

Supplemental Material:

Spatial delocalization of cold oscillators in hollow-core fibers

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This supplemental material details: (I) detailed covariance matrix based analysis of the particle's motion under dissipative thermal noise, (II) theoretical and numerical analysis of the motional squeezing in the polarization-adaptive methods (frequency jumps), (III) analytical and numerical investigations on the particle's motional dynamics in the dark harmonic potential, (IV) extension of the motional analysis for the anharmonic optical potential, and (V) analytical comparison of the recoil heating rates during polarization-adaptive squeezing and dark-fringe trap based squeezing.

I. STOCHASTIC THERMAL EVOLUTION

We consider a spherical, isotropic dielectric particle of mass m confined in a standing-wave optical trap formed by two counterpropagating laser beams. For simplicity, the beams here are assumed to have plane-wave like transverse profiles. The exact treatment of the transverse fiber mode profile is discussed in the next subsection [IA](#). The beams have identical frequency ω_ℓ and equal but opposite wave vectors $\pm k_\ell$ along z -direction. The positive-frequency part of the electric field is a coherent superposition of $\hat{e}_L \mathcal{E}_0 e^{ik_\ell z} e^{-i\omega_\ell t}$ and $\hat{e}_R \mathcal{E}_0 e^{-ik_\ell z} e^{-i\omega_\ell t} e^{-2i\phi}$ counterpropagating waves, where $\mathcal{E}_0$ denotes the common field amplitude, $\phi(t)$ is the relative phase between the optical fields, $\hat{e}_L$ and $\hat{e}_R$ are the polarization unit vectors of the left- and right-propagating beams, respectively. To eliminate the rapid optical oscillation, we transform to a frame rotating at the optical frequency ω_ℓ . In this rotating frame, the total electric field is given by [\[1\]](#)

$$E(z, t) = \mathcal{E}_0 \left[\hat{e}_L e^{ik_\ell z} + \hat{e}_R e^{-ik_\ell z} e^{-2i\phi(t)} \right]. \quad (\text{S1})$$

This field gives rise to a position- and time-dependent potential energy $U(z, t) = -\alpha' |E(z, t)|^2 / 4$, where $\alpha' \equiv \Re(\alpha)$ is the real part of the particle's complex polarizability. For small deviations from the trap minimum, the optical force is well approximated by a dipole gradient force acting on the particle's center of mass. Assuming the particle remains localized at the potential minimum and ignoring the constant shift, this force takes the form

$$F(z) = -\frac{\partial U(z, t)}{\partial z} = \frac{\alpha'}{4} \frac{\partial |E(z, t)|^2}{\partial z} \approx -\kappa_{\text{trap}} z \cos[2\phi(t)], \quad (\text{S2})$$

$$\text{with axial stiffness } \kappa_{\text{trap}} = 2k_\ell^2 \alpha' |\mathcal{E}_0|^2 \cos(\theta_p) \quad (\text{S3})$$

where θ_p is the relative polarization angle defined by $\hat{e}_L \cdot \hat{e}_R = \cos(\theta_p)$. Recognizing that $\kappa_{\text{trap}} = \kappa_o \cos(\theta_p)$ with $\kappa_o = 2k_\ell^2 \alpha' |\mathcal{E}_0|^2 = m\Omega_o^2$, the axial resonant frequency becomes

$$\Omega = \Omega_o \sqrt{\cos(\theta_p)}. \quad (\text{S4})$$

Crucially, this reveals that the axial frequency Ω can be tuned directly by adjusting the relative polarization angle θ_p between the two beams. Unlike power-based tuning, which requires changing the laser intensity and may introduce recoil heating issues, polarization control offers a fast, in-situ mechanism for modifying the trapping potential without altering the material properties m and α' , or the overall laser power. Furthermore, the relative phase of the counterpropagating fields can shift the standing-wave pattern, thereby relocating the particle from bright to dark fringes. Accordingly, at $\phi = 0$ the particle experiences a restoring force in the harmonic potential at the bright fringe, while at $\phi = \pi/2$ it experiences a repulsive force in the inverted trap at the dark fringe. For polarization-adaptive delocalization (see Main Text) θ_p is changed from 0° to 88.3° , therefore reducing the axial frequency from 55 kHz to 10.2 kHz.

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A. On-axis intensity and radial stiffness

When the relative polarization orientation between the trapping beams θ_p is switched from 0° to 88.3° the radial frequency also shifts to 0.92 kHz from 1.3 kHz. Here we explain the reasoning behind it. To account for the transverse fiber mode profile, we use [2]

$$E(r) = AJ_0\left(\frac{2u_{01}r}{d_{\text{core}}}\right), \quad (\text{S5})$$

where A is the maximum electric field amplitude at the center of the core, d_{core} is the core diameter and $u_{01} = 2.4048$ is the first zero of J_0 , such that $J_0(u_{01}) = 0$. Including this mode profile in Eq. (S1), the intensity is

$$|E(r, z, t)|^2 = 2|A|^2 J_0^2\left(\frac{2u_{01}r}{d_{\text{core}}}\right) \left[1 + \cos(\theta_p) \cos[2k_\ell z + 2\phi(t)]\right]. \quad (\text{S6})$$

At a bright fringe ($z = 0$, $\phi = 0$) and on axis ($r = 0$),

$$|E|_{\text{bright}}^2(0) = 2|A|^2[1 + \cos(\theta_p)], \quad (\text{S7})$$

so that

$$\frac{|E|_{\text{bright}}^2(90^\circ)}{|E|_{\text{bright}}^2(0^\circ)} = \frac{1}{2}. \quad (\text{S8})$$

Near the axis,

$$J_0^2\left(\frac{2u_{01}r}{d_{\text{core}}}\right) \simeq 1 - \frac{2u_{01}^2 r^2}{d_{\text{core}}^2}. \quad (\text{S9})$$

For a dispersive optical potential, $U(r) \propto -|E(r)|^2$, giving radial stiffness as

$$\kappa_r \propto \frac{2u_{01}^2}{d_{\text{core}}^2} |A|^2 [1 + \cos(\theta_p)]. \quad (\text{S10})$$

Thus, assuming the transverse mode profile and each trapping beam intensity are unchanged with polarization,

$$\frac{\kappa_r(\theta_p)}{\kappa_r(0)} = \frac{1 + \cos(\theta_p)}{2}, \quad (\text{S11})$$

$$\frac{\Omega_r(\theta_p)}{\Omega_r(0)} = \sqrt{\frac{1 + \cos(\theta_p)}{2}}. \quad (\text{S12})$$

For $\theta_p = 88.3^\circ$, Eq. (S12) predicts

$$\frac{\Omega_r(88.3^\circ)}{\Omega_r(0)} \simeq 0.711, \quad (\text{S13})$$

in agreement with the measured ratio

$$\frac{0.92}{1.3} \simeq 0.708. \quad (\text{S14})$$

B. Equations of motion

To capture the stochastic evolution of the particle's center-of-mass motion, we employ the theoretical framework detailed in Refs. [1, 3]. By setting $\phi = 0$, the dynamics is formulated as a system of coupled stochastic difference equations:

$$dq = \Omega p dt, \quad (\text{S15a})$$

$$dp = -\gamma p dt - \Omega q dt + \sqrt{2\gamma n_{\text{th}}} dW^{\text{th}}(t). \quad (\text{S15b})$$

The variables q and p appearing above denote the dimensionless position and momentum quadratures, defined respectively as $q = z/q_{\text{zpm}}$ and $p = p_z/p_{\text{zpm}}$, where $q_{\text{zpm}} = \sqrt{\hbar/(m\Omega)}$ and $p_{\text{zpm}} = \sqrt{m\hbar\Omega}$ are the zero-point position and momentum scales. The stochastic forcing term arises from thermal collisions with surrounding gas molecules, effectively coupling the oscillator to a reservoir at temperature T_{th} with mean occupancy n_{th} . This interaction manifests as a pressure-dependent friction coefficient γ and a Gaussian white-noise process represented by the Wiener increments $dW^{\text{th}}(t)$, which obey $\langle dW^{\text{th}}(t) \rangle = 0$ and $\langle (dW^{\text{th}}(t))^2 \rangle = dt$.

C. Covariance matrix analysis

To obtain an analytical treatment of the system, we discretize the interval $[0, t]$ into n equal segments of length $dt = t/n$. This procedure transforms the stochastic differential equations (Eqs. (S15)) into an equivalent set of recurrence relations, which explicitly account for thermal fluctuations at each discrete timestep. In the underdamped limit ($\gamma \ll \Omega$), recursive back-substitution in conjunction with a long-time asymptotic analysis yields the steady-state solutions for the position and momentum quadratures. These solutions can be expressed compactly as

$$\mathbf{u}(t) = e^{-\gamma\tau/2} \mathcal{R}(\Omega, \tau) \mathbf{u}(t_0) + \mathbf{N}_\Omega(t, t_0), \quad (\text{S16})$$

where $\mathbf{u}(t) = [q(t), p(t)]^T$ is the mechanical quadrature vector. Using the particle's axial eigenfrequency Ω as defined in the Eq. (S4), the rotation matrix in Eq. (S16) is given by

$$\mathcal{R}(\Omega\tau) = \begin{bmatrix} \cos(\Omega\tau) & \sin(\Omega\tau) \\ -\sin(\Omega\tau) & \cos(\Omega\tau) \end{bmatrix}, \quad (\text{S17})$$

with $\tau = t - t_0$. The diffusion vector in Eq. (S16) is defined as $\mathbf{N}_\Omega(t, t_0) = [\mathcal{N}_q^\Omega(t, t_0), \mathcal{N}_p^\Omega(t, t_0)]^T$, whose components are

$$\mathcal{N}_q^\Omega(t, t_0) = -\frac{i}{2} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (\lambda_-^j - \lambda_+^j) dW_{n-j-1}^{\text{th}}, \quad (\text{S18})$$

$$\mathcal{N}_p^\Omega(t, t_0) = -\frac{i}{4\Omega} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (\gamma_+ \lambda_+^j - \gamma_- \lambda_-^j) dW_{n-j-1}^{\text{th}}, \quad (\text{S19})$$

where $\lambda_\pm = 1 - \gamma_\pm dt/2$ are the eigenvalues of the evolution matrix, with $\gamma_\pm = \gamma \pm \bar{\Omega}$ and $\bar{\Omega} = \sqrt{\gamma^2 - 4\Omega^2}$.

In the presence of thermal noise Wiener increments, the above equations yield the phase-space distribution, whose covariance matrix is given by

$$V(t) = e^{-\gamma\tau} \mathcal{R}(\Omega\tau) V(0) \mathcal{R}^T(\Omega\tau) + \mathcal{Q}(t, t_0). \quad (\text{S20})$$

The first term arises from the deterministic dynamics and accounts for the rotation and relaxation of the initial distribution, characterized by the covariance

$$V(0) = \langle \mathbf{u}(0) \mathbf{u}^T(0) \rangle = n_{\text{th}} \mathcal{I}, \quad (\text{S21})$$

where $\mathcal{I}$ is the 2×2 identity matrix. The second term,

$$\mathcal{Q}(t, t_0) = \langle \mathbf{N}_\Omega(t, t_0) \mathbf{N}_\Omega^T(t, t_0) \rangle = \begin{bmatrix} \langle (\mathcal{N}_q^\Omega(t, t_0))^2 \rangle & \langle \mathcal{N}_q^\Omega(t, t_0) \mathcal{N}_p^\Omega(t, t_0) \rangle \\ \langle \mathcal{N}_q^\Omega(t, t_0) \mathcal{N}_p^\Omega(t, t_0) \rangle & \langle (\mathcal{N}_p^\Omega(t, t_0))^2 \rangle \end{bmatrix}, \quad (\text{S22})$$

represents the contribution of thermal noise diffusion to the covariance evolution. Here, we have invoked the symmetry of the covariance matrix, which implies that the cross-correlation terms satisfy $\langle \mathcal{N}_q^\Omega(t, t_0) \mathcal{N}_p^\Omega(t, t_0) \rangle = \langle \mathcal{N}_p^\Omega(t, t_0) \mathcal{N}_q^\Omega(t, t_0) \rangle$. Using the Wiener increment property $\langle dW_{n-j-1}^{\text{th}} dW_{n-j'-1}^{\text{th}} \rangle = \delta_{jj'} dt$ and evaluating the resulting geometric sums, the diffusion contribution to the covariance matrix in the underdamped regime becomes

$$\mathcal{Q}(t, t_0) = n_{\text{th}} (1 - e^{-\gamma\tau}) \mathcal{I}. \quad (\text{S23})$$

Finally, using the rotation matrix [Eq. (S17)] and Eq. (S23), the simplified form of the covariance matrix is

$$V(t) = n_{\text{th}} \mathcal{I}. \quad (\text{S24})$$

II. ANALYSIS OF SQUEEZING VIA SUDDEN FREQUENCY QUENCH

We now analyze the mechanical squeezing generated by rapid modulation of the trapping potential. The analysis is formulated in terms of the covariance matrix of the motional quadratures, allowing the evolution of the principal variances and the resulting squeezing to be determined directly. We first consider a single sudden frequency jump, which establishes the basic mechanism of transient squeezing. The frequency jump is experimentally performed by rapidly rotating the relative polarization orientation (θ_p) between the trapping beams from 0° to 88.3° [see Eq. (S4)]. We then extend the analysis to a finite square pulse, where the frequency is switched for a controlled duration before being restored to its initial value.

A. Sudden frequency jump

We first consider the elementary case of a sudden change in the relative polarization of the trapping beams leading to sudden quench in the mechanical frequency [4]. Such rapid frequency jumps provide one of the simplest and most efficient methods for generating mechanical squeezing. The abrupt change in the oscillator frequency modifies the relative scaling of the position and momentum quadratures, resulting in a redistribution of fluctuations between the two conjugate variables while preserving the total phase-space area. Immediately after the quench, one quadrature exhibits reduced fluctuations below its equilibrium value, whereas the conjugate quadrature experiences an increased variance, producing a squeezed state. In the absence of dissipation, the squeezed state undergoes periodic rotation in phase space at the new oscillation frequency while maintaining constant principal variances. In realistic systems, however, damping and thermal noise gradually restore thermal equilibrium, causing the squeezing to decay over time.

To visualize mechanical squeezing via a rapid frequency jump, we model the process as an instantaneous transition from Ω_0 to Ω_1 at the switching time t_1 , where $\Omega_0 \neq \Omega_1$. The resulting time-dependent frequency is thus given by the piecewise function

$$\Omega(t) = \begin{cases} \Omega_0, & 0 \leq t < t_1, \\ \Omega_1, & t \geq t_1, \end{cases}, \quad (\text{S25})$$

In Fig. S1, we depict a specific case of such a jump at a switching time $t = t_1$.

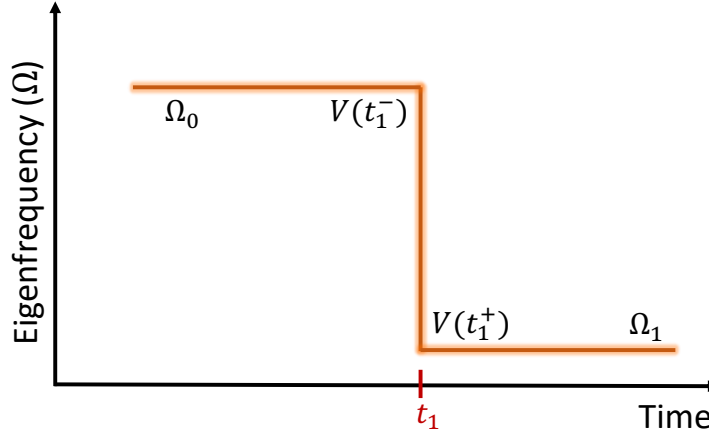


FIG. S1. Instantaneous axial eigenfrequency Ω as a function of time. The oscillator operates at its initial frequency Ω_0 for $0 \leq t < t_1$, and at the switching instant t_1 it undergoes an abrupt step to Ω_1 , remaining constant thereafter. The plot illustrates the idealized step discontinuity described by the piecewise frequency model. $V(t_1^-)$ and $V(t_1^+)$ are the covariance matrices of the particle motion just before and after the frequency jump, respectively.

1. Just before jump

To establish the covariance matrix immediately before the frequency quench, we evaluate Eq. (S20) for the pre-quench evolution $0 \leq t < t_1$. At $t = t_1^-$, the covariance matrix is

$$V(t_1^-) = e^{-\gamma t_1} \mathcal{R}(\Omega_0 t_1) V(0) \mathcal{R}^T(\Omega_0 t_1) + \mathcal{Q}(t_1, 0). \quad (\text{S26})$$

For an initially thermal state, $V(0) = n_{\text{th}}^{\Omega_0} \mathcal{I}$, where $n_{\text{th}}^{\Omega_0} = k_B T_{\text{th}} / \hbar \Omega_0$ is the thermal occupation corresponding to the initial mechanical frequency Ω_0 . Substituting the rotation matrix from Eq. (S17) and the corresponding diffusion contribution from Eq. (S23), the dissipative evolution preserves the thermal covariance, yielding

$$V(t_1^-) = n_{\text{th}}^{\Omega_0} \mathcal{I}. \quad (\text{S27})$$

Thus, immediately before the frequency jump, the dimensionless position and momentum quadratures have equal variances and zero cross-correlation. The motional state is therefore isotropic, corresponding to a circular phase-space distribution. This provides a well-defined thermal initial state for the subsequent nonadiabatic evolution following the frequency quench.

Dimensional covariance matrix before the frequency jump: To relate the dimensionless covariance matrix to the experimentally relevant position and momentum fluctuations, we introduce the dimensional phase-space vector

$$\mathbf{u}_z(t) = \begin{bmatrix} z(t) \\ p_z(t) \end{bmatrix} = \mathcal{D}(\Omega)\mathbf{u}(t) , \quad (\text{S28})$$

where $\mathbf{u}(t) = [q(t), p(t)]^T$ is the mechanical dimensionless quadrature vector and the scaling matrix is

$$\mathcal{D}(\Omega) = \begin{bmatrix} \sqrt{\frac{\hbar}{m\Omega}} & 0 \\ 0 & \sqrt{m\hbar\Omega} \end{bmatrix} . \quad (\text{S29})$$

Evidently, above scaling matrix is symmetric i.e. $\mathcal{D}^T(\Omega) = \mathcal{D}(\Omega)$. The corresponding inverse transformation is

$$\mathbf{u}(t) = \mathcal{D}^{-1}(\Omega)\mathbf{u}_z(t) , \quad (\text{S30})$$

with

$$\mathcal{D}^{-1}(\Omega) = \begin{bmatrix} \sqrt{\frac{m\Omega}{\hbar}} & 0 \\ 0 & \frac{1}{\sqrt{m\hbar\Omega}} \end{bmatrix} . \quad (\text{S31})$$

Under this transformation, the covariance matrix in dimensional variables is

$$V_z(t) = \langle \mathbf{u}_z(t)\mathbf{u}_z^T(t) \rangle = \mathcal{D}(\Omega)V(t)\mathcal{D}^T(\Omega) . \quad (\text{S32})$$

Immediately before the frequency jump, the dimensionless covariance matrix is given by Eq. (S27). Thus, evaluating the transformation at the pre-quench frequency Ω_0 gives

$$V_z(t_1^-) = \mathcal{D}(\Omega_0)V(t_1^-)\mathcal{D}^T(\Omega_0) = n_{\text{th}}^{\Omega_0} \begin{bmatrix} \frac{\hbar}{m\Omega_0} & 0 \\ 0 & m\hbar\Omega_0 \end{bmatrix} . \quad (\text{S33})$$

Thus, the isotropic thermal state in the dimensionless phase space corresponds to the expected thermal position and momentum variances in physical units, with their respective scales set by the mechanical frequency Ω_0 .

2. At the frequency jump

At $t = t_1$, the trapping frequency is suddenly quenched from Ω_0 to Ω_1 . Because the quench is assumed to be instantaneous, the dimensional position and momentum remain unchanged during the switching process. Consequently, the dimensional phase-space vector and its covariance matrix are continuous across the jump,

$$\mathbf{u}_z(t_1^+) = \mathbf{u}_z(t_1^-), \quad V_z(t_1^+) = V_z(t_1^-) . \quad (\text{S34})$$

Although the physical position and momentum remain unchanged, their dimensionless representations change because their natural scaling depends on the mechanical frequency. This change in scaling generates the squeezing transformation associated with the frequency quench.

3. Just after the frequency jump

Using the dimensional-to-dimensionless transformation at the new frequency Ω_1 , together with the continuity of $\mathbf{u}_z$, the post-quench dimensionless quadratures are related to their pre-quench values by

$$\mathbf{u}(t_1^+) = \mathcal{D}^{-1}(\Omega_1)\mathcal{D}(\Omega_0)\mathbf{u}(t_1^-) = \mathcal{S}_{10}\mathbf{u}(t_1^-) , \quad (\text{S35})$$

where

$$\mathcal{S}_{10} = \begin{bmatrix} \sqrt{\frac{\Omega_1}{\Omega_0}} & 0 \\ 0 & \sqrt{\frac{\Omega_0}{\Omega_1}} \end{bmatrix}. \quad (\text{S36})$$

This diagonal transformation stretches one dimensionless quadrature while compressing its conjugate. Introducing the squeezing parameter

$$r = \frac{1}{2} \ln \left(\frac{\Omega_1}{\Omega_0} \right), \quad (\text{S37})$$

the transformation takes the standard squeezing form

$$\mathcal{S}_{10} = \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix}. \quad (\text{S38})$$

Thus,

$$q(t_1^+) = e^r q(t_1^-), \quad p(t_1^+) = e^{-r} p(t_1^-). \quad (\text{S39})$$

The corresponding covariance matrix immediately after the jump is

$$V(t_1^+) = \mathcal{S}_{10} V(t_1^-) \mathcal{S}_{10}^T = n_{\text{th}}^{\Omega_0} \begin{bmatrix} e^{2r} & 0 \\ 0 & e^{-2r} \end{bmatrix}. \quad (\text{S40})$$

Thus, the sudden frequency quench transforms the initially isotropic thermal covariance into an anisotropic Gaussian covariance, with one quadrature amplified and its conjugate compressed. This provides the initial squeezed state for the subsequent evolution at the new mechanical frequency.

4. At a later time after the jump

For times $t > t_1$, the covariance matrix evolves under the post-quench mechanical frequency Ω_1 . Defining the elapsed time after the jump as $\tau_1 = t - t_1$, we have

$$V(t) = \langle \mathbf{u}(t) \mathbf{u}^T(t) \rangle = e^{-\gamma \tau_1} \mathcal{R}(\Omega_1 \tau_1) V(t_1^+) \mathcal{R}^T(\Omega_1 \tau_1) + \mathcal{Q}_2(t, t_1), \quad (\text{S41})$$

where $\mathcal{Q}_2(t, t_1) = n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma \tau_1}) \mathcal{I}$. Expressing the covariance matrix as

$$V(t) = \begin{bmatrix} V_{qq}(t) & V_{qp}(t) \\ V_{pq}(t) & V_{pp}(t) \end{bmatrix}. \quad (\text{S42})$$

Here, the diagonal elements, $V_{qq}(t)$ and $V_{pp}(t)$, describe the variances of the position and momentum quadratures, while the off-diagonal elements $V_{qp}(t) = V_{pq}(t)$ quantify the correlation between them. Substituting the covariance matrix immediately after the jump and evaluating the matrix products gives the covariance matrix elements for $t > t_1$

$$V_{qq}(t) \Big|_{t \geq t_1} = e^{-\gamma \tau_1} n_{\text{th}}^{\Omega_0} [\cosh(2r) + \cos(2\Omega_1 \tau_1) \sinh(2r)] + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma \tau_1}), \quad (\text{S43})$$

$$V_{pp}(t) \Big|_{t \geq t_1} = e^{-\gamma \tau_1} n_{\text{th}}^{\Omega_0} [\cosh(2r) - \cos(2\Omega_1 \tau_1) \sinh(2r)] + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma \tau_1}), \quad (\text{S44})$$

$$V_{qp}(t) \Big|_{t \geq t_1} = V_{pq}(t) \Big|_{t \geq t_1} = -e^{-\gamma \tau_1} n_{\text{th}}^{\Omega_0} \sin(2\Omega_1 \tau_1) \sinh(2r). \quad (\text{S45})$$

Eqs. (S43)-(S45) provide the complete analytical time evolution of the covariance matrix following the sudden frequency jump. Since the phase-space distribution remains Gaussian throughout the evolution, the covariance matrix completely characterizes its shape and orientation. The main features of these solutions are summarized below.

1. **Evolution of the quadrature variances:** The diagonal covariance elements describe the fluctuations of the position and momentum quadratures. Immediately after the sudden frequency jump, the initially isotropic thermal distribution is transformed into an anisotropic state, resulting in unequal fluctuations along the two quadratures. This anisotropy is the direct signature of mechanical squeezing generated by the sudden change in the oscillator frequency.
2. **Periodic exchange of quadrature fluctuations:** As evident from Eqs. (S43) and (S44), the position and momentum variances oscillate at twice the post-jump oscillator frequency with opposite phase, indicating a periodic exchange of fluctuations between the two quadratures. The oscillation amplitude is governed by the squeezing parameter r , while the exponential factor describes the gradual suppression of these oscillations due to damping.
3. **Emergence of position–momentum correlation:** Unlike the equilibrium thermal state, where the covariance matrix is diagonal, the sudden frequency jump generates a finite off-diagonal covariance, $V_{qp} = V_{pq}$. This correlation arises because the squeezed phase-space distribution is generally rotated with respect to the position and momentum axes, causing the two quadratures to become statistically correlated.
4. **Rotation of the squeezing axis:** As shown by Eq. (S45), the position–momentum correlation oscillates sinusoidally at twice the post-jump oscillator frequency while decaying exponentially with the damping rate. Physically, the non-zero covariance indicates that the principal axes of the covariance ellipse are no longer aligned with the coordinate axes. Instead, the squeezing axis continuously rotates in phase space until the correlation is gradually erased by dissipation.
5. **Relaxation to thermal equilibrium:** In the long-time limit ($\tau_1 \rightarrow \infty$), the exponentially decaying terms vanish, yielding

$$V_{qq} = V_{pp} = n_{\text{th}}^{\Omega_1}, \quad V_{qp} = 0. \quad (\text{S46})$$

Consequently, the covariance matrix becomes diagonal and isotropic, corresponding to the thermal equilibrium state associated with the final oscillator frequency Ω_1 . All transient squeezing, anisotropy, and quadrature correlations generated by the sudden frequency jump are therefore completely removed by the interaction with the thermal environment.

5. Squeezing characterization

To quantify the anisotropy generated by the sudden frequency jump, we diagonalize the covariance matrix. For the covariance matrix $V(t)$, the two principal variances are

$$\Lambda_{\text{max},\text{min}}(t) = \frac{V_{qq}(t) + V_{pp}(t)}{2} \pm \frac{1}{2} \sqrt{[V_{qq}(t) - V_{pp}(t)]^2 + 4V_{qp}^2(t)}. \quad (\text{S47})$$

Here, Λ_{max} and Λ_{min} denote the maximum and minimum variances, respectively, along the principal axes of the phase-space covariance ellipse, as shown schematically in Fig. S2. Substitution of the covariance elements derived above gives

$$\Lambda_{\text{max}}(t) = e^{-\gamma\tau_1} n_{\text{th}}^{\Omega_0} e^{2|r|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\tau_1}), \quad (\text{S48})$$

$$\Lambda_{\text{min}}(t) = e^{-\gamma\tau_1} n_{\text{th}}^{\Omega_0} e^{-2|r|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\tau_1}). \quad (\text{S49})$$

The two eigenvalues are the principal variances of the Gaussian phase-space distribution, with corresponding covariance-ellipse semi-axis lengths proportional to $\sqrt{\Lambda_{\text{min}}}$ and $\sqrt{\Lambda_{\text{max}}}$. The associated eigenvectors determine the orientation of these principal axes. The coherent contribution retains the anisotropy generated by the frequency jump, while damping attenuates this contribution and thermal diffusion drives both principal variances toward the same equilibrium value.

The normalized eigenvectors are

$$\mathbf{v}_{\text{max}} = \begin{bmatrix} \cos(\Omega_1\tau_1) \\ -\sin(\Omega_1\tau_1) \end{bmatrix}, \quad \mathbf{v}_{\text{min}} = \begin{bmatrix} \sin(\Omega_1\tau_1) \\ \cos(\Omega_1\tau_1) \end{bmatrix}. \quad (\text{S50})$$

Thus, the eigenvector associated with Λ_{min} defines the instantaneous squeezed quadrature, whereas that associated with Λ_{max} defines the anti-squeezed quadrature. Their explicit dependence on $\Omega_1\tau_1$ shows that the principal axes

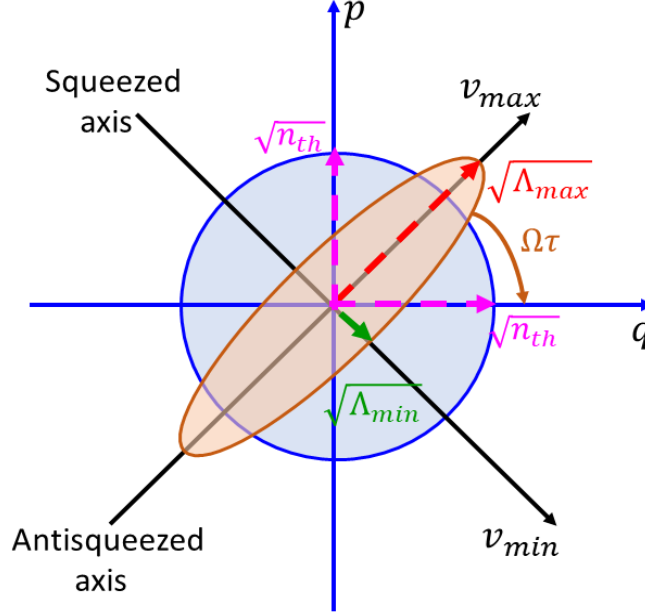


FIG. S2. **Phase-space schematic under a single frequency quench.** The schematic illustrates the transformation of the motional state in phase space. Initially, the oscillator is in an isotropic thermal state, represented by a circle of radius $\sqrt{n_{\text{th}}}$, where n_{th} is the mean thermal occupancy. An instantaneous frequency quench at t_1 deforms this circle into a squeezed ellipse with principal axes along the squeezed and antisqueezed quadratures, having variances Λ_{min} and Λ_{max} , respectively. The semi-minor and semi-major axis lengths are $\sqrt{\Lambda_{\text{min}}}$ and $\sqrt{\Lambda_{\text{max}}}$. Subsequently, the ellipse rotates clockwise in phase space at the trap frequency Ω , as described by the harmonic rotation $\mathcal{R}(\Omega\tau)$. Damping and thermal diffusion gradually restore the isotropic thermal state at long times.

rotate in phase space at the post-quench mechanical frequency. Thermal diffusion changes the magnitudes of the principal variances but does not alter these eigenvectors.

To characterize the anti-squeezing of the position quadrature in decibel units, we express the ratio of minimum variance to the initial thermal occupation $n_{\text{th}}^{\Omega_0}$ on a logarithmic scale as

$$\mathcal{S}(t) = -10 \log_{10} \left(\frac{\Lambda_{\text{min}}(t)}{n_{\text{th}}^{\Omega_0}} \right). \quad (\text{S51})$$

Pre-quench dynamics: $0 \leq t < t_1$

For completeness, before the frequency jump the oscillator evolves at the initial frequency Ω_0 . Starting from the thermal covariance $V(0) = n_{\text{th}}^{\Omega_0} \mathcal{I}$, the rotational evolution does not alter the isotropic covariance, while the diffusion contribution maintains thermal equilibrium. Consequently, throughout the interval $0 \leq t < t_1$,

$$V(t) = n_{\text{th}}^{\Omega_0} \mathcal{I}. \quad (\text{S52})$$

The two eigenvalues are therefore degenerate,

$$\Lambda_{\text{max}}(t) = \Lambda_{\text{min}}(t) = n_{\text{th}}^{\Omega_0}, \quad 0 \leq t < t_1. \quad (\text{S53})$$

Thus, the phase-space distribution remains circular and has no preferred principal direction before the frequency jump. The corresponding squeezing is $\mathcal{S}(t) = 0$.

Limiting cases after the jump

The squeezing expression has two particularly transparent limiting cases.

1. **Immediately after the frequency jump**, $\tau_1 = 0$: The diffusion contribution has not yet accumulated, so the principal variances are $\Lambda_{\max}(0) = n_{\text{th}}^{\Omega_0} e^{2|r|}$ and $\Lambda_{\min}(0) = n_{\text{th}}^{\Omega_0} e^{-2|r|}$. And the squeezing is

$$\mathcal{S}_{\max} \Big|_{\tau_1=0} = 10 \left| \log_{10} \left(\frac{\Omega_1}{\Omega_0} \right) \right|. \quad (\text{S54})$$

This is the maximum squeezing immediately following the quench.

2. **Long-time limit**, $\tau_1 \rightarrow \infty$: The contribution retaining the memory of the initial state is exponentially suppressed, while thermal diffusion drives both principal variances toward the equilibrium value $n_{\text{th}}^{\Omega_1}$. Hence, $\Lambda_{\max} = \Lambda_{\min} = n_{\text{th}}^{\Omega_1}$, giving $\mathcal{S}(\infty) = 0$. The covariance ellipse consequently becomes circular, corresponding to thermal equilibrium at the final mechanical frequency.

Overall, the analytical result captures the competition between quench-induced coherent anisotropy and thermalization: the former is exponentially attenuated by damping, while the latter drives both principal variances toward the same thermal equilibrium value. The resulting anisotropy consequently decreases from its maximum immediately after the frequency jump to zero at long times.

B. Squeezing analysis for a square pulse

Above we have described a detailed squeezing analysis for a single frequency jump. Now, we will extend the analysis when there are two successive frequency quenches

$$\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_2, \quad (\text{S55})$$

with switching times t_1 and t_2 , respectively. Fig. S3 depicts the schematic of such a pulse.

Importantly, we need to drive the covariance matrix $V(t)$ for $t > t_2$ and obtain its eigenvalues to characterize squeezing.

1. First frequency quench $\Omega_0 \rightarrow \Omega_1$

Let us start from the initial covariance matrix

$$V(0) = n_{\text{th}}^{\Omega_0} \mathcal{I}, \quad (\text{S56})$$

and defining $\tau_1 = t_1$, we can write its evolution upto a time just below the first quench

$$V(t_1^-) = e^{-\gamma\tau_1} \mathcal{R}(\Omega_0\tau_1) V(0) \mathcal{R}^T(\Omega_0\tau_1) + n_{\text{th}}^{\Omega_0} (1 - e^{-\gamma\tau_1}) \mathcal{I} = n_{\text{th}}^{\Omega_0} \mathcal{I}. \quad (\text{S57})$$

Note that just before the first quench, the covariance matrix is diagonal with variances $n_{\text{th}}^{\Omega_0}$. This shows that the covariance matrix remains proportional to the identity. Consequently, the position and momentum quadratures remain uncorrelated and the thermal distribution stays circular in phase space despite the deterministic phase-space rotation. The situation will be drastically different at the second frequency quench where the covariance matrix will involve off-diagonal contributions.

Now, the covariance matrix just above the first quench is governed by

$$V(t_1^+) = S_{10}(r_{10}) V(t_1^-) S_{10}^T(r_{10}), \quad (\text{S58})$$

where the squeezing matrix and the associated parameter with the first quench are

$$S_{10} = \begin{bmatrix} e^{r_{10}} & 0 \\ 0 & e^{-r_{10}} \end{bmatrix}, \quad r_{10} = \frac{1}{2} \log_e \left(\frac{\Omega_1}{\Omega_0} \right). \quad (\text{S59})$$

Then the covariance matrix for any time t over the interval $[t_1, t_2]$ and with the notation $\tau_2 = t - t_1$ is

$$V(t) \Big|_{t_1 \leq t < t_2} = e^{-\gamma\tau_2} \mathcal{R}(\Omega_1\tau_2) V(t_1^+) \mathcal{R}^T(\Omega_1\tau_2) + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\tau_2}) \mathcal{I}. \quad (\text{S60})$$

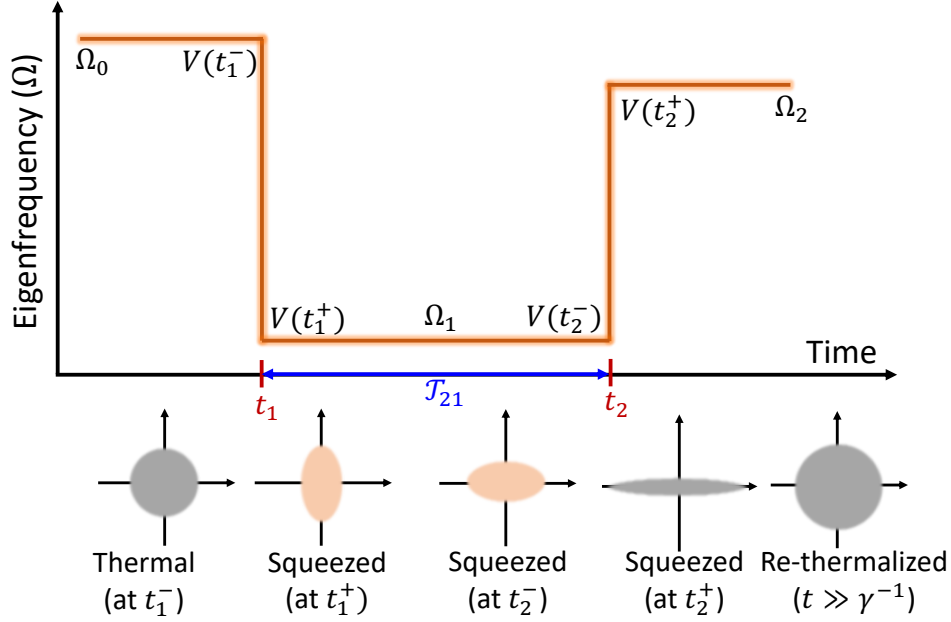


FIG. S3. **Squeezing via a square frequency pulse.** The schematic depicts the time-dependent trap frequency (top) and the corresponding phase-space distributions (bottom). Two rapid polarization rotations at t_1 and t_2 produce a square-pulse frequency modulation, $\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_2$. Experimentally, $\Omega_2 = \Omega_0$ as considered in Eq. (S80). The covariance matrices $V(t_1^-)$ and $V(t_1^+)$ denote the states immediately before and after the first quench, respectively; similarly, $V(t_2^-)$ and $V(t_2^+)$ denote the states immediately before and after the second quench. The first quench squeezes the initial thermal state; the quarter-period evolution at Ω_1 rotates the squeezed ellipse by $\pi/2$; and the second quench, acting on this rotated state, further enhances the squeezing. In the long-time limit, damping and thermal diffusion erase the anisotropy, and the system rethermalizes to an isotropic thermal distribution. The protocol demonstrates deterministic, feedback-free control of thermal noise.

2. Second frequency quench $\Omega_1 \rightarrow \Omega_2$

Just before the second quench, we can write the covariance matrix by defining $\mathcal{T}_{21} = t_2 - t_1$

$$V(t_2^-) = e^{-\gamma \mathcal{T}_{21}} \mathcal{R}(\Omega_1 \mathcal{T}_{21}) V(t_1^+) \mathcal{R}^T(\Omega_1 \mathcal{T}_{21}) + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma \mathcal{T}_{21}}) \mathcal{I}. \quad (\text{S61})$$

In general, $V(t_2^-)$ contains both unequal diagonal elements and finite off-diagonal correlations, indicating that the phase-space distribution is simultaneously squeezed and rotated with respect to the canonical quadratures. Just after the second frequency quench, the symplectic transformation connects the solution to the jump before it

$$V(t_2^+) = S_{21}(r_{21}) V(t_2^-) S_{21}^T(r_{21}), \quad (\text{S62})$$

where the updated squeezing matrix and associated parameter are

$$S_{21} = \begin{bmatrix} e^{r_{21}} & 0 \\ 0 & e^{-r_{21}} \end{bmatrix}, \quad r_{21} = \frac{1}{2} \log_e \left(\frac{\Omega_2}{\Omega_1} \right). \quad (\text{S63})$$

Unlike the first quench, the second symplectic transformation acts on a covariance matrix that is already squeezed and rotated. Consequently, the resulting covariance cannot, in general, be diagonalized analytically by inspection, and the principal variances must be obtained from the eigenvalues of $V(t)$.

Finally, the covariance matrix for $t > t_2$ is governed by

$$V(t) \Big|_{t_2 \leq t} = e^{-\gamma \tau_3} \mathcal{R}(\Omega_2 \tau_3) V(t_2^+) \mathcal{R}^T(\Omega_2 \tau_3) + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma \tau_3}) \mathcal{I}, \quad (\text{S64})$$

where $\tau_3 = t - t_2$.

3. Covariance matrix elements

Now let us find the matrix elements of above covariance matrix. For this, just before the second frequency quench, the covariance matrix can be expressed in terms of its elements as

$$V(t_2^-) = \begin{bmatrix} V_{qq}(\mathcal{T}_{21}) & V_{qp}(\mathcal{T}_{21}) \\ V_{qp}(\mathcal{T}_{21}) & V_{pp}(\mathcal{T}_{21}) \end{bmatrix}, \quad (\text{S65})$$

where the covariance elements are obtained from the dissipative evolution over the interval $t_1 \leq t < t_2$.

Applying the second symplectic transformation gives

$$V(t_2^+) = S_{21} V(t_2^-) S_{21}^T = \begin{bmatrix} e^{2r_{21}} V_{qq}(\mathcal{T}_{21}) & V_{qp}(\mathcal{T}_{21}) \\ V_{qp}(\mathcal{T}_{21}) & e^{-2r_{21}} V_{pp}(\mathcal{T}_{21}) \end{bmatrix}. \quad (\text{S66})$$

For $t \geq t_2$ the covariance matrix evolves as

$$V(t) = e^{-\gamma\tau_3} \mathcal{R}(\Omega_2, \tau_3) V(t_2^+) \mathcal{R}^T(\Omega_2, \tau_3) + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma\tau_3}) \mathcal{I}. \quad (\text{S67})$$

The individual covariance elements are therefore

$$V_{qq}(t) = e^{-\gamma\tau_3} \left[e^{2r_{21}} V_{qq}(\mathcal{T}_{21}) \cos^2(\Omega_2 \tau_3) + e^{-2r_{21}} V_{pp}(\mathcal{T}_{21}) \sin^2(\Omega_2 \tau_3) + V_{qp}(\mathcal{T}_{21}) \sin(2\Omega_2 \tau_3) \right] + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma\tau_3}), \quad (\text{S68})$$

$$V_{pp}(t) = e^{-\gamma\tau_3} \left[e^{2r_{21}} V_{qq}(\mathcal{T}_{21}) \sin^2(\Omega_2 \tau_3) + e^{-2r_{21}} V_{pp}(\mathcal{T}_{21}) \cos^2(\Omega_2 \tau_3) - V_{qp}(\mathcal{T}_{21}) \sin(2\Omega_2 \tau_3) \right] + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma\tau_3}), \quad (\text{S69})$$

$$V_{qp}(t) = \frac{e^{-\gamma\tau_3}}{2} \left[\left(e^{-2r_{21}} V_{pp}(\mathcal{T}_{21}) - e^{2r_{21}} V_{qq}(\mathcal{T}_{21}) \right) \sin(2\Omega_2 \tau_3) + 2V_{qp}(\mathcal{T}_{21}) \cos(2\Omega_2 \tau_3) \right]. \quad (\text{S70})$$

The principal variances of the covariance ellipse are obtained from the eigenvalues of $V(t)$ and can be obtained by using Eq. (S47). The covariance elements $V_{qq}(\mathcal{T}_{21})$, $V_{pp}(\mathcal{T}_{21})$, and $V_{qp}(\mathcal{T}_{21})$ in Eqs. (S68)-(S70) can be explicitly expressed as

$$V_{qq}(\mathcal{T}_{21}) = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} \left[\cosh(2r_{10}) + \cos(2\Omega_1 \mathcal{T}_{21}) \sinh(2r_{10}) \right] + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}), \quad (\text{S71})$$

$$V_{pp}(\mathcal{T}_{21}) = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} \left[\cosh(2r_{10}) - \cos(2\Omega_1 \mathcal{T}_{21}) \sinh(2r_{10}) \right] + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}), \quad (\text{S72})$$

$$V_{qp}(\mathcal{T}_{21}) = -e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} \sin(2\Omega_1 \mathcal{T}_{21}) \sinh(2r_{10}). \quad (\text{S73})$$

4. Pulse duration $\mathcal{T}_{21} = nT_1/4$

The general square-pulse solution derived above is valid for an arbitrary pulse duration $\mathcal{T}_{21} = t_2 - t_1$. In general, the covariance matrix immediately before the second quench contains non-vanishing position-momentum correlations (Eq. (S73)), which originate from the continuous phase-space rotation of the squeezed thermal state during its evolution under the trap frequency Ω_1 . Consequently, the principal axes of the covariance ellipse are generally rotated with respect to the position and momentum quadratures, leading to the more complicated expressions derived above.

A particularly important simplification occurs when the pulse duration satisfies

$$\mathcal{T}_{21} = \frac{n\pi}{2\Omega_1} = \frac{nT_1}{4}, \quad n = 0, 1, 2, \dots \quad (\text{S74})$$

At these stroboscopic times, the covariance ellipse realigns with the position and momentum axes, giving $V_{qp}(\mathcal{T}_{21}) = 0$. Physically, this corresponds to the absence of instantaneous position-momentum correlations even though the state remains squeezed. In other words, the squeezing axes coincide with the canonical quadratures, so that the covariance matrix becomes diagonal immediately before the second frequency quench.

Applying Eq. (S74) to the covariance matrix immediately before the second quench, as defined in Eq. (S65), yields a diagonal matrix,

$$V(t_2^-) = \begin{bmatrix} V_{qq}(\mathcal{T}_{21}) & 0 \\ 0 & V_{pp}(\mathcal{T}_{21}) \end{bmatrix}. \quad (\text{S75})$$

Under this condition, the matrix elements in Eqs. (S71)–(S72) simplify to:

$$V_{qq}(\mathcal{T}_{21}) = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{\pm 2r_{10}} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}), \quad (\text{S76})$$

$$V_{pp}(\mathcal{T}_{21}) = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{\mp 2r_{10}} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}), \quad (\text{S77})$$

where the upper (lower) sign corresponds to even (odd) values of n .

Since the covariance matrix is diagonal at $t = t_2^-$, its principal axes coincide with the canonical position and momentum quadratures. The second frequency quench therefore simply rescales the two principal variances according to

$$V(t_2^+) = S_{21} V(t_2^-) S_{21}^T = \begin{bmatrix} e^{2r_{21}} V_{qq}(\mathcal{T}_{21}) & 0 \\ 0 & e^{-2r_{21}} V_{pp}(\mathcal{T}_{21}) \end{bmatrix}, \quad (\text{S78})$$

The subsequent dissipative evolution acts independently on each principal variance. Consequently, the eigenvalues [Eq. (S47)] of the covariance matrix evaluate to:

$$\Lambda_{\text{max}}(t) = e^{-\gamma\tau_3} e^{2r_{21}} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{\pm 2r_{10}} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) \right] + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma\tau_3}), \quad (\text{S79a})$$

$$\Lambda_{\text{min}}(t) = e^{-\gamma\tau_3} e^{-2r_{21}} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{\mp 2r_{10}} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) \right] + n_{\text{th}}^{\Omega_2} (1 - e^{-\gamma\tau_3}). \quad (\text{S79b})$$

Representing a direct extension of the single-quench results [Eqs. (S49)–(S48)], Eq. (S79) describes a square-pulse protocol under the principal-axis revival condition. Experimentally, this regime is realized by synchronizing the second frequency jump with integer multiples of a quarter oscillation period at the intermediate trapping frequency Ω_1 . Furthermore, the equation naturally separates the dynamics into three distinct contributions: (i) the initially squeezed thermal state prepared by the first quench and damped throughout the pulse duration; (ii) the thermal fluctuations accumulated during evolution under Ω_1 that are subsequently transformed by the second quench; and (iii) the post-pulse relaxation toward thermal equilibrium at the final trapping frequency Ω_2 .

5. Return pulse: $\Omega_2 = \Omega_0$

A special case of significant experimental utility is the return-pulse protocol, wherein the trap frequency is restored to its initial value following a finite duration,

$$\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_0. \quad (\text{S80})$$

Under these conditions, setting $r_{21} = -r_{10}$ yields the principal variances for even n :

$$\Lambda_{\text{max}}^{\text{even}}(t) = e^{-\gamma\tau_3} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{-2|r_{10}|} \right] + n_{\text{th}}^{\Omega_0} (1 - e^{-\gamma\tau_3}), \quad (\text{S81a})$$

$$\Lambda_{\text{min}}^{\text{even}}(t) = e^{-\gamma\tau_3} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{2|r_{10}|} \right] + n_{\text{th}}^{\Omega_0} (1 - e^{-\gamma\tau_3}). \quad (\text{S81b})$$

For odd n , the corresponding expressions reduce to

$$\Lambda_{\text{max}}^{\text{odd}}(t) = e^{-\gamma\tau_3} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{4|r_{10}|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{2|r_{10}|} \right] + n_{\text{th}}^{\Omega_0} (1 - e^{-\gamma\tau_3}), \quad (\text{S82a})$$

$$\Lambda_{\text{min}}^{\text{odd}}(t) = e^{-\gamma\tau_3} \left[e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{-4|r_{10}|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{-2|r_{10}|} \right] + n_{\text{th}}^{\Omega_0} (1 - e^{-\gamma\tau_3}). \quad (\text{S82b})$$

Note that for the experimental choices $\Omega_0/2\pi = 55$ kHz and $\Omega_1/2\pi = 10.2$ kHz, the squeezing parameter is negative [$r_{10} = \frac{1}{2} \ln(\Omega_1/\Omega_0) < 0$], confirming the physical ordering $\Lambda_{\text{min}}^{\text{odd}}(t) < \Lambda_{\text{max}}^{\text{odd}}(t)$.

Finally, the squeezing level expressed in decibels follows directly from Eq. (S51).

6. Limiting cases

(i) *Immediately after the second quench.* At $t = t_2^+$, (i.e., $\tau_3 = 0$), the eigenvalues reduce to the following explicit forms. For even n ,

$$\Lambda_{\text{max}}^{\text{even}} \Big|_{\tau_3=0} = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{-2|r_{10}|}, \quad (\text{S83})$$

$$\Lambda_{\text{min}}^{\text{even}} \Big|_{\tau_3=0} = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{2|r_{10}|}. \quad (\text{S84})$$

For odd n ,

$$\Lambda_{\max}^{\text{odd}} \Big|_{\tau_3=0} = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{4|r_{10}|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{2|r_{10}|}, \quad (\text{S85})$$

$$\Lambda_{\min}^{\text{odd}} \Big|_{\tau_3=0} = e^{-\gamma\mathcal{T}_{21}} n_{\text{th}}^{\Omega_0} e^{-4|r_{10}|} + n_{\text{th}}^{\Omega_1} (1 - e^{-\gamma\mathcal{T}_{21}}) e^{-2|r_{10}|}. \quad (\text{S86})$$

These expressions highlight that the squeezing magnitude is directly governed by the phase accrued between the two quenches.

(ii) *Vanishing pulse duration*, $\mathcal{T}_{21} \rightarrow 0$. In this limit, for even n , $\Lambda_{\max}^{\text{even}} = \Lambda_{\min}^{\text{even}} = n_{\text{th}}^{\Omega_0}$, so that following Eq. (S51), the squeezing in dB scale becomes $\mathcal{S}_{\text{even}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} \rightarrow 0$. Thus, when the return quench occurs without allowing the oscillator to evolve at Ω_1 , the two inverse symplectic transformations cancel.

For odd n , the extremal eigenvalues of the phase-space distribution for $\tau_3 = 0$ and $\mathcal{T}_{21} \rightarrow 0$ reduces to $\Lambda_{\max}^{\text{odd}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} = n_{\text{th}} e^{4|r_{10}|}$ and $\Lambda_{\min}^{\text{odd}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} = n_{\text{th}} e^{-4|r_{10}|}$. Using the squeezing parameter $r_{10} = (1/2) \ln(\Omega_1/\Omega_0)$ yields

$$\Lambda_{\max}^{\text{odd}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} = n_{\text{th}} \exp \left[2 \left| \ln \left(\frac{\Omega_1}{\Omega_0} \right) \right| \right], \quad \Lambda_{\min}^{\text{odd}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} = n_{\text{th}} \exp \left[-2 \left| \ln \left(\frac{\Omega_1}{\Omega_0} \right) \right| \right]. \quad (\text{S87})$$

The amount of squeezing in dB scale for odd n then becomes

$$\mathcal{S}_{\text{odd}} \Big|_{\tau_3=0}^{\mathcal{T}_{21} \rightarrow 0} = 20 \left| \log_{10} \left(\frac{\Omega_1}{\Omega_0} \right) \right|. \quad (\text{S88})$$

7. Numerical results

To reveal squeezing for a polarization square pulse frequency quench protocol, we solve Eqs. (S15) numerically via an integrator [5] with step size $dt = 4 \times 10^{-5}/\gamma$, ensuring stability. The Wiener increments are taken as $dW_{\text{th}}(t) = \sqrt{dt} \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ is the standard normal distribution.

Fig. S4 shows the numerically simulated phase-space evolution of the motional state throughout the square-pulse protocol. Before the first quench, the oscillator occupies an isotropic thermal distribution with zero squeezing (a). At the switching instant t_1 , the instantaneous frequency jump deforms the state into a squeezed ellipse, yielding $\mathcal{S} = 7.33$ dB (b) and $\mathcal{S} = 7.31$ dB immediately after (c). During the intermediate interval $\mathcal{T}_{21} = T_1/4$, the ellipse rotates coherently at Ω_1 by $\pi/2$, misaligning the squeezing axis with the laboratory frame and reducing the measured value to 4.64 dB just before the second quench (d). At t_2 , the return quench restores the original trap frequency and applies the inverse symplectic transformation. Immediately after this second quench, the operation amplifies the anisotropy, producing peak transient squeezing of $\mathcal{S} = 11.96$ dB (e) and $\mathcal{S} = 11.95$ dB (f). Throughout the entire temporal evolution, the numerically computed squeezing values agree excellently with the analytical covariance matrix predictions given by Eqs. (S82b) and (S51). Subsequent evolution at Ω_0 preserves the squeezed orientation (g), while long-time dynamics ($t \gg \gamma^{-1}$) gradually erase the anisotropy through damping and thermal diffusion, returning the system to the isotropic thermal state (h).

III. SQUEEZING VIA INVERTED DARK OPTICAL POTENTIAL

We consider the axial dynamics of the nanoparticle after rapid translation of the standing-wave optical potential from a stable bright fringe to a neighboring dark fringe. The relative optical phase determines the position of the standing-wave pattern along the fiber axis and therefore provides a means of rapidly switching the particle between the two types of trapping regions. On the timescale of the phase switch, which is short compared with the mechanical evolution, the particle's phase-space coordinates can be taken to remain unchanged. The subsequent dynamics are consequently determined by the local curvature of the optical potential at the dark fringe.

In the vicinity of a stable bright fringe, the axial potential is locally harmonic and the conservative force is restoring. At a dark fringe, the local curvature is inverted, such that the conservative force changes sign while the damping and stochastic forces retain the same form. Accordingly, the coupled finite-difference equations used to describe the axial motion in the stable trap can be transformed to the dark-fringe configuration by reversing the sign of the conservative force term by setting $\phi = \pi/2$ in Eq. (S2). In dimensionless quadratures, the resulting equations are

$$dq = \Omega p dt, \quad (\text{S89})$$

$$dp = -\gamma p dt + \Omega q dt + \sqrt{2\gamma n_{\text{th}}} dW^{\text{th}}. \quad (\text{S90})$$

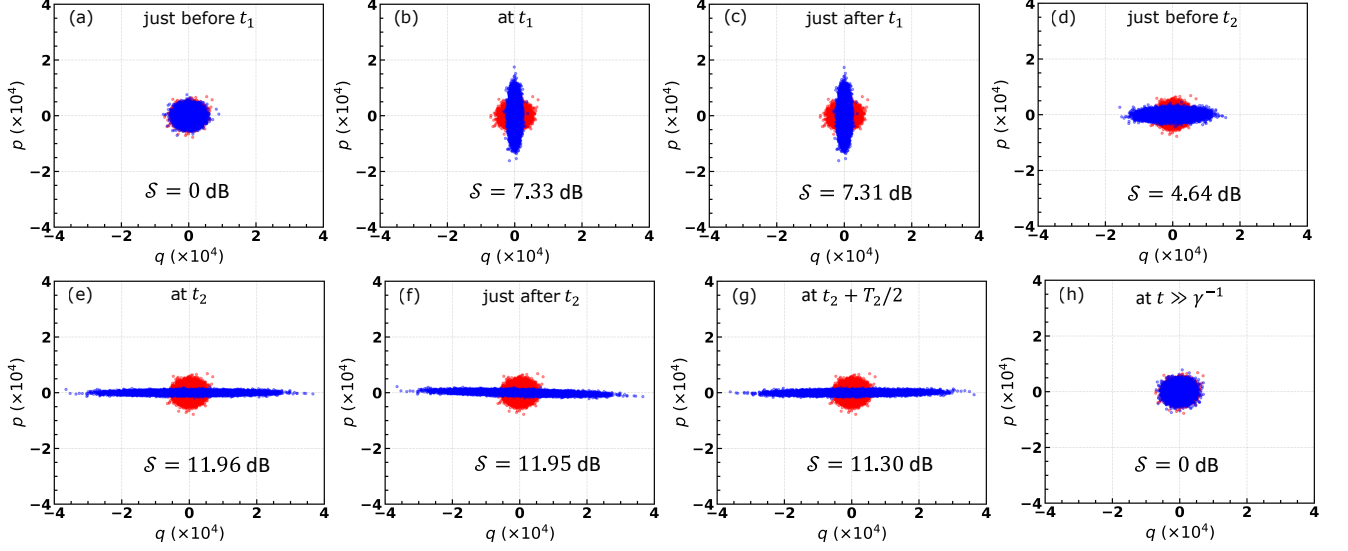


FIG. S4. **Numerical simulation of transient squeezing dynamics under a square-pulse protocol.** Phase-space distributions of the motional state at key times during the $\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_0$ frequency-quench sequence. (a) Initial isotropic thermal state ($S = 0$ dB). (b,c) The first quench at t_1 instantaneously generates momentum squeezing ($S \approx 7.3$ dB), which subsequently rotates coherently at Ω_1 . (d) Just before t_2 , the ellipse has rotated by $\pi/2$, reducing squeezing to $S = 4.64$ dB. (e,f) The return quench at t_2 restores Ω_0 and amplifies the anisotropy along the position quadrature, yielding peak squeezing $S \approx 11.9$ dB. (g) Subsequent rotation at Ω_0 preserves the squeezed axis ($S = 11.30$ dB). (h) Long-time evolution ($t \gg \gamma^{-1}$) restores the isotropic thermal distribution ($S = 0$ dB) via damping and thermal diffusion. Axes show dimensionless position q and momentum p (scaled by 10^4). The snapshots are generated from 20,000 trajectories, with initial conditions drawn randomly from a Boltzmann distribution describing a cold thermal state with occupation number n_{th} . The oscillator parameters are $\Omega_0/2\pi = 55$ kHz, $\Omega_1/2\pi = 10.2$ kHz, and $\gamma/2\pi = 200$ Hz, with corresponding periods $T_0 = 2\pi/\Omega_0$ and $T_1 = 2\pi/\Omega_1$. The frequency quenches occur at $t_1 = 0.1T_0$ and $t_2 = t_1 + T_1/4$, where $T_{21} = T_1/4$ is the pulse duration. The CoM temperature is maintained at 7.8 K.

Following the same machinery as introduced in Sec. IC, we obtain the solution by mechanical quadratures for under-damped oscillators in an inverted potential as follows:

$$q(t) = e^{-\gamma t/2} \cosh(\Omega t) q_0 + e^{-\gamma t/2} \sinh(\Omega t) p_0 + \frac{1}{2} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} [\lambda_u^j - \lambda_d^j] dW_{n-j-1}^{\text{th}}, \quad (\text{S91})$$

$$p(t) = e^{-\gamma t/2} \sinh(\Omega t) q_0 + e^{-\gamma t/2} \cosh(\Omega t) p_0 + \frac{1}{2\Omega} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} [\gamma_u \lambda_u^j + \gamma_d \lambda_d^j] dW_{n-j-1}^{\text{th}}. \quad (\text{S92})$$

The solution naturally separates into a deterministic evolution of the initial quadratures and a stochastic contribution arising from thermal fluctuations. The deterministic part in Eqs. (S91)-(S92) follows a hyperbolic propagator whose amplitude decays exponentially, a characteristic of the inverted harmonic potential. In contrast to the oscillatory evolution of a stable trap, the hyperbolic functions describe exponential separation of the phase-space trajectories. This behavior can be understood in terms of the underlying unstable and decaying eigenmodes, characterized by $\lambda_u = 1 + \gamma_u dt$ and $\lambda_d = 1 - \gamma_d dt$, with $\gamma_u = (\bar{\Omega}_D - \gamma)/2$ and $\gamma_d = (\bar{\Omega}_D + \gamma)/2$, where $\bar{\Omega}_D = \sqrt{\gamma^2 + 4\Omega^2}$. The unstable mode amplifies exponentially at a rate γ_u , whereas the decaying mode is attenuated at a rate γ_d ; their combined evolution produces the hyperbolic deterministic dynamics appearing in the solution above. The final terms in both equations account for the thermal diffusion accumulated throughout this unstable evolution.

A. Covariance matrix analysis

The stochastic solution above provides a direct route to the full statistical description of the nanoparticle's axial motion in inverted dark potential. We therefore characterize the evolution of the motional state through its covariance

matrix, which captures both the deterministic hyperbolic transformation of the initial fluctuations and the diffusion accumulated from thermal noise. This framework allows the separate contributions of the inverted dynamics and stochastic forces to be identified explicitly.

To make the subsequent covariance analysis compact, we express the solutions of Eqs. (S91) and (S92) in matrix form as

$$\mathbf{u}_D(t) = e^{-\gamma t/2} \mathcal{O}_D(\Omega t) \mathbf{u}_D(0) + \mathbf{N}_D(t, t_0), \quad (\text{S93})$$

where

$$\mathbf{u}_D(t) = \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}, \quad \mathcal{O}_D(\Omega t) = \begin{bmatrix} \cosh(\Omega t) & \sinh(\Omega t) \\ \sinh(\Omega t) & \cosh(\Omega t) \end{bmatrix}. \quad (\text{S94})$$

The matrix $\mathcal{O}_D(\Omega t)$ is the hyperbolic propagator of the inverted harmonic potential. It describes the deterministic phase-space evolution in the dark fringe, with the hyperbolic functions reflecting the exponential stretching and contraction of the two conjugate dynamical directions. The vector

$$\mathbf{N}_D(t, t_0) = \begin{bmatrix} \mathcal{N}_D^q(t, t_0) \\ \mathcal{N}_D^p(t, t_0) \end{bmatrix}, \quad (\text{S95})$$

collects the stochastic contribution accumulated during the dark-fringe evolution. Its components are

$$\mathcal{N}_D^q(t, t_0) = \frac{1}{2} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} [\lambda_u^j - \lambda_d^j] dW_{n-j-1}^{\text{th}}, \quad (\text{S96})$$

$$\mathcal{N}_D^p(t, t_0) = \frac{1}{2\Omega} \sqrt{2\gamma n_{\text{th}}} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} [\gamma_u \lambda_u^j + \gamma_d \lambda_d^j] dW_{n-j-1}^{\text{th}}. \quad (\text{S97})$$

The covariance matrix then follows directly from the phase-space solution. Assuming that the stochastic noise has zero mean and is uncorrelated with the initial state, we obtain

$$V_D(t) = \langle \mathbf{u}_D(t) \mathbf{u}_D^T(t) \rangle = e^{-\gamma t} \mathcal{O}_D(\Omega t) V_D(0) \mathcal{O}_D^T(\Omega t) + \mathcal{Q}_D(t, t_0), \quad (\text{S98})$$

where $V_D(0)$ is the covariance matrix at the beginning of the dark-fringe excursion and

$$\mathcal{Q}_D(t, t_0) = \langle \mathbf{N}_D(t, t_0) \mathbf{N}_D^T(t, t_0) \rangle = \begin{bmatrix} \langle (\mathcal{N}_D^q(t, t_0))^2 \rangle & \langle \mathcal{N}_D^q(t, t_0) \mathcal{N}_D^p(t, t_0) \rangle \\ \langle \mathcal{N}_D^q(t, t_0) \mathcal{N}_D^p(t, t_0) \rangle & \langle (\mathcal{N}_D^p(t, t_0))^2 \rangle \end{bmatrix}. \quad (\text{S99})$$

is the covariance accumulated from thermal diffusion during the evolution. The first term therefore describes the deterministic hyperbolic transformation of the initial motional state, including mechanical damping, whereas $\mathcal{Q}_D$ accounts for the stochastic broadening generated by thermal noise. This decomposition provides the basis for analyzing the evolution of the principal variances and the resulting dark-fringe thermomechanical squeezing contribution to the covariance matrix.

The diffusion covariance matrix is obtained by evaluating the second moments of the stochastic contributions in $\mathbf{N}_D(t, t_0)$. Using the zero-mean, uncorrelated Wiener increments and the Itô relation $\langle dW_i^{\text{th}} dW_j^{\text{th}} \rangle = \delta_{ij} dt$, each matrix element follows from the corresponding products of the noise kernels in $\mathcal{N}_D^q$ and $\mathcal{N}_D^p$. This yields the diffusion contribution to the variances of the position and momentum quadratures and their cross-correlation, which can then be evaluated explicitly from the sums appearing above. Their final expressions take the following form

$$\langle (\mathcal{N}_D^q(t, t_0))^2 \rangle = n_{\text{th}} e^{-\gamma t} \left[\left(\frac{\gamma}{2\Omega} \right) \sinh(2\Omega t) + \left(\frac{\gamma}{2\Omega} \right)^2 \cosh(2\Omega t) \right] - n_{\text{th}} [1 - e^{-\gamma t}], \quad (\text{S100})$$

$$\langle (\mathcal{N}_D^p(t, t_0))^2 \rangle = n_{\text{th}} e^{-\gamma t} \left[\left(\frac{\gamma}{2\Omega} \right) \sinh(2\Omega t) - \left(\frac{\gamma}{2\Omega} \right)^2 \cosh(2\Omega t) \right] + n_{\text{th}} [1 - e^{-\gamma t}], \quad (\text{S101})$$

$$\langle \mathcal{N}_D^q(t, t_0) \mathcal{N}_D^p(t, t_0) \rangle = (n_{\text{th}} e^{-\gamma t}) \left(\frac{\gamma}{2\Omega} \right) [\cosh(2\Omega t) - 1]. \quad (\text{S102})$$

For an initial thermal state, the covariance matrix is isotropic and given by $V_D(0) = n_{\text{th}}\mathcal{I}$. The full covariance matrix in the dark fringe is then obtained by combining the deterministic hyperbolic evolution with the diffusion contribution through the covariance-matrix relation above. Substituting the initial thermal covariance and the corresponding deterministic and diffusion terms yields the individual covariance elements for an underdamped oscillator

$$V_D^{qq}(t) = n_{\text{th}}e^{-\gamma t} \cosh(2\Omega t) + n_{\text{th}}e^{-\gamma t} \left(\frac{\gamma}{2\Omega}\right) \sinh(2\Omega t) - n_{\text{th}}[1 - e^{-\gamma t}], \quad (\text{S103})$$

$$V_D^{pp}(t) = n_{\text{th}}e^{-\gamma t} \cosh(2\Omega t) + n_{\text{th}}e^{-\gamma t} \left(\frac{\gamma}{2\Omega}\right) \sinh(2\Omega t) + n_{\text{th}}[1 - e^{-\gamma t}], \quad (\text{S104})$$

$$V_D^{qp}(t) = n_{\text{th}}e^{-\gamma t} \sinh(2\Omega t) + n_{\text{th}}e^{-\gamma t} \left(\frac{\gamma}{2\Omega}\right) \left[\cosh(2\Omega t) - 1 \right]. \quad (\text{S105})$$

The limiting behavior of these covariance elements provides useful consistency checks on the dark-fringe solution. At the initial time $t = 0$, Eqs. (S103)–(S105) give

$$V_D^{qq}(0) = V_D^{pp}(0) = n_{\text{th}}, \quad V_D^{qp}(0) = 0, \quad (\text{S106})$$

and hence recover the assumed thermal covariance

$$V_D(0) = n_{\text{th}}\mathcal{I}. \quad (\text{S107})$$

For the long-time behavior, $\Omega t \gg 1$, in the dark-trap regime, the unstable dynamics overcome the overall damping. The leading asymptotic behavior is therefore

$$V_D^{qq}(t) \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{(2\Omega - \gamma)t} \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{2\Omega t}, \quad (\text{S108})$$

$$V_D^{pp}(t) \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{(2\Omega - \gamma)t} \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{2\Omega t}, \quad (\text{S109})$$

while

$$V_D^{qp}(t) \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{(2\Omega - \gamma)t} \simeq \frac{n_{\text{th}}}{2} \left(1 + \frac{\gamma}{2\Omega}\right) e^{2\Omega t}. \quad (\text{S110})$$

Thus, all three covariance elements acquire the same dominant exponential scaling, $e^{2\Omega t}$, reflecting the unstable hyperbolic evolution generated by the inverted dark potential. This exponential growth characterizes the thermomechanical evolution of the covariance matrix during the dark-fringe excursion.

We analyze the generation and evolution of phase-space squeezing produced by rapidly transferring the particle between the stable bright fringe and the inverted dark fringe of the optical potential. We first consider an instantaneous switching to dark potential, which provides the basic mechanism and analytical benchmark for squeezing. We then extend the treatment to a finite-duration dark-fringe pulse, where the squeezed state evolves under the inverted potential before returning to the stable configuration. Finally, we consider sequences of dark-fringe pulses and show how repeated switching can control the transient squeezing dynamics.

B. Instantaneous switching to inverted dark potential

We first consider an instantaneous transfer of the particle from the stable bright trap to the inverted dark potential at time $t = t_1$. This is achieved through adaptive phase control by setting the relative phase $\phi = \pi/2$ between counter-propagating fields. The oscillator evolves in the bright trap for $0 \leq t < t_1$, while the restoring force is reversed for $t \geq t_1$. Since the switching is instantaneous, the phase-space coordinates q and p do not change discontinuously at the switching time. Consequently, the covariance matrix is continuous across the switching event, $V(t_1^+) = V(t_1^-)$.

Before the switch, the covariance evolves under stable-trap dynamics. With $\tau_1 = t - t_0$, the propagator for the bright interval is

$$V(t)|_{0 \leq t < t_1} = e^{-\gamma \tau_1} \mathcal{R}(\Omega \tau_1) V(0) \mathcal{R}^T(\Omega \tau_1) + n_{\text{th}}(1 - e^{-\gamma \tau_1}) I, \quad (\text{S111})$$

where $\mathcal{R}$ describes the coherent harmonic rotation of duration τ_1 at frequency Ω .

For an initially thermal state, $V(0) = n_{\text{th}}I$. Since $\mathcal{R}(\Omega \tau_1) \mathcal{R}^T(\Omega \tau_1) = I$, the thermal covariance is invariant under this rotation. Hence,

$$V(t)|_{0 \leq t < t_1} = n_{\text{th}}I. \quad (\text{S112})$$

Thus, the stable trap preserves the circular thermal phase-space distribution: harmonic rotation alone does not generate squeezing, while thermal diffusion maintains the equilibrium covariance.

Since the covariance matrix remains continuous across switching from bright to dark fringe, we have

$$V(t_1^-) = V(t_1^+) = n_{\text{th}} I . \quad (\text{S113})$$

For the subsequent dark-trap evolution over $t \geq t_1$, we define $\tau_2 = t - t_1$. The covariance matrix is then

$$V(t)|_{t \geq t_1} = e^{-\gamma \tau_2} \mathcal{O}_D(\Omega \tau_2) V(t_1^+) \mathcal{O}_D^T(\Omega \tau_2) + \mathcal{Q}_D(\tau_2) , \quad (\text{S114})$$

where $\mathcal{O}_D(\Omega \tau_2)$ is the deterministic hyperbolic propagator of the inverted trap [Eq. (S94)] and $\mathcal{Q}_D(\tau_2)$ represents the thermal covariance accumulated during the dark-trap evolution [Eq. (S99)]. In the underdamped regime, the covariance matrix elements for $t \geq t_1$ are given by Eqs. (S103)-(S105).

The essential difference between the bright and dark traps is therefore the change from trigonometric to hyperbolic phase-space evolution. In the bright trap, the distribution undergoes ordinary rotation, whereas in the dark trap one phase-space direction is exponentially amplified while the conjugate direction is compressed. An initially circular thermal distribution consequently develops an elliptical shape.

C. Finite dark fringe pulse

We now derive the covariance dynamics for a single finite-duration dark-trapping pulse. The protocol consists of three stages: bright trap ($0 \leq t < t_1$), dark trap ($t_1 \leq t < t_2$), and bright trap ($t \geq t_2$). The dark-pulse duration is

$$\mathcal{T}_{21} = t_2 - t_1 . \quad (\text{S115})$$

We assume instantaneous switching at both t_1 and t_2 , so the covariance matrix is continuous across each switching event:

$$V(t_1^+) = V(t_1^-), \quad V(t_2^+) = V(t_2^-) . \quad (\text{S116})$$

1. Bright-trap evolution before the pulse

Before the first switching, the system evolves in the bright fringe. For an initially thermal state, $V(0) = n_{\text{th}} I$, the covariance remains

$$V(t) = n_{\text{th}} I, \quad 0 \leq t < t_1 . \quad (\text{S117})$$

Thus, immediately before the first switching,

$$V(t_1^-) = n_{\text{th}} I , \quad (\text{S118})$$

and hence $V(t_1^+) = n_{\text{th}} I$.

2. Dark-trap dynamics

During the dark interval $t_1 \leq t < t_2$, the inverted potential amplifies displacements. With $\tau_2 = t - t_1$, the covariance evolves as

$$V(t) = e^{-\gamma \tau_2} \mathcal{O}_D(\Omega \tau_2) V(t_1^+) \mathcal{O}_D^T(\Omega \tau_2) + \mathcal{Q}_D(\tau_2) , \quad (\text{S119})$$

where $\mathcal{O}_D$ is the hyperbolic propagator of the inverted trap and $\mathcal{Q}_D$ accounts for thermal noise during the dark interval.

At the end of the dark pulse, we define

$$V_D(\mathcal{T}_{21}) \equiv V(t_2^-) = \begin{bmatrix} V_D^{qq}(\mathcal{T}_{21}) & V_D^{qp}(\mathcal{T}_{21}) \\ V_D^{pq}(\mathcal{T}_{21}) & V_D^{pp}(\mathcal{T}_{21}) \end{bmatrix} . \quad (\text{S120})$$

The covariance elements are

$$V_D^{qq}(\mathcal{T}_{21}) = n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \cosh(2\Omega \mathcal{T}_{21}) + n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \frac{\gamma}{2\Omega} \sinh(2\Omega \mathcal{T}_{21}) - n_{\text{th}} (1 - e^{-\gamma \mathcal{T}_{21}}) , \quad (\text{S121})$$

$$V_D^{pp}(\mathcal{T}_{21}) = n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \cosh(2\Omega \mathcal{T}_{21}) + n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \frac{\gamma}{2\Omega} \sinh(2\Omega \mathcal{T}_{21}) + n_{\text{th}} (1 - e^{-\gamma \mathcal{T}_{21}}) , \quad (\text{S122})$$

$$V_D^{qp}(\mathcal{T}_{21}) = n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \sinh(2\Omega \mathcal{T}_{21}) + n_{\text{th}} e^{-\gamma \mathcal{T}_{21}} \left(\frac{\gamma}{2\Omega} \right) \left[\cosh(2\Omega \mathcal{T}_{21}) - 1 \right] . \quad (\text{S123})$$

3. Bright-trap evolution after the pulse

The second switching restores the harmonic trap in the bright fringe, with $V(t_2^+) = V_D(\mathcal{T}_{21})$. For $t \geq t_2$, with $\tau_3 = t - t_2$, the covariance evolves as

$$V(t) = e^{-\gamma \tau_3} \mathcal{R}(\Omega \tau_3) V_D(\mathcal{T}_{21}) \mathcal{R}^T(\Omega \tau_3) + n_{\text{th}} (1 - e^{-\gamma \tau_3}) I , \quad (\text{S124})$$

where $\mathcal{R}(\Omega \tau_3)$ is the stable-trap rotation matrix.

The resulting covariance elements are

$$V_{qq}(t) = e^{-\gamma \tau_3} \left[V_D^{qq}(\mathcal{T}_{21}) \cos^2(\Omega \tau_3) + V_D^{pp}(\mathcal{T}_{21}) \sin^2(\Omega \tau_3) + V_D^{qp}(\mathcal{T}_{21}) \sin(2\Omega \tau_3) \right] + n_{\text{th}} (1 - e^{-\gamma \tau_3}) , \quad (\text{S125})$$

$$V_{pp}(t) = e^{-\gamma \tau_3} \left[V_D^{qq}(\mathcal{T}_{21}) \sin^2(\Omega \tau_3) + V_D^{pp}(\mathcal{T}_{21}) \cos^2(\Omega \tau_3) - V_D^{qp}(\mathcal{T}_{21}) \sin(2\Omega \tau_3) \right] + n_{\text{th}} (1 - e^{-\gamma \tau_3}) , \quad (\text{S126})$$

$$V_{qp}(t) = e^{-\gamma \tau_3} \left[\frac{V_D^{pp}(\mathcal{T}_{21}) - V_D^{qq}(\mathcal{T}_{21})}{2} \sin(2\Omega \tau_3) + V_D^{qp}(\mathcal{T}_{21}) \cos(2\Omega \tau_3) \right] . \quad (\text{S127})$$

The above equations show that the dark-trapping pulse determines the covariance matrix $V_D(\mathcal{T}_{21})$, while the subsequent bright-trap evolution rotates and damps this covariance ellipse. In particular, the quantities $V_D^{qq}(\mathcal{T}_{21})$, $V_D^{pp}(\mathcal{T}_{21})$, and $V_D^{qp}(\mathcal{T}_{21})$ contain the complete information about the squeezing generated during the dark pulse.

4. Squeezing and relaxation

The principal variances are obtained from the eigenvalues $\Lambda_{\mp}$ of the covariance matrix, with the squeezing level defined in Eq. (S51). At $t = t_2^+$, the covariance is $V_D(\mathcal{T}_{21})$. For $t > t_2$, the stable dynamics rotate the squeezed ellipse as its anisotropy decays to equilibrium. In the long-time limit,

$$V(t) \xrightarrow[t \rightarrow \infty]{} n_{\text{th}} I, \quad \mathcal{S}_{\text{dB}} \xrightarrow[t \rightarrow \infty]{} 0 . \quad (\text{S128})$$

5. Numerical simulation

To illustrate the squeezing effect via the hyperbolic evolution of the particle in the dark fringe, the numerical solution [5] of Eqs. (S89) and (S90) is obtained using a time step $dt = 4 \times 10^{-5}/\gamma$, selected for numerical stability. The Wiener increments are implemented as $dW_{\text{th}}(t) = \sqrt{dt} \mathcal{N}(0, 1)$, with $\mathcal{N}(0, 1)$ being the standard normal distribution.

The dark-fringe protocol generates squeezing through a fundamentally different mechanism than the frequency-quench protocol. Rather than an instantaneous symplectic transformation, the inverted optical potential at the dark fringe drives continuous hyperbolic evolution in which one quadrature is exponentially compressed while the other is exponentially stretched. Fig. S5 tracks this process in real time. At the start of the pulse (a), the distribution

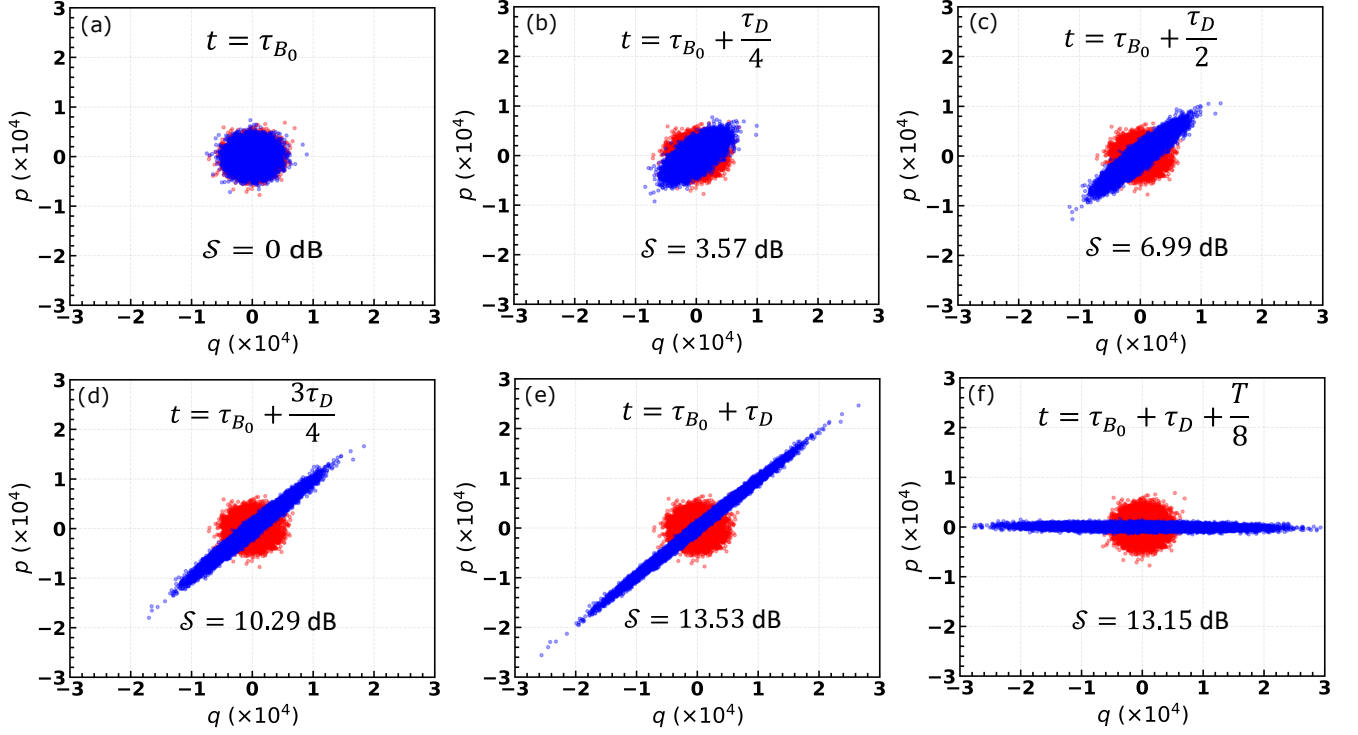


FIG. S5. **Hyperbolic phase-space compression during a dark-fringe excursion.** The inverted optical potential at the dark fringe produces exponential squeezing that grows monotonically over the pulse duration $\tau_D = T/4$. (a) Isotropic thermal state at $t = \tau_{B_0}$. (b) 25% traversal ($t = \tau_{B_0} + \tau_D/4$), $\mathcal{S} = 3.57$ dB. (c) 50% ($t = \tau_{B_0} + \tau_D/2$), $\mathcal{S} = 6.99$ dB. (d) 75% ($t = \tau_{B_0} + 3\tau_D/4$), $\mathcal{S} = 10.29$ dB. (e) Pulse end ($t = \tau_{B_0} + \tau_D$), maximum $\mathcal{S} = 13.53$ dB. (f) $T/8$ post-pulse delay, $\mathcal{S} = 13.15$ dB, reflecting the onset of dissipative decay. The snapshots are generated from 20,000 trajectories, with initial conditions randomly sampled from a Boltzmann distribution for a cold thermal state with occupation number n_{th} .

is isotropic. As the particle traverses the dark fringe, the squeezing builds continuously: $\mathcal{S} = 3.57$ dB at 25% (b), 6.99 dB at 50% (c), 10.29 dB at 75% (d), and 13.53 dB at the pulse end (e). The near-linear growth in dB reflects the exponential compression in phase space. Following the return to the bright fringe, the phase-space distribution rotates under the influence of damping and thermal diffusion. One such snapshot, taken at $T/8$ after the pulse, shows a squeezing of 13.15 dB (f). Throughout this evolution, the numerically computed squeezing values agree excellently with the analytical predictions for hyperbolic dynamics.

D. Sequence of dark fringe pulses

We now generalize the single dark-pulse protocol to a sequence of N finite dark-trap pulses. The protocol consists of alternating dark and bright trapping intervals,

$$B \rightarrow D_1 \rightarrow B_1 \rightarrow D_2 \rightarrow \cdots \rightarrow B_{N-1} \rightarrow D_N \rightarrow B,$$

where D_j denotes the j -th dark pulse and B_j denotes the bright interval between the j -th and $(j+1)$ -th dark pulses. The duration of the j -th dark pulse is $\tau_{D,j}$, while the subsequent bright interval has duration $\tau_{B,j}$.

Let V_j^- and V_j^+ denote the covariance matrices immediately before and after the j -th dark pulse, respectively. The covariance evolves according to the general form

$$V(t) = e^{-\gamma\tau} \mathcal{O}(t, t_0) V(t_0) \mathcal{O}^T(t, t_0) + \mathcal{Q}(t, t_0), \quad (\text{S129})$$

where $\tau = t - t_0$, $\mathcal{O}$ describes the deterministic evolution, and $\mathcal{Q}$ accounts for the covariance generated by stochastic noise.

1. Dark-trap evolution

During a dark interval, the deterministic propagator is the hyperbolic matrix

$$\mathcal{O}_D(\Omega\tau_D) = \begin{bmatrix} \cosh(\Omega\tau_D) & \sinh(\Omega\tau_D) \\ \sinh(\Omega\tau_D) & \cosh(\Omega\tau_D) \end{bmatrix}. \quad (\text{S130})$$

The covariance immediately after the j -th dark pulse is therefore

$$V_j^+ = e^{-\gamma\tau_{D,j}} \mathcal{O}_D(\Omega\tau_{D,j}) V_j^- \mathcal{O}_D^T(\Omega\tau_{D,j}) + \mathcal{Q}_D(\tau_{D,j}), \quad (\text{S131})$$

where $\mathcal{Q}_D(\tau_{D,j})$ is the thermal covariance accumulated during the dark interval.

2. Bright-trap evolution

Following the j -th dark pulse, the system is switched to the stable bright trap. The corresponding deterministic propagator is the rotation matrix

$$\mathcal{O}_B(\Omega\tau_B) = \begin{bmatrix} \cos(\Omega\tau_B) & \sin(\Omega\tau_B) \\ -\sin(\Omega\tau_B) & \cos(\Omega\tau_B) \end{bmatrix}. \quad (\text{S132})$$

The covariance after the subsequent bright interval is

$$V_{j+1}^- = e^{-\gamma\tau_{B,j}} \mathcal{O}_B(\Omega\tau_{B,j}) V_j^+ \mathcal{O}_B^T(\Omega\tau_{B,j}) + \mathcal{Q}_B(\tau_{B,j}), \quad (\text{S133})$$

where the bright-trap noise covariance is isotropic:

$$\mathcal{Q}_B(\tau_{B,j}) = n_{\text{th}} (1 - e^{-\gamma\tau_{B,j}}) I. \quad (\text{S134})$$

3. Cycle recurrence

For a sequence of identical dark pulses of duration τ_D separated by identical bright intervals of duration τ_B , we define one complete dark-bright cycle by the deterministic map

$$\mathcal{M} = \mathcal{O}_B(\Omega\tau_B) \mathcal{O}_D(\Omega\tau_D), \quad (\text{S135})$$

where the ordering reflects that the dark propagator acts first, followed by the bright propagator.

Combining the dark and bright evolutions and including the accumulated noise, the covariance after one complete cycle is

$$V_{j+1} = A V_j A^T + \mathcal{Q}_{\text{cyc}}, \quad (\text{S136})$$

where we define

$$A = e^{-\gamma(\tau_D + \tau_B)/2} \mathcal{M}, \quad (\text{S137})$$

and the total noise generated during one cycle is

$$\mathcal{Q}_{\text{cyc}} = e^{-\gamma\tau_B} \mathcal{O}_B(\Omega\tau_B) \mathcal{Q}_D(\tau_D) \mathcal{O}_B^T(\Omega\tau_B) + \mathcal{Q}_B(\tau_B). \quad (\text{S138})$$

4. Closed-form solution for N cycles

The recurrence in Eq. (S136) can be iterated to obtain a closed-form expression for the covariance after N cycles:

$$V_N = A^N V_0 (A^T)^N + \sum_{k=0}^{N-1} A^k \mathcal{Q}_{\text{cyc}} (A^T)^k. \quad (\text{S139})$$

Substituting $A = e^{-\gamma(\tau_D + \tau_B)/2} \mathcal{M}$, this becomes

$$V_N = e^{-N\gamma(\tau_D + \tau_B)} \mathcal{M}^N V_0 (\mathcal{M}^T)^N + \sum_{k=0}^{N-1} e^{-k\gamma(\tau_D + \tau_B)} \mathcal{M}^k \mathcal{Q}_{\text{cyc}} (\mathcal{M}^T)^k. \quad (\text{S140})$$

The physical interpretation is clear: the first term represents the initial covariance after N deterministic cycles, while the sum represents the accumulated stochastic noise, with earlier noise contributions propagated through more subsequent cycles. Thus, the effect of repeated pulses cannot be obtained simply by multiplying single-pulse squeezing by N .

5. Bright-interval phase matching

The bright-trap duration τ_B is a crucial control parameter. During the bright interval, the covariance ellipse rotates by

$$\theta_B = \Omega \tau_B. \quad (\text{S141})$$

For $\tau_B = T = 2\pi/\Omega$, the rotation angle is 2π , returning the ellipse to its original orientation. For $\tau_B = T/2$, the rotation angle is π , which also preserves the ellipse orientation (since an ellipse is invariant under a π rotation). For $\tau_B = T/4$, the rotation angle is $\pi/2$, which interchanges the squeezed and antisqueezed directions, generally reducing the effectiveness of subsequent dark pulses.

6. Squeezing generated by the pulse sequence

The squeezing is determined by the smallest eigenvalue of the covariance matrix and can be obtained using Eq. (S47). The squeezing in decibels is then governed by Eq. (S51). In the ideal limit of negligible damping and noise, each dark pulse reduces the squeezed variance by $e^{-2\Omega\tau_D}$. With appropriate timing between pulses, the squeezed variance after N pulses scales as

$$\Lambda_{\min}^{(N)} \simeq n_{\text{th}} e^{-2N\Omega\tau_D}, \quad (\text{S142})$$

corresponding to

$$\mathcal{S}_N \simeq -\frac{20\Omega\tau_D}{\ln 10} N. \quad (\text{S143})$$

In the realistic dissipative system, this ideal scaling is modified by damping and stochastic diffusion. The deviation from exact N -fold squeezing is determined by the competition between the exponential amplification during each dark pulse and the covariance added by $\mathcal{Q}_D(\tau_D)$ and $\mathcal{Q}_B(\tau_B)$ throughout the sequence.

IV. NONLINEAR OPTOMECHANICS AND PREPARATION OF NON-GAUSSIAN STATE

To preserve the full optical nonlinearity, we avoid expanding the standing-wave intensity about the equilibrium position, as was done in the earlier force equation (Eq. (S2)). Instead, we keep all orders of the expansion and write the most general form of the axial gradient force as

$$F(z) = \frac{\alpha'}{4} \frac{\partial |E(z, t)|^2}{\partial z} = -\alpha' |\mathcal{E}_0|^2 k_l \sin(2k_l z). \quad (\text{S144})$$

This expression is the exact gradient force within the standing-wave model and therefore retains the complete spatial nonlinearity contained in the sinusoidal term.

We characterize the optical nonlinearity with the dimensionless Lamb–Dicke-like parameter

$$\eta \equiv k_l z_{\text{zpm}} = k_l \sqrt{\frac{\hbar}{m\Omega}}, \quad (\text{S145})$$

which compares the zero-point displacement to the standing-wave period. The force prefactor can then be written in terms of frequency as

$$\frac{\alpha' |\mathcal{E}_0|^2 k_l}{p_{\text{zpm}}} = \frac{\Omega_0}{2\eta}. \quad (\text{S146})$$

Using this, the dimensionless equations of motion therefore take the compact form

$$dq = \Omega_0 p dt, \quad (\text{S147})$$

$$dp = \left[-\frac{\Omega_0}{2\eta} \sin(2\eta q) - \gamma p \right] dt + dW^{\text{th}}(t), \quad (\text{S148})$$

where $dW^{\text{th}}(t)$ thermal Wiener increments. In the small-displacement limit, $\eta q \ll 1$, we retain the usual harmonic Langevin dynamics. The sine dependence in Eq. (S148), retains all higher-order nonlinear corrections of the optical force.

1. Expansion about bright and dark fringes

The distinction between bright- and dark-fringe operation is most transparent when the local optical phase is used as the expansion variable. Defining

$$2k_l z + 2\phi(t) = \tilde{z} \quad (\text{S149})$$

for a bright fringe, the force $F = -K \sin \tilde{z}$, with $\kappa_{\text{trap}} \equiv \alpha' |\mathcal{E}_0|^2 k_l$, has the expansion

$$F_{\text{bright}} = -\kappa_{\text{trap}} \sin \tilde{z} = -\kappa_{\text{trap}} \tilde{z} + \frac{\kappa_{\text{trap}}}{6} \tilde{z}^3 + \mathcal{O}(\tilde{z}^5). \quad (\text{S150})$$

Thus, the leading nonlinear correction is cubic and has the opposite sign to the linear restoring term.

At a dark fringe, the local phase is shifted by π , so that

$$2k_l z + 2\phi(t) = \pi + \tilde{z}. \quad (\text{S151})$$

Using $\sin(\pi + \tilde{z}) = -\sin \tilde{z}$, one obtains

$$F_{\text{dark}} = \kappa_{\text{trap}} \sin \tilde{z} = \kappa_{\text{trap}} \tilde{z} - \frac{\kappa_{\text{trap}}}{6} \tilde{z}^3 + \mathcal{O}(\tilde{z}^5). \quad (\text{S152})$$

Hence both the linear and cubic terms change sign between the two operating points. For $\alpha' > 0$, this reflects the reversal of the local curvature of the optical potential: the bright fringe is locally stable, whereas the dark fringe is locally unstable. Importantly, these polynomial expansions are only local descriptions; the full sinusoidal force in Eq. (S144) remains valid beyond the small-displacement regime.

V. RECOIL HEATING

A. For polarization-adaptive squeezing

A key advantage of the polarization-adaptive squeezing protocol is the ability to control the trap stiffness without changing the total optical trapping power. At the same time, this control introduces an important consideration for operation at lower pressures: the expected increase in backaction heating as the pressure is reduced toward ultra-high vacuum [6]. The recoil heating is determined by photon scattering from the local optical field [6]. Assuming plane-wave like core modes, the cycle-averaged intensity is related to the complex field amplitude $E(z)$ as given in Eq. (S1):

$$I(z) = \frac{1}{2} \epsilon_0 c |E(z)|^2. \quad (\text{S153})$$

Near the equilibrium bright fringe position $z = 0$, Eq. (S153) therefore gives

$$I_b = \epsilon_0 c |\mathcal{E}_0|^2 (1 + |\cos \theta_p|). \quad (\text{S154})$$

Thus, for fixed power P in each beam,

$$I_b \propto P(1 + |\cos \theta_p|). \quad (\text{S155})$$

the total scattered power is

$$P_{\text{sc}} = \sigma_{\text{sc}} I_b, \quad (\text{S156})$$

where σ_{sc} is the photon-scattering cross section. For a Rayleigh particle the scattering cross-section can be written as [6]:

$$\sigma_{\text{sc}} = \frac{k_\ell^4 |\alpha|^2}{6\pi\epsilon_0^2}. \quad (\text{S157})$$

Since σ_{sc} is independent of θ_p for the particle considered here, Eqs. (S155) and (S156) give

$$P_{\text{sc}} \propto P(1 + |\cos \theta_p|). \quad (\text{S158})$$

The corresponding photon-scattering rate is

$$R_{\text{sc}} = \frac{P_{\text{sc}}}{\hbar\omega_\ell}. \quad (\text{S159})$$

which produces momentum diffusion and hence recoil heating of the mechanical mode. Following Ref. [6], the recoil heating rate of the axial mode, expressed in phonons per unit time, is

$$\Gamma_{\text{rec}} = \frac{1}{5} \frac{P_{\text{sc}}}{mc^2} \frac{\omega_\ell}{\Omega}, \quad (\text{S160})$$

where m is the particle mass, c is the speed of light, and Ω is the axial trap frequency.

Combining Eqs. (S158), (S160), and (S4), we find

$$\begin{aligned} \Gamma_{\text{rec}} &\propto \frac{P(1 + |\cos \theta_p|)}{\Omega} \\ &\propto \sqrt{P} \frac{1 + |\cos \theta_p|}{\sqrt{|\cos \theta_p|}}. \end{aligned} \quad (\text{S161})$$

Therefore, at fixed optical power,

$$\Gamma_{\text{rec}} \propto \frac{1 + |\cos \theta_p|}{\sqrt{|\cos \theta_p|}}. \quad (\text{S162})$$

Normalizing to the parallel-polarization configuration, $\theta_p = 0$, gives

$$\frac{\Gamma_{\text{rec}}}{\Gamma_{\text{rec},0}} = \frac{1 + |\cos \theta_p|}{2\sqrt{|\cos \theta_p|}}. \quad (\text{S163})$$

From this, we can find that rotating the relative polarization from parallel to $\theta_p = 88.3^\circ$ reduces the axial trap frequency to approximately 11.1% of its value for parallel polarization (denoted with subscript 0), while increasing the recoil heating rate, expressed in axial phonons per unit time, by a factor of approximately 3. This behavior is distinct from conventional squeezing protocols based on reducing the trapping intensity, for which both the trap stiffness and the recoil energy-diffusion rate decrease linearly with the optical power P . The standing-wave geometry, however, offers an additional degree of control that can be used to suppress recoil heating spatially.

B. Recoil suppression near a dark fringe

An alternative to polarization-based control is to operate the standing-wave trap near a dark fringe, where the local optical intensity is suppressed while the optical potential retains finite curvature. For parallel polarizations, $\theta_p = 0$, the standing-wave intensity near the dark fringe is

$$|E(z)|^2 = 4|\mathcal{E}_0|^2 \sin^2(k_\ell z) \simeq 4|\mathcal{E}_0|^2 k_\ell^2 z^2, \quad (\text{S164})$$

where the last equality holds for a feedback-cooled particle in the absence of squeezing as $k_\ell|z| \ll 1$.

The recoil heating rate is proportional to the photon-scattering rate and hence to the local intensity. Using

$$I(z) = \frac{1}{2}\epsilon_0 c |E(z)|^2,$$

the mean intensity sampled near the dark fringe is

$$\langle I_d(z) \rangle \simeq 2\epsilon_0 c |\mathcal{E}_0|^2 k_\ell^2 \langle z^2 \rangle, \quad (\text{S165})$$

where $\langle z^2 \rangle$ is the mean-square axial displacement. At a bright fringe,

$$I_b = 2\epsilon_0 c |\mathcal{E}_0|^2. \quad (\text{S166})$$

Since the scattered power is $P_{\text{sc}} = \sigma_{\text{sc}} I$, the cross section cancels in the ratio, giving

$$\frac{\Gamma_d}{\Gamma_b} = \frac{\langle I_d(z) \rangle}{I_b} \simeq k_\ell^2 \langle z^2 \rangle \quad (\text{S167})$$

where Γ_d and Γ_b are the recoil heating rates at the dark and bright fringes, respectively.

Thus, in the Lamb-Dicke regime,

$$k_\ell^2 \langle z^2 \rangle \ll 1, \quad (\text{S168})$$

recoil heating is strongly suppressed while the trap curvature remains finite. During accelerated squeezing, the increasing motional variance $\langle z^2(t) \rangle$ progressively increases the scattering rate according to

$$\Gamma_d(t) \simeq \Gamma_b k_\ell^2 \langle z^2(t) \rangle. \quad (\text{S169})$$

During accelerated squeezing, however, the growth of $\langle z^2(t) \rangle$ progressively increases the recoil rate. The resulting increase in recoil heating therefore establishes a trade-off between increasing the degree of squeezing/delocalization and maintaining low recoil heating. In Fig. S6(a), we indicate theoretical recoil heating for polarization-adaptive squeezing as a function of θ_p . In Fig. S6(b) the ratio of the recoil-heating rates in the dark and bright trapping configurations, $\Gamma_{\text{rec,d}}/\Gamma_{\text{rec,b}}$, as a function of the number of dark-trapping events N . The red circle and the black arrow indicates our present experimental conditions.

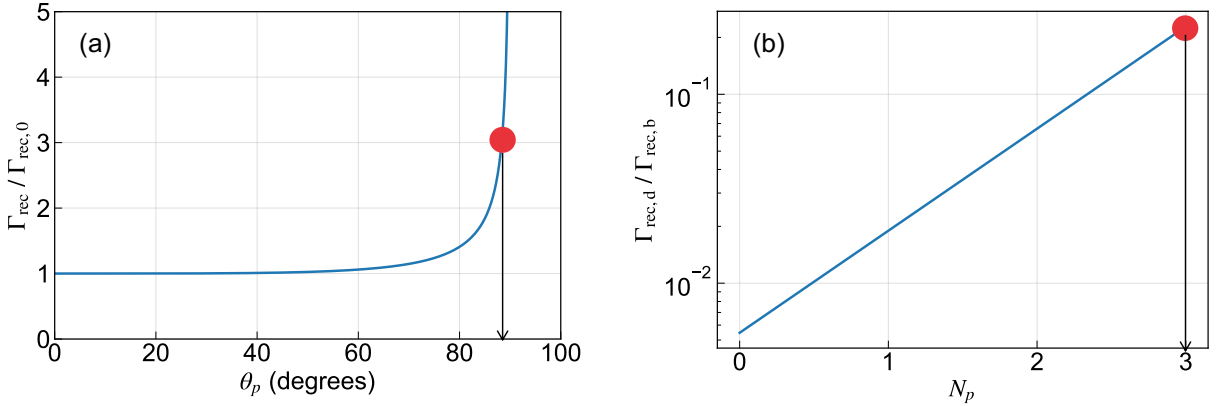


FIG. S6. **Numerically calculated recoil heating;** (a) Analytical recoil heating rate during polarization-adaptive delocalization as a function of the relative polarization angle θ_p between the trapping beams. The recoil rate at the red circle ($\theta_p = 88.3^\circ$) is 3 times higher than that of at $\theta_p = 0^\circ$. (b) Ratio of the analytical recoil-heating rates in the dark and bright fringe configurations, $\Gamma_{\text{rec,d}}/\Gamma_{\text{rec,b}}$, as a function of the number of dark-pulse events N_p . Here $N_p = 0$ means the particle is just transferred from bright to dark fringe for the first time. The red circle and the black arrow indicate our experiment conditions highlighting the fact that the recoil rate in the dark fringe trap at the maximum delocalization instance of 13 dB is ~ 0.25 times smaller than that of in the bright fringe trap.

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