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On Base , Normal and Near-normal Sequences

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Abstract

A base sequence $\mathbf{BS}(n+1, n)$ consists of four sequences of ± 1 entries with lengths $n+1$, $n+1$, n , and n , respectively, whose combined nonperiodic autocorrelation sums vanish. The base sequence conjecture asserts that $\mathbf{BS}(n+1, n)$ exists for every positive integer n ; this has previously been verified for $n \leq 40$. In this paper, we present an algorithm for constructing base sequences and construct $\mathbf{BS}(n+1, n)$ for $n = 41, 42$, and 43 for the first time.

Normal sequences $\mathbf{NS}(n)$ and near-normal sequences $\mathbf{NNS}(n)$ are subclasses of base sequences. Yang's conjecture asserts that an $\mathbf{NNS}(n)$ exists for every even positive integer n ; this has previously been verified for $n \leq 40$. An exhaustive computer search shows that no $\mathbf{NNS}(n)$ exists for $n = 42$ or 44 , providing the first counterexamples to Yang's conjecture. Exhaustive computer search also shows that no $\mathbf{NS}(n)$ exists for $n = 41, 42, 43, 44$, or 45 . We also prove that no $\mathbf{NS}(n)$ exists when $n = 8k - 2$ for any positive integer k .

Keywords: Base Sequence, Normal Sequence, Near-normal Sequence, Yang Conjecture

MSC Classification: 05B20 , 05B05

1 Introduction

A base sequence consists of four sequences of ± 1 entries, of lengths m, m, n, n , respectively, whose combined nonperiodic autocorrelation sums vanish. Base sequences were first introduced by Turyn [1] in the construction of Hadamard matrices. The base sequence conjecture (BSC) [2](P319) asserts that $\mathbf{BS}(n+1, n)$ exists for every $n \in \mathbb{Z}_+$.

As pointed out by Ćoković [3], the BSC is stronger than the famous Hadamard conjecture. To date, the BSC has been verified for all $n \leq 40$ and for all Golay numbers $n = 2^a 10^b 26^c$, $a, b, c \geq 0$ [4].

Definition 1 The base sequences, denoted by $\text{BS}(m, n)$, are quadruples (A, B, C, D) of ± 1 sequences of lengths m, m, n, n , respectively, whose combined nonperiodic autocorrelation sums satisfy

$$N_A(i) + N_B(i) + N_C(i) + N_D(i) = \begin{cases} 2m + 2n, & i = 0, \\ 0, & i = 1, \dots, n-1. \end{cases} \quad (1)$$

Definition 2 A base sequence $(A, B, C, D) \in \text{BS}(n+1, n)$ is called a normal sequence, denoted by $\text{NS}(n)$, if

$$\begin{cases} b_i = a_i, & 1 \leq i \leq n, \\ a_{n+1} = 1, & b_{n+1} = -1. \end{cases} \quad (2)$$

Definition 3 A base sequence $(A, B, C, D) \in \text{BS}(n+1, n)$ is called a near-normal sequence, denoted by $\text{NNS}(n)$, if n is even and

$$\begin{cases} b_i = (-1)^{i-1} a_i, & 1 \leq i \leq n, \\ a_{n+1} = 1, & b_{n+1} = -1. \end{cases} \quad (3)$$

Normal and near-normal sequences can be viewed as subclasses of $\text{BS}(n+1, n)$. They were introduced by Yang in the construction of T -sequences [5]. If normal and near-normal sequences of length t and base sequences $\text{BS}(m, n)$ exist, then a T -sequence of length $(m+n)(2t+1)$ can be constructed. We call $y = 2t+1$ the Yang number. The near-normal sequences is conjectured to exist for all even integer lengths, which is called Yang conjecture[2](P320).

Theorem 1 ([6, 7]) *If base sequences $\text{BS}(m, n)$, a Yang number y , Williamson-type matrices $\text{WT}(w)$, and Baumert–Hall–Welch arrays $\text{BHW}(4h)$ exist, then a Hadamard matrix $\text{HM}(4N)$ exists, where $N = yh(m+n)w$.*

Base, Normal and Near-normal Sequences are important in the construction of the Hadamard matrix, the method is described in Theorem 1. For example, $\text{BS}(71, 36)$ is used for the construction of Hadamard matrix of order 428 [8], it is also believed that this may be the most promising approach to construct $\text{H}(668)$, the smallest unknown case for Hadamard matrix[9]. Therefore, the study of Base, Normal and Near-normal Sequences is of interest and has been studied by many previous works. Proposition 2 lists the state of the art for Base, Normal and Near-normal Sequences.

Proposition 2

1. $BS(n+1, n)$ exists for all $n \leq 40$ and for all Golay numbers ($n = 2^a 10^b 26^c$ with $a, b, c \geq 0$). The first unknown case is $BS(42, 41)$ [4].
2. $NNS(n)$ exists for all even integers $n \leq 40$. The first unknown case is $NNS(42)$ [7].
3. $NS(n)$ exist for all the Golay numbers and $n = 3, 5, 7, 9, 11, 12, 13, 25, 29$, $NS(n)$ do not exist for all the other $n \leq 40$. The first unknown case is $NS(41)$. [10].
4. All odd numbers $n \leq 81$ are the Yang numbers except 35, 43, 47, 55, 63, 67, 71, 75, 79.

In this paper, we first review the properties of base sequences that are useful for our search. We then introduce an algorithm that makes extensive use of the known constraints. Most previous search methods directly search for all four sequences. In contrast, our method first enumerates all pairs of sequences that satisfy the relevant properties and then searches for the remaining two sequences for each candidate pair. Finally, we apply the algorithm to search for base, normal, and near-normal sequences. We present the first constructions of $BS(n+1, n)$ for $n = 41, 42$, and 43 . We also perform exhaustive searches for $NNS(n)$ with $n \leq 44$ and for $NS(n)$ with $n \leq 46$. These searches show that $NNS(n)$ does not exist for $n = 42$ or 44 , providing the first counterexamples to Yang conjecture, and that $NS(n)$ does not exist for $41 \leq n \leq 45$. In addition, we prove mathematically that $NS(n)$ does not exist when $n = 8k - 2$ for any $k \in \mathbb{Z}_+$.

2 Properties on Base Sequences($n+1, n$)

In this section, we introduce the properties of base sequences $BS(n+1, n)$ used in our search algorithm.

For a given sequence $A=(a_1, a_2, \dots, a_n)$, the negated sequence $-A$, reversed sequence A' and alternated sequence A^* are defined by

$$\begin{aligned} -A &= -a_1, -a_2, \dots, -a_n \\ A' &= a_n, a_{n-1}, \dots, a_1 \\ A^* &= a_1, -a_2, a_3, -a_4, \dots, (-1)^{n-1}a_n \end{aligned} \tag{4}$$

Two base sequences are equivalent if one can be transformed to the other by applying one or more isomorphic transformations. There are five isomorphic transformations of $(A; B; C; D) \in BS(n+1, n)$ [11]:

- (i) Negate one or more sequences of A, B, C, D .
- (ii) Reverse one or more sequences of A, B, C, D .
- (iii) Interchange A and B , or interchange C and D .
- (iv) Alternate all four sequences of A, B, C, D .
- (v) Replace $\begin{bmatrix} c_i & c_{n+1-i} \\ d_i & d_{n+1-i} \end{bmatrix}$ that are $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ with $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, respectively.

We next introduce properties involving the sums of the individual sequences.

Let $A(z) = a_1 + a_2z + \dots + a_nz^{n-1}$, we refer to the Laurent polynomial $N(A) = A(z)A(z^{-1})$ as the norm of A , then

$$N(A) = A(z)A(z^{-1}) = \sum_{i \in \mathbb{Z}} N_A(i)(z^i + z^{-i}), z \neq 0 \quad (5)$$

By Definition 1, $\text{BS}(m, n)$ satisfies

$$N(A) + N(B) + N(C) + N(D) = 2(m + n) \quad (6)$$

Theorem 3 Let $(A, B, C, D) \in \text{BS}(n + 1, n)$, and let a, b, c, d and a^*, b^*, c^*, d^* denote the sums of the entries in A, B, C, D and A^*, B^*, C^*, D^* , respectively. Then

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 4n + 2 \\ a^{*2} + b^{*2} + c^{*2} + d^{*2} &= 4n + 2 \\ a \equiv b \equiv a^* \equiv b^* \equiv n + 1 \pmod{2} \\ c \equiv d \equiv c^* \equiv d^* \equiv n \pmod{2} \end{aligned} \quad (7)$$

and the modular 4 properties:

$$\begin{aligned} c \equiv d, c^* \equiv d^* &\pmod{4} \text{ when } n \text{ even} \\ a \equiv b + 2, a^* \equiv b^* + 2 &\pmod{4} \text{ when } n \text{ odd} \end{aligned} \quad (8)$$

$$\begin{aligned} a \equiv a^*, b \equiv b^*, c \equiv c^*, d \equiv d^*, &\text{ when } n \equiv 0 \pmod{4} \\ a \equiv a^* + 2, b \equiv b^* + 2, c \equiv c^*, d \equiv d^*, &\text{ when } n \equiv 1 \pmod{4} \\ a \equiv a^* + 2, b \equiv b^* + 2, c \equiv c^* + 2, d \equiv d^* + 2, &\text{ when } n \equiv 2 \pmod{4} \\ a \equiv a^*, b \equiv b^*, c \equiv c^* + 2, d \equiv d^* + 2, &\text{ when } n \equiv 3 \pmod{4} \end{aligned} \quad (9)$$

Proof Evaluating (6) at $z = 1$ and $z = -1$ gives the first two equalities in (7). The congruences modulo 2 follow from the fact that all entries of the sequences are equal to ± 1 .

To prove (8), we use the fact that the square of every odd integer is congruent to 1 modulo 8.

If n is even, then a and b are odd, whereas c and d are even.

Since $a^2 + b^2 + c^2 + d^2 = 4n + 2 \equiv 2 \pmod{8}$

We have $c^2 + d^2 \equiv 0 \pmod{8}$, thus $c \equiv d \pmod{4}$

Similarly, if n is odd, then a and b are even, whereas c and d are odd.

Since $a^2 + b^2 + c^2 + d^2 = 4n + 2 \equiv 6 \pmod{8}$

We have $a^2 + b^2 \equiv 4 \pmod{8}$, thus $a \equiv b + 2 \pmod{4}$

The same method can be applied to a^*, b^*, c^* and d^* .

To prove (9), let a_o and a_e denote the sums of the entries in the odd and even positions of A , respectively. Then $a = a_o + a_e$, $a^* = a_o - a_e$.

Consequently, $a \equiv a^* \pmod{4}$ when a_e is even, and $a \equiv a^* + 2 \pmod{4}$ when a_e is odd.

Applying this observation to each of A, B, C , and D , and considering the four possible residue classes of n modulo 4, proves (9). $\square$

Corollary 4 For a near-normal sequence $\text{NNS}(n)$, in addition to the conclusions of Theorem 3, we have

$$a = b^* + 2, \quad b = a^* - 2. \quad (10)$$

For a normal sequence $\text{NS}(n)$, in addition to the conclusions of Theorem 3, we have

$$\begin{cases} a = b + 2, & a^* = b^* - 2, & \text{if } n \text{ is odd,} \\ a = b + 2, & a^* = b^* + 2, & \text{if } n \text{ is even.} \end{cases} \quad (11)$$

Proof For $\text{NNS}(n)$, n is even. By Definition 3, we have

$$a_o = b_o + 2, \quad a_e = -b_e.$$

Therefore,

$$a = a_o + a_e = b_o + 2 - b_e = b^* + 2,$$

and

$$b = b_o + b_e = a_o - 2 - a_e = a^* - 2.$$

For $\text{NS}(n)$, if n is odd, By Definition 2, we have

$$a_o = b_o, \quad a_e = b_e + 2.$$

Thus,

$$a = a_o + a_e = b_o + b_e + 2 = b + 2,$$

and

$$a^* = a_o - a_e = b_o - b_e - 2 = b^* - 2.$$

If n is even, then

$$a_o = b_o + 2, \quad a_e = b_e.$$

Consequently,

$$a = a_o + a_e = b_o + b_e + 2 = b + 2,$$

and

$$a^* = a_o - a_e = b_o + 2 - b_e = b^* + 2.$$

□

Theorem 3, which originates from [3], is useful for reducing the search space. Our search for $\text{BS}(n+1, n)$ begins with one of the possible sets of values for $a, b, c, d, a^*, b^*, c^*, d^*$ obtained from Theorem 3. For $\text{NS}(n)$ and $\text{NNS}(n)$, we begin by determining all possible sets of these values using Theorem 3 and Corollary 4. In Section 5, we also prove that no such set exists for $\text{NS}(n)$ when $n = 8k - 2$, where $k \in \mathbb{Z}_+$.

Theorem 5 [1, 12] Let A, B, C, D be $\text{BS}(n+1, n)$, then

$$\begin{aligned} a_i + b_i + a_{n+2-i} + b_{n+2-i} &= \begin{cases} 2 \bmod 4, & i = 1 \\ 0 \bmod 4, & i = 2, \dots, [(n+1)/2] \end{cases} \\ c_i + d_i + c_{n+1-i} + d_{n+1-i} &= 0 \bmod 4, i = 2, \dots, [n/2] \end{aligned} \quad (12)$$

Theorem 5 is a well known property and shows that for each $\begin{bmatrix} a_i & a_{n+2-i} \\ b_i & b_{n+2-i} \end{bmatrix}$ and $\begin{bmatrix} c_i & c_{n+1-i} \\ d_i & d_{n+1-i} \end{bmatrix}$, there are only 8 possible cases. This property has been widely used in previous work.

Theorem 6 [12] *Let A, B, C, D be $BS(n+1, n)$ and given $m \in 2, 3, \dots, n+1$, denote*

$$\begin{aligned} k_{im} &= \sum_{j \equiv i \pmod{m}} a_j, & r_{im} &= \sum_{j \equiv i \pmod{m}} b_j, \\ p_{im} &= \sum_{j \equiv i \pmod{m}} c_j, & q_{im} &= \sum_{j \equiv i \pmod{m}} d_j. \end{aligned} \quad (13)$$

$$\begin{aligned} N_K(s) &= \sum_{i=1}^{m-s} k_{im} k_{i+s, m}, & N_R(s) &= \sum_{i=1}^{m-s} r_{im} r_{i+s, m}, \\ N_P(s) &= \sum_{i=1}^{m-s} p_{im} p_{i+s, m}, & N_Q(s) &= \sum_{i=1}^{m-s} q_{im} q_{i+s, m}. \end{aligned} \quad (14)$$

then for the given $m \in 2, 3, \dots, [(n+1)/2]$, we have

$$\begin{aligned} N_K(0) + N_R(0) + N_P(0) + N_Q(0) &= k_{1m}^2 + \dots + k_{mm}^2 + r_{1m}^2 + \dots + r_{mm}^2 \\ &+ p_{1m}^2 + \dots + p_{mm}^2 + q_{1m}^2 + \dots + q_{mm}^2 = 4n + 2 \\ N_K(s) + N_R(s) + N_P(s) + N_Q(s) &+ N_K(m-s) + N_R(m-s) \\ &+ N_P(m-s) + N_Q(m-s) = 0, \quad s = 1, \dots, [m/2]. \end{aligned} \quad (15)$$

and for the $k_{im}, r_{im}, p_{im}, q_{im}$, we can also get the following range and module 4 property:

$$\begin{aligned} |k_{im}| &\leq \left\lfloor \frac{n+1-i}{m} \right\rfloor + 1, & |r_{im}| &\leq \left\lfloor \frac{n+1-i}{m} \right\rfloor + 1, & i = 1, \dots, m \\ |p_{im}| &\leq \left\lfloor \frac{n-i}{m} \right\rfloor + 1, & |q_{im}| &\leq \left\lfloor \frac{n-i}{m} \right\rfloor + 1, & i = 1, \dots, m \\ k_{im} &\equiv r_{im} \equiv \left\lfloor \frac{n+1-i}{m} \right\rfloor + 1, & p_{im} &\equiv q_{im} \equiv \left\lfloor \frac{n-i}{m} \right\rfloor + 1, & \pmod{2} \end{aligned} \quad (16)$$

$$\begin{aligned} k_{1m} + r_{1m} + k_{n+1, m} + r_{n+1, m} &\equiv \begin{cases} 2 \pmod{4} & \text{if } n \not\equiv 0 \pmod{m}, \\ 0 \pmod{4} & \text{if } n \equiv 0 \pmod{m}, \end{cases} \\ k_{jm} + r_{jm} + k_{n+2-j, m} + r_{n+2-j, m} &\equiv 0 \pmod{4}, \quad j = 2, \dots, m, \\ p_{jm} + q_{jm} + p_{n+1-j, m} + q_{n+1-j, m} &\equiv 0 \pmod{4}, \quad j = 1, \dots, m. \end{aligned} \quad (17)$$

Theorem 6 describes the sum-of-squares property obtained when the sequences are partitioned according to their indices modulo m . The proof can be found in [12]. This property can be used to search for base sequences as follows. First, we determine values of

$$k_{1m}, \dots, k_{mm}, \quad r_{1m}, \dots, r_{mm}, \quad p_{1m}, \dots, p_{mm}, \quad q_{1m}, \dots, q_{mm}$$

that satisfy (15)–(17). We then determine the corresponding quantities for modulus $2m$, and repeat this process until $m \geq n + 1$. Finally, we check whether the resulting sequences form a base sequence.

A limitation of this method is that, as the order of the base sequence and the modulus m increase, the number of combinations of $k_{1m}, \dots, k_{mm}$, $r_{1m}, \dots, r_{mm}$, $p_{1m}, \dots, p_{mm}$, and $q_{1m}, \dots, q_{mm}$ grows exponentially and eventually becomes computationally infeasible. In our algorithm, for a small number of moduli, we determine only the values of

$$k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}$$

or

$$p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}.$$

Although this does not use all the available constraints, it keeps the overall number of combinations relatively small. The resulting information can then be used to identify candidate pairs (A, B) or (C, D) for $BS(n + 1, n)$. These candidate sequences can be further filtered by the following property.

For a sequence A , let its associated Hall polynomial be

$$h_A(t) = \sum_{i=0}^{n-1} a_i t^i.$$

Define the real-valued function

$$f_A(\theta) = |h_A(e^{i\theta})|^2.$$

Then

$$f_A(\theta) = N_A(0) + 2 \sum_{j=1}^{n-1} N_A(j) \cos(j\theta). \quad (18)$$

By (18) and Definition 1, we obtain the following result.

Theorem 7 *Let A, B, C, D be $BS(n+1, n)$ and f_A be defined as (18), then*

$$(f_A + f_B + f_C + f_D)(\theta) = 4n + 2, \text{ for all } \theta \quad (19)$$

Theorem 7 can be used to eliminate unsuitable sequences. It is also a standard tool in searches for Williamson matrices and Turyn-type sequences [8, 9]. In particular, a pair (A, B) or (C, D) can be rejected as a candidate for a base sequence if

$$f_A(\theta) + f_B(\theta) > 4n + 2$$

or

$$f_C(\theta) + f_D(\theta) > 4n + 2$$

for some value of θ .

3 Search algorithm for BS(n+1,n) and results

In this Section we describe our algorithm for searching BS(n+1,n) and present constructions for n=41,42, and 43.

The main idea of our algorithm is to search first for pairs of sequences, either (A, B) or (C, D) , that can be extended to a base sequence, rather than searching simultaneously for all four sequences. We use the properties established in Section 2 to filter the possible candidates. Finally, we complete the remaining pair of sequences using a backtracking procedure.

The algorithm proceeds as follows.

1. Select values (a, b, c, d) and (a^*, b^*, c^*, d^*) satisfying Theorem 3.
2. Find all values $k_{13}, k_{23}, k_{33}, r_{13}, r_{23}, r_{33}, p_{13}, p_{23}, p_{33}, q_{13}, q_{23}, q_{33}$ satisfying the conditions in Theorem 6.
3. For each solution obtained in Step 2, find all tuples $(p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66})$ for which there exists at least one tuple $(k_{16}, \dots, k_{66}, r_{16}, \dots, r_{66})$ such that $(k_{16}, \dots, k_{66}, r_{16}, \dots, r_{66}, p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66})$ satisfies Theorem 6. Record all such tuples $(p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66})$.
4. For every tuple $(p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66})$ obtained in Step 3, generate all pairs of sequences (C, D) satisfying Theorem 5. Then, using Theorem 7, eliminate every pair (C, D) for which

$$f_C(\theta) + f_D(\theta) > 4n + 2$$

for some

$$\theta = \frac{j\pi}{100}, \quad j = 1, 2, \dots, 200.$$

5. For each pair (C, D) that remains after Step 4, search for a corresponding pair (A, B) using a backtracking procedure. During the search, we use Definition 1, Theorem 5, and the isomorphic transformations of the sequences A and B to prune the search tree. If the backtracking procedure finds no solution, we continue with the next pair (C, D) .

Using this algorithm, we get BS($n+1, n$) for $n = 41, 42$, and 43 for the first time. The resulting sequences are listed in Table 1.

Table 1:

BS(42,41)	X=+ - - + + - - + + - - - + - - + + - + +
	+ - - - + + - + - - + + - + - - + - +
	Y=+ + + + + + - - - + + - + - - + + - + -
	- - - - + + - + - + - + + - + - - - - + -
	Z=+ + + - + + + - - - - + - + + + - - - +
BS(43,42)	- + + + - + + - - + + + + - + + + +
	W=+ - - - + + + + - - + + - - + - + - + - +
	- - + + + + + + + + - + + - + - - + -
BS(43,42)	X=+ + - - + + - - + - + + + - + - - + - + -

Continued

	$- + - - - - + + + - + - + + + + + + +$ $Y = + + + + + + + - - - + - + - + + + - - +$ $+ - + - + - + + + - - + + - + + - - + -$ $Z = + + + + - - - - + + + - - + + + - + + - +$ $- - + - - - - + - - + - - + + + - - + -$ $W = + + - - - - - + - - - + + + + - - - + - +$ $- + + + - - + - + + - + - + + - + + + -$
BS(44,43)	$X = + + + - + + + + - - - + - - + - + + - + +$ $- - - + + + - + + - + - + - + - + + + - - +$ $Y = + + + - + + - - + + - + - + - + - + - - +$ $+ + + - - - + + - + - - - - - + + + + - - -$ $Z = + + - - - + - - + + + + - + - + - + - - -$ $+ - - + - + - + + - + + + + + - + - + + +$ $W = - - + + + + - - - + + - - + + + + + - + + -$ $- + - + + + + + + + + + - + + - - + - - -$

Proposition 8 *The Base Sequence Conjecture is verified for all $n \leq 43$ and all Golay numbers ($n = 2^a 10^b 26^c$ with $a, b, c \geq 0$).*

4 Near-normal sequences and Yang conjecture

In this section and the following section, we describe our searches for near-normal and normal sequences. Since $NS(n)$ and $NNS(n)$ are subclasses of $BS(n+1, n)$, the algorithm described in Section 3 can be applied to both types of sequences. Taking into account the synthesis of A,B sequences, we begin both searches with A,B pairs.

We first determine all tuples $a, b, c, d, a^*, b^*, c^*, d^*$ that satisfy Theorem 3 and Corollary 4. We also take the isomorphic transformations into account. If one tuple can be obtained from another by one or more of the following transformations, we retain only one representative:

1. Negate one or both of the sequences C and D .
2. Reverse one or both of the sequences C and D .
3. Interchange the sequences C and D .
4. Negate both A and B , and then interchange A and B .
5. Alternate all four sequences A, B, C , and D .

These transformations preserve the conditions $a_{n+1} = 1$ and $b_{n+1} = -1$.

Table 2 lists all sets of $(a, b, c, d, a^*, b^*, c^*, d^*)$ that fit the above restrictions for $NNS(42)$ and $NNS(44)$

Table 2:

	a, b, c, d	a^*, b^*, c^*, d^*		a, b, c, d	a^*, b^*, c^*, d^*
NNS(42)	3, -5, 6, 10	-3, 1, 4, 12	NNS(44)	11, -7, 2, 2	-5, 9, 6, 6
	3, -5, 6, 10	-3, 1, 12, 4		13, 3, 0, 0	5, 11, 4, 4
	13, 1, 0, 0	3, 11, 2, 6		7, 5, 2, 10	7, 5, 2, 10
	11, 3, 2, 6	5, 9, 0, 8		7, 5, 2, 10	7, 5, 10, 2
	11, 3, 2, 6	5, 9, 8, 0		9, -9, 0, 4	-7, 7, 4, 8
	9, 5, 0, 8	7, 7, 6, 6		9, -9, 0, 4	-7, 7, 8, 4
	11, 7, 0, 0	9, 9, 2, 2		5, 3, 0, 12	5, 3, 0, 12
				5, 3, 0, 12	5, 3, 12, 0
				7, 1, 8, 8	3, 5, 0, 12
				7, -7, 4, 8	-5, 5, 8, 8
				5, -5, 8, 8	-3, 3, 4, 12

By combining Definition 3 with Theorem 6, we obtain for even m ,

$$r_{im} = \begin{cases} k_{lm} - 2, & i \equiv n + 1 \pmod{m}, \\ (-1)^{i-1} k_{im}, & i \not\equiv n + 1 \pmod{m}, \end{cases} \quad 1 \leq i \leq m. \quad (20)$$

Thus, once the values $k_{1m}, \dots, k_{mm}$ are known, the values $r_{1m}, \dots, r_{mm}$ are determined. For each tuple listed in Table 2, we search for all tuples $k_{16}, \dots, k_{66}$ for which there exists at least one tuple $(p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66})$ satisfying the conditions of Theorem 6.

Here we also consider the following isomorphic transformation for NNS(n)[13]:

1. Replace A with $\bar{A} = -a_1, a_2, -a_3, a_4, \dots, -a_{n-1}, a_n, a_{n+1}$
2. Replace A with $\hat{A} = a_{n-1}, a_2, a_{n-3}, a_4, \dots, a_1, a_n, a_{n+1}$
3. Replace A with $A^* = a_1, -a_2, a_3, -a_4, \dots, -a_n, a_{n+1}$

If one tuple $(k_{1m}, \dots, k_{mm})$ can be obtained from another by one or more of these transformations, we retain only one representative.

For each tuple $(k_{16}, \dots, k_{66})$, we generate all pairs (A, B) satisfying Definition 3. We then eliminate every pair (A, B) for which

$$f_A(\theta) + f_B(\theta) > 4n + 2$$

for some

$$\theta = \frac{2j\pi}{\ell}, \quad j = 1, 2, \dots, \ell.$$

We first apply this test with $\ell = 50$. For the remaining pairs, we apply the test again with $\ell = 1000$. This filtering procedure is highly effective for both near-normal and normal sequences.

Finally, for each remaining pair (A, B) , we search for a corresponding pair (C, D) using a backtracking procedure. The isomorphic transformations described in Section 2 are then used to remove duplicate sequences.

We performed an exhaustive search for $\text{NNS}(n)$ for every even integer $n \leq 44$. Our results for $n \leq 40$ agree with the previous results reported in [7, 13, 14]. The search also shows that $\text{NNS}(n)$ does not exist for $n = 42$ or $n = 44$.

Yang's conjecture [2](P320) asserts that a near-normal sequence $\text{NNS}(n)$ exists for every even positive integer n . Therefore, the nonexistence results for $n = 42$ and $n = 44$ provide the first counterexamples to Yang's conjecture.

Proposition 9 *There is no $\text{NNS}(n)$ for $n=42$ and 44 , all other orders of even integers $n \leq 40$ exist. $\text{NNS}(46)$ is the first unknown case.*

5 Normal Sequences

We first determine all tuples $(a, b, c, d, a^*, b^*, c^*, d^*)$ that satisfy Theorem 3 and Corollary 4. As in the case of near-normal sequences, if one tuple can be obtained from another by one or more of the transformations listed in Section 4, we retain only one representative.

Table 3 lists all the remaining tuples for $a, b, c, d, a^*, b^*, c^*, d^*$ for $\text{NS}(n)$ when $n = 41, 42, 43, 44$, and 45 .

Table 3:

	a, b, c, d	a^*, b^*, c^*, d^*		a, b, c, d	a^*, b^*, c^*, d^*
NS(41)	2, 0, 9, 9	0, 2, 9, 9	NS(43)	-6, -8, 5, -7	-10, -8, 3, -1
	2, 0, 9, 9	8, 10, 1, 1		10, 8, 1, -3	-10, -8, -1, 3
	2, 0, 9, 9	-4, -2, 5, -11		10, 8, 1, -3	-10, -8, 3, -1
	-2, -4, 5, -11	-4, -2, -11, 5	NS(44)	7, 5, 10, 2	7, 5, 10, 2
	-2, -4, 5, -11	-4, -2, 5, -11		7, 5, 10, 2	7, 5, 2, 10
	-2, -4, 5, -11	8, 10, 1, 1		7, 5, 10, 2	-5, -7, 10, 2
NS(42)	10, 8, 1, 1	8, 10, 1, 1		7, 5, 10, 2	-5, -7, 2, 10
	3, 1, 4, 12	9, 7, 2, 6		5, 3, 0, 12	5, 3, 0, 12
	3, 1, 4, 12	9, 7, 6, 2		5, 3, 0, 12	5, 3, 12, 0
	3, 1, 4, 12	-7, -9, 2, 6		5, 3, 0, 12	-3, -5, 0, 12
	3, 1, 4, 12	-7, -9, 6, 2		5, 3, 0, 12	-3, -5, 12, 0
	3, 1, 4, 12	5, 3, 6, 10	NS(45)	2, 0, 3, 13	0, 2, 3, 13
NS(43)	3, 1, 4, 12	5, 3, 10, 6		2, 0, 3, 13	0, 2, -13, -3
	3, 1, 4, 12	-3, -5, 6, 10		2, 0, 3, 13	-4, -2, -9, 9
	3, 1, 4, 12	-3, -5, 10, 6		2, 0, 3, 13	4, 6, 3, -11
	2, 0, 1, 13	-2, 0, -1, -13		2, 0, 3, 13	4, 6, 11, -3
	2, 0, 1, 13	-2, 0, -13, -1		2, 0, 3, 13	4, 6, 7, 9
	2, 0, 1, 13	-2, 0, 7, 11		2, 0, 3, 13	4, 6, -9, -7
	2, 0, 1, 13	-2, 0, 11, 7		2, 0, 3, 13	-8, -6, -1, 9

Continued

	a, b, c, d	a^*, b^*, c^*, d^*		a, b, c, d	a^*, b^*, c^*, d^*
	2, 0, 1, 13	-6, -4, -1, 11		2, 0, 3, 13	-8, -6, -9, 1
	2, 0, 1, 13	-6, -4, 11, -1		2, 0, 3, 13	8, 10, 3, -3
	2, 0, 1, 13	6, 8, -5, 7		-2, -4, 9, 9	-4, -2, 9, 9
	2, 0, 1, 13	6, 8, 7, -5		-2, -4, 9, 9	4, 6, -3, -11
	2, 0, 1, 13	-10, -8, -1, 3		-2, -4, 9, 9	4, 6, -7, 9
	2, 0, 1, 13	-10, -8, 3, -1		-2, -4, 9, 9	-8, -6, 1, 9
	2, 0, -7, -11	-2, 0, -7, -11		-2, -4, 9, 9	8, 10, -3, -3
	2, 0, -7, -11	-2, 0, -11, -7		6, 4, 3, 11	4, 6, 3, 11
	2, 0, -7, -11	-6, -4, -1, 11		6, 4, 3, 11	4, 6, 11, 3
	2, 0, -7, -11	-6, -4, 11, -1		6, 4, 3, 11	4, 6, 7, -9
	2, 0, -7, -11	6, 8, -5, 7		6, 4, 3, 11	4, 6, -9, 7
	2, 0, -7, -11	6, 8, 7, -5		6, 4, 3, 11	-8, -6, -1, -9
	2, 0, -7, -11	-10, -8, -1, 3		6, 4, 3, 11	-8, -6, -9, -1
	2, 0, -7, -11	-10, -8, 3, -1		6, 4, 3, 11	8, 10, 3, 3
	6, 4, 1, -11	-6, -4, -1, 11		6, 4, 7, 9	4, 6, 7, 9
	6, 4, 1, -11	-6, -4, 11, -1		6, 4, 7, 9	4, 6, -9, -7
	6, 4, 1, -11	6, 8, -5, 7		6, 4, 7, 9	-8, -6, -1, 9
	6, 4, 1, -11	6, 8, 7, -5		6, 4, 7, 9	-8, -6, -9, 1
	6, 4, 1, -11	-10, -8, -1, 3		6, 4, 7, 9	8, 10, 3, -3
	6, 4, 1, -11	-10, -8, 3, -1		-6, -8, 1, 9	-8, -6, 1, 9
	-6, -8, 5, -7	6, 8, -5, 7		-6, -8, 1, 9	-8, -6, 9, 1
	-6, -8, 5, -7	6, 8, 7, -5		-6, -8, 1, 9	8, 10, -3, -3
	-6, -8, 5, -7	-10, -8, -1, -3		10, 8, 3, 3	8, 10, 3, 3

This step also allow us to prove that no normal sequence exists when $n = 8k - 2$, where $k \in \mathbb{Z}_+$. Normal sequences are already known not to exist for $n = 2^{2a-1}(8b + 7)$, $a, b \in \mathbb{Z}_+$ [15]. The following theorem provides a stronger nonexistence result.

Theorem 10 *There is no normal sequence $NS(n)$ for $n = 8k - 2, k \in \mathbb{Z}_+$.*

Proof Suppose that $NS(n)$ exists. By (7), (8), (9), and (11), there must exist integers $a, b, c, d, a^*, b^*, c^*, d^*$ satisfying

$$\begin{aligned}
a^2 + b^2 + c^2 + d^2 &= 4n + 2 \\
a^{*2} + b^{*2} + c^{*2} + d^{*2} &= 4n + 2 \\
a &= b + 2, a^* = b^* + 2 \\
a \equiv b \equiv a^* \equiv b^* &\equiv 1 \pmod{2} \\
c \equiv d \equiv c^* \equiv d^* &\equiv 0 \pmod{2} \\
a \equiv a^* + 2, b \equiv b^* + 2, c \equiv c^* + 2, d \equiv d^* + 2 &\pmod{4} \\
c \equiv d, c^* \equiv d^* &\pmod{4}
\end{aligned} \tag{21}$$

Due to the modular 4 property of c, d, c^*, d^* , we can turn the existence of $a, b, c, d, a^*, b^*, c^*, d^*$ into the following number theory problem: Let x and y be odd integers satisfying $x = y + 2$, and let z and w be even integers, and

$$x^2 + y^2 + z^2 + w^2 = 4n + 2 \tag{22}$$

Then for both $z \equiv w \equiv 0 \pmod{4}$ and $z \equiv w \equiv 2 \pmod{4}$ cases, such x, y, z, w exist.

However we can prove that

- 1) For $n = 16k - 10, k \in \mathbb{Z}_+$, such x, y, z, w do not exist for $z \equiv w \equiv 2 \pmod{4}$
- 2) For $n = 16k - 2, k \in \mathbb{Z}_+$, such x, y, z, w do not exist for both $z \equiv w \equiv 0 \pmod{4}$ and $z \equiv w \equiv 2 \pmod{4}$

For 1), assume x, y, z, w exists, then we can write

$$x = 2k_1 + 3, y = 2k_1 + 1, z = 4k_2 + 2, w = 4k_3 + 2, k_1, k_2, k_3 \in \mathbb{Z}$$

Substitute it to (22), we have:

$$4(16k - 10) + 2 = (2k_1 + 3)^2 + (2k_1 + 1)^2 + (4k_2 + 2)^2 + (4k_3 + 2)^2$$

Reduce the equation, we have:

$$8(k - 1) + 1 = k_1(k_1 + 2) + 2k_2(k_2 + 1) + 2k_3(k_3 + 1)$$

The left-hand side is congruent to 1 modulo 4. Since k_1 is odd, the first term on the right-hand side is congruent to 3 modulo 4, while the remaining two terms are divisible by 4. Thus, the right-hand side is congruent to 3 modulo 4, which is a contradiction.

For 2), if $z \equiv w \equiv 2 \pmod{4}$, we write

$$x = 2k_1 + 3, y = 2k_1 + 1, z = 4k_2 + 2, w = 4k_3 + 2, k_1, k_2, k_3 \in \mathbb{Z}$$

similarly, we have

$$8(k - 1) + 5 = k_1(k_1 + 2) + 2k_2(k_2 + 1) + 2k_3(k_3 + 1)$$

The left is congruent to 1 mod 4 and the right is congruent to 3 mod 4, which is impossible.

If $z \equiv w \equiv 0 \pmod{4}$, we write

$$x = 2k_1 + 3, y = 2k_1 + 1, z = 4k_2, w = 4k_3, k_1, k_2, k_3 \in \mathbb{Z}$$

Substitute it to (22) gives:

$$8(k - 1) + 6 = k_1^2 + 2k_1 + 2k_2^2 + 2k_3^2$$

The left is even, so here k_1 must be even. Writing $k_1 = 2r$, we obtain

$$4(k - 1) + 3 = 2r(r + 1) + k_2^2 + k_3^2$$

The term $2r(r + 1)$ is divisible by 4. Moreover, each square is congruent to either 0 or 1 modulo 4. Therefore, the right-hand side cannot be congruent to 3 modulo 4, which is a contradiction.

Thus, in all cases, the integers $a, b, c, d, a^*, b^*, c^*, d^*$ satisfying (21) do not exist. This proves the theorem. $\square$

As in the case of near-normal sequences, Definition 2 and Theorem 6 imply

$$r_{im} = \begin{cases} k_{lm} - 2, & i \equiv n + 1 \pmod{m}, \\ k_{im}, & i \not\equiv n + 1 \pmod{m}, \end{cases} \quad 1 \leq i \leq m. \quad (23)$$

Consequently, once the values $k_{16}, \dots, k_{66}$, are known, the values $r_{1m}, \dots, r_{mm}$ are determined. For every tuple listed in Table 3, we search for all tuples $k_{16}, \dots, k_{66}$ for which there exists at least one tuple $p_{16}, \dots, p_{66}, q_{16}, \dots, q_{66}$ satisfying the conditions of Theorem 6.

Here we also consider the following isomorphic transformation for $NS(n)$ [10]:

1. Replace A with $\bar{A} = -a_1, -a_2, -a_3, -a_4, \dots, -a_{n-1}, -a_n, a_{n+1}$
2. Replace A with $\hat{A} = a_n, a_{n-1}, a_{n-2}, a_{n-3}, \dots, a_2, a_1, a_{n+1}$
3. Replace A with $A^* = a_1, -a_2, a_3, -a_4, \dots, (-1)^{n-1}a_n, a_{n+1}$

If one tuple $(k_{1m}, \dots, k_{mm})$ can be obtained from another by one or more of these transformations, we retain only one representative.

For each $k_{16}, \dots, k_{66}$, we generate all pairs (A, B) satisfying 2. The remaining steps are exactly the same as the search for Near-normal Sequence.

We performed an exhaustive search for $NS(n)$ for every $n \leq 46$. The results for $n \leq 40$ agree with those reported in [10]. The search also shows that $NS(n)$ does not exist for $n=41, 42, 43, 44, 45, 46$.

Proposition 11 *$NS(n)$ exist for all the Golay numbers $(n = 2^a 10^b 26^c, a, b, c \geq 0)$ and $n = 3, 5, 7, 9, 11, 12, 13, 25, 29$. $NS(n)$ do not exist for $n = 8k - 2, k \in \mathbb{Z}_+$ and all the other $n \leq 46$, the first unknown case is $NS(47)$.*

6 Conclusion

In this paper, we developed a computational search algorithm for base sequences $BS(n+1, n)$ and its two important subclasses, namely near-normal sequences $NNS(n)$ and normal sequences $NS(n)$. The algorithm combines the sum and congruence conditions from Section 2, modular decompositions of the sequences, isomorphic transformations, spectral filtering, and a backtracking procedure.

Using this approach, we constructed base sequences $BS(n+1, n)$ for $n = 41, 42$, and 43. These constructions extend the known range of base sequences and verify the base sequence conjecture for all $n \leq 43$.

We also performed exhaustive searches for near-normal sequences of every even order $n \leq 44$. The results agree with previously known results for $n \leq 40$ and show that near-normal sequences do not exist for $n = 42$ or $n = 44$. Consequently, these two orders provide counterexamples to Yang's conjecture, which asserts that near-normal sequences exist for every even positive integer.

For normal sequences, we established the nonexistence of $NS(n)$ whenever $n = 8k - 2, k \in \mathbb{Z}_+$. This result strengthens previously known nonexistence conditions. In addition, our exhaustive search shows that normal sequences do not exist for $n =$

41, 42, 43, 44, 45, or 46. Normal sequences were verified for all Golay numbers and for the additional orders $n=3,5,7,9,11,12,13,25,29$.

The existence of $BS(n+1, n)$ for $n \geq 44$, $NNS(n)$ for even n $n \geq 46$ $NNS(n)$ for $n \geq 47$ remains open. The search techniques developed here can be applied to these cases and to other related sequences as well.

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