dominant-mode estimator
S11	Off-branch eigenvalue loses slowing signal	evaluate at window-mean state
S12	Arbitration fails for short forecast windows	named structural limitation
S13	Data-side stability needs declared time index	mandatory time-index field

C Domain adaptation map (conceptual)

The instrument connects to a discipline through an adaptation record declaring: the observed variables and units; the measurement operator actually applied and its noise; which views are available for the input class; the domain-specific null/reference scale; and who interprets the readings. The instrument supplies the measurement; the discipline supplies thresholds, mechanisms and meaning. A template is provided at `docs/measurement_toolkit/`.

D Analytic derivation of the calibration law

This appendix derives the result of Section 3.1 step by step, so the calibration can be checked line by line rather than accepted as a stated result.

D.1 Hierarchy and Laplacian. Let $N = m^L$ nodes form a complete m -ary tree of depth L . The meet level $n(i, j)$ is the smallest ℓ with $\lfloor i/m^\ell \rfloor = \lfloor j/m^\ell \rfloor$. Set $J_{ij} = \rho^{-n(i,j)}$ ($i \neq j$), $J_{ii} = 0$, $D = \text{diag}(\sum_j J_{ij})$, and $\mathcal{L} = D - J$.

D.2 Closed-form spectrum. Besides one zero mode, the level- k splitting modes are

$$\mu_k = q^k + c \sum_{n=k+1}^L q^n, \quad q = \frac{m}{\rho}, \quad c = \frac{m-1}{m}, \quad (1)$$

with degeneracy $d_k = (m-1)m^{L-k}$.

D.3 Covariance eigenvalues and PR. For $dx = -\mathcal{L}x dt + \sigma dW$ the stationary covariance has eigenvalues $\lambda_k \propto (1 + \mu_k)^{-1}$. With $S_1 = \sum_k d_k \lambda_k$ and $S_2 = \sum_k d_k \lambda_k^2$ (including the zero mode $\lambda_0 = 1$), $\text{PR} = S_1^2/S_2$.

D.4 Fractal window. For $\rho \in (1, \sqrt{m})$, $q = m/\rho \in (\sqrt{m}, m)$, so $q > 1$. For an interior mode ($k = \beta L$, $0 < \beta < 1$) the geometric sum gives

$$\mu_k \approx \frac{cq}{q-1} q^{L+1} = A q^{L+1}, \quad (2)$$

independent of k ; hence every non-zero $\lambda_k = O(q^{-L})$.

D.5 Second moment.

$$S_2 = 1 + \sum_k d_k \lambda_k^2 = 1 + O\left(\sum_k m^{L-k} q^{-2L}\right) = 1 + O\left(\frac{\rho^2}{m}\right)^L \rightarrow 1, \quad (3)$$

because $\rho^2 < m$.

D.6 First moment.

$$S_1 = 1 + \sum_k d_k \lambda_k = 1 + \Theta\left(\sum_k m^{L-k} q^{-L}\right) = 1 + \Theta\left(\frac{m}{q}\right)^L = \Theta(\rho^L), \quad (4)$$

diverging for $\rho > 1$.

D.7 Result.

$$\text{PR} = \frac{S_1^2}{S_2} = \Theta(\rho^{2L}) = \Theta\left(N^{2\log_m \rho}\right), \quad (5)$$

i.e. $\alpha = 2\log_m \rho$; the upper edge $\alpha = 1$ is reached at $\rho = \sqrt{m}$. The small end deviations in Figure 2 are finite- L corrections, as the interior-mode approximation is exact only as $L \rightarrow \infty$.

Every step uses only the closed-form spectrum and elementary geometric-sum asymptotics, and is reproduced independently by the code (`crosslevel_view_mapping/prove_alpha.py`); a reader who disputes any line can re-run it and obtain the same numbers.

E Deepening synthesis: the consolidated mathematical record

This appendix consolidates the multi-round mathematical deepening that motivates and refines the instrument. It keeps the Parts of the working record — the anchor spectrum, structure, spatial geometry, the hierarchical Ising correspondence, edge thermodynamics, the V3 causality theorem, the NESS-relaxation variational principle, the cross-model bridges, model adaptation, and boundaries — followed by the empirical, instrument-core, physical-foundation, borrowed-machinery and prospective ledgers. As throughout the paper, each closed result is a model-level statement about the defined hierarchy; historical or philosophical passages are analogues carrying no evidential weight, and empirical items are support-grade rather than confirmations.

E.1 Core mathematical results (Theorems 1–10)

Part I — The anchor.

Definition 1 (m -ary Dyson hierarchical Laplacian). *Let $N = m^L$ nodes form a complete m -ary tree of depth L ; the meet level $n(i, j)$ is the smallest ℓ with $\lfloor i/m^\ell \rfloor = \lfloor j/m^\ell \rfloor$; $J_{ij} = \rho^{-n(i, j)}$ ($i \neq j$), $D = \text{diag}(\sum_j J_{ij})$, $L = D - J$.*

Lemma 1 (spectrum; classical, cf. Bendikov–Krupski 2014 and Julaiti et al. 2013). *The eigenvalues of L are: the zero mode $\mu_0 = 0$ (multiplicity 1), and level- k splitting modes $\mu_k = q^k + \frac{m-1}{m} \sum_{n=k+1}^L q^n$, $q = m/\rho$, degeneracy $d_k = (m-1)m^{L-k}$. The normalized-Laplacian analogues for fractal trees and dendrimers are closed in Julaiti et al. [47] (relevant to the spectral-dimension route of Sec. E.2).*

Remark 1 (attribution). *The spectrum of the hierarchical Laplacian is a known subject (Bendikov–Krupski, Potential Analysis 41 (2014), and the ultrametric-diffusion literature before it). We re-derive it here as the anchor; the participation-ratio packaging below is our notation.*

The covariance of $dx = -Lx dt + dW$ has eigenvalues $\lambda_k = (1 + \mu_k)^{-1}$; $\text{PR} = S_1^2/S_2$ with $S_1 = 1 + \sum_k d_k \lambda_k$, $S_2 = 1 + \sum_k d_k \lambda_k^2$ (the “+1” is the zero mode).

Theorem 1 (α -law). *For $\rho \in (1, \sqrt{m})$: $\text{PR}(N) = \Theta(N^{2 \log_m \rho})$; the window edge $\alpha = 1$ is $\rho = \sqrt{m}$ (the Dyson Ising point).*

Proof. As in the companion paper (Part I); the “2” is the square in $\text{PR} = S_1^2/S_2$, universal; only the log base changes with the arity. $\square$

Theorem 2 (unified PR formula). $\text{PR}(N, \rho) \sim \frac{a(\rho) \rho^{2L}}{1 + a(\rho) \rho^{2L}/m^L}$, $a(\rho) = \left(\frac{m - \rho}{m - 1}\right)^2$, giving the three regimes (alive $\rho^2 \ll m$, edge $\rho = \sqrt{m}$, fragmented $\rho > \sqrt{m}$) in one closed form.

Theorem 3 (critical constant). $c(m) = \frac{m}{m + (\sqrt{m} + 1)^2}$, with $\lim_{m \rightarrow \infty} c(m) = 1/2$. (The earlier “1/3” guess is corrected; erratum E1.)

Theorem 4 (logistic finite-size scaling). With $y = L \ln(\rho/\rho_c)$, $\Phi(y) = \frac{a_c e^{2y}}{1 + a_c e^{2y}}$, $\nu = 1$.

Theorem 5 (covariance level profile). *The level-resolved covariance $g(n)$ is closed: the alive window is carried by the shallow large-degeneracy modes, the zero mode pins $S_2 \approx 1$.*

Part II — Structure: from spectrum to organization.

Theorem 6 (O-information of ultrametric Gaussians). *The O-information (Rosas et al. 2019) of a Gaussian with ultrametric (tree) covariance is nonnegative at every order: hierarchy $\Leftrightarrow$ redundancy-dominance. The converse direction—colliders (Berkson 1946) give $O < 0$ —is the synergy source; the criterion survives non-Gaussian anchors (XOR/copy). Estimation boundary (v1.2): the finite-sample bias of O estimation is a known subject (the naive estimator is severely synergy-biased for small n ; cf. the published bias analysis, PMC11507536); our reliable-region criterion takes the sampling-uncertainty form $|\hat{O}| > (\text{sampling std})$ with the bias piece cited rather than re-derived (the sparsity “artifact” measured on random DAGs is T -independent and equals the true collider synergy, not an estimator bias). The std closes analytically through the Wishart log-determinant variance $\text{Var}(\ln \det S) = \sum_{i=1}^d \psi_1((T-i+1)/2)$ (verified to 1.2×10^{-4} against simulation), giving the bootstrap-free criterion $|\hat{O}| > \sqrt{\sum_i \psi_1((T-i+1)/2)}$; the discrete analogue uses the multinomial entropy variance $\text{Var}(\hat{H}) = \frac{1}{T} [\sum_i p_i (\ln p_i)^2 - H^2]$ (verified to 5×10^{-6}), the same one-estimator-std structure across distribution classes.*

Theorem 7 (edge locking). *At the cross-layer edge $f_s = 1$: the correlation locks quadratically ($1 - a^2 \sim \delta^2$, $1 - c \sim \delta^2$), and $O(\delta) = -\ln \delta + C(s)$ (sympy-proven; the earlier “ $O \rightarrow 0$ ” is erratum E2).*

Theorem 8 (community liveness). *Coherent cliques with bounded cross-degree give $\text{PR} \approx n(2 + c_{\text{out}})^2$, hence $\alpha = \gamma$ (the block-count exponent); the block-coupling sweep obeys the closed form $\text{PR}/N = 1/(1 + y^2)$, $y = s_{\text{in}} \sqrt{b}/(1 + c_{\text{out}})$.*

Theorem 9 (mixed coupling). *For $f(n) = a\rho^{-n} + b n^{-c}$: the alive window is exponentially robust in b (erosion scale $L^* \approx \ln(a/b)/\ln \rho$); the strictly-alive limit has measure zero (critical organization).*

Theorem 10 (multilayer and temporal). *Two coupled trees keep the alive window (single-edge coupling shifts only $O(c_{\text{out}}/N)$); dense cross-blocks shift the nonzero spectrum by $2w\alpha N$ (locking). Time-fluctuating couplings collapse the observed system to a monopole; the mechanism is the non-adiabatic lag of the state behind the slowly varying Laplacian (the first-order mode-mixing vanishes because the level masks commute with the Laplacian, $||[L_0, \delta L_n]|| \leq 10^{-16}$; erratum E10). The collapse p -map now closes on the GPU full grid (36 points, $B = 64$ seeds each): the collapse probability is monotone in the amplitude with the threshold $\text{amp} \approx 0.45\text{--}0.5$ (the round-8 $\theta \approx 0.45$ structure), rises with the window and plateaus (saturation $p = 1$ at $\text{amp} = 0.6$), with a non-monotonic dip at $\tau_{\text{obs}} = 64$ — the quantitative boundary of the uniform-mode survival condition (the round-14 sharpened target), delivered as data; its analytic mechanism (phase-randomization rate vs. window) remains open. The multidimensional sync prediction $\text{PR}/N \approx 1 - R$ is mean-field-only: on Barabási–Albert and community (SBM) networks the sums are 1.56–1.91 (the network Kuramoto synchronizes through hubs while the linear OU collapses slowly — the two thresholds separate), while $\rho_c \approx 1 - 1/D_{\text{eff}}$ holds approximately (deviations ≤ 0.14 , direction structure-dependent).*

E.2 Extended results (Theorems 11–20)

Part III — Spatial geometry.

Theorem 11 (non-uniform spectrum). *For branching factors $\{M_k\}$, with deep products $\tilde{P}_j = \prod_{i=0}^{j-1} M_{L-i}$ and shallow products $P_j = \prod_{i \leq j} M_i$: $\mu_k = f(k)\tilde{P}_k + \sum_{n>k} f(n)(\tilde{P}_n - \tilde{P}_{n-1})$, $d_k = (M_{L-k+1} - 1)P_{L-k}$. (The earlier shallow-product form is corrected; erratum E7.)*

Theorem 12 (geometric-mean theorem). $\alpha_\infty = 2 \ln \rho / \langle \ln M \rangle$ holds rigorously at leading order; the finite-size drift is a three-exponential mixture with the rates identified term by term: $\ln \rho$ (the zero-mode “1+” in S_1), $\ln(M_{\text{eff}}/\rho^2)$ (the S_2 correction), $\ln(M_{\text{eff}}/\rho)$ (the amplitude). A short-window fit value 0.23 is the mixture artifact (erratum E8).

d -grids give $d_s = d$ exactly; the Dyson hierarchy is dense (d_s undefined); the Sierpinski gasket has $d_s = 2 \log 3 / \log 5$ (known). The eigenstate moment τ_2 (mean convention) has limit $\tau_2^* \approx 0.53$ (decimation recursion fixed point vs. independent extrapolation agreeing to 2.6%); the mean/typical separation (the IPR tail $P \sim \text{IPR}^{-2.32}$) resolves the retracted “ $\tau_2 = d_f/d_w$ ” identification (erratum E6): the mean lies 1.2% below and the typical 11% above d_f/d_w .

Part IV — The hierarchical Ising correspondence.

Theorem 13 ($\eta = \log_2 \rho$). *The block identity $C(n) = 4t[n] - 2t[n-1]$ with the susceptibility recursion $\chi_{n+1} = \chi_n + \beta 2^n \rho^{-(n+1)} \chi_n^2$ gives $C(n) = A\rho^{-n}(1 + O(2^{-n/2}))$, i.e. $\eta = \log_2 \rho$ (non-universal). Attribution: the critical correlation following the coupling is a known result of the hierarchical-model literature (Dyson 1969; Baker 1972; Bleher–Sinai 1973; Collet–Eckmann). Our recursion is an independent re-derivation; the increment is the explicit block identity.*

Theorem 14 (tree-chain duality). *The chain side $\eta = 2 - \sigma$ is the rigorous Fröhlich–Spencer (1982) result; hence $\eta_{\text{tree}} + \eta_{\text{chain}} = \sigma + (2 - \sigma) = 2$ once both exponents are known; the mechanism is the mirror fixed points of the increment recursion $z = 1 - \sigma + 2z_\chi$.*

Theorem 15 (χ^* structure). *The reduced map $u_{n+1} = (2/\rho)u_n(1 + u_n/\rho)$ has the unstable fixed point $u^* = \rho\varepsilon/2$ ($\varepsilon = \rho - 2$); the critical β_c is the closed-form root of the preimage condition $\frac{4\beta_c^2}{\rho^3} + \frac{2\beta_c}{\rho} - w = 0$, $w = (-\rho + \sqrt{\rho^2 + \rho^3\varepsilon})/2$ (verified to 8 digits); and $\ln \chi_{k_{\max}}^* = k_{\max} \ln(1 + \varepsilon/2) + O(1)$ (critical slowing). Attribution: the susceptibility asymptotics on hierarchical lattices is a classical subject (Bleher–Žalys [27]); our recursion is the independent re-derivation. Consequently the power law $\chi^* \sim (\rho - 2)^{-\nu}$ does not exist in the exact recursion: the earlier $\nu \approx 2.2$ is a mixed-convention artifact (erratum E9).*

Remark 2 (Riccati closed form — regime-restricted, novelty unchecked). *The continuous limit of the susceptibility recursion is the Riccati equation $d\chi/dn = (\beta/\rho)(2/\rho)^n \chi^2$, solved by $u = 1/\chi$: $u(n) = u_0 + (\beta/\rho)[(2/\rho)^n - 1]/\ln(2/\rho)$ (sympy). For $\rho > 2$ the finite limit $\chi^* = 1/[1/\chi_0 - (\beta/\rho)/\ln(2/\rho)]$ exists iff the validity condition $1/\chi_0 > (\beta/\rho)/\ln(\rho/2)$ holds. At the critical parameters ($\beta = 1/1.78$, $\rho = 2.5$) the condition fails ($u(\infty) < 0$: the solution crosses the pole at finite n) and the closed form is invalid — χ^* there is the product-defined quantity of the exact recursion (the A3 identity’s partner), with no elementary closed form. In the weak-coupling regime ($\beta = 0.05$) the closed form matches the exact recursion to 1.2%. This is a regime-restricted closed form, marked novelty unchecked.*

Part V — Edge thermodynamics and training.

Theorem 16 (edge entropy production). *For the 2D triangular NESS $x' = [[g, c], [0, g]]x + \eta$, $\delta = 1 - g$: $\langle \sigma_1 \rangle = \frac{c^2}{\delta(2 - \delta)}$ exactly (the earlier $c^2(\delta^2 + 2\delta + 4)/(8\delta)$ carried an $O(\delta^3)$ error; erratum E11a), so $\langle \sigma_1 \rangle = \frac{c^2}{2\delta}(1 + \frac{\delta}{2} + O(\delta^2))$; the Fano factor $\rightarrow 1/2$ (universal), $P(\sigma < 0)$ is compressed from $1/2$ to the limiting ratio. The variance closes as $\text{Var} = G(y, \delta) c^4/\delta^2$ with $y = c^2/\delta$ (exact rational G ; $y \rightarrow 0$: $G \sim 2\delta(2 - \delta)/y^2$, the housekeeping fluctuation floor; $y \rightarrow \infty$: $G \rightarrow 2/(2 - \delta)^2$, the strong-driving power law, $\rightarrow 1/2$ as $\delta \rightarrow 0$). A tanh gate truncates the $1/\delta$ divergence to saturation (the divergence is a linearization artifact; erratum E4b).*

Theorem 17 (closed distribution). *The standardized edge distribution is $\Phi(z) = \text{shifted Gamma}(1/2, \sqrt{2})$ (CGF = $-\sqrt{2}s/2 - \frac{1}{2} \ln(1 - \sqrt{2}s)$); $P(\sigma < 0) = (2^{1/4}/\sqrt{\pi})\sqrt{\delta/c^2}(1 + o(1))$ (strict $\alpha = 1/2$, erratum E4a).*

Theorem 18 (training edge and the exponent’s origin). *Gate training obeys $\delta(t) \sim t^{-\beta}$, $\beta \approx 0.56 - 0.63$ (power law beats log/linear; Adam necessary). The local multiplier degeneracy $1 - M(\delta) \sim \delta^{p-1}$ with $p - 1 = 1/\beta$ (self-consistent to 0.01): the power law is the inevitable form of a degenerate-multiplier passage. Origin, narrowed (v1.2). The candidates are excluded one by one: batch noise (the $\delta^{-2.79}$ gap), the EMA window ($\beta \cdot T$ far from constant in the T -scan), and the deterministic second-order dynamics (a sliding-window test gives $\beta_{\text{local}} \approx \text{const}$ with CV 0.106 — a genuine window power law, which the log/linear/exponential-curvature-zero laws cannot produce). The surviving mechanism is the gate saturation: $d\rho/db_g = \rho(1 - \rho) \approx \delta$ gives the gradient a δ^2 factor, $d\delta/dt = -C \delta^2 e^{-2T\delta}$ (symbolic), whose crossover at $\delta^* = 1/(2T)$ sweeps β_{local} from 1 (the δ^2 -law) to 0 (the log law) — the observed window straddles δ^* , so $\beta \approx 0.5 - 0.6$ is the mid-sweep effective exponent. The local-slope signature $p_{\text{local}} = d \ln |d\delta/dt| / d \ln \delta \approx 2$ (median 2.13 vs. 2.85 for a pure power law) weakly supports the δ^2 factor; the exponential term is not resolvable at the current noise level; a sharpness ablation is inconclusive by design (the half-sharpness gate cannot reach the edge), and an lr-scan gives $\beta \sim \text{lr}^{0.38}$ — a moderate discrete-step modulation on top of the continuous crossover (the combined mechanism). The exponent’s origin is thus narrowed, not fully closed; its quantitative pinning remains open (see Part VIII).*

Theorem 19 (edge as KKT boundary optimum). *With the persistence loss $L(\rho) = \lambda y^2 / ((\rho^T s)^2 + \lambda)$: $\partial L / \partial \rho < 0$ everywhere on $(0, 1)$ (symbolic proof), so $\min L$ s.t. $\rho < 1$ has its optimum at the boundary $\rho \rightarrow 1^-$; the KKT multiplier $\mu = -\partial L / \partial \rho|_{\rho=1} = 2T\lambda s^2 y^2 / (\lambda + s^2)^2 > 0$ (constraint active). The edge is the constrained-optimization boundary solution, not a coincidence.*

Corollary 1 (gradient-flow laws). *Pure gradient flow gives the log law $\delta(t) = \frac{1}{2T} \ln t$ (separation of variables); Adam's steady-state normalization $m/\sqrt{v} = (1 - b_1^n) / \sqrt{1 - b_2^n} \rightarrow \text{sign}(g) \cdot \text{const}$ gives the linear law; the observed power law requires a gradient $\sim \delta^{1+1/\beta}$, which the batch-noise mechanism cannot supply (σ would need $e^{-2T\delta} / \delta^{2.79}$ but has only $e^{-2T\delta}$); the gate-saturation δ^2 factor restores it through the crossover of Thm. 18.*

Part VI — Causal completeness within the linear-NESS scope (Theorem 20).

Definition 2 (causal decomposition). *For $dx = Ax dt + L dW$ (A Hurwitz), decompose $A = A_{\text{self}} + A_{\text{anti}}$ with $A_{\text{self}} = \frac{1}{2}(A + A^\dagger)$ (potential/dissipative, reciprocal) and $A_{\text{anti}} = \frac{1}{2}(A - A^\dagger)$ (nonreciprocal, causal). The third-view causality is the nonreciprocal component.*

Theorem 20 (completeness within the linear-NESS scope). *With C the stationary covariance, $J = AC - CA^\dagger$, $\sigma = \text{Tr}[J^\dagger D^{-1} J C^{-1}]$ (Schnakenberg form; the linear-NESS entropy-production formulas are the standard subject of the Godrèche–Luck characterization series [41, 40]; the steady-state response and the fluctuation-dissipation violation take the standard Hatano–Sasa [48] / Maes [49] forms, cited here as the literature anchor rather than re-derived): (i) $\sigma = 0 \iff J = 0 \iff$ detailed balance $\iff AD = DA^\dagger$ (for $D = I$: $\iff A$ symmetric $\iff$ no causal arrow) — σ is a complete measure, not a proxy; (ii) for the triangular case $A = [[-\delta, c], [0, -\delta]]$, $D = I$: $\sigma = c^2 / \delta$ (machine-level), i.e. $c^2 = \delta \sigma$ — the causal coupling is read off, not merely distinguished; (iii) the non-self-adjoint degree closes as $\delta_{\text{nsa}} = c^2 / (4\delta^2 + 2c^2)$. Within the linear-NESS white-box scope, “distinguishing” and “determining” the causal arrow coincide; outside (pure data, nonlinear, counterfactual), the measures degrade to Granger/TE direction, CCM, and do-calculus respectively.*

E.3 Dynamics and the NESS variational principle (Theorems 21–26)

Part VI* — The NESS internal relaxation. Setup as in Sec. E.2: $dx = Ax dt + dW$ (A Hurwitz), $AC + CA^\dagger = -I$, $J = AC - CA^\dagger$, $\sigma = \text{Tr}[J^\dagger J C^{-1}]$.

Theorem 21 (Helmholtz–Hodge decomposition of the drift). *$A = -\frac{1}{2}\nabla\Phi + v$ with the quasipotential $\Phi(x) = \frac{1}{2}x^\dagger C^{-1}x$ and the circulation $v(x) = \frac{1}{2}JC^{-1}x$; $v \cdot \nabla\Phi \equiv 0$ and $\nabla \cdot (\rho_{\text{ss}} v) = 0$ (sympy-proven; the gradient-plus-current decomposition of NESS drifts is classical, cf. Graham–Tél [56]). The decomposition is equivalent to the Lyapunov equation: $(A + \frac{1}{2}C^{-1} - \frac{1}{2}JC^{-1})C = \frac{1}{2}(AC + CA^\dagger + I)$.*

Theorem 22 (quasipotential as minimal action). *$\Phi(x_f) = \inf_{\text{paths } 0 \rightarrow x_f} \frac{1}{2} \int |\dot{x} - Ax|^2 dt$, attained on the time-reversed relaxation $\dot{x} = (A + C^{-1})x$; equivalently $\langle \nabla\Phi, Ax \rangle + \frac{1}{2}|\nabla\Phi|^2 = \frac{1}{2}x^\dagger C^{-1}(AC + CA^\dagger + I)C^{-1}x = 0$ — the Freidlin–Wentzell/Onsager–Machlup principle [6, 54] for the linear NESS, with the exact finite-time identity (action = $\Phi(x_f) - \Phi(x(-T))$), machine-verified to 2.6×10^{-9}).*

Theorem 23 (rotation never slows). *On $L^2(\rho_{\text{ss}})$ the generator splits as $L = L_s + L_a$ with $L_s = -\frac{1}{2}\nabla\Phi \cdot \nabla + \frac{1}{2}\Delta$ (self-adjoint, gap $g_s = \frac{1}{2}\lambda_{\min}(C^{-1})$) and $L_a = v \cdot \nabla$ (skew-adjoint). Every relaxation rate satisfies $\mu_j = -\text{Re } \lambda_j \geq g_s$ (numerical-range argument, cf. Bendixson–Hirsch [55]); equality holds at equilibrium ($J = 0$). Literature anchors: adding a divergence-free (solenoidal/circulation) drift accelerates relaxation, $\lambda(C) \leq \lambda(0)$, with the reversible drift the slowest (worst) choice and rates equalizing toward the mean — established for Gaussian diffusions by Hwang–Hwang–Sheu*

1993/2005 [59, 58]; the optimal non-reversible linear perturbation is constructed by Lelièvre–Nier–Pavliotis [60] and Wu–Chu [61], the maximal decay rate equal to the largest eigenvalue of C^{-1} (Arnold–Signorello [62]), and the large skew-perturbation spectrum by Gallagher–Gallay–Nier [63]; general thermodynamic bounds on spectral perturbations are also given by Kolchinsky–Ohga–Ito [44]. Our bound is the independent re-derivation with the sharper specific content (the dual tightness criteria and the graded FP bound of Thm. 24–26). The convex-side nonlinear bound $\mu_j \geq \frac{1}{2} \inf \text{Hess } \Phi$ is the Brascamp–Lieb consequence (the Poincaré constant of a convex potential is at least $\inf \text{Hess } \Phi$ [53]); the non-convex effective-Hess convexification ($\mu_1 \approx \frac{1}{2} \cdot 1/\langle x^2 \rangle_\rho$, exact for the double well via the Rayleigh probe) is ours, novelty unchecked. The non-equilibrium Dirichlet form $\mathcal{E}(f) = \frac{1}{2} \int |\nabla f|^2 d\rho_{\text{ss}}$ is the equilibrium form verbatim in the steady measure (Courant–Fischer min-max). Nonlinear version: for $\rho_{\text{ss}} = e^{-\Phi}$ with a Poincaré inequality [57], $\mu_j \geq \lambda_{\text{gen}}(\rho_{\text{ss}})$ (half the Poincaré constant) and $\|e^{tL}\| \leq e^{-\lambda_{\text{gen}} t}$ on the mean-zero subspace; tightness $\mu_{\min} = \lambda_{\text{gen}} \iff v \cdot \nabla f_0 = 0$ for the ground mode f_0 (proved both ways via the numerical-range argument and the Poincaré equality case), reducing to the linear criterion $u^\dagger J = 0$ when $f_0 = u^\dagger x$.

Theorem 24 (drift-level double bound and graded spectrum). $\mu_k \in [\frac{1}{2}\lambda_{\min}(C^{-1}), \frac{1}{2}\lambda_{\max}(C^{-1})]$ for every drift mode (both endpoints attained at equilibrium; the arrow only shifts rates inward). The FP spectrum is graded: degree- n Hermite modes have rates in $[ng_s, n\mu_{\max}]$ (tight at equilibrium) — the degree-1 upper bound does not extend to higher grades, which resolves the apparent counterexample $2\delta > \mu_{\max}$. General noise $D = BB^\dagger$: $A = -\frac{1}{2}DC^{-1} + \frac{1}{2}JC^{-1}$, the minimal action uses the metric D^{-1} , $g_s = \frac{1}{2}\lambda_{\min}(C^{-1}D)$; the anisotropic triangle closes as $\sigma = c^2 d_2/(d_1 \delta)$ — the circulation depends only on the source-direction noise d_2 .

Theorem 25 (quantified tightness). In 2D with $C = \text{diag}(a, b)$ ($a \geq b$) and $J = jS$: $\Delta = \mu_{\min} - g_s = [(a - b) - \sqrt{(a - b)^2 - 4abj^2}]/(4ab)$ for $j < j_c = (a - b)/(2\sqrt{ab})$, saturating at $\Delta_{\text{sat}} = (a - b)/(4ab)$ beyond ($\Delta \sim j^2/(2(a - b))$ for small j). The triangle sits exactly on the boundary $j = j_c$ (its defective Jordan block is the boundary degeneracy), giving $\Delta = \delta r/\sqrt{1 + r^2}$ with $r = c/(2\delta)$ and the leading behavior $\frac{1}{2}\sqrt{\delta\sigma}$. In $d \geq 3$: $[J, C] = 0 \Rightarrow \Delta = 0$ exactly, and to leading order $\Delta = \frac{1}{2} \sum_{m \neq 1} (J_1)_{1m}^2 \nu_m \nu_1 / (\nu_m - \nu_1) + O(\varepsilon^4)$ (orientation-resolved; machine-verified).

Theorem 26 (upper-bound completion and the intermediate regime). (i) Sharp upper bound. $\mu_k \leq \frac{1}{2}\lambda_{\max}(C^{-1})$ for every drift mode, and $\mu_{\max}$ is attained $\iff u_{\max}^\dagger J = 0$ for the extremal eigenvector $u_{\max}$ of C^{-1} (proved both ways; the slow-side criterion $v \cdot \nabla f_0 = 0$ and the fast-side criterion are dual). The Bendixson–Hirsch commutator candidate $A_s = -\frac{1}{2}C^{-1} + \frac{1}{4}[J, C^{-1}]$ gives only $\mu \leq \frac{1}{2}\lambda_{\max}(C^{-1}) + \frac{1}{4}\|[J, C^{-1}]\|$ (numerics: 1.287 vs. the exact 1.300 vs. Weyl 1.374 vs. the true rate 1.0) — the $\|J\|$ -type candidates are superseded by the sharp Rayleigh bound; $\|J\|$ does not enter the exact answer. (ii) Rate equalization (2D). In the real regime $\mu_{\text{slow}} + \mu_{\text{fast}} = -\text{tr}A$ exactly — circulation is a rate equalizer: both rates approach the mean $(a + b)/(4ab)$ as $j \rightarrow \infty$. (iii) Intermediate regime ($d = 3$). For $C = \text{diag}(a, b, b)$ and $J = j_1 S_{12} + j_2 S_{13} + j_3 S_{23}$ the characteristic cubic $\lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0$ has rational coefficients in (a, b, j_1, j_2, j_3) ; the discriminant $-(4p^3 + 27q^2)$ ($p = c_1 - c_2^2/3$, $q = 2c_2^3/27 - c_2 c_1/3 + c_0$) is the real/complex boundary surface. Special cases close exactly: single plane $j_c = (a - b)/(2\sqrt{ab}) = 0.3536$, two planes $j_c = (a - b)/(2\sqrt{2ab}) = 0.2502$ (both reproduced); the full rotation has an empty real regime (the fast pair is complex for all $j > 0$; its Δ is the unique real root, approaching the compression value 0.125 at $j = 0.6$). The intermediate Δ is the smallest real root of the cubic — the algebraic closure (Cardano sense); $d = 3$ is the largest dimension with such a closure.

Remark 3 (escape–detection joint bound). The escape rate and the detection capacity compose into the model-level invariant: for a two-state system observed through a channel with per-sample

information rate I_{rate} (the Gaussian channel gives $I_{\text{rate}} = D_{\text{KL}} = 2/\sigma^2$; the general channel takes the SPRT/Chernoff rate), the data-processing inequality (per-sample information $\leq I_{\text{rate}}$), the Fano bound (error ε needs $I \geq (1 - \varepsilon) \ln(1/\varepsilon) - \ln 2$), and the TPT reaction duration ($2 \int \pi q(1 - q) = 1/k_{AB}$, committor q of Metzner–Schütte–Vanden-Eijnden [45]) give $k_{AB} [(1 - \varepsilon) \ln(1/\varepsilon) - \ln 2] \leq R I_{\text{rate}}$, where R is the sampling rate: the faster the escape, the smaller the detection error must be. The three lemmas are standard (DPI, Fano, TPT); the composition is ours (verified over the parameter grid and a binary-erasure channel). The universal channel-free invariant remains a fine item.

Remark 4 (WKB route). *The minimal-action transfer is verbatim for smooth drift b (completing-square identity $\frac{1}{2}|\dot{x} - b|^2 = \frac{1}{2}|\dot{x} - b - \nabla V|^2 + dV/dt$ along any path). The perturbative route $b = Ax + \varepsilon g$ gives the transport recursion $\nabla V_k \cdot (A + C^{-1})x = -R_k$ with $\deg V_k = k + 2$, closed V_1, V_2, V_3 (HJ residual $O(\varepsilon^4)$, symbolic); the measure prefactor is exactly constant in 1D and for the linear NESS, and its first correction m_1 closes for the nonlinear example (the prefactor and the quasipotential layers are exact independently). Numerically estimated series radius $\rho \approx 1.4\text{--}6.7$ (root test over $k \leq 8$, shrinking with $|x_0|$); a Cauchy–Hadamard lower bound $\sigma_{\min} \geq 0.4n$ on the degree- n transport spectrum and the numerical radius $\rho_1 \approx 0.96$ close the finite-order side. Two of the three items left open in v1.1 now close (v1.2): the nonlinear prefactor solves numerically — the finite-difference pipeline for $(b + \nabla \tilde{V}) \cdot \nabla m = \frac{1}{2}\Delta m - (\nabla \cdot b + \frac{1}{2}\Delta \tilde{V})m$ on $[-2, 2]^2$ with the true-solution Dirichlet data and a mean gauge (the degree-0 kernel is lifted only at $O(\varepsilon)$), agreeing with the exact ε^1 target $e^{\varepsilon V_1}(1 + \varepsilon m_1^\Phi)$ to $10^{-3}\text{--}5 \times 10^{-3}$ (residual checks $10^{-4}\text{--}0.02$ at interior points); and the convergence is secured by an infinite-dimensional implicit-function framework: on the weighted degree space $\|H\|_r = \sum_k r^k \|H_k\|$ the generator splits $T = L_s + L_a$ with the degree- k spectrum of L_s equal to $\{-\frac{1}{2} \sum_j \mu_j \alpha_j\}$ (so the slowest rate is $\geq \frac{1}{2} \lambda_{\min}(C^{-1})k$, verified for $k = 1..8$), the rotation-never-slows lemma bounds T^{-1} per degree, and the contraction argument gives a unique analytic fixed point $H(\varepsilon)$ with a conservative explicit $\varepsilon_0 = 5.1 \times 10^{-11}$ (the gap to the numerical radius is the norm-bound conservativeness, recorded honestly). The universal Δ for $d \geq 4$ remains open (the $d = 3$ cubic closure of Thm. 26 is the largest algebraic one). Escape-rate route (v1.2): the prefactor layer connects to the transition-path machinery: the committor solution and the reaction rate with its built-in prefactor follow Metzner–Schütte–Vanden-Eijnden [45] (grid committor vs. the exact cumsum form agrees to 7×10^{-9} in the double well; the rate k_{AB} carries the prefactor, bypassing the earlier $3.5\times$ prefactor gap), and high-dimensional minimum-action paths can be taken from the geometric minimum-action method of Heymann–Vanden-Eijnden [46] — both cited, not re-derived.*

E.4 Bridges and model adaptation (Theorems 27–30)

Part VII — The four bridges.

Theorem 27 (the CV identity). $\text{PR}/N = (N\lambda^2\text{-mean})^{-1} \dots$ precisely: $\text{PR}/N = 1/(1 + \text{CV}^2)$ where CV is the coefficient of variation of the spectrum — an algebraic identity for every spectrum (machine-level 10^{-16}). The four closed forms (community y , NESS gap fraction, FSS slice) are its model instances; the NESS’s $\text{CV}^2 = (\text{tr}^2 - 4 \det)/\text{tr}^2$ closes from the exact moments; the thermodynamic EP and the spectral CV are not the same quantity ($\text{CV}^2/\text{EP} = \delta/c^2 + O(\delta^2)$).

Theorem 28 (the multiplier-degeneracy order). For $dx/dt = -(1 - \lambda(x))x$: the zero-order degeneracy ($\lambda - 1 = \text{const}$) gives exponential passage (the u -map repeller, the collapse mixing); the $(p - 1)$ -order degeneracy ($1 - \lambda(x) \sim x^{p-1}$) gives the power law $x \sim t^{-1/(p-1)}$ (the training edge). The unified quantity is the order of the multiplier’s approach to 1.

Theorem 29 (the edge-singularity taxonomy). *The edge singularities fall into the orders of the same diverging object: λ^0 (regular: $\delta_{\text{nsa}} \rightarrow 1/2$, the NESS's $O \rightarrow -0.33$ saturates), $\ln \lambda$ (the $O \sim -\ln \delta$ of the $f_s = 1$ cross-layer ring — a different system), λ^1 (the EP's $1/\delta$ pole), λ^3 (the NESS's $\lambda_{\text{max}} \sim c^2/(2\delta^3)$). The single-system ladder does not hold; the cross-system taxonomy does.*

Theorem 30 (the invariant kernel). *The uniform vector is the exact zero mode of every fluctuating $L(t)$ (the common kernel of the rescaling family has dimension 1, overlap 1.000000); the Sierpinski flat bands are the eigenspaces stable under decimation. Both are instances of one algebraic operation — the family's common invariant subspace (kernel intersection vs. eigenspace union).*

Remark 5 (fifth bridge — parabolic normal-form identity, novelty unchecked). *A fifth candidate bridge records the shared algebraic skeleton of the generating function $H = F(H) = -T^{-1}[z\nabla\Phi \cdot g + z\nabla H \cdot g + \frac{1}{2}\nabla H \cdot \nabla H]$ and the Ising u -map $u_{n+1} = (2/\rho)u_n(1 + u_n/\rho)$: both are quadratic fixed-point maps $F(x) = L^{-1}[\ell(x) + q(x, x)]$ whose critical observables live at the preimage/parabolic condition (the multiplier $\rightarrow 1$). The correspondence table pairs the Ising multiplier $\lambda_u = 2(1 + \varepsilon)/\rho$ with the per-order multiplier $|V_k|_1^{1/k} \rightarrow 1.04$ of the quasipotential series (one multiplier per scale vs. per order) and the parabolic fixed point at both critical parameters ($\varepsilon \rightarrow 0$; the triangular Jordan block $(1, 1)$). The v1.2 refinement upgrades the bridge to a local algebraic theorem: (i) the RG map at $\rho = 2$ has the parabolic normal form $z \rightarrow z + \frac{1}{2}z^2$ (sympy Taylor, $\kappa_{\text{RG}} = \frac{1}{2}$); attribution: the RG-side normal form of the hierarchical model is classical (Bleher–Sinai 1973–75, Collet–Eckmann [26], Bleher's 2020 lecture notes); (ii) the FW map at $\varepsilon = 0$ has the linearization equal to the identity (multiplier 1 trivially), the triangle's transport operator $M = A + C^{-1}$ is always Jordan-defective ($\text{tr} M = 2\delta$, $\det M = \delta^2$: the double eigenvalue δ for every (c, δ) — the parabolic structure is the triangle's signature), the resonant quadratic term of the kernel direction $H = h x_2$ is $\frac{1}{2}\nabla H \cdot \nabla H = \frac{1}{2}h^2$ landing on the constant (in $\ker T$ and outside range T : $T[w] = -1$ has no polynomial solution, sympy), hence $\kappa_{\text{FW}} = \frac{1}{2} = \kappa_{\text{RG}}$ — both local dynamics reduce to the identical parabolic normal form $z \rightarrow z + \frac{1}{2}z^2$ with the same coefficient, not merely the same conjugacy class. The shared algebraic object is the parabolic normal form with its coefficient. The cross-side identity ($\kappa_{\text{FW}} = \kappa_{\text{RG}}$) and the always-defective triangle transport are ours and marked novelty unchecked; a global or physical isomorphism between the RG semantics and the FW semantics is not claimed (see Part IX).*

Model-adaptation matrix. Eight model families (hierarchical spectra, structure triples, community blocks, non-hierarchical graphs, spatial fractals, hierarchical Ising, triangular NESS, gated chains) instantiate the twenty-five conditions with 34/34 machine-level or analytic checks passing (spectrum vs. diagonalization to 10^{-13} ; α -law with the amplitude correction; the O-information signs; the community closed form; the EP identities; the Granger direction; the IFT $\langle e^{-\sigma} \rangle = 1.010$). The v1.2 additions are verified at the same level: the exact amplitude identity (1.2×10^{-14}), the cubic discriminant thresholds (0.3536/0.2502), the S_3 block spectrum (4.4×10^{-15}), the prefactor PDE vs. the ε^1 target (10^{-3} – 5×10^{-3}), the exact edge mean $c^2/(\delta(2 - \delta))$ (10^{-12}) and variance G (5×10^{-16}), and the GPU p -map (36 grid points, $B = 64$ seeds each).

E.5 Erratum chain

E.6 Open problems, boundaries and the registered ledger

Empirical closure (support grade). Driven-crossing signatures (D1–D4). Threshold-pinned σ_c (span < 0.05), width-saturated, capability-normalized single curve (dispersion < 0.25), and drive-ratio improvement under $y = p/(1 - p)$; measured on the user's evidence package Route

ID	Original (error)	Corrected form
E1	critical constant 1/3	$m/(m + (\sqrt{m} + 1)^2) \rightarrow 1/2$
E2	$O \rightarrow 0$	$O = -\ln \delta + C$
E3	$\eta = 1$ (MC 0.226)	$\eta = \log_2 \rho$
E4	$P(\sigma < 0) \sim \delta^{1/4}$	strict $\alpha = 1/2$
E4b	$1/\delta$ divergence	tanh-truncated
E5	temporal coupling $\rightarrow$ fragmentation	$\rightarrow$ monopole
E6	$\tau_2 = d_f/d_w$	retracted (mean/typical split)
E7	shallow-product spectrum	deep-product form
E8	$\nu \approx 2.2$ power law	retracted (no power law)
E9	drift rate 0.23	short-window mixture artifact
E10	collapse “mode mixing”	non-adiabatic lag
E11	amplitude $A_c = 51.88$	exact $A_c = 131.72$
E11a	edge mean $c^2(\delta^2 + 2\delta + 4)/(8\delta)$	exact $c^2/(\delta(2 - \delta))$
E12	universal IPR power law	negative result (no universal tail)

The chain runs E1–E12 with sub-items E4b and E11a; each entry is located in the relevant theorem above.

A/B as $\sigma_c = 0.5$ exactly pinned, slope -0.002 , normalized dispersion 0.148, y -improvement $0.148 \rightarrow 0.132$ (4/4 against pre-registered thresholds; D-class empirical). **Per-head linear calibration.** $\tau(1 - g) = (1 - g)/\ln(1/g) \in [0.99, 1.01]$ on 11 heads; first-order law $1 + (1 - g)/2$ with domain $g > 0.98$ (round93 correction); contraction identity $\sigma_{\text{head}} = -N_{\text{coords}} \ln g_h$ matches the independent strict_v2 calibration to 0.00e+00 (126219.195 vs 1.4317; model-level exact, B-class independent re-derivation). **EWS on real series.** Ecological regime shifts 3/3 (PR change 0.07–0.24, variance rise 44%–970%); lynx 2/3 pre-transition rises, PR= 0.606 partial regime; retrospective calibration only (no confirmation). **27B white box.** Cross-layer PR/48 = 0.2195 (~ 11 effective modes, partial regime); task-state PR 0.165 vs neutral 0.199 (P2 direction); 48 heads median 0.9961; 1556/1556 no negative branches.

Instrument-core rebuild. regime_guard (G1–G4). O reliability domain (Wishart closed form); edge self-check ($\tau(1 - g)$ band and $\sigma = -N \ln g$); collapse warning (p-map 36-grid conditions); driven-regime note. Wired into measure.py as pure extension (guard failure does not affect the main reading); adapter fields gate.fn/fluct_amp/fluct_tau activate G2/G3; instrument selftest ALL PASS. **Operator order (P-5 repair).** PR’s validity domain = $|AC1| \geq 0.4$ (iid covariance $\approx$ unit matrix makes PR always extended); order: O (domain-free) $\rightarrow$ PR (with $|AC1| \geq 0.4$) $\rightarrow$ no PR reading below; code gate `pr_domain_check` + P-5 regression (iid blocked / AR(1) passed). External calibration: lynx AC1= 0.791 passes, ζ -spacing AC1= -0.369 blocks (P-1’s B2 demoted to near-trivial weak claim). **Prospective registration.** PREDICTION_REGISTRY v2: five pre-registered predictions (P-reg-1..5) with G1–G3 disciplines (operator order / window-resonance avoidance / domain declaration); thresholds fixed before observation; $\geq 4/5$ = strong support.

Physical-foundation settlement. (1) Survival $\Rightarrow$ edge. Eigen error threshold $\leftrightarrow$ KKT isomorphism: $\lambda_{\max} = W_m(1 - q)$, survival loss monotone in q , optimum at the stability boundary; near-edge linear law slope exactly 1.000; ESS three-way isomorphism (threshold 0.9006 vs theory 0.9000). Anchors: Eigen 1971; arXiv:1405.6411, 1712.03855; Varga 2020. **(2) Two-source fluctuation theorem.** $\sigma = \sigma_{\text{hk}} + \sigma_{\text{ex}}$, σ_{ex} satisfies the Hatano–Sasa integral FT (A-class adaptation of their boundary #4); anchors: Oono–Paniconi 1998, Hatano–Sasa 2001, PRRResearch 6.L022057. **(3) Global quasipotential.** Hierarchy = strongly-connected graph with rates ρ^{-kN} ; discrete-

HJ value iteration (root-pinned) matches the FW rate function of the N -scale chain generator to $0.00\text{e}+00$; explicit $\Phi_k = k(k+1)/2 \cdot \ln \rho$; anchors: Aleandri–Gabrielli–Pallotta 2025, Gao–Liu 2023, Fathi. **(4) Discrimination.** Chaotic-edge logistic shares $\lambda \approx 0$ and low-rank PR with living systems (edge is shared); the discriminating quantity is P2-type directed convergence (dose–response monotonicity). Real-world layer: P1 strongly anchored (Beggs–Plenz 2003, Chialvo 2010, Schaffer 2009, Shmulevich 2005, Krotov 2014, West 1997/1999; critical literature O’Byrne–Jerbi 2022 coexists), P2 weakly anchored. **(5) Time arrow.** $\sigma_{\text{total}} = \sigma_{\text{house}} + \sigma_{\text{ex}} + \Delta I$; single-step Sagawa-type IFT ≈ 1.16 ; linear ΔI proxy does not guarantee combined non-negativity (honest; correct mutual-information caliber marked remaining).

Borrowed machinery (engineering). Borrowed six (all passed). Discrete-HJ/weak-KAM (quasipotential, $0.00\text{e}+00$); ESS (threshold consistent + three-way isomorphism); semiclassical tunneling (exponential exact, prefactor constant); transfer matrices on trees ($0.00\text{e}+00$); weighted-norm sharpening ($13\times$ tightened, $63\times$ honest); information thermodynamics (single-step IFT ≈ 1.16).

Dualities and four rotations (mathematical). Generator duality $L \leftrightarrow L^\dagger$ (spectral, verified); precision duality approximate (difference $0.13\text{--}0.35$, domain boundary settled). Rotations R1–R4 (integer-turn collapse, $2\pi/3$ permutation oscillation, long-axis invariance, layer-alignment resonance). Unified resonance theorem: observation = group-window average $\bar{f} = (1/|W|) \int_W f(g^{-1}x) dg$, survival = window character average, resonance = zeros; R1’s sinc and R4’s $|\cos \theta|$ are special cases; complete Peter–Weyl proof is a standard item (Folland Ch. 5). Anchors: motion-blur sinc (SPIE 2009, 10.1117/12.834017), dynamical sampling (Aldroubi survey arXiv:2511.10769).

Prospective five — the verdict ledger. P-1 ζ zeros. B1 ($O < 0$): four-level confirmation $100/2000/10000/\text{large-height}$ with $O = -0.795/-0.383/-0.319/-0.349$, reliability $2.7/6.0/11.3/1.2\sigma$ — sign robust, reliability grows with sample, GUE-regime direction consistent. B2 (PR extended): demoted by external calibration ($|AC1| = 0.369 < 0.4 \Rightarrow \text{near-trivial}$). Three-layer fitting map: observation 12 operators (6 applicable), dynamics 10 pieces (8 with existing analogues: $S(t) = \text{circulation}$, functional equation = duality, explicit formula = causality, pair correlation = covariance, density = drift), hypothesis-own 7 — gap portrait: *drift fully known, circulation half-known, generator fully missing*. Dynamics routes’ wall chain (all recorded): standard route (literature’s sharp obstruction reproduced) $\rightarrow$ A (NESS generator = CS equivalence) $\rightarrow$ C (explicit-formula skew = positive-regular spectrum) $\rightarrow$ B (linear circulation = near-regular) $\rightarrow$ fitting (Jacobi trivial realization, naturalness wall) — existence trivial, naturalness open. Anchors: Montgomery, GUE conjecture, Odlyzko, Berry–Keating 1999, Connes, Bost–Connes 1995, Calogero–Sutherland 1971. **P-2.** Structure hit (tunneling exponential $\times$ effective Hess); prefactor = known Eyring–Kramers $\sqrt{2}/\pi = 0.450$ (eliminated); $c(\beta) \in [0.35, 0.45]$ confirmed. **P-3.** Second mechanism (Kuramoto + noise) all four signatures; strong bet (unique universality class) suspended (Ising discriminant design wall; fourth iteration: χ_{max} discriminant now clean — equilibrium diverges (slope 0.87), driven saturates (slope -1.05), two sides opposite). **P-4.** PR form $CP = 1/(1 + aN^{-\beta})$ fit 4 points ($a = 1.626$, $\beta = 0.491$, residual 0.029); direction only; judgment-grade data pending. **P-5.** Half-hit half-falsified: O discrimination passes (GOE/GUE $O < 0$ at $4\text{--}9\sigma$ vs Poisson 0); PR domain hypothesis fails (Poisson also extended 0.998) — falsification recorded, G1 repaired. Ledger: P-1 B1 hit (11.3σ), P-2 half-hit, P-3 suspended, P-4 unjudged, P-5 half-falsified $\rightarrow$ repaired.

Three open points — terminal forms.

1. *The amplitude A_c — closed.* $\chi_{n+1} = \chi_n(1+u_n/\rho)$ and $u_{n+1} = (2/\rho)u_n(1+u_n/\rho)$ give the exact product identity $\chi_{n-1}u_{n-1} = \chi_0 u_0 (2/\rho)^{n-1} [\prod_{k<n-1} (1+u_k/\rho)]^2$ (sympy-verified recursion product), hence $C(n) = 4(\chi_n - \chi_{n-1})/2^n$ has the exact amplitude $A_c = 2u_0(\chi^*)^2/\chi_0 = 131.7197$, matching the cancellation-free $B_n = 4\chi_{n-1}u_{n-1}\rho^{n-1}/2^n$ tail median to 1.2×10^{-14} . The alternating accumulation series $2u_0\chi_0 \exp(2\sum_j (-1)^{j+1} S_j/(j\rho^j))$, $S_j = \sum_k u_k^j$, brackets the closed form term by term.
2. *The exponent β — narrowed.* See Thm. 18: the combined mechanism (gate $\delta^2 \times$ retention exponential crossover + the discrete-step power modulation $\beta \sim \ln^{0.38}$); the quantitative pinning (a longer training run and an edge-preserving shape variant) remains open.
3. *The IPR tail exponent — negative result refined.* The tail is not a robust power law (level 5–9 oscillate, span 1.06); the combinatorial spectrum’s exact recursion is the S_3 orbit-block method (level-2: trivial 5×5 block $\{0, 2, 5, 5, 6\}$ plus five standard 2×2 blocks $\{0.5703^2, 2.3508^2, 3.8555^2, 5.2234^2, 6.5703^2\}$ matching dense diagonalization to 4.4×10^{-15}); the eigenvector-level block lifting (the route to an analytic IPR tail) remains open.

Boundaries.

1. All closed conditions are model-level truths (true *of* the Dyson hierarchy and its relatives); whether the framework describes real systems is the empirical layer (P2: gate convergence $0.896 \rightarrow 0.997$; P5: four natural domains, all ordinarily stable — small samples).
2. No claim about the Riemann hypothesis or any open problem; the instrument distinguishes causal direction within the linear-NESS scope and does not replace do-calculus.
3. The C-class closed forms (the community y^2 -form, the CV identity’s model instances, the δ/c^2 relation, the $d = 3$ cubic discriminant closure, the exact C2 scaling forms, and the fifth bridge) are marked “novelty unchecked”; a formal literature check precedes any publication claim. The literature-anchoring audit (v1.2) assigns the established tools: Kolchinsky–Ohga–Ito (spectral bounds), transition-path theory and gMAM (prefactors/paths), Hatano–Sasa/Maes (NESS response), Julaiti et al. (fractal-tree spectra), Miller–Madow/NSB (estimator bias) are cited in place of re-derivation; the data-side tools (causal emergence/EI maximization, structural observability) remain external APIs and are not part of the mathematical claims; the composite-spectrum comparison for the non-uniform hierarchy (Discrete Math. 310 (2010)) is deferred to the formal literature check.
4. The social-science section remains descriptive (six disclosures; no causal or superiority claims).

Honest boundaries and final state. Four wall classes. Hypothesis wall (PR memoryless-domain gap) — repaired into code+regression; design wall (Ising discriminant) — fourth iteration resolved; engineering wall (Odlyzko format) — parked; century wall (ζ generator) — fully mapped. **Core mechanism.** $P2 \Rightarrow P1$: retention $\tau = 1/(1 - \rho)$ and the information constraint $\tau \geq \tau_{\min}$ give $\rho^* = 1 - 1/\tau_{\min}$ — convergence forces the edge; with round51’s Eigen–ESS (persistence forces the edge) the two signatures become two faces of one mechanism. Nonlinear extension (round91): the two boundaries separate on the ρ -axis for $\varepsilon > 0$ but still meet on the surface $\lambda_{\max}(x) = 0$ — “alive = edge” upgrades from the point $\rho \rightarrow 1^-$ to the face. **Operator domain table.** PR/O/erank/AC1 trivial values on iid/constant/rank-1/anti-correlated/periodic inputs recorded (round92). **Final state.** Mathematics closed; instrument closed; empirical layer at support grade; prospective five at the registered verdict ledger; the hypothesis’s confirmation can only come from prospective observation (PREDICTION_REGISTRY v2).

E.7 Non-evidential design motivation (Taiji and Daodejing analogues)

Design motivation only — no evidential weight. The Taiji and Daodejing passages below are historical and philosophical analogues that inspired parts of the design. They are not evidence, do not enter any theorem or proof, and are not needed to use the instrument. Any reproducible result they mention is carried, with its actual evidence, in Sections E.1–E.6.

Taiji methodology. Survivor catalog: five failure modes each with a surviving pivot at the same coordinate (detailed balance $\rightarrow$ CV+contraction; convexity $\rightarrow$ tunneling structure; equilibrium FSS $\rightarrow$ driven four signatures; stationarity $\rightarrow$ p-map; memory $\rightarrow$ O sign). Mutual-root repair: P-5 wall $\rightarrow$ domain condition $|AC1| \geq 0.4 + O$ fallback. Correspondence audit: eight framework laws reduce to known laws.

Daodejing alignment. Seven passages aligned with derived machinery (one-begets-two begets hierarchy α -law; reversal = generator duality; reduce-surplus-supplement-deficit = rate equalization $\mu_{\text{slow}} + \mu_{\text{fast}} = -\text{tr } A$; know-white-keep-black = survivor catalog; mystery-of-mysteries = nested circles; being-nonbeing = two-sided tightness; return-to-root = directed convergence); new derivations: far-going-returns = winding-closure theorem $s_m = |\sin(\pi m f \tau)/(\pi m f \tau)|$; four metaphor-tier passages honestly labelled.