

Laplace Transform Inversion Techniques for the Ruin Probability of the Compound Poisson Process Perturbed by Diffusion

Raphael Waller

University of Bern

Riccardo Gatto

`riccardo.gatto@unibe.ch`

University of Bern

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Raphael Waller and Riccardo Gatto

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Abstract

The probability of ultimate ruin of the insurer's compound Poisson surplus process subject to an additive Brownian perturbation is approximated using three important techniques of numerical inversion of the Bromwich integral of the Laplace transform: Talbot method, Euler method and the saddlepoint approximation. While the Talbot and Euler methods necessitate the numerical integration of an inverse Laplace transform integral, the saddlepoint approximation circumvents this numerical step by employing an optimal, problem-specific selection of the integration contour. The regularity conditions required for the application of these approximation are specified and a detailed error analysis of Euler method is given. A concise numerical investigation indicates that all three methods produce small relative errors when approximating extremely small ruin probabilities. Nevertheless, the Talbot approximation achieves the highest accuracy with respect to relative error, followed by the Euler method and subsequently the saddlepoint approximation. These findings provide support for prospective applications to related classes of risk processes.

Key words and phrases

Bromwich integral, discretization error, Euler inversion method, Talbot inversion method, trapezoidal rule, saddlepoint approximation, Wiener process

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Address Institute of Mathematical Statistics and Actuarial Science, University of Bern, Alpeneggstrasse 22, 3012 Bern, Switzerland

raphael.waller2@unibe.ch – orcid.org/0009-0000-8075-8513

riccardo.gatto@unibe.ch – orcid.org/0000-0001-8374-6964

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1 Introduction

The Cramér-Lundberg risk process is the classical stochastic model for the fluctuations of the insurer surplus over time. Denote by $u \geq 0$ the initial surplus and by $c > 0$ the constant premium rate. At any time $t \geq 0$, $S(t)$ denotes the aggregate claim amount and $W(t)$ a Wiener diffusion term, acting as perturbation once rescaled by $\sigma > 0$. The Cramér-Lundberg risk process perturbed by diffusion is the insurer surplus model

$$U(t) = u + ct - S(t) + \sigma W(t), \quad \forall t \geq 0, \quad (1)$$

with the following components. The aggregate claim amount takes the form $S(t) = \sum_{j=0}^{N(t)} X_j$, where $X_0 = 0$ and $X_1, X_2, \dots > 0$ are independent individual claim amounts with common distribution function (d.f.) F_X and density f_X over $(0, \infty)$ and where $\{N(t)\}_{t \geq 0}$ is a homogeneous Poisson process with rate $\lambda > 0$. The stochastic processes $\{X_j\}_{j \geq 1}$, $\{N(t)\}_{t \geq 0}$ and $\{W(t)\}_{t \geq 0}$ are assumed independent. This diffusion represents the uncertainties inherent to the claim amounts or to the premium incomes. Let $\mu = \mathbb{E}[X_1] \in (0, \infty)$ and define the relative security loading as $q = (c - \lambda\mu)/c$. A quantity of interest is the probability of ruin in the infinite time horizon, precisely the probability that $\{U(t)\}_{t \geq 0}$ ever falls below the null line. Define the time of ruin

$$T = \begin{cases} \inf\{t \in (0, \infty) \mid U(t) \leq 0\}, & \text{if the infimum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

then the probability of ruin is $\psi(u) = \mathbb{P}(T < \infty)$. We assume $q > 0$, otherwise ruin occurs almost surely. The perturbed risk process (1) was introduced by Gerber [1] and Dufresne and Gerber [2].

An exact formula for the probability of ruin was proposed by [2], however limited to individual claim amounts following the linear combination of exponential distributions. For general light-tailed individual claim amount distributions, one often uses the Cramér-Lundberg asymptotic approximation, valid for large values of the initial capital u and due to [2, 3]. Four accurate approximations to the probability of ruin were introduced by [4, 5]. The first is based on the saddlepoint approximation of asymptotic analysis, the second is based on recursive upper and lower approximations, the third method is Monte Carlo importance sampling and the Fast Fourier transform is the fourth method. This work extends the four approximations presented in [5] by introducing Talbot's and Euler's methods of Laplace inversion for the numerical evaluation of the probability of ruin. These approaches constitute direct alternatives to the saddlepoint approximation, insofar as they

all rely on the inversion of the Laplace transform. The saddlepoint approximation is presented again, here within the framework of Talbot's and Euler's methods. The numerical results highlight two principal findings. First, all three methods yield high accuracy, particularly when estimating extremely small ruin probabilities. Second, the Talbot and Euler methods demonstrate superior accuracy relative to the saddlepoint approximation. This outcome is consistent with expectations, given that the Talbot and Euler methods rely on the trapezoidal rule of numerical integration, whereas the saddlepoint approach produces a comparatively compact formula, simple to evaluate.

Concerning the literature on the perturbed risk process, there are several general references, for instance the books [6] or [7] as well as the review article [8]. Some results of special interest are [9], who consider the perturbed risk process under random force of interest. Then, [3] examines the renewal, the Cox and the general Markovian risk perturbed processes, that are inbedded into an exponential class of stochastic processes through exponential tilting. We mention also [10], who considers the more general α -stable perturbation. Then, [11] examine the expected discounted value of the penalty at ruin, a concept introduced for the classical Cramér-Lundberg model by [12]. For the classical Cramér-Lundberg model, [13] obtain formulae for the probability of ruin under phase-type individual claim amount distributions. These light-tailed distributions are characterized as matrix exponential distributions and include mixtures of exponential and gamma distributions; see e.g. [14, Section 8.2]. A phase-type individual claim amount leads to a computable formula for the probability of ruin of the Cramér-Lundberg model; see in particular [14, Theorem 8.3.1]. However, these phase-type distributions are rather sophisticated and involve a large number of parameters.

This article has the following structure. Section 2 gives a short summary of results pertaining with the compound Poisson risk process perturbed by the Wiener diffusion. In particular, the Laplace transform of the probability of ruin is given. Section 3 addresses the Talbot technique for approximating the probability of ruin, Section 4 presents the Euler approximation and Section 5 summarizes the saddlepoint approximation. A numerical comparative study, based on an example where the exact probability of ruin is available, is provided in Section 6. Some concluding remarks about the three methods and future research can be found in Section 7.

2 The probability of ruin and its Laplace transform

This section presents additional results from [2] relating to the probability of ruin and in particular the Laplace transforms that are inverted by the method of Talbot and Euler.

The probability of ruin can be re-expressed as the survival function of the maximal aggregate loss. The aggregate loss process is $L(t) = S(t) - ct - \sigma W(t)$, $t \geq 0$, and the maximal aggregate loss is given by

$$L = \sup_{t \geq 0} \{L(t)\}. \quad (2)$$

One can show that the probability of ruin satisfies $\psi(u) = P(L > u)$, $u \geq 0$. From $P(L = 0) = 1 - \psi(0)$ we deduce that L has: a proper absolutely continuous distribution, if $\sigma > 0$; a defective distribution with $P(L = 0) > 0$, if $\sigma = 0$ and $q > 0$. Further, the probability of ruin can be decomposed as $\psi(u) = \psi^{(1)}(u) + \psi^{(2)}(u)$, where $\psi^{(1)}(u) = P(T < \infty, U(T) = 0)$ is the probability that the null line is first crossed by an oscillation of the path of the risk process, i.e. by creeping, and $\psi^{(2)}(u) = P(T < \infty, U(T) < 0)$ is the probability that the null line is first crossed by a jump of the path risk process, i.e. by an individual claim amount. From the regularity of the Wiener process follows that $\psi(0) = \psi^{(1)}(0) = 1$ and thus $\psi^{(2)}(0) = 0$. Ruin due to claim is more important than ruin due to oscillation, because in the first case the deficit at ruin can be substantial, whereas it is zero in the second case, from the continuity of the Wiener process.

Let $\zeta = 2c/\sigma^2$, $H_1(x) = 1 - e^{-\zeta x}$ and $H_2(x) = \int_0^x \{1 - F_X(y)\} dy/\mu$, $x \geq 0$. The defective renewal equation for the survival probability $R(u) = 1 - \psi(u)$ is given in (2.9) of [2] and is as follows,

$$R(u) = qH_1(u) + (1 - q)(R * h_1 * h_2)(u), \quad (3)$$

$u \geq 0$, where h_1 and h_2 are the densities of H_1 and H_2 , respectively. The inversion of (3) yields (3.4) of [2], which is

$$R(u) = \sum_{n=0}^{\infty} q(1 - q)^n H_1^{*(n+1)} * H_2^{*n}(u), \quad (4)$$

$u \geq 0$. In (4), H^{*n} represents the n -th convolution power of any d.f. H . We note that, although (4) has a compact formulation, its numerical evaluation is generally very difficult. Here we rather proceed by inverting the Laplace transform of $R(u)$, which can be obtained by inverting (3). This Laplace transform is given in (3.3) of [2]. In terms of $\psi(u) = 1 - R(u)$, it is given by

$$\mathcal{L}(\psi)(s) = \frac{1}{s} \left(1 - \frac{qs}{s(1 + \frac{s}{\zeta}) + \frac{\lambda}{c} \{\mathcal{L}(f_X)(s) - 1\}} \right). \quad (5)$$

The Laplace transform (5) is inverted in Section 3 using Talbot’s method, with explicit inversion formula given in (11). It is then inverted in Section 4 using the Euler method, with explicit formula provided in (22). Then, in Section 5, the inversion is carried out with the saddlepoint approximation.

3 Talbot approximation to the probability of ruin

3.1 Derivation of the approximation

This section summarizes a numerical inversion technique introduced by Talbot in 1979 [15] for evaluating an inverse Laplace transform. It consists in performing numerical integration along a deformation of the path of integration of the Laplace integral. Let $f : [0, \infty) \rightarrow \mathbb{R}$ satisfy $f(t) = O(e^{\gamma t})$, as $t \rightarrow \infty$, for some $\gamma > 0$. The Laplace transform of f is given by $\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt$, for all $s \in \mathbb{C}$ in the domain of convergence, which is typically characterized by $\operatorname{Re}(s) > \gamma$. Assume that all singularities of $\mathcal{L}(f)$ lie on the left of $\operatorname{Re}(s) = \gamma$ and define the contour $\mathfrak{B} = \{\gamma + it \mid t \in \mathbb{R}\}$. Denoting $\hat{f}(s) = \mathcal{L}(f)(s)$, the inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}(\hat{f})(t) = \frac{1}{2\pi i} \int_{\mathfrak{B}} e^{st} \hat{f}(s) ds = \frac{1}{2\pi i} \int_{\mathfrak{B}} e^{st} \mathcal{L}(f)(s) ds, \quad (6)$$

for $t > 0$. This integral is known as Mellin’s inversion formula or Bromwich integral and the integration path $\mathfrak{B}$ is known as Bromwich contour.

A direct numerical computation of the Bromwich integral often leads to significant practical difficulties. Indeed, the factor e^{st} causes oscillations over the entire integration contour. The central idea of Talbot’s method is to substitute the contour $\mathfrak{B}$ with an equivalent contour $\mathfrak{B}'$ in the negative half plane that has a “ $\supset$ ” shape, thus forcing $\operatorname{Re} s \rightarrow -\infty$ at each end. Thus, the oscillatory behavior of e^{st} is attenuated at the endpoints of the integration path.

We follow the approach of Abate and Valkó [16], which is a slight modification of Talbot’s original contour. Consider

$$\begin{aligned} s &: (-\pi, \pi) \rightarrow \mathbb{C}, \\ \theta &\mapsto r\theta(\cot \theta + i), \end{aligned} \quad (7)$$

for some scaling parameter $r > 0$. Because $\theta \cot \theta \rightarrow 1$, as $\theta \rightarrow 0$, we define $s(0) = r$, which makes s a continuous curve in $\mathbb{C}$. Moreover, as $\theta \rightarrow \pm\pi$, $\theta \cot \theta \rightarrow -\infty$ and $r\theta i \rightarrow \pm\pi r i$. Thus, the new integration contour $\mathfrak{B}_r = s((-\pi, \pi))$ has the desired “ $\supset$ ”

shape, with asymptotes $\pm\pi ri$ and its endpoints have real part $-\infty$. These characteristics ensure a rapid convergence of the inversion integral.

Let F be some Laplace transform. In order to prove that the integral along $\mathfrak{B}_r$ is equal to the one along $\mathfrak{B}$, we make the following assumptions about $\hat{f}$.

(A1) All singularities of $\hat{f}$ are enclosed within $\mathfrak{B}_r$. In particular, $\hat{f}$ is analytic (holomorphic) in the region enclosed between $\mathfrak{B}$ and $\mathfrak{B}_r$.

(A2) As $|s| \rightarrow \infty$, $|\hat{f}(s)| \rightarrow 0$, uniformly for all s in the region enclosed between $\mathfrak{B}$ and $\mathfrak{B}_r$.

Note that (A1) must be considered together with another implicit assumption, which holds automatically in most cases. As mentioned, the integral defining the Laplace transform is usually only (absolutely) convergent on some right half plane, i.e. $\hat{f}(s)$ is only defined for $\text{Re}(s) > \gamma$. However, we need $\hat{f}$ to also be defined for complex numbers s with $\text{Re}(s) \rightarrow -\infty$. It is usually possible to analytically extend F to such a region and the uniqueness of the analytic continuation solves this issue. Precisely, denote by $\tilde{F}$ the analytic continuation of $\hat{f}$, then

$$\int_{\mathfrak{B}_r} \tilde{f}(s) e^{st} ds = \int_{\mathfrak{B}} \tilde{f}(s) e^{st} ds = \int_{\mathfrak{B}} \hat{f}(s) e^{st} ds \propto \mathcal{L}^{-1}(\hat{f})(t).$$

The justification of the first equality requires a detailed analysis based on Cauchy's Theorem and it is beyond the scope of this presentation. The second equality is due to the uniqueness of the analytic continuation. We slightly abuse notation and consider $\hat{f}(s)$ defined for all required $s \in \mathbb{C}$.

These developments lead to the following Talbot's inversion formula,

$$f(t) = \frac{1}{2\pi i} \int_{\mathfrak{B}_r} e^{st} \mathcal{L}(f)(s) ds = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{s(\theta)t} \mathcal{L}(f)(s(\theta)) s'(\theta) d\theta, \quad (8)$$

for $t > 0$, where the function s is given in (7) and $\mathfrak{B}_r = s((-\pi, \pi))$

Note that an alternative contour to $\mathfrak{B}_r$ could be considered, as long as the assumptions (A1-2) remain valid. We refer to [17, Section 6.5] for other paths.

We can approximate the second integral in (8) by applying trapezoidal rule after reformulating it as follows. Let $g(\theta) = \theta + (\theta \cot \theta - 1) \cot \theta$, then $s'(\theta) = ir(1 + ig(\theta))$. Also,

$s(-\theta) = \overline{s(\theta)}$, $g(-\theta) = -g(\theta)$ and $\mathcal{L}(f)(s(-\theta)) = \overline{\mathcal{L}(f)(s(\theta))}$. All these facts lead to

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{s(\theta)t} \mathcal{L}(f)(s(\theta)) r(1 + ig(\theta)) d\theta \\ &= \frac{r}{2\pi} \left(\int_{-\pi}^0 e^{s(\theta)t} \mathcal{L}(f)(s(\theta))(1 + ig(\theta)) d\theta + \int_0^{\pi} e^{s(\theta)t} \mathcal{L}(f)(s(\theta))(1 + ig(\theta)) d\theta \right) \\ &= \frac{r}{2\pi} \left(\overline{\int_0^{\pi} e^{s(\theta)t} \mathcal{L}(f)(s(\theta))(1 + ig(\theta)) d\theta} + \int_0^{\pi} e^{s(\theta)t} \mathcal{L}(f)(s(\theta))(1 + ig(\theta)) d\theta \right) \\ &= \frac{r}{\pi} \int_0^{\pi} \operatorname{Re} \left\{ e^{s(\theta)t} \mathcal{L}(f)(s(\theta))(1 + ig(\theta)) \right\} d\theta. \end{aligned}$$

The trapezoidal rule with $n \geq 2$ steps and step size π/n yields the approximation to f in the following form,

$$\begin{aligned} f_T(t; n, r) &= \frac{r}{\pi} \frac{\pi}{n} \left(\frac{1}{2} e^{s(0)t} \mathcal{L}(f)(s(0)) \operatorname{Re}(1 + ig(0)) \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \operatorname{Re} \left\{ e^{s(\theta_k)t} \mathcal{L}(f)(s(\theta_k))(1 + ig(\theta_k)) \right\} \right. \\ &\quad \left. + \frac{1}{2} \operatorname{Re} \left\{ e^{s(\theta_n)t} \mathcal{L}(f)(s(\theta_n))(1 + ig(\theta_n)) \right\} \right) \\ &= \frac{r}{n} \left(\frac{1}{2} e^{rt} \mathcal{L}(f)(r) + \sum_{k=1}^{n-1} \operatorname{Re} \left\{ e^{s(\theta_k)t} \mathcal{L}(f)(s(\theta_k))(1 + ig(\theta_k)) \right\} \right), \quad (9) \end{aligned}$$

where $\theta_k = \pi k/n$, for $k = 1, \dots, n$. Note that $g(0) = 0$ and, from (A2), the last term at $\theta_n = \pi$ vanishes. The choice of the scale parameter $r = 2n/(5t)$ is a rule of thumb suggested by [16]. With it, (9) becomes

$$f_T(t; n) = \frac{2}{5t} \left(\frac{1}{2} e^{\frac{2n}{5}} \mathcal{L}(f) \left(\frac{2n}{5t} \right) + \sum_{k=1}^{n-1} \operatorname{Re} \left\{ e^{s(\theta_k)t} \mathcal{L}(f)(s(\theta_k))(1 + ig(\theta_k)) \right\} \right). \quad (10)$$

3.2 Adaptation for the ruin probability

Applying the approximation (10) to $\mathcal{L}(\psi)$ in (5) yields the following approximation to $\psi(u)$,

$$\psi_T(u; n) = \frac{2}{5u} \left(\frac{1}{2} e^{\frac{2n}{5}} \mathcal{L}(\psi) \left(\frac{2n}{5u} \right) + \sum_{k=1}^{n-1} \operatorname{Re} \left\{ e^{s(\theta_k)u} \mathcal{L}(\psi)(s(\theta_k))(1 + ig(\theta_k)) \right\} \right), \quad (11)$$

for $u > 0$, with $r = 2n/(5u)$ and $\mathcal{L}(\psi)(s(\theta_k))$ given by (5). We then set $\psi_T(0; n) = 1$.

We need to verify (A1-2). Concerning (A1), $s = 0$ is a removable singularity that does not cause any trouble. The poles are the non-null solutions of

$$D(s) = \left\{ s \left(1 + \frac{s}{\zeta} \right) \right\} + \frac{\lambda}{c} \{ \mathcal{L}(f_X)(s) - 1 \} = 0 \quad (12)$$

Considered as real functions, the two functions within the two brackets in (12) are convex. Thus (12) has at most one non-null negative (real) solution. But other complex solutions

may exist. There is thus at most one real pole and possible complex poles. The real pole is clearly within $\mathfrak{B}_r$, while the complex poles require an appropriate choice of the scale parameter r , in order to lie within $\mathfrak{B}_r$. Then, (A2) can be simplified to: $1/|D(s)| \rightarrow 0$, as $|s| \rightarrow \infty$, uniformly for s in the region between $\mathfrak{B}$ and $\mathfrak{B}_r$.

4 Euler approximation to the probability of ruin

4.1 Derivation of the approximation

The method of Euler for inverting Laplace transforms starts also with a re-expression of the Bromwich contour inversion integral given in (6). We follow [18], who presents the method for $f : [0, \infty) \rightarrow [-1, 1]$. This restriction, which holds for probabilities, is later used for the error analysis. For any given $\gamma > 0$, we can re-express (6) as

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}(\hat{f})(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma+iu)t} \hat{f}(\gamma+iu) du \\ &= \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} \left\{ \operatorname{Re} \hat{f}(\gamma+iu) \cos ut - \operatorname{Im} \hat{f}(\gamma+iu) \sin ut \right\} du \\ &= \frac{2e^{\gamma t}}{\pi} \int_0^{\infty} \operatorname{Re} \hat{f}(\gamma+iu) \cos ut du. \end{aligned}$$

The last equality is due to $\hat{f}(\gamma-iu) = \overline{\hat{f}(\gamma+iu)}$ and to $\int_0^{\infty} \operatorname{Re} \hat{f}(\gamma+iu) \cos ut du = -\int_0^{\infty} \operatorname{Im} \hat{f}(\gamma+iu) \sin ut du$. Applying the trapezoidal numerical integration rule with step size $h > 0$ yields

$$f(t) \simeq \frac{he^{\gamma t}}{\pi} \left(\operatorname{Re} \hat{f}(\gamma) + 2 \sum_{k=1}^{\infty} \operatorname{Re} \hat{f}(\gamma+ihk) \cos(hkt) \right). \quad (13)$$

By choosing $h = \pi/(2t)$ and defining $\gamma = a/(2t)$, for fixed $a > 0$, one obtains the series

$$f_S(t; a) = \frac{e^{\frac{a}{2}}}{2t} \left\{ \operatorname{Re} \hat{f}\left(\frac{a}{2t}\right) + 2 \sum_{k=1}^{\infty} (-1)^k a_k(t) \right\},$$

where

$$a_k(t) = \operatorname{Re} \hat{f}\left(\frac{a+2\pi ki}{2t}\right).$$

If the signs of $a_k(t)$, $k = 1, 2, \dots$ were constant, the resulting expression would be an alternating series and this could exhibit better convergence. These signs are indeed constant for sufficiently large values of k , whenever f is nondecreasing and under further mild conditions.

To see this, note that

$$\operatorname{Re} \hat{f}(u+iv) = v^{-2} \{u - f'(0) + o(1)\}, \quad \text{as } v \rightarrow \infty, \quad (14)$$

which follows from two partial integrations; cf. [18] for details. Thus $\text{Re}\hat{f}(u+iv)$ is eventually positive, provided $f'(0) < u$, which is true with $u > 0$.

Let us denote the truncated approximations by

$$f_S(t; a, n) = \frac{e^{\frac{a}{2}}}{2t} \left\{ \text{Re}\hat{f}\left(\frac{a}{2t}\right) + 2 \sum_{k=1}^n (-1)^k a_k(t) \right\}, \quad \text{for } n = 1, 2, \dots$$

We now apply Euler's acceleration method for alternating series, defined as the binomially weighted average of the m subsequent partial sum approximations, starting from the n -th one:

$$f_E(t; n, m) = 2^{-m} \sum_{j=0}^m \binom{m}{j} f_S(t; a, n+j), \quad \text{for } m = 0, 1, \dots \quad (15)$$

A typical choice is $n = 15$ and $m = 11$.

4.2 Error analysis

Euler's approximation error $\varepsilon(t; a, m, n) = f(t) - f_E(t; n, m)$ can be decomposed into the three components $\varepsilon_d, \varepsilon_t$ and ε_r : The discretization error ε_d of replacing the integral with a series, the truncation error ε_t of truncating the series to a finite sum and the rounding error ε_r attributable to finite-precision computer arithmetic. Thus $\varepsilon = \varepsilon_d + \varepsilon_t + \varepsilon_r$. In this section, we examine these three error terms in detail and provide guidelines for selecting the parameters a, m and n to achieve an accuracy that is approximately consistent with a prescribed target. In addition, we outline a potential strategy for handling ε_r .

Discretization Error

It holds that $f(t) = f_S(t; a) + \varepsilon_d(t; a)$, where

$$\varepsilon_d(t; a) = \sum_{k=1}^{\infty} e^{-ka} f([1+2k]t). \quad (16)$$

This is established in [18] and the proof proceeds by: wrapping the exponentially damped function $e^{-bt}f(t)$, with $b > 0$, around the circle of circumference $2\pi/h$, so to obtain a periodic function, say $p(t)$; obtaining the Fourier series of $p(t)$, say $p_F(t)$; writing

$$f(t) = \left\{ e^{bt} p_F(t) \right\} - \left\{ e^{bt} p(t) - f(t) \right\} \quad (17)$$

with $h = \pi/t$ and $b = a/(2t)$. Then, it is shown that the first bracket of (17) coincides with $f_S(t; a)$ and the second one with $\varepsilon_d(t; a)$. Here, our assumption $|f(t)| \leq 1$ ensures the convergence of the series resulting from wrapping $e^{-bt}f(t)$ around the circle.

If f is nonincreasing and non-negative (as it happens with the ruin probability or any survival probability), then we get

$$\varepsilon_d(t; a) \leq \frac{e^{-a}}{1 - e^{-a}} f(3t) \simeq e^{-a} f(3t), \quad (18)$$

for a large, which is decreasing in t . This result is consistent with the construction: in the trapezoidal rule (13), the chosen step size of $h = \pi/(2t)$ is decreasing in t , thus making the discretization better for larger t . Without assuming monotonicity of f , the bound $|\varepsilon_d(t; a)| \leq e^{-a}/(1 - e^{-a}) \simeq e^{-a}$ holds, now independently of t .

For the nonincreasing case, since f is unknown, we can estimate (an upper bound for) the discretization error by

$$\hat{\varepsilon}_d(t; a) = e^{-a} f_E(3t; n', m'), \quad (19)$$

for preferably large values of n', m' . Hence, if we desire a discretization error of at most $\delta/2$, then a sensible choice for the parameter a is

$$a \left(t, \frac{\delta}{2} \right) = \log \left(\frac{2f_E(3t)}{\delta} \right).$$

Truncation error

The truncation error can be estimated by $f_E(t; n+1, m) - f_E(t; n, m)$, as proposed by [18]. Substituting the above definitions and simplifying correctly yields the estimator

$$\hat{\varepsilon}_t(t; a, n, m) = \frac{e^{\frac{a}{2}}}{2^m t} \sum_{k=0}^m \binom{m}{k} a_{n+k+1}(t) (-1)^{n+k+1}. \quad (20)$$

The dependence of $\hat{\varepsilon}_t$ on a is twofold: The exponential factor in front of the summation as well as the term $a_k(t)$, both depend on a . Specifically, $a_k(t)$ is eventually vanishing as $a \rightarrow \infty$, by the Riemann-Lebesgue Lemma. However, the decrease rate of $a_k(t)$ is typically too slow to counterbalance the growth of $e^{a/2}$. Overall, $\hat{\varepsilon}_t$ exhibits an increasing trend w.r.t. a . Consequently, while increasing a does reduce the discretization error ε_d , it also leads to an increase in the truncation error ε_t . Moreover, the ratio in front of the sum in (20) decreases w.r.t. t , but $a_k(t)$ tends to increase w.r.t. t , again by Riemann-Lebesgue. In numerical experiments, we found that these two effects largely offset one another. Thus, the estimator (20) depends primarily on a, n and m , while largely independent of t . Practically, after fixing a to control ε_d , we can choose n, m such that the following holds:

$$\hat{\varepsilon}_t(t; a, n, m) \simeq \frac{\delta}{2}.$$

According to [18, Remark 2], n should grow along t in order to control the error. Our numerical experiments indicate otherwise: They show no significant variability of the error

estimate $\hat{\varepsilon}_t$ over a range of values of t (see Figure 1 in Section 6). Their suggestion is likely based on their incorrect representation of the error estimate. Moreover, in the numerical illustration of Section 6, the estimator $\hat{\varepsilon}_t$ appears adequate, though this needs not be the case in general.

Rounding error

There is no rigorous or general methodology for obtaining accurate estimates of the rounding error ε_r . However, ε_r can be controlled by using the multi-precision representation of numbers (i.e. use a higher precision than, say, double precision). In particular, (one of) the largest term(s) appearing in (15) is

$$T_0 = \frac{e^{\frac{a}{2}}}{2t} \operatorname{Re} \hat{f} \left(\frac{a}{2t} \right),$$

because it appears in each one of the Euler's sums, so its overall weight in the mean is 1, and because $|a_k(t)| \rightarrow 0$ as $k \rightarrow \infty$, from Riemann-Lebesgue. Hence, if we can compute T_0 up to D decimal digits correctly, the rounding error ε_r is of order $O(10^{-D}T_0)$. To control this, we can use the M decimal digit representation and estimate D by

$$-\log_{10} |T_0(M+k) - T_0(M)|, \quad (21)$$

for selected values of $k \geq 1$, where $T_0(M)$ is T_0 computed under the M digits representation.

4.3 Adaptation for the ruin probability

Applying the approximation (15) to $\mathcal{L}(\psi)$ in (5) yields Euler's approximation for $\psi(u)$ in the following form,

$$\psi_E(u; n, m) = \frac{e^{\frac{a}{2}}}{2^{m+1}t} \sum_{j=0}^m \binom{m}{j} \left\{ \operatorname{Re} \mathcal{L}(\psi) \left(\frac{a}{2u} \right) + 2 \sum_{k=1}^{n+j} (-1)^k \operatorname{Re} \mathcal{L}(\psi) \left(\frac{a + 2\pi k i}{2u} \right) \right\}, \quad (22)$$

for $u > 0$. We then set $\psi_E(0; n, m) = 1$. The only assumption for the validity of (22) is that all singularities of $\mathcal{L}(\psi)$ lie on the left of $\operatorname{Re}(s) = \gamma = a/(2u)$. As explained in Section 3.2, the poles of $\mathcal{L}(\psi)$ are the non-null solutions of (12). Also, note that because $\psi(u)$ is nonincreasing, $\varepsilon_d(u; a, n)$ is approximately bounded by $e^{-a}\psi(3u)$, according to (18).

5 Saddlepoint approximation to the probability of ruin

5.1 Main ideas of the derivation of the approximation

The saddlepoint method or method of steepest descent provides approximations to contour integrals of the form

$$\int_{\mathfrak{P}} h(s) e^{\nu g(s)} ds,$$

for large values of $\nu > 0$, where $h : \mathbb{C} \rightarrow \mathbb{C}$ and $g : \mathbb{C} \rightarrow \mathbb{C}$ are analytic functions on a domain containing the path $\mathfrak{P}$ and suitable deformations of it. This general form includes the Bromwich integral of the inverse Laplace transform $f(t) = \mathcal{L}^{-1}(\hat{f})(t)$ given in (6), with f satisfying the regularity conditions presented at the beginning of Section 3.1. With large values of ν , the major contribution to the integral is obtained at values of s close to the saddlepoint of the surface $\operatorname{Re} g(s)$, which is denoted s_0 . Precisely, the integration path $\mathfrak{P}$ is deformed so as to coincide with the steepest descent contour emanating from s_0 , on that surface. Then, an asymptotic expansion of the integrand about s_0 is integrated term by term, along the path of steepest descent. The first order term is called saddlepoint approximation. The relative error of the saddlepoint approximation vanishes asymptotically. Despite its asymptotic nature, this approximation remains very accurate even with very small values of ν . Chapters 7 and 8 of [19] provide a detailed presentation. In [20], Daniels obtained the saddlepoint expansion to the density of the mean of ν i.i.d. random variables, in the form of an asymptotic expansion in powers of ν^{-1} . Lugannani and Rice [21] obtained a saddlepoint approximation to the tail probabilities of this sample mean. Two important monographs on saddlepoint approximations for stochastic models are [22, 23], whereas a short review is [24]. A recent application to the distribution of a first passage time is given in [25].

In contrast to the inversion methods of Talbot and Euler, the saddlepoint approximation does not rely on numerical integration. Instead, it optimizes the integration contour and expands the integral in a neighborhood of the saddlepoint. Talbot's method likewise entails a deformation of the integration contour; however, the resulting contour is a general one and not optimized for the specific integrand. Since the (costly) numerical integration is performed along the deformed contour, determining the optimal integration path of steepest descent is unnecessary.

Let us give the computational steps of Lugannani and Rice's approximation to the upper tail probability $P(L > t)$, for a random variable L with cumulant generating function (c.g.f.)

$K(s) = \log \mathbb{E}[e^{sL}]$, $s \in \mathbb{R}$. At a given $t \in \mathbb{R}$, solve

$$K'(s) = t \quad (23)$$

w.r.t. $s \in \mathbb{R}$ and denote by s_t the solution, which is the saddlepoint. Alternatively, compute

$$s_t = \operatorname{argsup}_{s \in \mathbb{R}} st - K(s).$$

Denote the supremum

$$I_t = s_t t - K(s_t)$$

and define

$$\nu_t = \operatorname{sgn}\{s_t\} \sqrt{2I_t} \quad \text{and} \quad \xi_t = s_t \sqrt{K''(s_t)}.$$

Compute the saddlepoint approximation to $\mathbb{P}(L > t)$ by

$$S(t) = 1 - \Phi(\nu_t) - \phi(\nu_t) \left(\frac{1}{\nu_t} - \frac{1}{\xi_t} \right), \quad (24)$$

where Φ and ϕ denote the standard Gaussian d.f. and density.

A sufficient condition for the existence of the saddlepoint, viz. the solution to (23), is the steepness of the c.g.f. K . For the convex function $g : \mathbb{R} \rightarrow [-\infty, \infty]$, define $\operatorname{dom} g = \{s \in \mathbb{R} \mid g(s) < \infty\}$, then g is steep at s , a boundary point of $\operatorname{dom} g$, if

$$|g'(s_n)| \xrightarrow{n \rightarrow \infty} \infty, \quad \text{whenever } s_n \xrightarrow{n \rightarrow \infty} s, \quad \text{for any } \{s_n\}_{n \geq 0} \text{ in } \operatorname{int} \operatorname{dom} g. \quad (25)$$

5.2 Adaptation for the ruin probability

Although the application of saddlepoint approximation to the probability of ruin is already introduced in [4, 5], we present it again in order to relate it directly with our previous formulae. By taking the random variable L in the above formula as the maximal aggregate loss (2) and by taking $t = u > 0$ as the initial capital, (24) becomes the saddlepoint approximation to the probability of ruin $\psi(u)$: $\psi_S(u) = S(u)$. We need to find the c.g.f. of the maximal aggregate loss L , denoted $K(s)$. We first note that the Laplace transform of the density of L is given by $l(s) = 1 - s\mathcal{L}(\psi)(s)$. This and the formula (5) for $\mathcal{L}(\psi)(s)$ yield

$$l(s) = \frac{qs}{s(1 + \frac{s}{\zeta}) + \frac{\lambda}{c}\{\mathcal{L}(f_X)(s) - 1\}}.$$

This, $K(s) = \log l(-s)$ and $M_X(s) = \mathcal{L}(f_X)(-s)$, the common moment generating function of the individual claim amounts, yield

$$K(s) = \log \frac{qs}{s(1 - \frac{s}{\zeta}) - \frac{\lambda}{c}\{M_X(s) - 1\}}. \quad (26)$$

The saddlepoint approximation exists under steepness of K . The denominator of the ratio in (26) is equal to $-D(-s)$, for the function D given in (12). We saw that (12) viz. $D(s) = 0$ has at most one negative solution (which is within $\mathfrak{B}_r$). Let $-s^* < 0$ be that solution, then $s^* > 0$ is the pole of K . Thus $\lim_{s \rightarrow s^*} K(s) = \infty$ and, from differentiability of K , $\lim_{s \rightarrow s^*} K'(s) = \infty$. To conclude, if (12) admits the solution $-s^* < 0$, then s^* is the steepness point of K , the saddlepoint equation (23) can be uniquely solved in $\mathbb{R}$ giving $s_t \in (-\infty, s^*)$, for any $t = u > 0$. We note that s^* is often called Lundberg's exponent or adjustment coefficient and that this steepness condition was already established in [26, Result 2.3]. Clearly, the existence of the saddlepoint at any $t = u$ requires the existence of $M_X(s)$ for all s in some neighborhood of the origin. This condition indicates that the distribution of individual losses exhibits a light right tail.

6 Numerical study

We consider the same example in [5] where the individual claim amounts follow the linear combination of exponential distributions, because it offers a closed-form expression for the exact probability of ruin, which is crucial for our comparisons. A Monte Carlo substitute would entail its own approximation error that could not allow for reliable comparisons. Moreover, keeping the same example allows for comparisons with the other methods of [5]. We thus consider the following setting. The individual claim amount density is $f_X(x) = 12(e^{-3x} - e^{-4x})$, $x \geq 0$, giving $\mu = 7/12$, and the other parameters are: $\lambda = 1$, $c = 1$ and $\sigma^2 = 0.4$, yielding $\zeta = 5$ and $q = 5/12$. In this case the probability of ruin can be computed exactly; cf. [2]. We now give the details of the applications of Talbot's, Euler's and the saddlepoint approximations. Then, we show the numerical results and the comparisons. The comparisons are mainly based on the relative error of the three methods, defined by

$$\text{RE}_{\text{method}}(u) = \frac{\psi_{\text{method}}(u) - \psi(u)}{\min\{\psi(u), 1 - \psi(u)\}},$$

where “method” stands for T, E or S when respectively Talbot's, Euler's or the saddlepoint approximations are considered.

Talbot's method

In order to apply the method of Talbot, we first compute $\mathcal{L}(f_X)(s) = 12 \{(3+s)^{-1} - (4+s)^{-1}\}$, for $\text{Re } s > \max\{-3, -4\} = -3$ but can be analytically continued to $\mathbb{C} \setminus \{-3, -4\}$. Thus (A2) holds. We simplify (12) to $sp(s) = 0$, where $p(s) = s^3 + 12s^2 + 42s + 25$. The poles are

the roots of the cubic $p(s)$. We find the real and the two complex roots $s_1 \simeq -0.7433259$, $s_2 \simeq -5.628337 - 1.398015i$ and $s_3 = \overline{s_2}$. In order to have s_2 and s_3 within the “ $\supset$ ” of $\mathfrak{B}_r$, it is sufficient to select $r \geq 0.6$. If $r = 0.6$, then with $\theta_0 = 2.8$ we have $r\theta_0 \cot \theta_0 \simeq -4.7$ and $r\theta_0 i = 1.68i$. Since the imaginary part of $\mathfrak{B}_r$ grows when the real part decreases, we deduce that when the real part is about -5.628337 , the imaginary part is larger than 1.68 , thus s_2 and s_3 are enclosed in $\mathfrak{B}_r$. From the rule of thumb $r = 2n/(5u)$, we obtain $n \geq 3u/2$. Since [5] only consider $u \leq 40$, we choose $n = 60$ for all u . Thus (A1) holds. Thus the function to be inverted numerically with (11) is the rational function

$$\mathcal{L}(\psi)(s) = \frac{1}{s} \left(1 - \frac{\frac{5}{12}s}{s + \frac{1}{5}s^2 + \frac{12}{3+s} - \frac{12}{4+s} - 1} \right) = \frac{12s^2 + 119s + 329}{12s^3 + 144s^2 + 504s + 300}. \quad (27)$$

Euler’s method

To implement Euler’s method, it suffices to employ the Laplace transform (27) and to verify that all singularities are located to the left of the vertical line $\text{Re}(s) = a/(2u)$, with $a > 0$. As established in the analysis conducted for Talbot’s method, this condition is satisfied. We choose $a = 8 \log 10$ and $n = 20, m = 50$, for all considered values of $u \leq 40$. We also fix $M = 50$ decimal digit representation.

Saddlepoint approximation

To obtain the saddlepoint approximation to $\psi(u)$, we simply follow the steps given at the end of Section 5.1. The c.g.f. obtained with (26) gives

$$K(s) = \log \frac{\frac{5}{12}s}{s(1 - \frac{s}{5}) - (\frac{12}{3-s} - \frac{12}{4-s} - 1)} = \log \frac{-25s^2 + 175s - 300}{12s^3 - 144s^2 + 504s - 300}.$$

Regarding steepness, we obtain that the c.g.f. K is steep with steepness point $s^* = -s_1 \simeq 0.7433259$. It follows that the saddlepoint equation (23) can be solved at any $u = t > 0$ and so the saddlepoint approximation can be obtained at these values.

Results

Table 1 provides the exact probabilities of ruin $\psi(u)$ and the three approximations $\psi_T(u; 60)$ by Talbot, $\psi_E(u; 20, 50)$ by Euler and $\psi_S(u)$ by saddlepoint, for initial capitals u ranging from 0.1 to 40. We see that the exact ruin probabilities $\psi(u)$ do not differ from their Talbot approximation $\psi_T(u; 60)$ up to the displayed digits. Their difference can be seen through the relative errors in Table 2, which gives the relative errors of the three methods,

$\text{RE}_T(u; 60)$, $\text{RE}_E(u; 20, 50)$ and $\text{RE}_S(u)$. We see from Tables 1 and 2 that Talbot's approximation performs the best out of these three methods. The approximations obtained by the saddlepoint method are up to rounding the same as in [5]. The relative errors of Talbot's and Euler's methods are substantially smaller than those given in [5, Table 2], for the alternative techniques that are Fast Fourier transform, Monte Carlo importance sampling and for an ad hoc method based on lower and upper bounds. For instance, in that table Monte Carlo with importance sampling has relative error -0.02059 at $u = 40$.

Table 2 shows that, although the values of $\text{RE}_T(u; 60)$ increase w.r.t. u , they remain extremely small. The reason for which these relative errors increase is that the parameter n controls the error but not the relative error. The rate at which these relative errors increase is approximately equal to the rate at which the true ruin probability $\psi(u)$ decreases. Table 2 shows also that the values of $\text{RE}_E(u; 20, 50)$ decrease w.r.t. u . This indicates that it is not necessary to increase n along u , although it is suggested by [18, Remark 2]. Further, this is coherent with the error analysis of Section 4.2, in which (19) tells that the discretization error $\hat{\varepsilon}_d(u; 8 \log 10)$ should decrease together with $\psi(u)$, thus as u increases. But with the linear combination of exponentials claim amount, $\psi(u)$ takes the form of a sum of exponentials. This makes the estimated discretization error $\hat{\varepsilon}_d(u)$ decrease exponentially w.r.t. u . This phenomenon is illustrated in Figure 1, which shows the decomposition of the absolute error $|\varepsilon(u)| = |\psi(u) - \psi_E(u)|$ in terms of the estimated discretization error $\hat{\varepsilon}_d(u)$ and of the estimated truncation error $\hat{\varepsilon}_t(u; 8 \log 10, 20, 50)$; cf. (20). Eventually, for u large enough, $\hat{\varepsilon}_d(u)$ becomes sufficiently small to make $\hat{\varepsilon}_t(u)$ significant and to remain the only significant contribution to the error. This happens at about $u = 23$. The results indicate that the truncation error $\hat{\varepsilon}_t(u)$ remains of the same order of magnitude over the full domain of u values.

Finally, Table 3 analyzes the decomposition of Euler's method error $\psi_E(u; 20, 50)$, given by $\varepsilon = \varepsilon_d + \varepsilon_t + \varepsilon_r$. The true error ε and the estimators of the three errors of the decomposition are shown. We observe that the estimated upper bound for the discretization error $\hat{\varepsilon}_d$ decreases w.r.t. u , as explained before. For sufficiently small values of u , the global error ε closely matches the estimated discretization error $\hat{\varepsilon}_d$, whereas the truncation error is negligible. For larger values of u , the truncation error has the significant impact on the global error. Lastly, the estimate of the rounding error ε_r , given in the last column, indicates that the rounding error is negligible in this example. As mentioned before, the computations were done using $M = 50$ decimal digit representations, which seems sufficient. Then an estimate for D is obtained by (21). This behavior of the truncation and discretization errors

is reflected in the error in Figure 1.

To summarize, with all the three methods, the relative errors appear sufficiently small for nearly all practical applications. When very high precision is required, Talbot's approximation is the most recommended.

The computer programs of this study are written with the programming language Python and the code is available from the authors upon request. The `mpmath` package was employed to adjust numerical precision. This package was also used to compute the exact ruin probabilities.

u	$\psi(u)$	$\psi_T(u; 60)$	$\psi_E(u; 20, 50)$	$\psi_S(u)$
0.1	0.834698	0.834698	0.834698	0.818542
0.2	0.728087	0.728087	0.728087	0.723503
0.3	0.653506	0.653506	0.653506	0.659802
0.4	0.596584	0.596584	0.596584	0.606927
0.5	0.549721	0.549721	0.549721	0.560256
1.0	0.378619	0.378619	0.378619	0.381991
2.0	0.180406	0.180406	0.180406	0.181036
3.0	0.085790	0.085790	0.085790	0.086179
4.0	0.040795	0.040795	0.040795	0.041057
5.0	0.019399	0.019399	0.019399	0.019562
10.	0.000471	0.000471	0.000471	0.000479
20.	$2.789 \cdot 10^{-7}$	$2.789 \cdot 10^{-7}$	$2.789 \cdot 10^{-7}$	$2.864 \cdot 10^{-7}$
30.	$1.649 \cdot 10^{-10}$	$1.649 \cdot 10^{-10}$	$1.649 \cdot 10^{-10}$	$1.703 \cdot 10^{-10}$
40.	$9.750 \cdot 10^{-14}$	$9.750 \cdot 10^{-14}$	$9.750 \cdot 10^{-14}$	$1.011 \cdot 10^{-13}$

Table 1: Exact ruin probabilities $\psi(u)$ and approximations by Talbot $\psi_T(u; n = 60)$, Euler $\psi_E(u; n = 20, m = 50)$ and saddlepoint $\psi_S(u)$

7 Conclusion and future research

This article provides a focused numerical illustration of the accuracy attained by Talbot's, by Euler's and by the saddlepoint approximations to the ruin probabilities of the perturbed risk process. Section 6 demonstrates numerically that the accuracy achieved using Talbot's approximation exceeds that of Euler's approximation and is several orders of magnitude higher than the accuracy obtained with the saddlepoint approximation. Moreover,

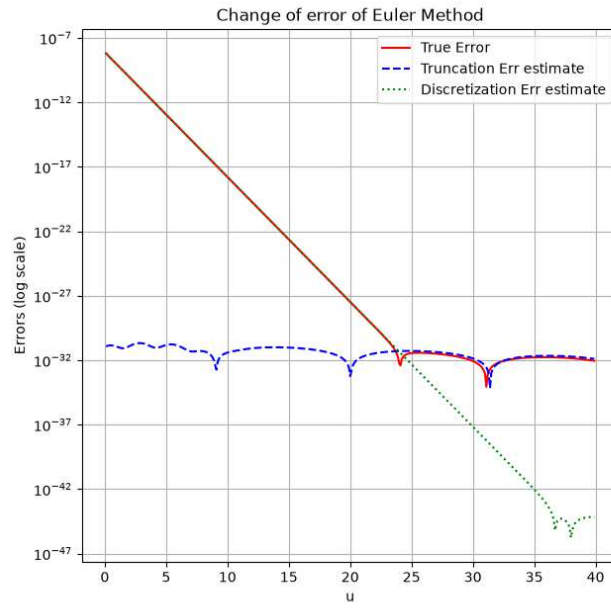


Figure 1: Estimators $\hat{\varepsilon}_d(u)$ and $\hat{\varepsilon}_t(u)$ of discretization and truncation errors together with absolute error $|\varepsilon(u)|$ of Euler's approximation $\psi_E(u)$, in logarithmic scale. Domination of discretization component for small u with truncation component becoming eventually noticeable

u	$\text{RE}_T(u; 60)$	$\text{RE}_E(u; 20, 50)$	$\text{RE}_S(u)$
0.1	$1.1495713 \cdot 10^{-36}$	$3.9534138 \cdot 10^{-8}$	-0.097739609
0.2	$7.0013288 \cdot 10^{-37}$	$1.8717940 \cdot 10^{-8}$	-0.016859252
0.3	$5.5043298 \cdot 10^{-37}$	$1.1761820 \cdot 10^{-8}$	0.018171117
0.4	$4.7362110 \cdot 10^{-37}$	$8.0973333 \cdot 10^{-9}$	0.025638612
0.5	$4.2508813 \cdot 10^{-37}$	$5.8086307 \cdot 10^{-9}$	0.023397207
1.0	$5.0996636 \cdot 10^{-37}$	$2.2658691 \cdot 10^{-9}$	0.0089076438
2.0	$1.0877664 \cdot 10^{-36}$	$5.1135109 \cdot 10^{-10}$	0.0034957922
3.0	$2.3210366 \cdot 10^{-36}$	$1.1562877 \cdot 10^{-10}$	0.0045444066
4.0	$4.9444856 \cdot 10^{-36}$	$2.6146962 \cdot 10^{-11}$	0.0064083199
5.0	$1.0516008 \cdot 10^{-35}$	$5.9125734 \cdot 10^{-12}$	0.0083830977
10.0	$4.4750788 \cdot 10^{-34}$	$3.4958524 \cdot 10^{-15}$	0.016727556
20.0	$7.5683754 \cdot 10^{-31}$	$1.2220884 \cdot 10^{-21}$	0.027021269
30.0	$1.2573854 \cdot 10^{-27}$	$-5.17625 \cdot 10^{-23}$	0.033307763
40.0	$2.1091502 \cdot 10^{-24}$	$8.8065764 \cdot 10^{-20}$	0.037706682

Table 2: Relative errors of approximations to $\psi(u)$ by Talbot $\text{RE}_T(u; n = 60)$, by Euler $\text{RE}_E(u; n = 20, m = 50)$ and by saddlepoint $\text{RE}_S(u)$ approximations

the accuracy of Talbot's and Euler's approximations are many orders of magnitude higher than that of the alternative methods of [5], that include the Fast Fourier transform and Monte Carlo importance sampling. Accordingly, Talbot's and Euler's methods represent a promising computational approach for actuarial use and for the analysis of alternative risk processes, in particular for those presented in [3]. We note, however, that the saddlepoint approximation offers a conceptual advantage over the competing numerical approaches: it can be evaluated directly via an explicit formula that requires neither numerical integration (as in Talbot's and Euler's methods) nor random variable generation (as in Monte Carlo importance sampling). An open issue concerns the extension of Talbot's approximation to risk processes with heavy-tailed individual claim amounts. We can also mention, as future research, the application of Talbot's approximation to the probability of ruin in finite time. For the perturbed risk process, the saddlepoint approximation to this probability and to related quantities is provided by [26].

u	$\varepsilon(u)$	$\hat{\varepsilon}_t(u; 8 \log 10, 20, 50)$	$\hat{\varepsilon}_d(u; 8 \log 10)$	$T_0(50) \cdot 10^{-\hat{D}}$
0.1	$6.5351 \cdot 10^{-9}$	$1.1707 \cdot 10^{-31}$	$6.5351 \cdot 10^{-9}$	$2.2674 \cdot 10^{-45}$
0.2	$5.0896 \cdot 10^{-9}$	$1.2761 \cdot 10^{-31}$	$5.0896 \cdot 10^{-9}$	$2.1732 \cdot 10^{-45}$
0.3	$4.0754 \cdot 10^{-9}$	$1.3505 \cdot 10^{-31}$	$4.0754 \cdot 10^{-9}$	$1.2791 \cdot 10^{-45}$
0.4	$3.2666 \cdot 10^{-9}$	$1.3921 \cdot 10^{-31}$	$3.2666 \cdot 10^{-9}$	$1.7577 \cdot 10^{-45}$
0.5	$2.6155 \cdot 10^{-9}$	$1.4012 \cdot 10^{-31}$	$2.6155 \cdot 10^{-9}$	$1.6594 \cdot 10^{-45}$
1.0	$8.5790 \cdot 10^{-10}$	$1.1000 \cdot 10^{-31}$	$8.5790 \cdot 10^{-10}$	$1.2092 \cdot 10^{-45}$
2.0	$9.2251 \cdot 10^{-11}$	$1.2238 \cdot 10^{-31}$	$9.2251 \cdot 10^{-11}$	$6.9329 \cdot 10^{-46}$
3.0	$9.9198 \cdot 10^{-12}$	$2.0798 \cdot 10^{-31}$	$9.9198 \cdot 10^{-12}$	$1.3982 \cdot 10^{-45}$
4.0	$1.0667 \cdot 10^{-12}$	$9.6145 \cdot 10^{-32}$	$1.0667 \cdot 10^{-12}$	$8.8438 \cdot 10^{-46}$
5.0	$1.1470 \cdot 10^{-13}$	$1.5177 \cdot 10^{-31}$	$1.1470 \cdot 10^{-13}$	$7.0158 \cdot 10^{-46}$
10.0	$1.6490 \cdot 10^{-18}$	$3.9001 \cdot 10^{-32}$	$1.6490 \cdot 10^{-18}$	$2.5492 \cdot 10^{-46}$
20.0	$3.4085 \cdot 10^{-28}$	$5.7942 \cdot 10^{-34}$	$3.4085 \cdot 10^{-28}$	$4.2795 \cdot 10^{-46}$
30.0	$8.5359 \cdot 10^{-33}$	$1.3754 \cdot 10^{-32}$	$7.0452 \cdot 10^{-38}$	$1.2597 \cdot 10^{-45}$
40.0	$8.5865 \cdot 10^{-33}$	$1.2327 \cdot 10^{-32}$	$6.7423 \cdot 10^{-45}$	$2.3353 \cdot 10^{-45}$

Table 3: Total error $\varepsilon(u)$, estimate of truncation error $\hat{\varepsilon}_t(u)$, estimate of discretization error $\hat{\varepsilon}_d(u)$ and estimate rounding error

Declaration of the use of generative AI and AI-assisted technologies During the preparation of this manuscript, the authors used Overleaf AI assistance for grammar improvements. The authors reviewed and edited the output as needed and take full responsibility for the content of the publication.

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