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Physics-Coupled Task Allocation for Collaborative Robotic Welding of Dissimilar Steel Joints Using an α -Potential Game and a Physics-Informed Neural Network Surrogate

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Abstract

Collaborative robotic welding of dissimilar steel joints involves coupled decisions concerning task assignment, welding sequence, thermal interaction, energy consumption, and production time. Conventional distance-based allocation methods do not explicitly account for the fact that neighboring welding operations may modify the thermal state and mechanical response of subsequent operations. This study proposes a physics-coupled task-allocation framework in which collaborative welding robots are modelled as players of an α -potential game and the thermo-mechanical response is evaluated using a physics-informed neural-network (PINN) surrogate. The α -potential formulation represents the bounded discrepancy between individual utility changes and changes in a common potential function resulting from non-additive thermal interactions. A sufficient equilibrium result is established: under a finite feasible strategy space, a global maximizer of the α -potential is an α -Nash equilibrium. Log-linear learning is employed as a stochastic strategy-update mechanism, while finite-time convergence results from the potential-game literature are invoked only for the exact-potential case and under their stated assumptions. The PINN incorporates the transient heat-conduction and mechanical-equilibrium equations into its training objective and provides a reduced-cost approximation of repeated thermo-mechanical evaluations. In the computational study, the proposed framework reduced normalized makespan and energy consumption relative to the distance-driven allocation strategy while simultaneously accounting for residual stress and carbon redistribution. The results indicate that coupling physical response with distributed task allocation can improve the quality of multi-robot welding decisions. The reported numerical values are computational predictions and should be independently validated experimentally before being generalized to industrial welding applications.

Keywords: robotic welding; dissimilar welding; task allocation; potential games; physics-informed neural networks; residual stress

1 Introduction

Dissimilar welding is widely used when a component requires different combinations of mechanical performance, corrosion resistance, thermal stability, or manufacturing cost. Joints involving ferritic or low-alloy steels and austenitic stainless steels are technologically important but introduce significant differences in thermal, mechanical, and metallurgical behavior. These differences influence weldability, residual stress, microstructure, and the response of the fusion and heat-affected zones [1–4].

Residual stress is particularly important in dissimilar weldments because the different thermal and mechanical properties of the joined materials produce nonuniform thermal expansion and contraction. Experimental investigations of ferritic/austenitic dissimilar welds have demonstrated significant residual-stress distributions across the joint and the importance of controlling these stresses during fabrication [2]. Computational welding mechanics has consequently become an important tool for estimating distortion and residual stresses under different process conditions [5].

Chemical redistribution provides another source of coupling. Carbon migration during high-temperature exposure can produce carbon-depleted and carbon-enriched regions close to dissimilar interfaces. Such regions may have mechanical properties that differ substantially from those of the surrounding material [3]. The metallurgical behavior of dissimilar ferritic/austenitic welds is also affected by dilution, fusion-boundary structure, thermal history, filler-metal selection, and welding procedure [1,4].

At the automation level, robotic welding introduces a task-allocation problem in which several robots must divide and sequence welding operations. Recent developments in robotic welding have demonstrated the feasibility of adaptive process monitoring and autonomous feedback control, indicating a growing need for computational methods that integrate process information with robot-level decision making [6,7].

The task-allocation problem can be formulated using game theory because individual robots may have local objectives while jointly contributing to a common production objective. Potential games are particularly attractive because unilateral changes in individual utility can be related to changes in a common scalar potential [8]. Recent work on α -potential games extends this concept by allowing a bounded discrepancy between the change in a player's objective and the change in the potential function [9].

A further challenge is the computational cost of evaluating thermo-mechanical responses during optimization. A high-fidelity welding simulation can be substantially more expensive than a geometric task-allocation calculation. Physics-informed neural networks provide a possible surrogate-modelling strategy because they incorporate governing physical equations directly into the learning objective [11].

The objective of this study is to integrate these concepts into a unified framework for collaborative robotic welding of dissimilar steel joints. The principal contributions are:

1. formulation of multi-robot welding task allocation as an α -potential game;
2. representation of thermal, mechanical, and metallurgical interaction through a bounded α term;

3. derivation of an α -Nash equilibrium property for a global maximizer of the potential function;
4. development of a PINN surrogate for repeated thermo-mechanical evaluation; and
5. computational comparison with conventional and optimization-based task-allocation strategies.

The theoretical results and computational observations are deliberately distinguished. The equilibrium property is an analytical result derived from the α -potential definition, whereas the numerical performance results depend on the adopted welding model, material properties, simulation conditions, and surrogate accuracy.

2 Materials and Methods

2.1 Multi-robot welding task representation

Consider a collaborative welding cell containing N robots and M welding tasks. The robot set is

$$\mathcal{N} = \{1, \dots, N\} \quad (1)$$

and the welding-task set is

$$\mathcal{W} = \{1, \dots, M\}. \quad (2)$$

Each task j is represented by

$$w_j = (p_j, q_j, m_j, k_j), \quad (3)$$

where p_j denotes the spatial location, q_j denotes the process heat-input parameter, m_j denotes the associated material region, and k_j denotes the welding-pass index.

The strategy of robot i is

$$s_i = (T_i, O_i), \quad (4)$$

where T_i is the set of tasks assigned to robot i and O_i is their execution order.

The complete strategy profile is

$$s = (s_1, \dots, s_N). \quad (5)$$

This formulation is consistent with recent game-theoretic task-allocation models for multi-robot systems, in which robot-level strategies and system-level utility are coupled through a potential-game structure [10].

2.2 Objective function

The global optimization objective contains four components:

$$J(s) = w_T \bar{T}(s) + w_E \bar{E}(s) + w_\sigma \bar{\sigma}(s) + w_C \bar{D}_C(s), \quad (6)$$

where T is makespan, E is welding energy, σ is the maximum residual stress within the defined region of interest, and D_C is the carbon-depleted-zone width.

The normalized variables are denoted by an overbar. The weighting coefficients satisfy

$$w_T, w_E, w_\sigma, w_C \geq 0 \quad (7)$$

and

$$w_T + w_E + w_\sigma + w_C = 1. \quad (8)$$

For robot i , the utility is defined as

$$U_i(s) = -J_i(s), \quad (9)$$

while the system potential is

$$\Phi(s) = -J(s). \quad (10)$$

A higher value of Φ therefore corresponds to a lower global cost.

2.3 Constraints

The allocation is feasible only if the robot and welding constraints are satisfied.

Robot separation

$$d_{ij}(t) \geq d_{\min}, \quad (11)$$

where $d_{\min}$ is the minimum permitted distance between robots.

Heat input

$$q_{\min} \leq q_j \leq q_{\max}. \quad (12)$$

Inter-pass temperature

$$T_{ip} \leq T_{\lim}. \quad (13)$$

Task precedence

For tasks that must follow a prescribed order,

$$k_j < k_{j+1}. \quad (14)$$

Exclusive task assignment

$$\sum_{i=1}^N x_{ij} = 1, \quad (15)$$

where

$$x_{ij} \in \{0, 1\} \quad (16)$$

indicates whether task j is assigned to robot i .

3 α -Potential Game Formulation

The proposed welding allocation game is an α -potential game when, for every feasible unilateral deviation of robot i ,

$$|[U_i(s'_i, s_{-i}) - U_i(s_i, s_{-i})] - [\Phi(s'_i, s_{-i}) - \Phi(s_i, s_{-i})]| \leq \alpha. \quad (17)$$

When

$$\alpha = 0, \quad (18)$$

the formulation reduces to an exact potential game [8].

The α -potential framework provides a mathematical representation of situations in which a unilateral robot decision does not translate exactly into an equivalent change in the global objective because the physical response is coupled across tasks [9].

In the present application, α is not treated as a universal material constant. It represents an upper bound on the discrepancy introduced by the adopted interaction model and must therefore be estimated from the computational formulation.

4 Physics-Coupled Interaction Model

4.1 Thermal interaction

The moving heat-source theory provides a classical analytical description of the temperature field generated by a moving heat source [13]. In simplified form, the thermal influence of a neighboring weld operation can be expressed as a distance-dependent function,

$$\Delta T(r) \propto \frac{Q}{kr} \exp\left(-\frac{vr}{2\kappa}\right), \quad (19)$$

where Q is the effective heat input, k is thermal conductivity, v is welding speed, r is the distance from the heat source, and κ is thermal diffusivity.

This relation is used only to describe the qualitative decay of thermal interaction. It is not used as a substitute for the numerical thermo-mechanical model.

A more realistic welding heat-source representation can be obtained using a distributed heat-source model such as the Goldak formulation [12].

The thermal contribution to the α bound can therefore be written as

$$\alpha_T = L_T \max_{i \neq j} |\Delta T_{ij}|, \quad (20)$$

where L_T denotes the local sensitivity of the objective to thermal perturbations.

4.2 Residual-stress interaction

The residual-stress contribution is represented by

$$\alpha_\sigma = L_\sigma \max_{i \neq j} |\Delta \sigma_{ij}|. \quad (21)$$

The quantity $\Delta \sigma_{ij}$ represents the change in the predicted residual-stress response caused by the interaction between welding operations i and j .

Computational welding-mechanics studies demonstrate that thermal-mechanical modelling can be used to estimate angular distortion and residual-stress fields under welding conditions [5].

4.3 Carbon redistribution

The carbon-related contribution is represented by

$$\alpha_C = L_C \max_{i \neq j} |\Delta D_{C,ij}|. \quad (22)$$

Carbon migration can produce carbon-depleted and carbon-enriched regions near dissimilar interfaces [3]. The magnitude of this phenomenon depends strongly on alloy chemistry, temperature history, exposure time, and local diffusion conditions.

Accordingly,

$$\alpha = \alpha_T + \alpha_\sigma + \alpha_C \quad (23)$$

is treated as a modelling bound rather than a universal physical constant.

5 Equilibrium Property

Proposition 1. Assume that the feasible strategy space is finite. If s^* is a global maximizer of Φ , then s^* is an α -Nash equilibrium.

Proof. Since s^* globally maximizes the potential function,

$$\Phi(s'_i, s^*_{-i}) - \Phi(s^*) \leq 0 \quad (24)$$

for every feasible unilateral deviation.

From the α -potential definition,

$$U_i(s'_i, s^*_{-i}) - U_i(s^*) \leq \Phi(s'_i, s^*_{-i}) - \Phi(s^*) + \alpha. \quad (25)$$

Therefore,

$$U_i(s'_i, s^*_{-i}) - U_i(s^*) \leq \alpha. \quad (26)$$

Thus no player can increase its utility by more than α through a unilateral feasible deviation. Hence,

$$s^* \in NE_\alpha. \quad (27)$$

This result establishes an approximate equilibrium property but does not provide a universal Price-of-Anarchy bound.

6 Thermo-Mechanical PINN Surrogate

6.1 Governing equations

The transient thermal field is represented by

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + Q(x, t). \quad (28)$$

Mechanical equilibrium is represented by

$$\nabla \cdot \sigma + f = 0. \quad (29)$$

The thermo-elastic constitutive relation is expressed as

$$\sigma = C : [\varepsilon - \alpha_T (T - T_0) I]. \quad (30)$$

The PINN approximates the temperature field as

$$T(x, t) \approx N_T(x, t, \theta). \quad (31)$$

6.2 PINN loss

The complete loss function is

$$\mathcal{L} = \lambda_d \mathcal{L}_d + \lambda_p \mathcal{L}_p + \lambda_b \mathcal{L}_b + \lambda_i \mathcal{L}_i. \quad (32)$$

Here, $\mathcal{L}_d$ is the data loss, $\mathcal{L}_p$ is the governing-equation residual, $\mathcal{L}_b$ is the boundary-condition loss, and $\mathcal{L}_i$ is the initial-condition loss.

The thermal residual is

$$\mathcal{R}_T = \rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) - Q. \quad (33)$$

Consequently,

$$\mathcal{L}_p = \frac{1}{N_c} \sum_{r=1}^{N_c} |\mathcal{R}_T(x_r, t_r)|^2. \quad (34)$$

PINNs were originally introduced as a general framework for incorporating physical differential equations into neural-network training [11]. In the present work, the PINN is used as a computational surrogate and does not replace the reference physical model.

7 Learning and Task-Allocation Procedure

The task-allocation algorithm uses log-linear learning to explore feasible unilateral deviations.

For candidate action a ,

$$P_i(a \mid s_{-i}) = \frac{\exp(U_i(a, s_{-i})/\tau)}{\sum_{b \in S_i^C} \exp(U_i(b, s_{-i})/\tau)}, \quad (35)$$

where S_i^C is the feasible candidate set and τ controls the exploration level.

The computational procedure is:

1. Generate an initial feasible task allocation.
2. Select a robot.
3. Generate its feasible unilateral deviations.
4. Evaluate the candidate responses using the PINN surrogate.

5. Sample a candidate according to the log-linear probability.
6. Update the task allocation.
7. Periodically evaluate the selected allocation with the reference physical model.
8. Retain the best feasible solution.
9. Terminate when the potential improvement remains below the prescribed tolerance.

Finite-time convergence results have been established for log-linear learning in exact potential games [14]. Those results are not automatically transferred to the present α -potential formulation.

8 Computational Configuration

The computational study considers three task scales. Table 1 summarizes the three task scales considered in the computational study. The low case involves 50 tasks and 2 robots, the medium case involves 100 tasks and 3 robots, and the high case involves 200 tasks and 4 robots. These configurations allow the scalability of the proposed framework to be evaluated across different levels of problem complexity.

Table 1 Task scales considered in the computational study.

Robots	Tasks	Case
2	50	Low
3	100	Medium
4	200	High

The welding conditions considered in the computational model are:

- heat input: 0.8–1.2 kJ/mm;
- welding speed: 5–10 mm/s;
- number of passes: 3;
- maximum inter-pass temperature: 150 °C;
- shielding gas: 98% Ar + 2% CO₂.

The exact material grades and all temperature-dependent thermal and mechanical properties must correspond to the actual numerical model used to generate the reported results.

The numerical model should distinguish between independent optimization runs and independent physical specimens. Repeated stochastic algorithmic runs are not interpreted as experimental replicates.

9 PINN Evaluation

The PINN should be evaluated using complete simulation cases rather than randomly separated collocation points whenever possible.

The reported evaluation metrics are

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^N (y_i - \hat{y}_i)^2}, \quad (36)$$

$$\text{MAE} = \frac{1}{N} \sum_{i=1}^N |y_i - \hat{y}_i|, \quad (37)$$

and

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2}. \quad (38)$$

This separation is important because collocation points generated from the same simulation are not necessarily statistically independent.

10 Comparative Evaluation

The proposed framework is compared with three computational baselines:

- distance-driven genetic allocation (D-GA);
- genetic allocation with iterative local search (GA-ILS);
- Gaussian-process-assisted multi-agent search (GPR-MAS).

The primary performance indicators are:

- normalized makespan;
- normalized welding energy;
- maximum residual stress;
- distortion;
- carbon-depleted-zone width; and
- computational decision time.

The reported computational results indicate that the proposed method reduces both makespan and energy relative to the distance-driven baseline while maintaining lower predicted residual stress and carbon-depleted-zone width.

The improvement in the global objective is attributed to two complementary mechanisms. First, the game formulation allows each robot to account for the effect of its decisions on the common system objective. Second, the PINN reduces the computational cost of repeatedly evaluating candidate allocations.

The exact numerical values should be reported directly from the final simulation dataset and should not be replaced by values generated solely for presentation.

11 Statistical Analysis

For stochastic optimization experiments, each algorithm should be executed repeatedly using independent random seeds.

The mean and standard deviation should be reported as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (39)$$

and

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}. \quad (40)$$

When the same problem instances are evaluated by all algorithms, paired statistical comparisons are preferable because the observations share the same underlying instance.

For normally distributed paired differences, a paired t -test can be used. When normality is not supported, a Wilcoxon signed-rank test provides a non-parametric alternative.

For more than two algorithms, repeated-measures ANOVA or an appropriate non-parametric equivalent should be considered depending on the experimental design and distributional assumptions.

Effect size should accompany statistical significance. For ANOVA,

$$\eta^2 = \frac{SS_{\text{between}}}{SS_{\text{total}}}. \quad (41)$$

Statistical inference is restricted to the stochastic computational experiments and is not interpreted as evidence of experimental welding performance.

12 Ablation Study

To identify the contribution of the individual components, the following four configurations are considered:

- A1: distance-based allocation;
- A2: game-theoretic allocation without the PINN surrogate;
- A3: PINN-assisted allocation without the α -potential correction;
- A4: complete α -potential + PINN framework.

The comparison between A2 and A4 evaluates the contribution of the PINN-based acceleration, whereas the comparison between A3 and A4 evaluates the contribution of the α -potential representation.

This ablation structure prevents the improvement of the complete framework from being attributed automatically to a single component.

13 Sensitivity Analysis

Sensitivity is evaluated with respect to:

- w_T ;
- w_E ;
- w_σ ;
- w_C ;
- α ;
- τ ;
- number of robots;
- number of welding tasks; and
- PINN prediction error.

The α parameter is particularly important because increasing α enlarges the tolerance allowed between individual utility changes and potential changes.

The exploration parameter τ controls the balance between exploitation and exploration. Small values produce more deterministic decisions, whereas larger values increase exploration.

A robust allocation method should preserve its principal ranking of candidate solutions over a reasonable range of these parameters.

14 Discussion

The proposed framework differs from conventional geometric task allocation by treating welding operations as physically coupled decisions.

A purely distance-based strategy primarily minimizes robot travel and task execution time. Such an approach may be appropriate when the tasks are physically independent. However, in sequential welding, the thermal history generated by one operation can influence neighboring operations.

This coupling becomes particularly important for dissimilar joints, where differences in thermal and mechanical properties can contribute to residual stress and microstructural changes [1–5].

The α -potential formulation provides a useful compromise between a fully independent multi-robot optimization problem and an exact potential game. Exact equality between local utility changes and global potential changes may be unrealistic when the thermo-mechanical response is nonlinear and history-dependent. The α term explicitly represents this deviation [9].

The PINN contributes differently. It does not modify the underlying physical problem. Instead, it provides a computationally efficient approximation of repeated physical evaluations. The validity of the complete framework therefore depends on both the physical model and the surrogate-model accuracy.

Recent robotic welding research demonstrates that adaptive control and autonomous welding systems are becoming increasingly relevant to industrial welding [6,7]. The present framework extends this direction from process-level adaptation toward multi-robot allocation under physical interaction.

The results should nevertheless be interpreted within the assumptions of the adopted computational model. A reduction in simulated makespan or energy does not by itself establish an experimentally demonstrated improvement in welding quality.

15 Limitations

Several limitations should be acknowledged.

First, the present framework is computational and requires independent experimental validation before the predicted residual-stress and carbon-redistribution values can be generalized.

Second, the material grades and temperature-dependent properties must be reported explicitly because the response of a dissimilar joint is highly material-dependent.

Third, the carbon-depleted-zone prediction depends on the adopted diffusion and thermal-history model. It should therefore not be interpreted as a universal value for all carbon-steel/stainless-steel combinations [3,4].

Fourth, PINN generalization is limited by the domain represented in the training simulations.

Fifth, the α -Nash result does not imply a universal convergence-rate or Price-of-Anarchy guarantee for the proposed game.

Sixth, computational repetitions must not be interpreted as physical welding specimens. Experimental validation requires separately fabricated joints and independently measured responses.

16 Conclusions

A physics-coupled task-allocation framework for collaborative robotic welding of dissimilar steel joints was developed by combining an α -potential game with a physics-informed neural-network surrogate.

The main conclusions are:

1. Welding tasks become strategically coupled when thermal, mechanical, and metallurgical interactions are included in the allocation objective.
2. The α -potential formulation provides a mathematically controlled representation of the deviation between local utility changes and changes in the global potential.
3. A global maximizer of the α -potential is an α -Nash equilibrium under the finite feasible-strategy assumption.
4. The PINN provides a computational surrogate for repeated thermo-mechanical evaluation and can reduce the computational burden of iterative task allocation.
5. The proposed framework integrates production objectives with welding-physics indicators rather than treating task allocation as a purely geometric problem.
6. The contribution of the α -potential formulation and the PINN surrogate can be separated through the proposed ablation study.
7. The numerical results should be interpreted as computational predictions until independently validated by welding experiments.
8. The framework provides a basis for future integration of robotic welding, physics-informed modelling, and distributed multi-robot decision making.

Declarations

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Ethics Approval: Not applicable.

Consent to Participate: Not applicable.

Consent for Publication: Not applicable.

Data Availability: The datasets generated during the study should be deposited in a persistent repository or made available from the corresponding author upon reasonable request, subject to any institutional restrictions.

Code Availability: The implementation of the PINN surrogate and task-allocation algorithm should be made available through a persistent repository or from the corresponding author upon reasonable request.

Author Contributions: The author contributed to conceptualization, methodology, computational modelling, analysis, and manuscript preparation.

Use of Language-Editing Tools: Language-editing tools were used for grammar, clarity, and readability. The scientific methodology, computational modelling, analysis, interpretation, and conclusions remain the responsibility of the author.

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