

Small numbers, big bubbles: Gambler's fallacy and hot-hand traders in a heterogeneous-agent market

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Small numbers, big bubbles: Gambler's fallacy and hot-hand traders in a heterogeneous-agent market

Abstract In this paper I build a simple heterogeneous-agent model of asset price bubbles. The behavioural traders in the model believe in the “law of small numbers” of Rabin (2002). They think of price changes as draws from an urn with up and down balls. Contrarian traders know the starting mix of the urn, so they expect reversals. This is the gambler's fallacy. Momentum traders do not know the mix and estimate it from the last price change, so they expect the change to continue. This is the hot-hand fallacy. These traders trade with fundamentalists, who know the fundamental value but have limited risk-bearing capacity, and with liquidity traders. With CARA utility and normal beliefs, the market-clearing price has a closed form. I show that the model has a unique steady state at the fundamental value. Near this steady state, the gap between the price and the fundamental value follows a simple second-order difference equation with one coefficient. This coefficient combines the share of momentum traders, the share of fundamentalists, fundamental risk and the precision of beliefs. It tells whether the market is stable and how large and long bubbles are. In simulations, one change in the fundamental value, or one liquidity shock, creates a bubble that overshoots, a crash below the fundamental value and smaller secondary bubbles. Trading volume peaks together with mispricing. The model also gives a psychological foundation for the extrapolative demand used in earlier bubble models.

Keywords Asset price bubbles · Law of small numbers · Heterogeneous agents · Momentum and contrarian trading · Gambler's fallacy · Trading volume

JEL Classification G12 · G41 · D84 · C62

1 Introduction

Asset price bubbles are episodes in which prices move far away from fundamental values and then collapse. They have a long history, from the Dutch tulip mania to the dot-com bubble (Kindleberger and Aliber 2005; Ofek and Richardson 2003). Bubbles misallocate capital, redistribute wealth and can hurt the real economy. So it is important to understand how they start, grow and burst.

There are two broad groups of models of bubbles. In rational bubble models, a bubble can exist because investors expect to sell the asset later at a higher price (Blanchard and

Watson 1982; Tirole 1985). These models say little about how a bubble starts and why it bursts. In behavioural and disagreement models, bubbles come from biased beliefs or from disagreement between investors (De Long et al. 1990; Scheinkman and Xiong 2003; Barberis et al. 2018; Liao et al. 2022). Many behavioural models use extrapolation: some investors expect recent price changes to continue. Extrapolation fits survey data on investor expectations well (Greenwood and Shleifer 2014). But in most models it is simply assumed.

There is also a large literature on heterogeneous-agent models (HAMs), where fundamentalists trade with chartists (Day and Huang 1990; Lux 1995; Brock and Hommes 1998; Hommes 2006). These models show that simple trading rules can create bubbles, crashes and complex price dynamics. They can also be estimated on real data (Boswijk et al. 2007; Chiarella et al. 2014; Franke and Westerhoff 2016). However, the chartist rules are again usually assumed, not derived from a model of how people form beliefs.

In this paper I derive both trend-following and contrarian demand from one well-known psychological bias. This is the belief in the “law of small numbers” (Tversky and Kahneman 1971; Rabin 2002). People with this bias expect even short random sequences to look like the population they come from. Rabin (2002) models them as if they draw balls from a small urn without replacement. Rabin and Vayanos (2010) show that the same bias can lead to two different errors. One is the gambler’s fallacy, when people expect a run to reverse. The other is the hot-hand fallacy, when they expect a run to continue. I use this idea to build two types of behavioural traders. Contrarian traders know the starting mix of the urn of price changes, so after a rise they expect a fall. Momentum traders do not know the mix. They estimate it from the last price change, so they expect the change to continue. These traders trade with fundamentalists, who know the fundamental value but have limited risk-bearing capacity (Shleifer and Vishny 1997), and with liquidity traders.

I keep the model simple. Each period, agents with CARA utility and normal beliefs choose positions in a risky asset, and the market clears. This gives a closed-form price. I have four main results. First, the model has a unique steady state. In it, the price equals the fundamental value and there is no trade. Second, near the steady state, the gap between the price and the fundamental value follows a second-order linear difference equation with one coefficient a . This coefficient is larger when there are more momentum traders and when fundamental risk is higher. It is smaller when there are more fundamentalists and when the momentum traders’ beliefs are more precise. The steady state is stable if and only if $a < 1$, and the price converges with oscillations. Third, one change in fundamentals or one liquidity shock creates a bubble that overshoots. Then the price crashes below the fundamental value,

and smaller secondary bubbles follow. This path looks like the phases of a bubble described by Kindleberger and Aliber (2005). Fourth, trading volume rises and falls together with mispricing. This agrees with the evidence that trading volume is high in bubbles (Ofek and Richardson 2003).

The paper adds to two literatures. For the behavioural literature on bubbles, it gives a psychological foundation for extrapolative and momentum demand. I show that the extrapolative beliefs of Barberis et al. (2018) and the momentum order flow of Hong and Stein (1999) are special cases of law-of-small-numbers beliefs. For the HAM literature, it gives a microfounded version of the fundamentalist–chartist model. In this version, the strength of trend-following has a clear behavioural meaning, and the stability condition has a closed form. The simulations use only the model itself and no external data.

The paper is organised as follows. Section 2 reviews the related literature. Section 3 presents the model. Section 4 derives the analytical results. Section 5 presents the simulations. Section 6 discusses the results, the policy implications and the limitations, and Section 7 concludes. The proofs are in the Appendix.

2 Related literature

Rational and disagreement models. In rational bubble models, a bubble can exist when the price grows at the required rate of return and investors expect to sell to someone else (Blanchard and Watson 1982; Tirole 1985). In these models, a bubble cannot start once the asset is traded, and the bubble bursts for exogenous reasons. Disagreement models explain bubbles by differences in beliefs together with short-sale constraints. In Scheinkman and Xiong (2003), overconfident investors disagree about fundamentals. The price then includes the value of the option to resell the asset to a more optimistic investor. Doblas-Madrid (2012) builds a rational model of bubbles in which agents receive private signals at different times, and the bubble bursts endogenously. These models explain several features of bubbles, but the source of disagreement is usually exogenous.

Extrapolation and behavioural models. De Long et al. (1990) show that rational speculators may buy ahead of positive-feedback traders and push prices further away from fundamentals. Barberis et al. (1998), Daniel et al. (1998) and Hong and Stein (1999) explain short-run momentum and long-run reversal with different behavioural mechanisms. Barberis et al. (2018) build a model of bubbles in which investors put weight on both a fundamental signal and an extrapolative signal and “waver” between them. This creates bubbles with high trading volume. Liao et al. (2022) get high volume from extrapolation together with

realisation utility. Bordalo et al. (2019) link overreaction to diagnostic expectations. In most of these papers, extrapolation itself is an assumption. I derive it from the law of small numbers.

The law of small numbers. Rabin (2002) models people who believe that random draws come from a small urn without replacement. So they expect too much reversal in short sequences. If they do not know the composition of the urn, they draw too strong conclusions from short streaks. Rabin and Vayanos (2010) develop this idea into a general model of the gambler's and hot-hand fallacies. They apply it to fund flows and to momentum and reversal in asset returns. Their model studies how a single investor forms beliefs about a signal process. Here I put two types of such investors into a simple market-clearing model and study the price dynamics they create. Laboratory evidence also shows that uninformed traders follow momentum and contrarian strategies (Bloomfield et al. 2009).

Heterogeneous-agent models. HAMs study markets where fundamentalists and chartists trade with each other (Day and Huang 1990; Lux 1995; Brock and Hommes 1998). In the influential model of Brock and Hommes (1998), traders switch between simple forecasting rules depending on past performance. This can lead to bifurcations and chaotic dynamics. The surveys by Hommes (2006) and He (2013) show that these models can reproduce excess volatility, bubbles and crashes. Estimated versions find strong evidence of behavioural heterogeneity in stock prices (Boswijk et al. 2007; Chiarella et al. 2014). Simple herding models can also match the stylised facts of daily returns (Franke and Westerhoff 2016). My model belongs to this family. But here the trading rules come from the law of small numbers, and the strength of trend-following depends on how many “balls” traders think are drawn each period. The population shares in my model are fixed. So it is closest to the simple fundamentalist–chartist models of Day and Huang (1990) and Chiarella (1992), not to models with strategy switching.

3 The model

3.1 Market setup

Time is discrete, $t = 0, 1, 2, \dots$. There are two assets. The risk-free asset has a net return of zero. The risky asset has a fundamental value F_t , which follows a random walk:

$$F_t = F_{t-1} + v_t, \quad v_t \sim N(0, \sigma_v^2) \quad (1)$$

Each period, agents choose a position X_t in the risky asset at price P_t and hold it for one period. This is the same as trading a one-period forward contract. It removes the need for

intertemporal optimisation, as in Barberis et al. (2018). All agents have CARA utility over next-period wealth with risk aversion γ , and they believe that the next price is normally distributed. Under these assumptions, the optimal position of an agent with expected price change μ_t and variance s_t^2 is

$$X_t = \frac{\mu_t}{\gamma s_t^2} \quad (2)$$

There are four types of traders. Fundamentalists have mass θ . Traders who believe in the law of small numbers (LSN traders) have mass $1 - \theta$. A share τ of them are momentum traders, and a share $1 - \tau$ are contrarian traders. Liquidity traders submit an exogenous demand X_t^L . The net supply of the risky asset is zero, and short sales are allowed.

3.2 Fundamentalists

Fundamentalists know F_t and expect the price to move towards it. So their expected price change is $F_t - P_t$, and their perceived variance is the fundamental variance σ_v^2 . By Eq. (2), their demand is

$$X_t^f = \frac{F_t - P_t}{\gamma \sigma_v^2} \quad (3)$$

Fundamentalists do not trade the price fully back to the fundamental value. Their positions are limited by risk aversion and by fundamental risk. This is a simple way to capture limits to arbitrage (Shleifer and Vishny 1997). This demand is the same as in Barberis et al. (2018) when the asset supply is zero.

3.3 Beliefs of LSN traders

LSN traders do not know the fundamental value. Following Rabin (2002), they think of price changes as draws from an urn with N balls. A share λ of the balls are “up” balls, which raise the price by one unit. The rest are “down” balls, which lower it by one unit. Each period, M balls are drawn without replacement, where M is large but much smaller than N . If M_t^+ up balls are drawn, the price change is

$$\Delta P_t = P_{t+1} - P_t = 2M_t^+ - M \quad (4)$$

Because $M \ll N$, the number of up balls is approximately binomial. Because M is large, it is also approximately normal, $M_t^+ \sim N(M\lambda, M\lambda(1 - \lambda))$. So an LSN trader with belief λ expects a price change of $M(2\lambda - 1)$ with variance $4M\lambda(1 - \lambda)$. By Eq. (2), the demand of an LSN trader is

$$X_t = \frac{1}{\gamma} h(\lambda_t), \quad h(\lambda) = \frac{2\lambda - 1}{4\lambda(1 - \lambda)} \quad (5)$$

The function h is increasing, equals zero at $\lambda = 1/2$, and has slope $h'(1/2) = 2$. An LSN trader buys when she thinks up balls are more likely and sells when she thinks down balls are more likely. The parameter M measures how “lumpy” the trader thinks price changes are. A larger M means that one observed price change says less about the urn, so beliefs move less. The two types of LSN traders differ in what they know about the urn.

3.4 Contrarian traders

Contrarian traders know that the urn starts with the same number of up and down balls, $\lambda_0 = 1/2$. This is the only starting value at which the expected price change is zero, which fits a zero interest rate. After they observe a price change ΔP_{t-1} , they infer from Eq. (4) that $(M + \Delta P_{t-1})/2$ up balls were taken out of the urn. Then the urn is refilled with M new balls in the original proportion. I also let beliefs return slowly to $1/2$ at rate $\kappa \in [0,1)$, so that contrarians cannot hold wrong beliefs forever. Their belief evolves as

$$\lambda_t^c = \lambda_{t-1}^c - \frac{\Delta P_{t-1}}{2N} + \kappa \left(\frac{1}{2} - \lambda_{t-1}^c \right) \quad (6)$$

After a price rise, fewer up balls are left in the urn. So contrarians expect a fall and sell. This is the gambler’s fallacy. Because N is large, one price change moves their belief only a little.

3.5 Momentum traders

Momentum traders do not know the composition of the urn. They estimate it from the last draw, using the share of up balls among the M balls drawn. By Eq. (4), this gives

$$\lambda_t^m = \frac{M_{t-1}^+}{M} = \frac{1}{2} + \frac{\Delta P_{t-1}}{2M} \quad (7)$$

After a price rise, they believe that the urn has more up balls, so they expect the rise to continue. This is the hot-hand fallacy. Their beliefs react much more strongly to price changes than those of contrarians, because they learn from M draws instead of updating an urn of N balls. If we substitute Eq. (7) into Eq. (5), we get the momentum demand

$$X_t^m = \frac{1}{\gamma} \frac{M \Delta P_{t-1}}{M^2 - \Delta P_{t-1}^2} \quad (8)$$

Eq. (7) requires $|\Delta P_{t-1}| < M$, so that λ_t^m is a probability. This is a natural limit of the model: a trader cannot see more up balls than the number of balls drawn. I come back to it in Sect. 6.

3.6 Liquidity traders and equilibrium

Liquidity traders trade for reasons that are not related to the asset, for example because they need cash. Their demand X_t^L is an exogenous process with mean zero. The market clears when total demand equals the zero net supply:

$$\theta X_t^f + (1 - \theta)[\tau X_t^m + (1 - \tau)X_t^c] + X_t^L = 0 \quad (9)$$

Proposition 1 (Equilibrium price). *For any beliefs $\lambda_t^m, \lambda_t^c \in (0,1)$ and any liquidity demand X_t^L , the market clears at a unique price*

$$P_t = F_t + \frac{\sigma_v^2}{\theta} \{ (1 - \theta)[\tau h(\lambda_t^m) + (1 - \tau)h(\lambda_t^c)] + \gamma X_t^L \} \quad (10)$$

The proof is in the Appendix. Only the fundamentalists' demand depends on the current price. So the market-clearing condition is linear in P_t and has exactly one solution. Fundamentalists absorb the net demand of the other traders, and the price moves away from F_t until they are willing to do this. The gap is larger when there are few fundamentalists (small θ) and when they see the asset as risky (large σ_v^2).

I define the bubble as the gap between the price and the fundamental value, $B_t = P_t - F_t$ (Brunnermeier 2001). I call an episode a bubble when $|B_t|$ is above some threshold. The risk-free rate is zero and the asset supply is zero, so B_t measures mispricing directly. I measure trading volume as the total size of long positions. This equals the total size of short positions:

$$V_t = \frac{1}{2} [\theta |X_t^f| + (1 - \theta)\tau |X_t^m| + (1 - \theta)(1 - \tau)|X_t^c| + |X_t^L|] \quad (11)$$

4 Analytical results

4.1 Steady state

Proposition 2 (Steady state). *Suppose that the fundamental value is constant, $F_t = F$, there is no liquidity demand, and $\kappa > 0$. Then the model has a unique steady state. In it, $\lambda^m = \lambda^c = 1/2$, the price equals the fundamental value, $P = F$, the expected price change is zero, and there is no trade.*

In the steady state, all price changes are zero. Then, by Eq. (7), momentum traders believe that the urn is balanced, and by Eq. (6), the contrarians' belief goes back to 1/2. Both LSN groups have zero demand. So fundamentalists must also hold zero positions, and this happens only at $P = F$. The steady state is therefore the efficient-market outcome: the price equals the fundamental value and grows at the risk-free rate of zero. In this model, bubbles are temporary deviations from this state, and they are caused by shocks.

207 4.2 Dynamics after a shock

208 Now consider a one-time change in the fundamental value. The market is in the steady
 209 state before period T . At T , the fundamental value jumps from F to $F + \Delta F$ and stays there.
 210 There is no liquidity demand.

211 In period T , beliefs are still at $1/2$, because they depend only on past price changes. So
 212 LSN traders do not trade, and the price jumps to the new fundamental value, $P_T = F + \Delta F$.
 213 Fundamentalists do not trade either, so there is no volume. Kindleberger and Aliber (2005)
 214 call this phase “displacement”. In period $T + 1$, momentum traders see the price change ΔF
 215 and become optimistic. If we ignore the small contrarian term, Eqs. (8) and (10) give the first
 216 bubble:

$$B_{T+1} = a \frac{\Delta F}{1 - (\Delta F/M)^2}, \quad a = \frac{\sigma_v^2(1 - \theta)\tau}{\theta M} \quad (12)$$

217 The coefficient a shows how strong trend-following is in the market. It is the price
 218 impact of momentum demand. It equals the share of momentum traders, $(1 - \theta)\tau$, times the
 219 price that fundamentalists need to absorb one unit of demand, $\gamma\sigma_v^2/\theta$, times the sensitivity of
 220 momentum demand to the last price change, $1/(\gamma M)$. After this first step, the bubble feeds on
 221 itself. The next price change is B_{T+1} . It changes momentum beliefs again, and so on.

222 **Proposition 3** (Local dynamics). *Near the steady state, and ignoring the contrarian*
 223 *term, which is of order $1/N$, the bubble follows*

$$B_t = a(B_{t-1} - B_{t-2}), \quad t \geq T + 2 \quad (13)$$

224 Then: (i) if $0 < a < 4$, the bubble changes sign over time, that is, convergence or
 225 divergence is oscillatory; (ii) the steady state is stable if and only if $a < 1$, and in this case
 226 $|B_t|$ shrinks by a factor of about $\sqrt{a}$ per period; (iii) the oscillations have a period of $2\pi/$
 227 $\arccos(\sqrt{a}/2)$.

228 The proof is in the Appendix. Eq. (13) has a simple meaning. Momentum traders buy
 229 when the price has just risen, and this pushes the price above the fundamental value. When
 230 the price stops rising, they lose their optimism, and fundamentalists push the price back. The
 231 fall in the price makes momentum traders pessimistic, so the price overshoots below the
 232 fundamental value. When $a < 1$, each swing is smaller than the one before. This mechanism
 233 creates sequential bubbles: after the first bubble comes a smaller negative bubble, then a
 234 smaller positive one, and so on. So the steady state of Proposition 2 is not automatically
 235 stable. Proposition 3 shows that it is stable only if $a < 1$, that is,

$$\sigma_v^2(1 - \theta)\tau < \theta M \quad (14)$$

So a market can be unstable if it has few fundamentalists, high fundamental uncertainty, many momentum traders or very reactive momentum beliefs (small M). In such a market, small shocks create growing swings. Contrarians work in the opposite direction. They add a term $2c \ell_t$ to Eq. (13), where $\ell_t = \lambda_t^c - 1/2$ is their belief deviation, which falls after price rises, and $c = \sigma_v^2(1 - \theta)(1 - \tau)/\theta$. Since ℓ_t moves by only $\Delta P_{t-1}/(2N)$ per period, this term is very small when N is large. In the baseline calibration below, it changes the price by less than 0.1.

Proposition 4 (Size of bubbles). *The first bubble B_{T+1} in Eq. (12) is increasing in the share of momentum traders τ , in fundamental risk σ_v^2 and in the size of the fundamental shock $|\Delta F|$, and decreasing in the share of fundamentalists θ and in the precision parameter M . It is more than proportional to ΔF : large changes in fundamentals create disproportionately large bubbles.*

This follows directly from Eq. (12). The last property comes from the variance term in the momentum demand. A large price change raises the expected price change of momentum traders. It also makes them more certain about it, because $\lambda_t^m(1 - \lambda_t^m)$ falls when λ_t^m moves away from 1/2.

Liquidity bubbles. Now let the fundamental value be constant, and let liquidity traders buy X^L units in only one period T_L . By Eq. (10), the price rises by $\gamma\sigma_v^2 X^L/\theta$ in that period. Then momentum traders see a price rise, and from period $T_L + 1$ the dynamics are the same as in Eq. (13). So a liquidity bubble is a bubble without any change in fundamentals. For an observer who sees only prices, it looks the same as a fundamental bubble. This is a difference from Barberis et al. (2018), where bubbles start from good news about fundamentals.

4.3 Volume

Proposition 5 (Volume and mispricing). *(i) In period T there is no trade. (ii) In every period, $V_t \geq \theta|B_t|/(\gamma\sigma_v^2)$, with equality when all other traders take positions opposite to the fundamentalists. (iii) When contrarian positions are negligible (large N) and there is no liquidity demand, volume is proportional to $|B_t|$, so it is highest when the bubble is largest.*

When momentum traders buy, fundamentalists must sell, and the size of their short position is proportional to the mispricing. So volume and mispricing rise and fall together. In the model, people trade because LSN traders and fundamentalists disagree, and they disagree most at the top of the bubble. This agrees with the high volume at the peak of the dot-com bubble (Ofek and Richardson 2003). It also agrees with the disagreement-based explanation

268 of volume in Scheinkman and Xiong (2003). Short sales are allowed, so contrarians can also
 269 trade against the bubble, and this adds volume at the peak.

270 4.4 Links to extrapolation models

271 The law-of-small-numbers beliefs nest two popular forms of trend-following demand.

272 **Corollary 1** (Extrapolation). *Suppose that momentum traders estimate λ^m as an*
 273 *exponentially weighted average of past draws, $\hat{\lambda}_t = (1 - \zeta) \sum_{k \geq 1} \zeta^{k-1} M_{t-k}^+ / M$ with $\zeta \in$*
 274 *(0,1). Then their expected price change is $M(2\hat{\lambda}_t - 1) = (1 - \zeta) \sum_{k \geq 1} \zeta^{k-1} \Delta P_{t-k}$, which is*
 275 *the extrapolative expectation of Barberis et al. (2018) with weight ζ .*

276 **Corollary 2** (Momentum order flow). *For small price changes, $|\Delta P_{t-1}| \ll M$, the*
 277 *momentum demand (8) is approximately linear, $X_t^m \approx \Delta P_{t-1} / (\gamma M)$. This is the order flow of*
 278 *momentum traders in Hong and Stein (1999) with a one-period horizon and slope $1/(\gamma M)$.*

279 Both results follow by substitution. They show that the “extrapolation parameter” of
 280 earlier models has a behavioural meaning here. It is the number of balls M that traders think
 281 are drawn each period. Traders who think that price changes come from many small
 282 independent draws (large M) react less to any single price change.

283 5 Simulations

284 5.1 Calibration

285 Now I simulate the model to show the bubble dynamics. I do not try to match a particular
 286 market. I only want to illustrate the mechanisms of Sect. 4. Table 1 shows the parameters. The
 287 share of fundamentalists is small, $\theta = 0.07$, and most LSN traders are momentum traders,
 288 $\tau = 0.9$, so trend-following matters. In period $T = 7$, the fundamental value rises from 10 to
 289 60. With $\Delta F = 50$ and $M = 100$, momentum beliefs stay inside (0,1). The coefficient a is
 290 0.299 in the baseline and 0.436 in the “sequential bubbles” case with higher fundamental risk.
 291 So both cases are stable.

292 **Table 1** Parameters of the simulations

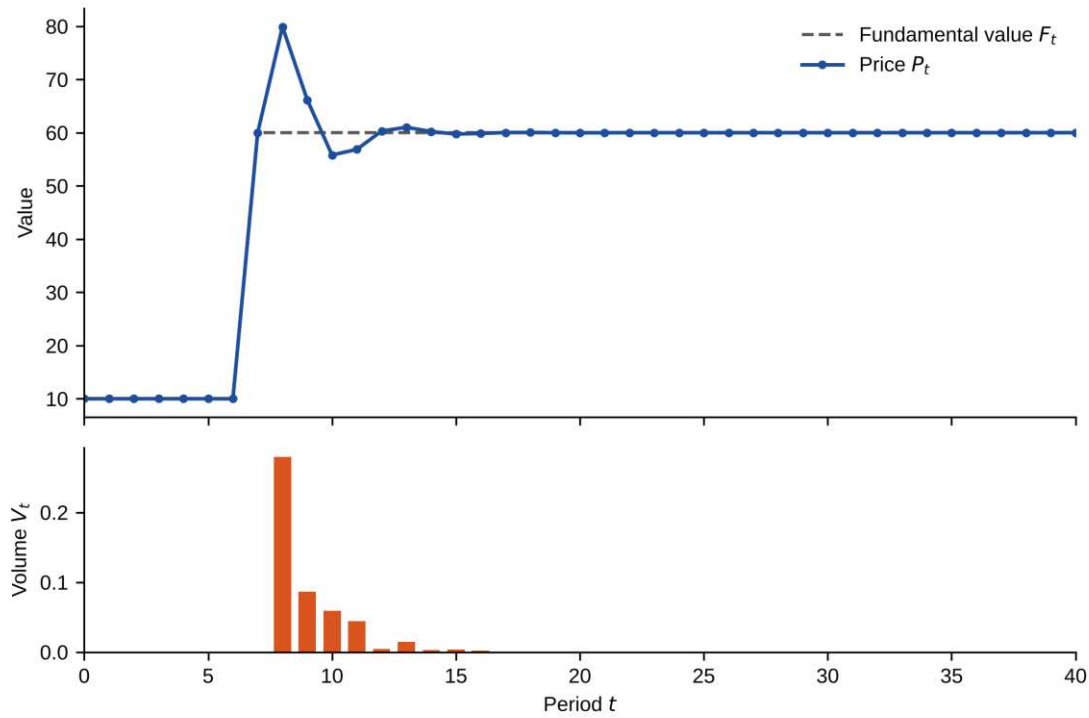
Parameter	Symbol	Baseline	Sequential bubbles	Liquidity bubble
Risk aversion	γ	2	2	2
Share of fundamentalists	θ	0.07	0.07	0.07
Share of momentum traders among LSN traders	τ	0.9	0.9	0.9
Balls drawn per period	M	100	100	100
Balls in the urn	N	10,000	10,000	10,000
Belief correction of contrarians	κ	0.05	0.05	0.05
Fundamental variance	σ_v^2	2.5	3.65	2.5
Fundamental value	F_t	10 for $t < 7$, 60 for $t \geq 7$	10 for $t < 7$, 60 for $t \geq 7$	10
Liquidity demand	X_t^L	0	0	0.25 at $t = 9$, 0 otherwise
Trend coefficient	a	0.299	0.436	0.299

293 The coefficient a is defined in Eq. (12). For any κ between 0 and 0.2, the price path changes by less
 294 than 0.02, because the contrarian term is small when N is large

295 I wrote the simulation code in Python, and it is provided as Online Resource 1.
 296 Generative AI tools were used to help write and check the simulation code. I checked all code
 297 and results.

298 5.2 A fundamental bubble

299 Fig. 1 shows the baseline simulation. In period 7, the fundamental value rises to 60, and
 300 the price jumps with it. There is no trade, because beliefs have not changed yet. In period 8,
 301 momentum traders react to the price rise. Their belief λ^m rises to 0.75, and the price
 302 overshoots to 79.9, about 33% above the new fundamental value. Volume reaches its
 303 maximum. In period 9, the price rise is smaller, momentum traders become less optimistic,
 304 and the price falls to 66.2. In periods 10 and 11, the price falls below the fundamental value,
 305 to 55.8 and 56.9, because momentum traders now extrapolate the fall. Volume rises again, but
 306 less, during this crash. After a small rebound above the fundamental value in period 13 (+1.0),
 307 the price settles at 60. The gap stays above 0.5 in absolute value for only six periods after the
 308 shock.



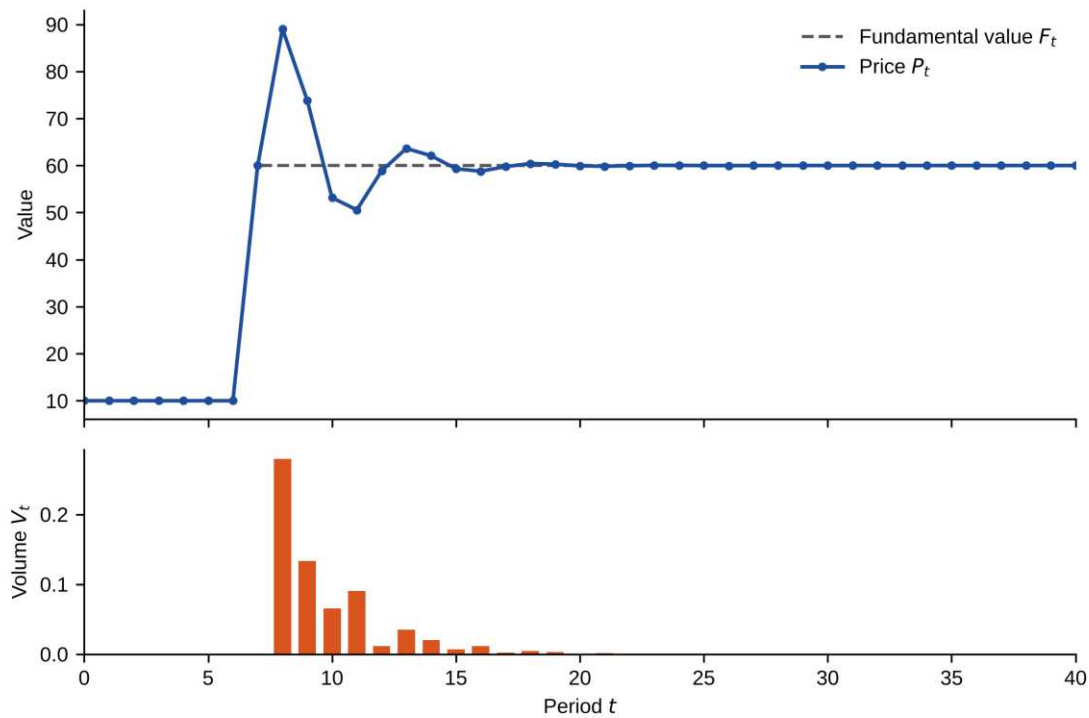
309

310 **Fig. 1** Baseline fundamental bubble. Top: price (solid line with markers) and fundamental
 311 value (dashed line). Bottom: trading volume V_t from Eq. (11). The fundamental value rises
 312 from 10 to 60 in period 7. Parameters as in Table 1

313 The path follows the phases of a bubble described by Kindleberger and Aliber (2005).
 314 The change in fundamentals is the displacement (period 7). The overshooting with high
 315 volume is the boom and euphoria (period 8). The fall in periods 9–11, when the price goes
 316 below its fundamental value, is the distress and revulsion. Then the market recovers. The
 317 small rebound in period 13 does not last, so it looks like a “bull trap” during the recovery.

318 5.3 Sequential bubbles

319 When fundamental risk is higher ($\sigma_v^2 = 3.65$), fundamentalists are less willing to trade
 320 against mispricing, and α rises to 0.436. Fig. 2 shows that the first bubble is larger: the price
 321 reaches 89.1, almost 50% above the fundamental value. The crash is also deeper, to 50.6 in
 322 period 11. After it comes a second positive bubble (+3.6 in period 13). The swings now die
 323 out more slowly, by a factor of about $\sqrt{0.436} = 0.66$ per period instead of $\sqrt{0.299} = 0.55$.
 324 The gap stays above 0.5 for nine periods. So one shock to fundamentals creates several
 325 bubbles in a row, as Proposition 3 predicts.



326

327 **Fig. 2** Sequential bubbles with higher fundamental risk ($\sigma_v^2 = 3.65$). Top: price and
 328 fundamental value. Bottom: trading volume. Other parameters as in Table 1

329 5.4 A liquidity bubble

330 Fig. 3 shows a bubble without any change in fundamentals. The fundamental value stays
 331 at 10, and in period 9 liquidity traders buy 0.25 units. The price jumps to 27.9, which is
 332 $\gamma\sigma_v^2 X^L/\theta = 17.9$ above the fundamental value. After that, momentum traders take over. The
 333 price falls back, overshoots below the fundamental value (by 3.8 in period 11) and converges
 334 with the same oscillations as in Fig. 1. So a liquidity shock that lasts one period creates a
 335 bubble that lasts several periods.

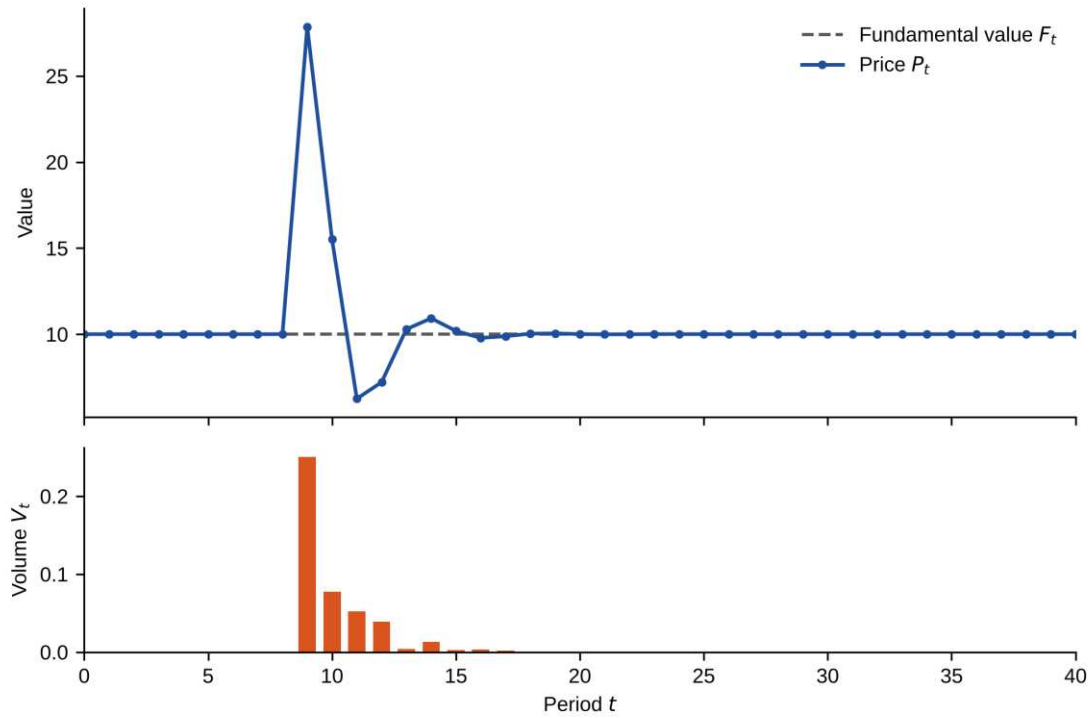


Fig. 3 Liquidity bubble. The fundamental value is constant at 10, and liquidity traders buy 0.25 units in period 9. Top: price and fundamental value. Bottom: trading volume. Other parameters as in Table 1

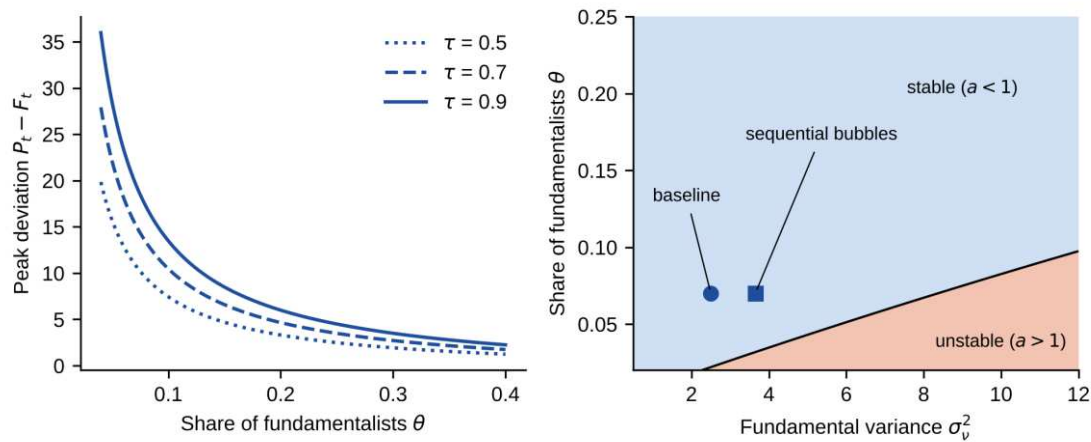
5.5 Comparative statics

Table 2 shows how the bubble changes with the parameters, and Fig. 4 shows the same results in graphs. If the share of fundamentalists doubles (to 0.14), the coefficient a halves and the first bubble falls from 19.9 to 9.2. A more balanced mix of LSN traders ($\tau = 0.5$) has a similar effect. If M doubles, momentum traders react half as strongly to each price change, and the bubble falls to 8.0. A smaller fundamental shock ($\Delta F = 25$ instead of 50) creates a bubble that is much less than half as large (8.0 instead of 19.9). This is the convexity in Proposition 4. The right panel of Fig. 4 shows the stability boundary $a = 1$. Both calibrations are in the stable region. But markets with fewer fundamentalists or higher fundamental risk can move into the unstable region.

350 **Table 2** Comparative statics of the fundamental bubble

Case	a	Peak bubble	Deepest crash	Periods with $ B_t \geq 0.5$	Peak volume	Half-life of swings	Period of swings
Baseline	0.299	19.91	-4.20	6	0.279	1.15	4.86
More fundamentalists ($\theta = 0.14$)	0.138	9.21	-1.11	3	0.258	0.70	4.54
Fewer momentum traders ($\tau = 0.5$)	0.166	10.99	-1.62	4	0.155	0.77	4.60
Higher fundamental risk ($\sigma_v^2 = 3.65$)	0.436	29.07	-9.44	9	0.279	1.67	5.09
Less reactive beliefs ($M = 200$)	0.149	7.95	-1.03	3	0.112	0.73	4.57
Smaller shock ($\Delta F = 25$)	0.299	7.96	-1.68	4	0.112	1.15	4.86

351 Peak bubble and deepest crash are the largest positive and negative values of $B_t = P_t - F_t$. Half-life
 352 and period (in periods) are computed from Proposition 3 as $\ln 0.5 / \ln \sqrt{a}$ and $2\pi / \arccos(\sqrt{a}/2)$. All
 353 other parameters as in the baseline of Table 1



354 **Fig. 4** Comparative statics. Left: peak deviation of the price from the fundamental value after
 355 a fundamental shock of 50, as a function of the share of fundamentalists, for three shares of
 356 momentum traders. Right: stability region of the steady state, $a < 1$ (light area), for $\tau = 0.9$
 357 and $M = 100$; the markers show the baseline and the sequential-bubbles calibration
 358

359 5.6 A stochastic version

360 Finally, I simulate the model with random shocks to see which stylised facts it can
 361 produce. The fundamental value follows Eq. (1) with standard deviation σ_v . Liquidity demand
 362 is normal with mean zero and standard deviation 0.02. I run 20,000 periods and drop the first
 363 100. Table 3 shows the results.

364 **Table 3** Statistics of the stochastic simulations

Statistic	Baseline	$\sigma_v^2 = 3.65$	$\tau = 0.5$	$\theta = 0.2$
Std. dev. of price changes	2.57	3.65	2.52	1.73
Std. dev. of fundamental changes	1.57	1.90	1.57	1.57
Autocorrelation of price changes, lag 1	-0.01	0.07	-0.14	-0.01
Autocorrelation of price changes, lag 2	-0.32	-0.42	-0.21	-0.11
Autocorrelation of price changes, lag 3	-0.09	-0.21	-0.01	-0.00
Correlation of volume with $ B_t $	0.93	0.91	0.97	0.97
Excess kurtosis of price changes	0.00	0.00	-0.01	0.02
Autocorrelation of absolute price changes, lag 1	0.01	0.02	0.03	0.00

365 20,000 periods after a burn-in of 100 periods. Fundamental shocks are normal with variance σ_v^2 ;
 366 liquidity demand is normal with standard deviation 0.02. Other parameters as in the baseline of Table
 367 1

368 The model produces two well-known features of asset prices. First, prices are more
 369 volatile than fundamentals. In the baseline, the standard deviation of price changes is 2.57,
 370 against 1.57 for the fundamental value. So excess volatility is about 60%. It falls when there
 371 are more fundamentalists. Second, price changes are negatively autocorrelated at short lags.
 372 This reflects the overshooting and correction after each shock. Volume is also strongly
 373 correlated with mispricing. However, the model does not produce fat tails or volatility
 374 clustering: excess kurtosis and the autocorrelation of absolute price changes are close to zero.
 375 This is expected, because the population shares are fixed. In HAMs, these features usually
 376 come from switching between strategies (Brock and Hommes 1998; Franke and Westerhoff
 377 2016). This is a natural extension of my model.

378 6 Discussion

379 6.1 What the model explains

380 The model explains bubbles with one psychological bias and a simple market structure.
 381 The same bias leads to two kinds of behaviour. Traders who know the composition of the urn
 382 expect reversals, and traders who must estimate it expect continuation (Rabin 2002; Rabin
 383 and Vayanos 2010). In this market, momentum traders create the bubble, and fundamentalists
 384 end it with some help from contrarians. Momentum traders react to price changes, not to price
 385 levels. So the bubble bursts on its own: when the price stops rising quickly, their optimism
 386 disappears. The model therefore does not need an exogenous event to end the bubble. Here it
 387 differs from rational bubble models, where the bubble bursts for exogenous reasons

(Blanchard and Watson 1982). It is close to Barberis et al. (2018), where the bubble ends when the demand of extrapolators weakens.

The model also explains why bubbles often come in a series. After the first bubble, momentum traders extrapolate the fall, and the price drops below the fundamental value. This creates a negative bubble and later a smaller positive one. Such oscillations are typical of fundamentalist–chartist models (Day and Huang 1990; Chiarella 1992; Hommes 2006). What is new here is that their size and speed depend on a coefficient a with a clear psychological meaning, and that the stability condition (14) has a closed form.

6.2 Policy implications

The model suggests three ways to reduce bubbles. First, the coefficient a falls when the share of informed traders θ rises. Policies that make information about fundamentals cheaper and easier to get, such as better disclosure, increase this share. Then bubbles become smaller and shorter. Second, a falls when fundamental uncertainty σ_v^2 falls. Clear communication about fundamentals reduces the size of the first bubble, for example communication about new technologies at the start of a boom. Third, a market participant who is ready to absorb the net demand of LSN traders could prevent bubbles. By Eq. (10), if such a stabiliser traded $-(1 - \theta)[\tau X_t^m + (1 - \tau)X_t^c]$ each period, the price would always equal F_t . In practice this requires knowing the fundamental value, which is difficult. But it shows why liquidity provision by informed market makers is stabilising. Liquidity shocks, on the other hand, can create bubbles by themselves (Sect. 5.4). So sudden large flows into or out of an asset can be destabilising even without news.

6.3 Limitations and extensions

The model has several limitations. First, momentum beliefs are defined only for price changes smaller than M in absolute value, because the estimated share of up balls must be between 0 and 1. So very large shocks need rescaling, for example by measuring price changes in percent, or a transformation such as a logistic function. Second, the shares of traders are fixed. If traders could switch between strategies depending on past profits, as in Brock and Hommes (1998), the dynamics would become nonlinear. This could produce fat tails and volatility clustering, which the current model does not produce (Sect. 5.6). Third, the risk-free rate is zero and positions are held for one period. A positive interest rate could be added by applying the urn model to deviations of returns from the risk-free rate. Fourth, short sales are allowed. A short-sale constraint, as in Scheinkman and Xiong (2003) and Barberis et

al. (2018), would remove contrarian selling at the peak. It would make bubbles larger and reduce the negative bubbles after the crash. Finally, I do not estimate the model. A natural next step would be to estimate α on market data, as other studies do for HAMs (Boswijk et al. 2007; Chiarella et al. 2014).

7 Conclusion

In this paper I build a heterogeneous-agent model of bubbles in which behavioural traders believe in the law of small numbers. Contrarian traders expect price changes to reverse, and momentum traders expect them to continue. Which one they expect depends on what they know about the “urn” that generates price changes. Together with fundamentalists and liquidity traders, the model has a closed-form price and a unique steady state at the fundamental value. Near this steady state, the bubble follows a simple second-order difference equation with one coefficient. This coefficient depends on the share of momentum traders, the share of fundamentalists, fundamental risk and the precision of beliefs. The steady state is stable only when the coefficient is below one, and convergence always has oscillations.

The simulations show that one shock to fundamentals or to liquidity creates a bubble that overshoots. Then the price crashes below the fundamental value, and smaller secondary bubbles follow. Trading volume peaks together with mispricing. The model also gives a psychological foundation for the extrapolative demand that many bubble models use. Future work could add switching between strategies, short-sale constraints and positive interest rates. It could also estimate the key coefficient on market data.

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Appendix: Proofs

Proof of Eq. (2). With CARA utility $-\exp(-\gamma W_{t+1})$, wealth $W_{t+1} = W_t + X_t \Delta P_t$ and $\Delta P_t \sim N(\mu_t, s_t^2)$, expected utility is $-\exp(-\gamma W_t - \gamma X_t \mu_t + 1/2 \gamma^2 X_t^2 s_t^2)$. Maximising the exponent over X_t gives the first-order condition $\mu_t - \gamma X_t s_t^2 = 0$, so $X_t = \mu_t / (\gamma s_t^2)$. Eqs. (3) and (5) follow by substituting $\mu_t = F_t - P_t$ and $s_t^2 = \sigma_v^2$ for fundamentalists, and $\mu_t = M(2\lambda_t - 1)$ and $s_t^2 = 4M\lambda_t(1 - \lambda_t)$ for LSN traders. ▀

515 **Proof of Proposition 1.** Substituting Eqs. (3) and (5) into the market-clearing condition
 516 (9) gives

$$517 \quad \theta \frac{F_t - P_t}{\gamma \sigma_v^2} + \frac{1 - \theta}{\gamma} [\tau h(\lambda_t^m) + (1 - \tau)h(\lambda_t^c)] + X_t^L = 0.$$

518 The beliefs depend only on past prices, so this equation is linear in P_t with slope
 519 $-\theta/(\gamma \sigma_v^2) \neq 0$. It therefore has exactly one solution, which is Eq. (10). ▀

520 **Proof of Proposition 2.** In a steady state $P_t = P$ for all t , so $\Delta P_t = 0$. Then Eq. (7) gives
 521 $\lambda^m = 1/2$, and Eq. (6) gives $\lambda_t^c - 1/2 = (1 - \kappa)(\lambda_{t-1}^c - 1/2)$, which converges to zero for
 522 $\kappa > 0$. With $h(1/2) = 0$ and $X^L = 0$, Eq. (10) gives $P = F$. All demands are zero, so there is
 523 no trade. Conversely, if $P_t \neq F$ in some period, Eq. (10) requires non-zero demand from LSN
 524 traders, so $\lambda_t^m \neq 1/2$ or $\lambda_t^c \neq 1/2$, and the price changes in the next period. Hence the steady
 525 state is unique. ▀

526 **Proof of Eq. (12) and Proposition 3.** Ignore the contrarian term and liquidity demand.
 527 For $t \geq T + 1$ the fundamental value is constant, so Eqs. (8) and (10) give

$$528 \quad B_t = \frac{\sigma_v^2(1 - \theta)\tau}{\theta} \frac{M\Delta P_{t-1}}{M^2 - \Delta P_{t-1}^2} = a \frac{\Delta P_{t-1}}{1 - (\Delta P_{t-1}/M)^2}.$$

529 At $t = T + 1$ the last price change is $\Delta P_T = \Delta F$, which gives Eq. (12). For $t \geq T + 2$,
 530 $\Delta P_{t-1} = B_{t-1} - B_{t-2}$. Linearising around $\Delta P = 0$ gives Eq. (13). Its characteristic equation is
 531 $r^2 - ar + a = 0$, with roots $r = (a \pm \sqrt{a^2 - 4a})/2$. (i) For $0 < a < 4$ the roots are
 532 complex, so the solution is $B_t = Ar^t \cos(\omega t + \phi)$ and changes sign. (ii) Complex roots have
 533 modulus $\sqrt{r_1 r_2} = \sqrt{a}$, so $|B_t|$ shrinks by the factor $\sqrt{a}$ per period and the steady state is stable
 534 if and only if $a < 1$. For $a \geq 4$ the roots are real and one of them is larger than one, so the
 535 steady state is unstable. (iii) The argument of the roots satisfies $\cos \omega = \text{Re}(r)/|r| =$
 536 $(a/2)/\sqrt{a} = \sqrt{a}/2$, so the period is $2\pi/\omega$. Including contrarians adds the term $c h(\lambda_t^c) \approx$
 537 $2c \ell_t$ with $\ell_t = (1 - \kappa)\ell_{t-1} - \Delta P_{t-1}/(2N)$, which is of order $1/N$. ▀

538 **Proof of Proposition 4.** In Eq. (12), a is increasing in τ and σ_v^2 and decreasing in θ and
 539 M , and the factor $\Delta F/(1 - (\Delta F/M)^2)$ is positive and increasing in $\Delta F \in (0, M)$, with
 540 $1/(1 - (\Delta F/M)^2) > 1$ increasing in $|\Delta F|$. So $B_{T+1}/\Delta F$ increases with $|\Delta F|$. The factor
 541 $1/(1 - (\Delta F/M)^2)$ is also decreasing in M , which strengthens the effect of M . ▀

542 **Proof of Proposition 5.** (i) In period T beliefs are $1/2$, so LSN demand is zero, and with
 543 no liquidity demand $P_T = F_T$, so fundamentalists' demand is also zero. (ii) Because the
 544 market clears, total long positions equal total short positions, and both equal V_t . The
 545 fundamentalists' position has size $\theta |X_t^f| = \theta |B_t|/(\gamma \sigma_v^2)$ and is on one side of the market, so

546 $V_t \geq \theta|B_t|/(\gamma\sigma_v^2)$. If all other traders are on the other side, the fundamentalists are the only
 547 traders on their side, and the inequality holds with equality. (iii) Without liquidity demand and
 548 with negligible contrarian positions, only fundamentalists and momentum traders trade, on
 549 opposite sides, so $V_t = \theta|B_t|/(\gamma\sigma_v^2)$. ▀

550 **Proof of Corollaries 1 and 2.** For Corollary 1, $M(2\hat{\lambda}_t - 1) = (1 - \zeta)\sum_{k \geq 1}$
 551 $\zeta^{k-1}(2M_{t-k}^+ - M)$, and $2M_{t-k}^+ - M = \Delta P_{t-k}$ by Eq. (4). For Corollary 2, a first-order
 552 expansion of Eq. (8) around $\Delta P_{t-1} = 0$ gives $X_t^m = \Delta P_{t-1}/(\gamma M) + O(\Delta P_{t-1}^3)$. ▀

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- [ESM1simulationcode.py](#)