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Theoretical prediction of masses and decay constants of bottomonium and bottom-charm mesons with spin dependent Cornell potential

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Abstract

We investigate the bound-state spectrum and decay properties of mesons using a spin-dependent Cornell potential within first-order perturbation theory. Analytical expressions for the energy eigenvalues are obtained, and the model is validated by considering the spin-independent limit, where the results reproduce the standard Cornell potential spectrum and show good agreement with existing calculations based on artificial neural networks, the asymptotic iteration method, and the matrix Numerov approach. The parameters of the spin-dependent potential are determined by fitting the resulting analytical mass-spectrum equation to experimental data. These parameters are then used to calculate the masses and decay properties of bottomonium and bottom-charm mesons for low-lying quantum states. For the bottomonium states, an average percentage absolute deviation of approximately 0.15% from the experimental masses is obtained, demonstrating an improvement over several previous theoretical predictions while the bottom-charm masses aligned with other theoretic values in the literature. The vector and pseudoscalar decay constants are calculated from the corresponding wave functions at the origin and exhibit a decreasing behavior with increasing principal quantum number. The pseudoscalar di-photon and vector leptonic decay widths are subsequently evaluated using the leading and first-order Van Royen–Weisskopf formula. The predicted decay constants are found to be generally consistent with available experimental data and previous theoretical results. Furthermore, incorporating the QCD running strong coupling at appropriate renormalization scales improves the description of the decay properties, emphasizing the sensitivity to the wave function at the origin. The results demonstrate that the perturbative spin-dependent Cornell potential provides a simple and effective phenomenological framework for describing the mass spectra and decay properties of bottomonium and bottom-charm mesons.

Keywords: Heavy mesons masses; Decay constants; Perturbation theory; Cornell potential.

1. Introduction

Our comprehension of the fundamental forces in nature, especially the strong interactions that unite quarks, is greatly aided by mesons, as they are essential components of hadronic matter. However with their seeming simplicity, meson mass spectra are difficult to predict precisely because of the intricate character of quantum chromodynamics (QCD) at low energies, where the

theory turns non-perturbative. Mass spectra, decay rates, spin configurations, and other characteristics of quarks inside mesons have all been described by several phenomenological models that have been created throughout time to better explain their dynamics. These models consider the quark-antiquark interaction as an effective potential, allowing quantum mechanical techniques to approximate mesons' bound state dynamics. The Cornell potential [1, 2] is the most well-known of these potentials, as it includes both a Coulomb-like component, which accounts for the short-range behaviour due to one-gluon exchange, and a linear confinement term, which reflects the long-range nature of quark confinement in QCD. While the Cornell potential provides a good approximation of meson masses, it ignores the significant contributions of spin-dependent forces, which are required for a more precise computation of the meson spectrum. The mass spectra of heavy and heavy-light mesons have been calculated for the Cornell potential by solving the Schrödinger equation using the Numerov approach [3]. Within the context of the spinless Salpeter equation (SSE), the mass spectra for heavy mesons have been determined using the Wentzel-Kramers-Brillouin (WKB) approximation for the Cornell potential [4] and the spin-dependent Cornell potential [5]. Recently, Nahool et al. [6, 7] employed some machine learning algorithms for predicting the mass spectra of heavy mesons with the Cornell potential in the presence and absence of spin-spin interactions. Furthermore, Malekhsosseini et al. predicted the mass and width of the ground and exotic mesons using a deep-learning model [8]. Moreover, Tariq et al. [9] and Mutuk [10] used an artificial neural network approach with effective potential to solve the Schrödinger wave equation for the mass spectrum of several charmonium states. The impact of the topological defect's gravitational field on heavy quarkonia mass spectra for the Cornell potential has been investigated using the Nikiforov-Uvarov (NU) and extended NU techniques [11, 12]. In the framework of the Schrödinger equation, the properties of heavy mesons have been examined for the Cornell potential using various techniques [13-20]. The NU approach has been applied to obtain the masses of heavy mesons within the Klein-Gordon equation using the linear plus modified Yukawa potential [21]. The spin-averaged mass spectra of heavy quarkonia with Coulomb plus quadratic potential have been investigated by Al-Jamel and Widyan [22] using the NU method. Using the Jacobi approach, Abou-Salem [23] has numerically calculated the masses and leptonic decay widths of heavy and heavy-light mesons. The decay constants of vector and pseudoscalar mesons have been determined using QCD sum rules [24, 25]. The WKB approximation method [26 - 28] was used to approximate the Schrödinger equation with the Cornell potential. The energy equation was applied to obtain the mass spectra of the heavy and heavy-light meson systems. Das et al. [29] used Dalgarno's perturbation approach to analyze the root mean square and charge radii of heavy-flavoured mesons for the static potential utilizing three-loop corrections. Spin-dependent interactions, such as spin-spin, spin-orbit, and tensor forces, are important for understanding the fine structure of meson spectra. As a result, recent works have focused on including these interactions into potential models as well as lattice QCD. Omugbe et al. [27] emphasized the crucial role of spin-spin and spin-orbit forces in predicting the mass splitting of heavy quarkonium states. Their analysis proved that the inclusion of these terms produces findings that closely match experimental data for mesons like charmonium and bottomonium. The Schrödinger equation with the Cornell potential and spin-spin interactions has been solved using the NU technique in [30] where the energy and wave function solutions have been used to derive the masses and root mean square radii of the heavy mesons. In the framework of the Dirac formalism [32] the mass spectra, weak decay constants, and Magnetic transitions of the ground and excited s-wave states of the B , B_s and B_c mesons have been explored using the relativistic

independent quark model. Behera and Panda [32] used the relativistic square root potential to compute the mass spectra and decay properties of B and D mesons. Abu-Shady and Khokha [33] solved the Dirac equation using the generalized Cornell potential. Furthermore, the energy eigenvalues have been used to predict the spin-averaged mass spectra of heavy and heavy-light mesons by employing the non-relativistic limit. Mansour et al. [34] examined the mass spectra of heavy meson within the framework of the Schrödinger equation for the phenomenological potential with spin-dependent corrections using the NU approach. The spectrum and electromagnetic transitions of bottomonium have been investigated within the framework of the non-relativistic screened potential quark model by solving the Schrödinger equation using the three-point difference central technique for the spin-dependent potentials [35]. More recently, Mistry et al. [36] studied the mass spectra, Regge trajectories and decay properties of heavy-flavour mesons by using the variational method with the Coulomb plus confining power potentials in the presence of spin-spin, spin-orbit, and tensor interactions. In further study, Garg et al. [37] investigated bottomonia mass spectra and their decay characteristics by numerically solving the Schrödinger equation with the quark-antiquark potential in the presence of spin-dependent interactions. In the context of the fractional Schrödinger equation under the spin-dependent Cornell potential, the masses of the heavy mesons have been obtained by using the conventional NU method [38]. The findings show that the mass spectra obtained with the fractional Schrödinger equation are superior to those obtained with other theoretical methods. The spectroscopic properties and Regge trajectories of quarkonia have also been investigated within the framework of the QCD-inspired relativistic quark model [39] where the predicted masses are close to experimental data. Within theoretical frameworks, the prediction of the meson masses is based on numerical and approximation methods since the interaction potentials do not admit the exact analytical solution of the wave equations. Approximation methods have been used within the nonrelativistic quantum mechanics to obtain the mass spectra solutions [4, 26, 34, 38, 40], while numerical approaches have also been utilized in other works [43–48]. Moazami et al. [49] investigated the masses and decay properties of the heavy-light mesons using a combination of Cornell, Gaussian and squared inverse quadratic potential within the framework of the Dalgarno perturbation theory. Following the perturbation method, the 2s and 3s state masses and decay constants of heavy-flavoured mesons have been calculated in [50]. The decay rates, and mass spectra of mesons were investigated theoretically through a variety of potential models and computational techniques. Krassnigg et al. [51] studied leptonic decays of D -wave vector quarkonia, while Patel and Vinodkumar [52] studied $Q\bar{Q}$ ($Q \in b, c$) mesons in a Coulomb plus power potential where they calculated decay constants such as di-photon, and dileptonic decay widths with and without radiative QCD corrections. Also, Lakhina and Swanson [53] conducted an assessment of the nonrelativistic quark models for charmonia by comparing theoretical predictions of decay constants, form factors, and widths with both experimental data and lattice gauge results. Li and Chao [54, 55] investigated bottomonium and higher charmonia spectra under a colour-screened potential, calculating the masses, electromagnetic decay rates and E1 dipole transitions. Ebert et al. [56] used a relativistic method to obtain two-photon decay rates of heavy quarkonia by including relativistic effects and one-loop radiative corrections. Godfrey and Isgur [57] used a relativized quark model that includes quantum chromodynamics, predicting and analysing strong electromagnetic, and weak meson couplings. Shah et al. [58] analysed leptonic and di-photon decays using a Martin-like potential. Kim et al. [59] used the relativistic Salpeter method to study the annihilation decay rates of heavy quarkonia. The spectroscopic properties of mesons have also been investigated in [60–62].

Accurate determination of heavy meson mass spectra and decay rates remains a central challenge in hadron spectroscopy, as commonly used analytic approaches often employ approximations that limit predictive accuracy. In particular, several recent studies [4, 27, 30, 34, 38, 40] have applied the Pekeris approximation to the spin-dependent Cornell potential, obtaining results that deviate from experimental data and other theoretical predictions. Also, investigation of the mesons masses have been carried out using non-exactly solvable potentials such as the coulomb plus logarithm function and the Born-Meyer-type potential [63,64]. In these recent works, the decay properties had not been studied. Therefore, we will address these shortcomings by employing the Cornell potential, solving the bound-state problem using perturbation theory in conjunction with the Matrix Numerov method. This approach would allow us to obtain improved and physically consistent results of the masses, vector and pseudo-scalar decay constants, dileptonic and di-photon decay constants for the bottomonium and bottom-charm mesons. The remainder of this paper is organized as follows. Section 2 outlines the application of perturbation theory to nonrelativistic bound-state solutions for the spin-dependent Cornell potential. Section 3 presents the formalism for calculating the mass spectra and decay constants. In Section 4, we report and discuss our numerical results, comparing them with other theoretical predictions and the values from the Particle Data Group (PDG). Concluding remarks are given in Section 5.

2. Theoretical framework

2.1 Perturbation Theory (PT)

The perturbation approach involves the division of the interacting Hamiltonian into disturbed and undisturbed parts where a part of the potential is taken as the parent and the remainder as the noise to the system. In this case, the solution of the parent potential may be obtained using any of the solution methods for bound states. Suppose an exactly solvable system is perturbed by a small bump, then the eigenvalues equation that describes the undisturbed system can be written as

$$H_0 R_n^{(0)} = E_n^{(0)} R_n^{(0)} \quad (1)$$

The disturbance to the system only contributes a small bump say λ which is constrained between 0 and 1. The total Hamiltonian for the system is given as

$$H = H_0 + \lambda H' \quad (2)$$

where H_0 and H' are the respective Hamiltonians for the unperturbed and the disturbed systems. The primary idea of the perturbation theory is to obtain bound state solutions in the form of convergent series [65] :

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \quad (3)$$

$$R_n = R_n^{(0)} + \lambda R_n^{(1)} + \lambda^2 R_n^{(2)} + \dots \quad (4)$$

The Schrödinger equation for the disturbed system may be written as

$$(H_0 + \lambda H') R_n = E_n R_n \quad (5)$$

Inserting Eq. (3) and (4) into (5) and comparing the coefficient of λ to the second degree yields the following equations:

$$H_0 R_n^{(0)} = E_n^{(0)} R_n^{(0)} \quad (6)$$

$$E_n^{(1)} R_n^{(0)} = H' R_n^{(0)} + (H_0 - E_n^{(0)}) R_n^{(1)} \quad (7)$$

$$E_n^{(2)} R_n^{(0)} = (H' - E_n^{(1)}) R_n^{(1)} + (H_0 - E_n^{(0)}) R_n^{(2)} \quad (8)$$

Equation. (6) corresponds to the unperturbed system, while Eq. (7) and (8) represent the first-order and second-order approximation respectively. The second-order approximation is only needed when the first-order scheme is not sufficient to obtain the bound state solutions. The wave function of the unperturbed problem is orthonormal such that the disturbed wave function may be written as a convergent series form [65]

$$R_n^{(1)} = \sum_m A_{nm} R_m^{(0)} \quad (9)$$

where A_{nm} are coefficients to be determined later. Using Eq. (9), Eq. (7) may be written as

$$E_n^{(1)} R_n^{(0)} = H' R_n^{(0)} + \sum_m A_{nm} (H_0 - E_n^{(0)}) R_m^{(0)} \quad (10)$$

Multiplying Eq. (10) with $R_n^{(0)*}$ from the left and performing integration taking into account that the terms $n \neq m$ would vanish results to

$$E_n^{(1)} = \langle R_n^{(0)*} | H' | R_n^{(0)} \rangle = \int R_n^{(0)*} | H' | R_n^{(0)} dr = H_{nn} \quad (11)$$

Also, multiplying (10) with $R_m^{(0)*}$ and performing integration allow for the determination of the coefficients A_{nm}

$$A_{nm} = \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}} \quad n \neq m \quad (12)$$

In the same vein, the second-order correction to the energy may be obtained from Eq. (8) as

$$E_n^{(2)} = \sum_{m \neq n} \frac{|H'_{mn}|^2}{(E_n^{(0)} - E_m^{(0)})} \quad (13)$$

2.2 Approximate energy spectra and wave function of the spin-dependent Cornell potential

The spin-dependent Cornell potential is given as [4, 27, 34, 40, 45]

$$V(r) = -\frac{4\alpha_s}{3r} + Br + \Lambda_s e^{-\sigma^2 r^2} - \frac{A_0^s}{r} + \frac{A_1^s}{r^3} \quad (14)$$

$$\text{where } A_0^s = \frac{B \langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle}{2m_q m_{\bar{q}}}, \quad A_1^s = \frac{2\alpha_s (2 \langle \bar{T} \rangle + \langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle)}{m_q m_{\bar{q}}}.$$

The notations Λ_s , $\langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle$ and $\langle \bar{T} \rangle$ are constants relating to the spin-spin, spin-orbit and tensor matrix elements given by

$$\Lambda_s = \frac{16\alpha_s\pi}{9m_q m_{\bar{q}}} \left(\frac{\sigma}{\sqrt{\pi}} \right)^3 \left(s(s+1) - \frac{3}{2} \right) \quad (15)$$

$$\langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle = \frac{1}{2} (j(j+1) - l(l+1) - s(s+1)) \quad (16)$$

$$\langle \bar{T} \rangle = - \frac{6(\langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle)^2 + 3\langle \bar{\mathbf{L}} \cdot \bar{\mathbf{S}} \rangle - 2s(s+1)l(l+1)}{6(2l-1)(2l+3)} \quad (17)$$

Also, the terms B, α_s and σ are constants that will be obtained later by fitting the mass spectra with the experimental data [66, 67]. The terms m_q and $m_{\bar{q}}$ are the masses of the quark and its anti-particle while s represents the spin quantum number. The quantum number J^{PC} denotes the hyperfine splitting between the spin quantum states with $P((-1)^{l+1})$ and $C((-1)^{l+s})$ being parity and charge conjugation respectively. The total angular momentum quantum number J is the sum of the orbital and spin quantum numbers. The spin-spin component is responsible for the s-wave ($l = 0$) hyperfine splitting between the triplet ($s = 1$) and singlet ($s = 0$) states. For $l > 0$, and $j = l \pm 1, j = l$, we have the splitting for the triplets quantum states. The n represents the principal quantum number. The spin-dependent potential in (14) gives the mass shifts and is obtained from leading-order perturbation [68]. The operator $\langle \bar{T} \rangle$ represents the non-vanishing diagonal matrix element for $l > 0$ and corresponds to the spin-triplet states. In this work, we take the Coulomb part of the potential plus the term $\frac{A_0^s}{r}$ given in Eq. (14) as the parent while the

other functions $Br + \Lambda_s e^{-\sigma^2 r^2} + \frac{A_1^s}{r^3}$ are taken as a perturbation. The solution of the Schrodinger equation with the parent potential may be obtained with any method for the bound state since the potential admits the exact energy spectra and wave function given by

$$E_{n,l}^{(0)} = - \frac{(4/3\alpha_s + A_0^s)^2 \mu}{2\hbar^2 (n_r + l + 1)^2} \quad (18)$$

$$R_{n,l}^{(0)}(r) = N_{n,l} r^{a+\frac{1}{2}} e^{-b_{n,l} r} L_{n_r}^{2a}(2br), \quad (19)$$

where

$$a = l + \frac{1}{2}, \quad b_{n_r} = \sqrt{-\frac{2\mu E_{n_r,l}}{\hbar^2}}$$

The normalization constant ($N_{n_r,l}$) in the wave function is obtained by solving the equation:

$$N_{n_r,l} \int_0^\infty \left| R_{n_r,l}^{(0)}(r) \right|^2 dr = 1 \quad (20)$$

$$N_{n_r,l} = \sqrt{\frac{(2b_{n_r})^{2a+2} n_r!}{\Gamma(n_r + 2a + 1)(2n_r + 2a + 1)}} \quad (21)$$

With the knowledge of the unperturbed bound state solutions, we can then calculate the first-order corrections to the energy eigenvalues using Eq. (11)

$$E_{n_r,l}^{(1)} = \left\langle R_{n_r,l}^{(0)*} \left| H' \right| R_{n_r,l}^{(0)} \right\rangle = \int_0^\infty R_{n_r,l}^{(0)*} \left| H' \right| R_{n_r,l}^{(0)} dr = I_{n_r,1} + I_{n_r,2} + I_{n_r,3} \quad (22)$$

where

$$I_{n_r,1} = B \int_0^\infty R_{n_r,l}^{(0)*} r R_{n_r,l}^{(0)} dr \quad (23)$$

$$I_{n_r,2} = \Lambda_s \int_0^\infty R_{n_r,l}^{(0)*} e^{-\sigma^2 r^2} R_{n_r,l}^{(0)} dr \quad (24)$$

$$I_{n_r,3} = A_1^s \int_0^\infty R_{n_r,l}^{(0)*} \frac{1}{r^3} R_{n_r,l}^{(0)} dr \quad (25)$$

Using the normalized wave function, the integrals in (23) -(24) can be evaluated with the Mathematica program for any quantum quantum n_r . For $n_r = 0$, we obtained

$$I_{0,1} = B \int_0^\infty R_{0,l}^{(0)*} r R_{0,l}^{(0)} dr = B \left(\frac{a+1}{b_{n_r}} \right), \quad (26)$$

$$I_{0,2} = \frac{\Lambda_s b_{n_r}}{\Gamma(2a+2)\sigma^2} \left(\frac{2b_{n_r}}{\sigma} \right)^{2a+1} \left(\sigma \Gamma(a+1) {}_1F_1 \left(a+1, \frac{1}{2}, \frac{b_{n_r}^2}{\sigma^2} \right) - 2b_{n_r} \Gamma \left(a + \frac{3}{2} \right) {}_1F_1 \left(a + \frac{3}{2}, \frac{3}{2}, \frac{b_{n_r}^2}{\sigma^2} \right) \right) \quad (27)$$

$$I_{0,3} = A_1^s \int_0^\infty R_{0,l}^{(0)*} \frac{1}{r^3} R_{0,l}^{(0)} dr = \frac{4b_{n_r}^3 A_1^s}{4a^3 - a} \quad (28)$$

For $n_r = 1$, the integrals in (23) - (24) are obtained as

$$I_{1,1} = \frac{(1+a)(7+2a)B}{(3+2a)b} \quad (29)$$

$$I_{1,2} = \frac{\Lambda_s b_{n_r}}{(3+2a)\sigma^4 \Gamma(2+2a)} \left(\frac{2b_{n_r}}{\sigma} \right)^{2a+1} \left(\sigma \Gamma(1+a)(\Gamma_1 + \Gamma_2) - 2b_{n_r} \Gamma\left(a + \frac{3}{2}\right)(\Gamma_3 + \Gamma_4) \right) \quad (30)$$

where

$$\begin{aligned} \Gamma_1 &= (\sigma + 2a\sigma)^2 {}_1F_1\left(a+1, \frac{1}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) \\ \Gamma_2 &= 4b_{n_r}^2 (a+1) \left({}_1F_1\left(a+2, \frac{1}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) + 2(2a+1) {}_1F_1\left(a+2, \frac{3}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) \right) \\ \Gamma_3 &= 2\sigma^2 (2a+1) {}_1F_1\left(a+\frac{3}{2}, \frac{1}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) + (\sigma + 2a\sigma)^2 {}_1F_1\left(a+\frac{3}{2}, \frac{3}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) \\ \Gamma_4 &= 2b_{n_r}^2 (3+2a) {}_1F_1\left(a+\frac{5}{2}, \frac{3}{2}, \frac{b_{n_r}^2}{\sigma^2}\right) \\ I_{1,4} &= \frac{4b_{n_r}^3 A_1^s}{4a^3 - a} \end{aligned} \quad (31)$$

By neglecting higher-order correction terms in the perturbation series, the total energy equation can be obtained from Eq. (3) as follows:

$$E_{n_r} = E_{n_r}^{(0)} + E_{n_r}^{(1)} \quad (32)$$

where, $E_{n_r}^{(0)}$ and $E_{n_r}^{(1)}$ represent the unperturbed energy and first-order correction.

Mass spectra and decay properties of Heavy and heavy-light mesons

The mass spectra of mesons can be obtained using the mass energy equation

$$M_{nl}(B, \alpha_s, \sigma, m_q, m_{\bar{q}}) = m_q + m_{\bar{q}} + E_{nl} \quad (33)$$

The decay constants within the framework of the nonrelativistic QCD theory are dependent on the wave function at the origin while the existence of the decay of mesons has been confirmed experimentally using large Hadron colliders [46]. The leptons are associated with weak decay

and their decay constants can be obtained via the Van Royen Weisskopf [14, 41, 42, 47, 69, 70, 71, 72] equation given by

$$f_{P/V} = \sqrt{\frac{12|\Psi(0)|^2}{M_{nsP/V}}} \left(1 - \frac{\alpha_s}{\pi} \left(2 - \frac{m_q - m_{\bar{q}}}{m_q + m_{\bar{q}}} \ln \left(\frac{m_q}{m_{\bar{q}}} \right) \right) \right) \quad (34)$$

The notation $|\Psi(0)|^2 (= |R(0)|^2 / 4\pi)$ is the s-state wave function at the origin which may be expressed in terms of the mass difference between the vector and pseudoscalar mesons [73].

$$|\Psi(0)|^2 = \frac{9m_q m_{\bar{q}} (M_V - M_P)}{32\pi\alpha_s} \quad (35)$$

Also, di-photon decay width provides information on the dynamics of the internal structure of mesons [46] where the decay widths for pseudoscalar mesons can be obtained from the Van Royen Weisskopf formula alongside radiative corrections within the framework of QCD theory [41, 42, 69, 70]

$$\Gamma(n^1S_0 \rightarrow \gamma\gamma) = \frac{3\alpha^2 e_q^4 |R_{nsP}(0)|^2}{m_q^2} \left(1 + \frac{\alpha_s}{\pi} \left(\frac{\pi^2 - 20}{3} \right) \right) \quad (36)$$

The leptonic annihilation decay width for vector meson is given [41, 42, 70] by

$$\Gamma(n^3S_1 \rightarrow e^+ e^-) = \frac{4\alpha^2 e_q^2 |R_{nsV}(0)|^2}{M_{nsV}^2} \left(1 - \frac{16\alpha_s}{3\pi} \right) \quad (37)$$

In Eq. (36) and (37), $\alpha(1/137)$ is the electromagnetic fine structure constant while e_q is the charge of the quark in units of electron charge. For bottomonium $e_q = (-1/3)e$. Also, the quantum number $n = n_r + 1$ with $n_r = 0, 1$.

3. Discussion of results

We have derived the mass spectra equation using the energy constructed from the perturbation theory discussed in section 2. The mass spectrum equation was used to fit the experimental masses allowing for the determination of the potential parameters given in Table 1. We fitted fifteen established experimental data points for the bottomonium meson by solving transcendental equations numerically with the help of the NLPSolve function in maple program. We chose the values of the parameters that closely predicted the masses, decay properties and also obtained the potential energy curve for the heavy and heavy-light mesons. In Fig. 1, the potential energy curve accounts for both Coulomb interaction and linear confinement component. To test the accuracy of the perturbation method, we have checked the energy eigenvalues of the standard Cornell potential in the absence of spin and tensor interactions. In Table 2, the energy eigenvalues for arbitrary parameters (B and α_s) for the ground and first excited quantum states have been predicted and compared with the values obtained by the Matrix Numerov method [43]. The results obtained here using the perturbation method agree with the exact values obtained by numerical approach. The energy eigenvalues for $n=1, \alpha=0.75, B=0.01 (E_1 = -0.2200)$ also matches those obtained by the artificial neural

network (-0.221030286) [10], asymptotic iteration method (-0.221031) [19] and numerical approach (-0.220591855) [43]

Table 1. Parameters of the spin-dependent Cornell potential.

Parameters	Bottomonium	Bottom-charm
Mass(GeV)	$m_b = m_{\bar{b}} = 4.9148$	$m_c = 1.80, m_b = 4.9148$
α_s	0.4486	0.5965
$B(\text{GeV}^2)$	0.0551	0.0418
$\sigma(\text{GeV})$	2.5844	2.5844

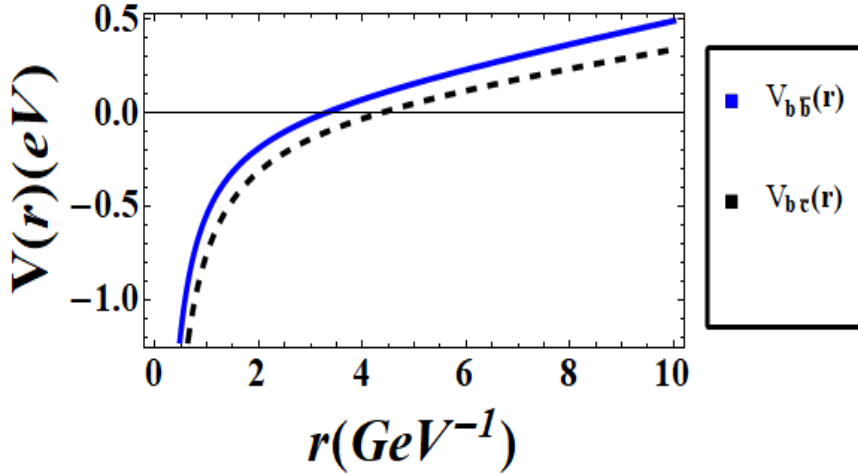


Figure 1. Potential function curves for bottomonium meson and bottom-charm mesons

Table 2. Comparison of the s-wave energy eigenvalues in atomic units for the Cornell potential

$(-\frac{4\alpha_s}{3r} + Br)$				
B	$n = 1, \alpha_s = 7.5$		$n = 2, \alpha_s = 7.5$	
	This work	Numerical [43]	This work	Numerical [43]
0.01	-24.99700000	-24.99658550	-6.23800000	-6.237942035
0.03	-24.99100000	-24.99058636	-6.21400000	-6.213984653
0.05	-24.98500000	-24.98458818	-6.19000000	-6.190068698
0.07	-24.97900000	-24.97859096	-6.16600000	-6.166193693
0.09	-24.97300000	-24.97259469	-6.14200000	-6.142359166
0.10	-24.97000000	-24.96959691	-6.13000000	-6.130456943
α	$n = 1, B = 0.001$		$n = 2, B = 0.001$	
0.75	-0.247000000	-0.247002376	-0.050500000	-0.050852733
2.25	-2.249000000	-2.248946451	-0.558500000	-0.558497609
3.75	-6.249400000	-6.249383380	-1.560100000	-1.560049166
5.25	-12.24957143	-12.24950758	-3.060785714	-3.060352141
6.75	-20.24966667	-20.24949234	-5.061166667	-5.060996539
8.25	-30.24972727	-30.24933858	-7.561409091	-7.561356085

In Tables 3 and 4, we have applied equation (33) to obtain the masses for the bottomonium and bottom-charm mesons respectively. We obtained the charm and bottom quarks masses of $m_c = 1.80$ and $m_b = 4.9148$ given in Table 1, which are within the range reported in the literature. In previous studies [4, 27], the Pekeris-type approximation [72] had been used to enable the analytical solution of the bound state while the Gaussian function in the spin-spin components was Taylor expanded to a quadratic function. The perturbation approach does not require this process and we found that it improves the meson masses compared to the results in [4, 27]. By calculating the absolute percentage average deviation (APAD) given by

$$APAD\% = \frac{100}{d} \sum_{i=1}^d \left| \frac{M_{nl}^{exp} - M_{nl}^{theo}}{M_{nl}^{exp}} \right| \quad (38)$$

where d , M_{nl}^{exp} and M_{nl}^{theo} are the numbers of established experimental data points, experimental masses and calculated masses respectively. The bottomonium masses increase with the principal quantum number. The obtained masses for the $\chi_{b_1}(1^3P_1)$, $\chi_{b_2}(1^3P_2)$, $\chi_{b_0}(1^3P_0)$ and $h_b(1^1P_1)$ states were found to deviate from the experimental values by 11.76MeV, 0.92MeV, 36.8MeV and 7.7MeV respectively. The APAD of 0.15% was obtained indicating an improvement over the results in [27, 34, 40, 46] and comparable to the results in [39]. The masses of the bottom-charm meson were calculated as presented in Table 4. We have used the σ value of the bottomonium meson to fit the mass spectra equation and obtained the potential parameters for the effective coupling constant (α_s) and linear confinement parameter (B). The masses obtained are in agreement with the results obtained in [4, 39, 34, 46].

Table 3. Low-lying mass spectrum of bottomonium meson in GeV .

State		This work	[27]	[75]	[39]	[40]	[34]	[35]	[46]	Expt. [66]
$n^{2s+1}L_j$	J^{PC}									
$\Upsilon(1^3S_1)^*$	1^{--}	9.4629	9.878	9.465	9.460	9.49081	9.525	9.4600	9.4600	$9.460 \pm (2.6 \times 10^{-4})$
$\eta_b(1^1S_0)^*$	0^{-+}	9.3965	9.951	9.402	9.398	9.43601	9.472	9.3900	9.4280	$9.399 \pm (2 \times 10^{-3})$
$\Upsilon(2^3S_1)$	1^{--}	9.9465	10.130	10.003	10.023	10.01257	10.049	10.0150	9.9790	$10.023 \pm (3.1 \times 10^{-4})$
$\eta_b(2^1S_0)$	0^{-+}	9.9390	10.205	9.976	9.990	9.99146	10.028	9.9900	9.9550	
$\chi_{b_1}(1^3P_1)$	1^{++}	9.9048	9.937	9.876	9.892	9.87371	9.875	9.9030	9.8190	$9.893 \pm (2.6 \times 10^{-4}) \pm (3.1 \times 10^{-4})$
$\chi_{b_2}(1^3P_2)$	2^{++}	9.9111	9.912	9.897	9.912	9.89083	9.903	9.921	9.825	$9.912 \pm (2.6 \times 10^{-4}) \pm (3.1 \times 10^{-4})$
$\chi_{b_0}(1^3P_0)$	0^{++}	9.8958	9.970	9.847	9.859	9.8432	9.840	9.864	9.806	$9.859 \pm (4.2 \times 10^{-4}) \pm (3.1 \times 10^{-4})$
$h_b(1^1P_1)$	1^{+-}	9.9067	9.998	9.882	9.900	9.87919	9.884	9.9090	9.8210	$9.899 \pm (8 \times 10^{-4})$
$\chi_{b_1}(2^3P_1)$	1^{++}	10.2493	10.198	10.246	10.255	10.21695	10.229	10.249	10.2170	$10.255 \pm (2.2 \times 10^{-4}) \pm (5 \times 10^{-4})$
$\chi_{b_2}(2^3P_2)$	2^{++}	10.2494	10.168	10.261	10.268	10.22961	10.254	10.246	10.224	$10.269 \pm (2.2 \times 10^{-4}) \pm (5 \times 10^{-4})$
$\chi_{b_0}(2^3P_0)$	0^{++}	10.2475	10.235	10.226	10.233	10.19625	10.202	10.220	10.205	$10.233 \pm (4 \times 10^{-4}) \pm (5 \times 10^{-4})$
$h_b(2^1P_1)^*$	1^{+-}	10.2491	10.259	10.250	10.260	10.22153	10.237	10.254	10.220	$10.260 \pm (1.2 \times 10^{-3})$
$\Upsilon_2(1^3D_2)$	2^{--}	10.1750	10.027	10.147	10.161	10.1126	10.096	10.153	10.075	$10.164 \pm (1.4 \times 10^{-3})$
$\Upsilon_3(1^3D_3)$	3^{--}	10.1729	9.990	10.155	10.166	9.73855	9.849	10.157	10.073	10.172[75]
$\Upsilon_1(1^3D_1)$	1^{--}	10.1761	10.064	10.138	10.154	9.72905	9.666	10.146	10.074	10.155[75]
$\eta_{b_2}(1^1D_2)$	2^{-+}	10.1742	10.085	10.148	10.163	9.7355	9.767	10.153	10.074	10.165[75]
$\Upsilon_2(2^3D_2)$	2^{--}	10.5905	10.300	10.449	10.443	9.94259	10.071	10.432	10.424	
$\Upsilon_3(2^3D_3)$	3^{--}	10.5860	10.257	10.455	10.449	9.94704	10.175	10.436	10.423	
$\Upsilon_1(2^3D_1)$	1^{--}	10.5934	10.344	10.441	10.435	9.93775	9.996	10.425	10.423	
$\eta_{b_2}(2^1D_2)$	2^{-+}	10.5890	10.359	10.450	10.445	9.94405	10.093	10.432	10.424	
$\chi_{b_3}(1^3F_3)$	3^{++}	10.4779	10.148	10.355	10.346	9.7361	9.754	10.340	10.287	
$\chi_{b_4}(1^3F_4)$	4^{++}	10.4726	10.103	10.358	10.349	9.74242	9.896	10.340	10.291	
$\chi_{b_2}(1^3F_2)$	2^{++}	10.4820	10.190	10.350	10.343	9.72948	9.642	10.338	10.283	
$\eta_{b_3}(1^1F_3)$	3^{-+}	10.4766	10.203	10.355	10.347	9.73759	9.778	10.339	10.288	
$\chi_{b_3}(2^3F_3)$	3^{++}	10.9946	10.440	10.619	10.614	9.94462	10.081		10.607	
$\chi_{b_4}(2^3F_4)$	4^{++}	10.9854	10.388	10.622	10.617	9.9508	10.219		10.609	
$\chi_{b_2}(2^3F_2)$	2^{++}	11.0015	10.491	10.615	10.610	9.93815	9.971		10.604	
$\eta_{b_3}(2^1F_3)$	3^{-+}	10.9922	10.495	10.619	10.615	9.94608	10.104		10.607	
$\Upsilon_4(1^3G_4)$	4^{--}	10.8444	10.298	10.531	10.512					
$\Upsilon_5(1^3G_5)$	5^{--}	10.8344	10.243	10.532	10.514					
$\Upsilon_3(1^3G_3)$	3^{--}	10.8526	10.338	10.529	10.511					
$\eta_{b_4}(1^1G_4)$	4^{-+}	10.8424	10.343	10.530	10.513					
APAD%		0.15	1.35	0.13	0.011	1.08	1.06	0.097	0.58	

Table 4. Low-lying mass spectrum of bottom-charmed meson in GeV .

State		This work	[4]	[39]	[34]	[46]	Expt.[66]
$n^{2s+1}L_j$	J^{PC}						
1^3S_1	1^{--}	6.3916	6.225	6.333	6.313	6.321	
$(1^1S_0)^*$	0^{-+}	6.2750	6.275	6.272	6.276	6.272	6.275
2^3S_1	1^{--}	6.8558	6.783	6.882	6.867	6.900	
$(2^1S_0)^*$	0^{-+}	6.8420	6.842	6.842	6.841	6.864	6.842
1^3P_1	1^{++}	6.8094	6.218	6.743	6.281	6.705	
1^3P_2	2^{++}	6.8171	6.248	6.761	6.366	6.712	
1^3P_0	0^{++}	6.7982	6.178	6.699	6.223	6.686	
1^1P_1	1^{+-}	6.8120	6.280	6.750	6.290	6.706	
2^3P_1	1^{++}	7.1709	6.777	7.134	6.836	7.165	
2^3P_2	2^{++}	7.1704	6.803	7.157	6.917	7.173	
2^3P_0	0^{++}	7.1690	6.743	7.146	6.782	7.146	
2^1P_1	1^{+-}	7.1703	6.847	7.147	6.846	7.168	
1^3D_2	2^{--}	7.0914	6.228	7.025	6.299	6.997	
1^3D_3	3^{--}	7.0878	6.278	7.029	6.429	6.990	
1^3D_1	1^{--}	7.0936	6.177	7.021	6.200	6.998	
1^1D_2	2^{-+}	7.0901	6.291	7.026	6.308	6.994	
2^3D_2	2^{--}	7.5339	6.787	7.399	6.852	7.403	
2^3D_3	3^{--}	7.5263	6.829	7.405	6.975	7.399	
2^3D_1	1^{--}	7.5389	6.742	7.392	6.759	7.403	
2^1D_2	2^{-+}	7.5313	6.856	7.400	6.861	7.401	
1^3F_3	3^{++}	7.4133	6.244	7.269	6.326	7.242	
1^3F_4	4^{++}	7.4043	6.312	7.277	6.501	7.244	
1^3F_2	2^{++}	7.4201	6.175	7.273	6.182	7.234	
1^1F_3	3^{+-}	7.4110	6.307	7.268	6.335	7.241	
2^3F_3	3^{++}	7.9656	6.801	7.616	6.876	7.615	
2^3F_4	4^{++}	7.9503	6.859	7.617	7.041	7.617	
2^3F_2	2^{++}	7.9773	6.740	7.618	6.741	7.607	
2^1F_3	3^{+-}	7.9618	6.870	7.615	6.885	7.614	
1^3G_4	4^{--}	7.8047	6.265	7.489			
1^3G_5	5^{--}	7.7880	6.353	7.482			
1^3G_3	3^{--}	7.8185	6.178	7.497			
1^1G_4	4^{-+}	7.8013	6.328	7.487			

In Tables 5 and 6, we obtained the vector and pseudoscalar decay constants using the leading order Van Royen Weisskopf relation and the effective coupling constant obtained from fitting the mass spectrum. We have applied the mass difference equation (35) to obtain the wave function at the origin which is then inserted into Eq. (34). We have applied this process in reproducing the decay constants for the charmonium mesons in [42] with accuracy giving us the motivation to use it in this present work. However, using (40) is limited to the s-state wave function at the origin because the equation of the decay widths for $l > 0$ includes the first and second derivative of the wave function at the origin. Alongside the leading-order decay properties, we also calculated the radiative corrections to first-order within QCD theory. We calculated the strong running coupling constant ($\alpha_s(m)$) to leading or one loop approximation at

an appropriate renormalization scale from the current experimental world value $\alpha_s(m_z) = 0.118$ [76]

$$\alpha_s(\bar{\mu}) = \frac{\alpha_s(m_z)}{1 - \left(11 - \frac{2n_f}{3}\right) \left(\frac{\alpha_s(m_z)}{2\pi}\right) \ln\left(\frac{m_z}{\bar{\mu}}\right)}, \quad (45)$$

where m_z , $\bar{\mu}(2m_q m_{\bar{q}}/m_q + m_{\bar{q}})$ and n_f are the respective mass of z-Boson with recent value of 91.187 GeV, energy scale and number of quark flavours. We found $\alpha_s(m_b) = 0.2173$ and $\alpha_s(m_{bc}) = 0.2650$. The results of the vector and pseudoscalar decay constant of bottomonium and bottom-charm mesons for the $1S$ and $2S$ quantum states (with the values in parentheses indicating radiative corrections) are tabulated in Table 5 and 6 alongside the results of other theoretical and available experimental data. The decay constants decrease with increasing principal quantum number. For the bottomonium meson 1^3S_1 and 2^3S_1 quantum states (see Table 5), the vector and pseudoscalar decay constants without QCD corrections are in agreement with other theoretical [41, 46, 51-53] and available experimental data [67]. In Table 6, the bottom-charm vector and pseudoscalar decay constants are in agreement with the results in [46, 52] for the 1^3S_1 the 2^3S_1 quantum states. Generally, the results obey the inequality $\sqrt{M_V} f_V \geq \sqrt{M_P} f_P$ to four decimal places. In Tables 7 and 8, the pseudoscalar di-photon and vector di-lepton decay widths were calculated and presented respectively. The di-photon decay widths for the pseudoscalar bottomonium meson with state 1^1S_0 were found to be 0.3286 KeV using the leading-order Van Royen Weisskopf formula with the effective coupling constant obtained from fitting and 0.5199 KeV using the strong running coupling with radiative corrections. These values are close to other results in [41, 46, 55, 56, 59]. For the 2^1S_0 state, the decay widths are underestimated but comparable with the results of Refs. [41, 53]. In Table 8, the results obtained for the 1^3S_1 di-leptonic vector decay widths for the bottomonium meson are close to the results obtained using other methods [41, 42, 62] and observed values [67] while the 2^3S_1 underestimate other theoretic values and observed data.

Table 5. Vector and pseudoscalar decay constant for bottomonium mesons in *MeV*.

Decay	State	This work		Others				
			[41]	[46]	[52]	[51]	[53]	Exp [67]
F_V	1^3S_1	637.1036 (849.7251)	640(530)	647.250	706	707	665	715±5
	2^3S_1	208.8496 (278.5493)	383(317)	519.436	547	393	475	498±8
F_P	1^1S_0	639.3507(852.7221)	639(529)	646.025	744	756	599	
	2^1S_0	208.9284(278.6544)	382(317)	518.803	577	285	411	

Table 6. Vector and pseudoscalar decay constant for bottom-charm meson in MeV .

		This work	Others
Decay	State		
F_V	1^3S_1	539.1261 (754.7128)	[46] 434.642 [52] 435
	2^3S_1	179.0838(250.6962)	356.435 337
F_P	1^1S_0	544.1120(761.6925)	432.955 456
	2^1S_0	179.2643(250.9489)	355.504 361

Table 7. Di-photon decay widths ($\Gamma_{\gamma\gamma}$) in KeV

State	Bottomonium							
	This work					Others		
		[46]	[55]	[57]	[53]	[56]	[41]	[59]
1^1S_0	0.3286(0.5199)	0.387	0.527	0.214	0.23	0.35	0.3364(0.2361)	0.384 ± 0.047
2^1S_0	0.0371(0.0587)	0.263	0.263	0.121	0.07	0.15	0.1270(0.0896)	0.191 ± 0.025

Table 8. Di-lepton decay widths ($\Gamma_{e^+e^-}$) in KeV

State	Bottomonium						
	This work	Others					
		[42]	[58]	[61]	[62]	[41]	Exp.[67]
1^3S_1	1.0637(1.3858)	1.095	1.20	1.33	0.98	1.074(0.582)	1.34 ± 0.018
2^3S_1	0.1087(0.1417)	0.670	0.52	0.62	0.41	0.363(0.197)	0.612 ± 0.011

4. Conclusion

We have investigated the bound-state properties of mesons using a spin-dependent Cornell potential within first-order perturbation theory. Analytical expressions for the energy eigenvalues were obtained, and in the absence of spin-dependent interactions, the results reduce to those of the standard Cornell potential. The corresponding energy eigenvalues show good agreement with results obtained using the artificial neural network [10], asymptotic iteration method [19], and matrix Numerov method [43], thereby providing a useful validation of the perturbative approach. The parameters of the spin-dependent Cornell potential were determined by fitting the analytical mass-spectrum equation to the well established experimental masses of the particle data group. These parameters were subsequently employed in the numerical calculation of the low-lying quantum states of the masses and decay properties of bottomonium and bottom-charm mesons. For the bottomonium states, an average percentage absolute deviation of approximately 0.15% from the experimental masses was obtained, representing an improvement over several previous works in [4, 27, 34, 40] and recent studies [63, 64]. This demonstrates that the proposed phenomenological framework provides a satisfactory description of the heavy-quark mass spectrum. The decay constants obtained for the vector and pseudoscalar states are comparable with values reported in previous theoretical studies. They exhibit a decreasing trend with increasing principal quantum number while obeying the inequality $\sqrt{M_V} f_V \geq \sqrt{M_P} f_P$. The pseudoscalar di-photon and vector leptonic decay widths were also calculated using the Van Royen–Weisskopf formalism. The resulting predictions are generally consistent with available experimental measurements and with values obtained using other theoretical approaches. An important feature of the present formulation is that the perturbative treatment permits the strong coupling entering the decay calculations to be considered within a running-coupling framework.

The use of the QCD running coupling at appropriate renormalization scales leads to improved predictions for several decay observables, highlighting the sensitivity of decay properties to the effective coupling (α_s) and to the wave function at the origin. Thus, the present approach provides a unified phenomenological framework in which the mass spectrum, wave function at the origin, and decay properties can be treated consistently. Overall, the results demonstrate that first-order perturbation theory applied to the spin-dependent Cornell potential provides an effective approach for describing the heavy and heavy-light quark bound states. In particular, the present calculation extends the treatment of decay properties beyond the Pekeris-type approximation and complements recent phenomenological studies [63, 64]. The agreement obtained with experimental data and previous theoretical calculations supports the applicability of the model to bottomonium and bottom-charm mesons.

Credit authorship contribution statement

E.O. Conceptualization, Data curation, Methodology, Writing – original draft, Writing – review & editing. E.P.I. Writing – review & editing, Visualization. E.M.K. Formal analysis, Writing – original draft, Writing – review & editing. All authors reviewed the manuscript.

Data Availability statement

The authors declare that the data supporting the findings of this study are available within the paper.

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