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A 2D discrete model with a phase-field regularisation applied to dynamic crack propagation

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Abstract

In the current study, we propose a phase-field regularisation of a two-dimensional discrete model previously developed by the author and co-workers, with the aim of reducing the dependence of dynamic crack propagation on the orientation of the lattice. The proposed model combines first-neighbour normal and transverse bond interactions with a nodal phase field calibrated with the tensile strength and the fracture energy. The gradient energy is expressed as a sum over the lattice bonds, and the equations of motion are integrated explicitly. Three numerical examples are analysed: the Kalthoff–Winkler test, a single-edge-notched plate under tension and the fragmentation of a thick cylinder under internal impulsive pressure. The first two tests are studied on lattices rotated by 0, 15 and 30 degrees, while centred and shifted lattices are considered for the cylinder. The results are presented in terms of the crack path, the propagation speed, the load–displacement response and the energies, and are compared with experimental observations and numerical results reported in the literature. The regularisation reduces the directional bias in the Kalthoff–Winkler and tension tests, while the cylinder illustrates the influence of lattice symmetry and regularisation on fragmentation. The differences between the energy removed by bond degradation and the regularised crack energy are also discussed.

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1 Introduction

The problem of dynamic crack propagation in brittle materials remains a difficult challenge. A numerical model for dynamic fracture should be able to capture the crack path, the crack speed and the branching of the crack, and its predictions should depend on the material and not on the discretisation. Numerous methods have been developed to this end, and they can be classified into two types: continuous and discrete methods (Braun and Fernández-Sáez 2016).

In continuum methods, the cracked solid is considered as a continuous medium. The extended finite element method (XFEM) was introduced by Belytschko and Black (1999) and Moës et al. (1999) to model cracks of arbitrary geometry with minimal remeshing, and was first applied to dynamic crack propagation by Belytschko et al. (2003). In the cohesive zone models (CZM), the fracture process is described by a cohesive law, and the cracks are represented by cohesive elements inserted between the finite elements (Xu and Needleman 1994). Peridynamic models provide

another approach to crack propagation, including comparisons with alternative fracture methods (Agwai et al. 2011). A comparative study of these methods for dynamic fracture was presented by Song et al. (2008). More recently, the phase-field models (PFM) have been developed as an alternative to analyse problems of brittle fracture. In these approaches, the crack is not introduced in the solid as a discontinuity; instead, it is approximated by a scalar field that smooths the crack over a small region of width ℓ . The method was first proposed by Francfort and Marigo (1998) as a variational problem, and later implemented by Bourdin et al. (2000). It has been applied to quasi-static (Miehe et al. 2010) and dynamic (Borden et al. 2012; Hofacker and Miehe 2012; Bleyer et al. 2017; Mandal et al. 2020) brittle fracture. Wu (2017) proposed the phase-field cohesive zone model (PF-CZM), in which the regularised crack is equivalent to a cohesive crack defined by the tensile strength and the fracture energy, and whose global response is insensitive to ℓ (Wu and Nguyen 2018); it has also been extended to dynamic cohesive fracture (Geelen et al. 2019).

In discrete methods, the solid is represented by a set of points that interact according to a certain law, and the crack is the natural outcome of the loss of these interactions. This category includes the particle methods (Cundall and Strack 1979), peridynamic theory (Silling 2000; Silling and Askari 2005) and the lattice models, whose origin goes back to the work of Hrennikoff (1941) (see also Ostoja-Starzewski 2002). Most of the lattice models have restrictions regarding the Poisson’s ratio of the material. In lattices with central (axial) interactions only, the Poisson’s ratio is constrained to $1/4$ in plane strain for the triangular central-force lattice considered here to guarantee the equivalence with the isotropic continuum (Kosteski et al. 2012). The non-central interaction of the Born model (Born and Huang 1954) removes this restriction, but it is not invariant under rigid rotations. Other authors introduced additional volumetric terms in the interaction potential (Chen et al. 2014; Chen and Liu 2016), or angular springs between consecutive bonds (Braun et al. 2024).

A 2D discrete model (DM) was presented by Braun and Fernández-Sáez (2014) based on the discretisation of Navier’s equations of elastodynamics after decomposing the total displacement

into the sum of normal and transverse components (Martín et al. 2000). This model is stable in the admissible range of Poisson’s ratio, and its parameters are a function only of the elastic constants. Braun and Fernández-Sáez (2016) incorporated a bilinear softening constitutive law calibrated with the fracture energy, which improved the prediction of the crack path and of the dissipated energy. Lattice models of this family have since been applied to the dependence of the stress intensity factor on the crack speed (Braun and González-Albuixech 2019), to mixed-mode fracture of polymers (Braun and Aranda-Ruiz 2021) and to the dynamic fracture of anisotropic and composite materials (Braun and Ariza 2019, 2020; Braun et al. 2021, 2024). However, both the DM and its softening version present a dependence of the crack path on the orientation of the lattice. In Braun and Fernández-Sáez (2014), the lattice was rotated from 0° to 50° in the crack branching and in the Kalthoff–Winkler problems, and the symmetry of the branches and the inclination of the crack were found to depend on the rotation angle. Since the crack advances by breaking bonds, it can only follow the six directions of the first-neighbour bonds, and the energy dissipated per unit crack length depends on the angle between the crack and the bonds.

The dependence of the crack path on the orientation of regular lattices is a well-known problem of discrete models (Schlangen and Garboczi 1997). It can be reduced by using irregular lattices, at the cost of a scaling of the interactions to recover the elastic uniformity (Bolander and Sukumar 2005), or by including more neighbours in the interactions. Chen et al. (2014) showed that a lattice spring model with three or four shells of neighbours eliminates the directional preference of the crack in quasi-static mode I, dynamic mode II and dynamic branching problems, with a computational cost that increases linearly with the number of neighbours. In peridynamics, the interactions within a horizon play the same role, although they introduce a surface effect near the free boundaries (Le and Bobaru 2018).

The phase-field approach suggests a different solution: if the crack is represented by a field defined on the nodes, its path is no longer restricted to the bond directions. Yue et al. (2023) recently proposed a phase-field lattice model

(PFLM), in which the PF-CZM is applied to a 3D body-centred cubic lattice of truss elements. In the PFLM, the phase field is discretised with one-dimensional finite elements along each bar, the crack driving force is the axial stress of the bar, the fracture toughness of the bars is calibrated for a crack parallel to the faces of the cubic module, and the problems are solved under quasi-static conditions with an implicit scheme. Since the bars only transmit axial forces, the equivalence with the isotropic continuum is exact only for $\nu = 0.25$. The authors observed that the orientation of the truss elements has a negligible influence on the crack evolution, but this was not assessed by rotating the lattice.

In the current study, we propose a phase-field regularisation of the DM. The elastic model is the first-neighbour DM of [Braun and Fernández-Sáez \(2014\)](#), interpreted here as a Born-type lattice with normal and transverse bond interactions and positive stiffness coefficients throughout the admissible Poisson-ratio range. The crack is described by a nodal phase field calibrated with the PF-CZM, which uses the same two fracture parameters as the bilinear softening law of [Braun and Fernández-Sáez \(2016\)](#), namely the tensile strength and the fracture energy. The main contributions are the following: (i) the gradient energy of the phase field is written as a sum over the bonds that is exact for linear fields on the hexagonal shell, so that the regularised crack is isotropic up to terms of order a/ℓ and no orientation-dependent calibration is needed; (ii) the crack driving force is computed from an objective nodal strain, since the transverse bond interaction of the DM is not invariant under rigid rotations; (iii) the equations of motion are integrated explicitly, which allows the analysis of dynamic fracture problems; and (iv) the lattice-orientation dependence is assessed systematically by rotating the lattice. The performance of the model is evaluated in three problems: the Kalthoff–Winkler test, the single-edge-notched tension test of [Chen et al. \(2014\)](#) and the fragmentation of a thick cylinder under internal impulsive pressure. The results are compared with experimental results and with numerical results reported by other authors.

The paper is organised as follows. Section 2 describes the DM, the bilinear softening law and the objectivity of the bond interactions. The phase-field regularisation is developed in

Section 3, where its relation to the PFLM of [Yue et al. \(2023\)](#) is also discussed. The numerical examples are presented in Section 4, and the conclusions are drawn in Section 5.

2 Description of the model

In this section, we briefly outline the DM developed by [Braun and Fernández-Sáez \(2014\)](#) and the bilinear softening constitutive law incorporated by [Braun and Fernández-Sáez \(2016\)](#), which are the starting point of the present work. We also discuss the objectivity of the bond interactions of the DM, which conditions the formulation of the crack driving force in Section 3.

2.1 Linear elastic discrete model

Consider a two-dimensional triangular lattice of spacing a (Fig. 1a) and a linear elastic, homogeneous, isotropic material of Young’s modulus E , Poisson’s ratio ν and density ρ . In plane strain, the longitudinal and transverse wave speeds are

$$c_l = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}, \quad c_t = \sqrt{\frac{E}{2\rho(1+\nu)}}; \quad (1)$$

in plane stress, c_l is replaced by $\sqrt{E/[\rho(1-\nu^2)]}$, which corresponds to the Lamé constant $\lambda^* = E\nu/(1-\nu^2)$. Navier’s equation of elastodynamics, $\ddot{\mathbf{u}} = c_l^2 \nabla^2 \mathbf{u} + (c_l^2 - c_t^2) \text{grad div } \mathbf{u}$, is decomposed ([Braun and Fernández-Sáez 2014](#)) by writing $\mathbf{u} = \mathbf{u}_n + \mathbf{u}_t$, where $\mathbf{u}_n$ is curl-free and $\mathbf{u}_t$ is divergence-free, which yields the uncoupled form

$$\ddot{\mathbf{u}} = c_l^2 \nabla^2 \mathbf{u}_n + c_t^2 \nabla^2 \mathbf{u}_t. \quad (2)$$

For each bond ij connecting node i to a first neighbour j (at distance a , unit vector $\mathbf{n}_{ij}$), the local normal and transverse relative displacements are (Fig. 1a)

$$\begin{aligned} \mathbf{u}_{ij}^n &= (\mathbf{u}_{ij} \cdot \mathbf{n}_{ij}) \mathbf{n}_{ij}, & \mathbf{u}_{ij}^t &= \mathbf{u}_{ij} - \mathbf{u}_{ij}^n, \\ \mathbf{u}_{ij} &= \mathbf{u}_j - \mathbf{u}_i. \end{aligned} \quad (3)$$

The discretisation of Eq. (2) is based on the weighted least-squares (WLS) paraboloid fit of [Martín et al. \(2000\)](#): given a smooth field f and a set of M neighbours $\{\mathbf{r}_j\}$ around $\mathbf{r}_i$, with $\mathbf{d}_j = \mathbf{r}_j - \mathbf{r}_i$, the fit minimises $\sum_j [f_i + \mathbf{A} \cdot \mathbf{d}_j + \frac{1}{2} \mathbf{B} :$

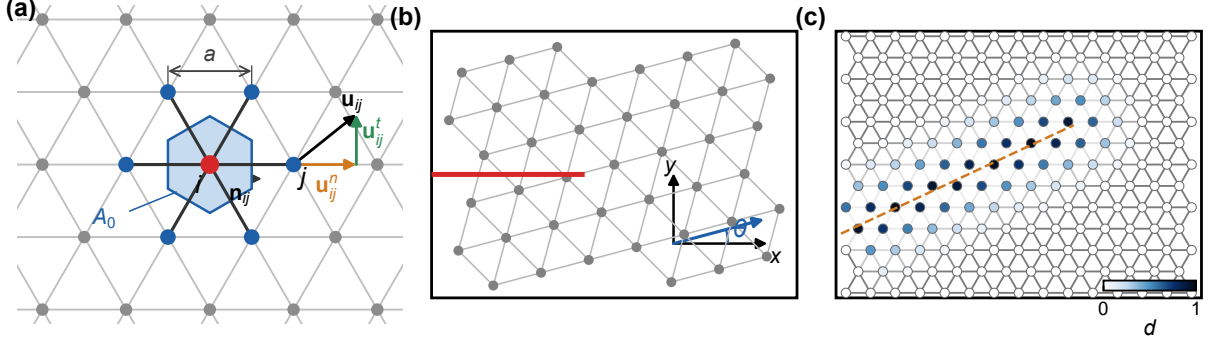


Fig. 1 (a) Triangular lattice of the DM: node i , its six first neighbours, the hexagonal cell of area A_0 and the decomposition of the relative displacement $\mathbf{u}_{ij}$ of a bond into its normal and transverse components. (b) Lattice rotated by an angle θ with respect to the specimen axes; no lattice row is then aligned with the notch (red). (c) Phase-field regularisation: the crack (dashed line) is described by the nodal field d_i , whose path is not restricted to the bond directions, and the stiffness of each bond is multiplied by $\omega(d_{ij})$ (bond shading)

$\mathbf{d}_j \mathbf{d}_j - f_j]^2$, giving $\mathbf{A} \approx \nabla f_i$ and $\text{tr } \mathbf{B} \approx \nabla^2 f_i$. The normal equations are $\mathbf{F}_1 = \mathbf{R}_2 \cdot \mathbf{A} + \frac{1}{2} \mathbf{R}_3 : \mathbf{B}$ and $\mathbf{F}_2 = \mathbf{R}_3 \cdot \mathbf{A} + \frac{1}{2} \mathbf{R}_4 : \mathbf{B}$, where $\mathbf{F}_1 = \sum (f_j - f_i) \mathbf{d}_j$, $\mathbf{F}_2 = \sum (f_j - f_i) \mathbf{d}_j \mathbf{d}_j$ and $\mathbf{R}_k = \sum \mathbf{d}_j^{\otimes k}$. For a regular shell the result takes a simple form, which is used below.

Lemma 1 (single regular shell) *Let $\{\mathbf{r}_j\}_{j=1}^N$, $N \geq 5$, be regularly distributed on a circle of radius ϱ centred at $\mathbf{r}_i$, i.e. $\mathbf{d}_j = \varrho(\cos \theta_j, \sin \theta_j)$, $\theta_j = \theta_0 + 2\pi j/N$. Then*

$$\begin{aligned} \nabla f_i &\approx \frac{2}{N\varrho^2} \sum_{j=1}^N (f_j - f_i) \mathbf{d}_j, \\ \nabla^2 f_i &\approx \frac{4}{N\varrho^2} \sum_{j=1}^N (f_j - f_i). \end{aligned} \quad (4)$$

Proof (sketch) For even N the point set is centrosymmetric, so that the odd-order moments $\mathbf{R}_1$ and $\mathbf{R}_3$ vanish (for odd $N \geq 5$ they vanish by the trigonometric sums). Moreover, $\mathbf{R}_2 = \frac{N}{2} \varrho^2 \mathbf{I}$ and, for $N \geq 5$, $\mathbf{R}_4$ is isotropic, $R_{4,klmn} = C(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})$ with $C = \frac{N}{8} \varrho^4$ (using $\sum \cos^4 \theta_j = \frac{3N}{8}$ and $\sum \cos^2 \theta_j \sin^2 \theta_j = \frac{N}{8}$). The normal equations then decouple, $\mathbf{A} = \mathbf{R}_2^{-1} \mathbf{F}_1$, and the trace of the second one gives $\text{tr } \mathbf{B} = \text{tr}(\mathbf{F}_2)/(2C)$; substituting $\mathbf{R}_2$, C and $\text{tr}(\mathbf{F}_2) = \varrho^2 \sum_j (f_j - f_i)$ gives Eq. (4). For $N = 6$ and $\varrho = a$ it reduces to Eqs. (14) and (16) of [Braun and Fernández-Sáez \(2014\)](#). $\square$

Following [Braun and Fernández-Sáez \(2014\)](#), we retain a first-neighbour Born-type lattice

whose equation of motion at node i is

$$\begin{aligned} \ddot{\mathbf{u}}_i &= \beta \sum_{j=1}^6 \mathbf{u}_{ij}^n + \alpha \sum_{j=1}^6 \mathbf{u}_{ij}^t, \\ \beta &= \frac{2c_l^2}{3a^2}, \quad \alpha = \frac{2c_t^2}{3a^2}. \end{aligned} \quad (5)$$

Equation (5) defines the mechanical lattice used in all comparisons of orientation. Its bond coefficients are positive for $\nu \in (-1, 0.5)$. The quantities c_l and c_t parameterise these coefficients through the nominal material data. The present study assesses crack paths, propagation speeds and fracture energies within this discrete model. It can be interpreted as the equation of motion of particles of mass $m = \rho A_0$, where $A_0 = \sqrt{3}a^2/2$ is the area of the hexagonal cell, interacting with their first neighbours through bonds of normal and transverse stiffness $m\beta$ and $m\alpha$. The equations are integrated explicitly with the Verlet algorithm, with the time step limited by the Courant–Friedrichs–Lewy condition $\Delta t \leq a/c_l$ ([Braun and Fernández-Sáez 2016](#)). The strain tensor at a node is evaluated as ([Braun and Fernández-Sáez 2014](#))

$$\gamma_i = \frac{1}{6a} \sum_{j=1}^6 \left[(\mathbf{u}_j - \mathbf{u}_i) \otimes \mathbf{n}_{ij} + \mathbf{n}_{ij} \otimes (\mathbf{u}_j - \mathbf{u}_i) \right]. \quad (6)$$

2.2 Bilinear softening constitutive law

The strain tensor of Eq. (6), averaged between the two nodes of a bond, is used to evaluate a bilinear

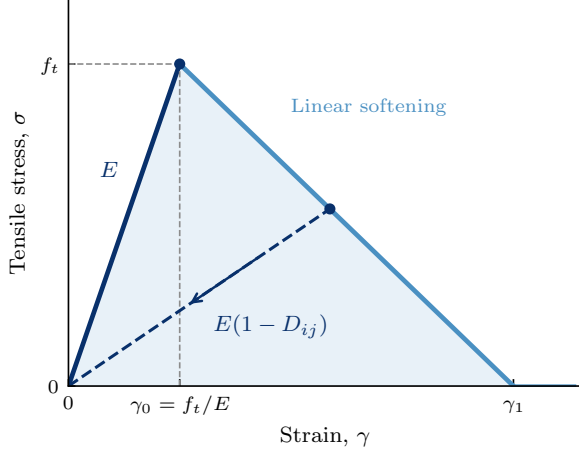


Fig. 2 Bilinear tensile stress-strain reference law: elastic loading with modulus E up to f_t at $\gamma_0 = f_t/E$, followed by linear softening to zero stress at γ_1 . The dashed line illustrates unloading from a point on the softening branch at fixed bond damage D_{ij} , with secant stiffness $E(1 - D_{ij})$. The shaded area corresponds to $G_f h_{ij}/\Omega_{ij}$. Schematic, not to scale; the linear-softening branch provides the target calibration for the phase-field formulation

softening damage law calibrated with the fracture energy G_f (Braun and Fernández-Sáez 2016): denoting by $\gamma_0 = f_t/E$ the strain at peak stress and by γ_1 the strain at complete softening,

$$\gamma_1 = \frac{2G_f h_{ij}}{E \gamma_0 \Omega_{ij}}, \quad \Omega_{ij} = \frac{a^2}{2\sqrt{3}}, \quad h_{ij} = \frac{a}{2\sqrt{3}}, \quad (7)$$

where Ω_{ij} is the tributary area of a bond and h_{ij} its characteristic length. The damage variable of bond ij , $D_{ij} \in [0, 1]$, multiplies the bond stiffness in Eq. (5) through the factor $(1 - D_{ij})$.

Under monotonic uniaxial tension, the bilinear reference law is

$$\sigma(\gamma) = \begin{cases} E\gamma, & 0 \leq \gamma \leq \gamma_0, \\ f_t \frac{\gamma_1 - \gamma}{\gamma_1 - \gamma_0}, & \gamma_0 < \gamma < \gamma_1, \\ 0, & \gamma \geq \gamma_1, \end{cases} \quad (8)$$

with $\gamma_1 > \gamma_0$. Its area is $\frac{1}{2}f_t\gamma_1 = G_f h_{ij}/\Omega_{ij} = G_f/a$. Figure 2 illustrates this reference law. The phase-field formulation below uses the corresponding linear-softening calibration; the schematic does not represent a measured homogeneous response of the hybrid lattice-phase-field model.

Since only six bond directions are available at each node, the crack, which advances by breaking bonds, can only follow one of six directions (60°

apart). Braun and Fernández-Sáez (2014) report a measurable dependence of the predicted crack path on the orientation of the lattice with respect to the loading axis (Fig. 1b). The phase-field regularisation of Section 3 addresses this directional restriction using first neighbours only, without an orientation-dependent recalibration of the fracture law.

2.3 Objectivity of the bond interactions

The transverse two-node interaction in Eq. (5) should not be confused with an objective bending interaction. Indeed, under an infinitesimal rigid rotation, $\mathbf{u}(\mathbf{r}) = \mathbf{W}\mathbf{r}$, $\mathbf{W}^T = -\mathbf{W}$, one has $\mathbf{u}_{ij} \cdot \mathbf{n}_{ij} = 0$ but $\mathbf{u}_{ij}^t = \mathbf{W}\mathbf{d}_{ij} \neq \mathbf{0}$. Consequently, a positive energy proportional to $|\mathbf{u}_{ij}^t|^2$ penalises a rigid-body rotation. This has no consequence for the linear elastic response of Eq. (5), which is exact for affine fields, but it conditions any fracture criterion based on the bond energies: a crack driving force derived from the transverse term would damage rotating, but unstrained, material. For this reason, the crack driving force of Section 3 is computed from an objective nodal strain tensor and not from the bond energies.

3 Phase-field regularisation of the DM

The bond-breaking criteria of Section 2 tie the crack to the lattice: a crack can only advance along bond directions, and the energy dissipated per unit crack length depends on the angle between the crack and the bonds it severs. For first-neighbour bonds of a triangular lattice this energy is proportional to $\sum_{k=0}^2 |\cos(\phi - k\pi/3)|$, which varies by 14% between $\phi = 0^\circ$ and $\phi = 30^\circ$. In this section, the elastic lattice is kept as the first-neighbour DM, Eq. (5), and the crack is represented by a nodal phase field $d_i \in [0, 1]$ of width ℓ , independent of the bond directions (Bourdin et al. 2000; Miehe et al. 2010) (Fig. 1c). The phase field is calibrated with the PF-CZM of Wu (2017), which uses the same two fracture parameters as the bilinear softening law, f_t and G_f , and whose global response is insensitive to ℓ (Wu and Nguyen 2018).

3.1 Energy

Per unit thickness, the total energy of the lattice is

$$\begin{aligned} \Pi(\mathbf{u}, d) = & \sum_{\langle ij \rangle} \omega(\bar{d}_{ij}) \mathcal{V}_{ij}(\mathbf{u}) + \frac{G_f}{\pi} \left[\frac{1}{\ell} \sum_i A_0 \varphi(d_i) \right. \\ & \left. + \ell \frac{1}{\sqrt{3}} \sum_{\langle ij \rangle} (d_j - d_i)^2 \right], \\ \bar{d}_{ij} = & \frac{1}{2}(d_i + d_j), \end{aligned} \quad (9)$$

where $\mathcal{V}_{ij} = \frac{1}{2}m(\beta|\mathbf{u}_{ij}^n|^2 + \alpha|\mathbf{u}_{ij}^t|^2)$ is the DM bond energy, $\varphi(d) = 2d - d^2$ the crack geometric function and $\omega(d)$ the degradation function. The bracket is the discrete counterpart of the regularised crack length $\Gamma_d = \int \pi^{-1}(\varphi/\ell + \ell|\nabla d|^2) dA$, whose one-dimensional minimiser is $d(x) = 1 - \sin(|x|/\ell)$ for $|x| \leq \pi\ell/2$ and 0 otherwise, with $\Gamma_d = 1$ per unit crack length.

Lemma 2 (bond-sum gradient energy) *On the six-point shell of a triangular lattice, for every linear field $d(\mathbf{r}) = \mathbf{g} \cdot \mathbf{r}$ and every orientation of $\mathbf{g}$,*

$$\frac{1}{2} \cdot \frac{1}{\sqrt{3}} \sum_{j=1}^6 (d_j - d_i)^2 = A_0 |\nabla d|^2. \quad (10)$$

Proof $\sum_j (d_j - d_i)^2 = \mathbf{g} \cdot \mathbf{R}_2 \mathbf{g} = 3a^2 |\mathbf{g}|^2$ by Lemma 1, and $\frac{1}{2} \cdot 3a^2/\sqrt{3} = A_0$. The factor $\frac{1}{2}$ assigns each bond to its two end nodes. $\square$

Equivalently, the gradient term of Eq. (9) has stationarity condition $-\frac{2}{\sqrt{3}} \sum_j (d_j - d_i) = -2A_0 \nabla^2 d_i$, i.e. it is the six-point Laplacian of Lemma 1. Because it is written as a bond sum, it yields a consistent natural (Neumann) condition in its discrete variational form on free edges and notch faces (whose crossing bonds are removed), in contrast with the horizon-truncation (surface) effect of non-local interactions near free boundaries (Le and Bobaru 2018).

3.2 Degradation function and crack driving force

Following Wu (2017),

$$\begin{aligned} \omega(d) = & \frac{(1-d)^p}{(1-d)^p + a_1 d (1 + a_2 d + a_2 a_3 d^2)}, \\ a_1 = & \frac{4}{\pi} \frac{\ell_{ch}}{\ell}, \quad \ell_{ch} = \frac{EG_f}{f_t^2}, \end{aligned} \quad (11)$$

with $(p, a_2, a_3) = (2, -\frac{1}{2}, 0)$ for linear softening, the counterpart of the softening branch of Braun and Fernández-Sáez (2016). The crack driving force is the Rankine-type effective energy

$$\begin{aligned} Y_i = & \frac{\langle \bar{\sigma}_{1,i} \rangle^2}{2E}, \quad H_0 = \frac{f_t^2}{2E}, \\ H_i(t) = & \max \left(H_0, \max_{\tau \leq t} Y_i(\tau) \right), \end{aligned} \quad (12)$$

where $\langle \cdot \rangle$ is the Macaulay bracket and $\bar{\sigma}_{1,i}$ the maximum principal stress at node i ,

$$\begin{aligned} \bar{\sigma}_{1,i} = & \frac{\sigma_{i,11} + \sigma_{i,22}}{2} \\ & + \sqrt{\left(\frac{\sigma_{i,11} - \sigma_{i,22}}{2} \right)^2 + \sigma_{i,12}^2}, \end{aligned} \quad (13)$$

with $\boldsymbol{\sigma}_i = \lambda \text{tr}(\boldsymbol{\gamma}_i) \mathbf{I} + 2\mu \boldsymbol{\gamma}_i$ (λ^* in plane stress). Here $\boldsymbol{\sigma}$ is an effective undegraded stress computed for the driving force; it is not obtained by differentiating the degraded bond energy. The nodal strain $\boldsymbol{\gamma}_i$ is the first-order reproducing gradient of the displacement, fitted with bond weights $w_{ij} = \omega(\bar{d}_{ij})$:

$$\begin{aligned} \boldsymbol{\gamma}_i = & \text{sym}(\mathbf{B}_i \mathbf{K}_i^{-1}), \\ \mathbf{B}_i = & \sum_j w_{ij} \mathbf{u}_{ij} \otimes \mathbf{d}_{ij}, \\ \mathbf{K}_i = & \sum_j w_{ij} \mathbf{d}_{ij} \otimes \mathbf{d}_{ij}, \end{aligned} \quad (14)$$

which reduces to Eq. (6) in an intact, complete neighbourhood and is exact for affine fields also at incomplete boundary neighbourhoods. A node next to an open crack thus measures the strain of its intact side and not the crack opening, which would otherwise raise H in the whole stencil and widen the damage band. The fit is accepted when $\det \mathbf{K}_i > 0.05 (\text{tr} \mathbf{K}_i)^2$ (the ratio is 0.25 for a complete shell); otherwise $Y_i = 0$. This condition is scale invariant, so uniform degradation of

a stencil does not invalidate it, but it rejects one-sided, nearly collinear fits. As in the original DM, the nodes of the boundaries where the velocity is prescribed are excluded from the driving force ($Y_i = 0$), since their damage would be an artefact of the boundary condition.

The phase field minimises

$$\begin{aligned} \Pi_d(d) = & \sum_i A_0 \left[\omega(d_i) H_i + \frac{G_f}{\pi \ell} \varphi(d_i) \right] \\ & + \frac{G_f \ell}{\pi \sqrt{3}} \sum_{\langle ij \rangle} (d_j - d_i)^2, \quad d_i^{\text{old}} \leq d_i \leq 1, \end{aligned} \quad (15)$$

whose stationarity condition at a node i strictly between its bounds is

$$\begin{aligned} A_0 \left[\omega'(d_i) H_i + \frac{2G_f}{\pi \ell} (1 - d_i) \right] \\ + \frac{2G_f \ell}{\pi \sqrt{3}} \sum_{j \in S_1(i)} (d_i - d_j) = 0, \end{aligned} \quad (16)$$

where $S_1(i)$ is the set of first neighbours of node i . At either bound, the corresponding complementarity condition replaces this equality.

Proposition 1 (onset and local convexity) *At the lower bound $d = 0$, the stationarity equation must be replaced by its Karush–Kuhn–Tucker inequality. Since $\omega'(0) = -a_1$, the homogeneous residual is $a_1(H_0 - H)$. Thus $d = 0$ is locally admissible for an unfloored driving energy $H \leq H_0$. With the history field of Eq. (12), $H = H_0$ below the tensile threshold and the residual is zero. Moreover, $\omega''(0) = 2a_1(a_1 - p - a_2)$, so strict local convexity at the history floor requires $a_1 > p + a_2 + \frac{1}{2}$. For linear softening this is $a_1 > 2$, or $\ell < 2\ell_{ch}/\pi$.*

If this condition is violated, the second derivative at $H = H_0$ is negative and the undamaged homogeneous state is locally unstable under admissible positive perturbations. This does not establish instability for every unfloored $H < H_0$, or global convexity of the constrained problem. Damage above the tensile threshold is therefore interpreted subject to the local stability condition and solver convergence.

The driving force (12) is computed from the objective nodal strain rather than from $\partial \Pi / \partial d$ of the bond energies $\mathcal{V}_{ij}$, for the reason given in Section 2.3. The resulting scheme is variational in

d for a given H but not jointly in $(\mathbf{u}, d)$, as in the hybrid formulation of Ambati et al. (2015).

3.3 Solution algorithm

The explicit integration of the DM is unchanged, with $\Delta t = 0.125 a/c_l$ in all the examples. At every time step:

1. the nodal strain, Eq. (14), and Y_i and H_i , Eq. (12), are updated;
2. problem (15) is solved by a primal–dual active-set method: on the free nodes (in practice the process zone only), a Newton step with the curvature $\omega'' H - 2G_f/(\pi \ell)$ floored to a positive value, projected on the bounds and globalised by backtracking on Π_d ;
3. every bond force of Eq. (5) is multiplied by $\omega(\bar{d}_{ij})$.

Step 2 involves only a sparse system with the connectivity of the lattice, and it is restricted to the candidate nodes ($H_i > H_0$, $d_i > 0$, or a neighbour with $d_j > 0$); elsewhere the residual (16) is exactly zero. Each accepted iterate decreases Π_d . In the Kalthoff–Winkler and the SEN tests, up to 30 iterations per step are allowed (relative residual tolerance 10^{-8}); in the cylinder test, whose lattices have up to 2.3×10^5 nodes, the number of iterations is limited to five per step (relative residual tolerance 10^{-6}), continued from the previous step (Section 4.3). The fracture energy is $G_f t \Gamma_d$, where t is the thickness, evaluated with the bracket of Eq. (9). The energy removed from the bonds by the degradation, $W_{\text{deg}} = \sum_{\text{steps}} \sum_{\langle ij \rangle} \max(\omega_{ij}^{\text{old}} - \omega_{ij}, 0) \mathcal{V}_{ij}$, is recorded separately, and the energy balance $W_{\text{ext}} = K + U + W_{\text{deg}}$, where K is the kinetic energy, U the stored elastic energy and W_{ext} the external work, is checked in every example.

3.4 Mechanical energy balance and fracture-energy consistency

The energy accounting distinguishes the work removed from the lattice by degradation, W_{deg} , from the regularised crack energy, $W_f = G_f t \Gamma_d$. At a fixed displacement $\mathbf{u}^n$, a damage update

changes the stored bond energy by

$$\begin{aligned}\Delta W_{\text{deg}}^n &= U(\mathbf{u}^n, d^{n-1}) - U(\mathbf{u}^n, d^n) \\ &= \sum_{\langle ij \rangle} (\omega_{ij}^{n-1} - \omega_{ij}^n) \mathcal{V}_{ij}(\mathbf{u}^n) \geq 0.\end{aligned}\quad (17)$$

The inequality follows from irreversibility, the decreasing degradation function and the non-negative bond energies for the positive bond coefficients used here. The positive-part operation in the implementation enforces this sign when accumulating the increments. This accounts for the energy removed by a damage update; it does not establish consistency between the Rankine driving force and the mechanical bond energy.

All simulations start from an unstressed lattice with $d = 0$; the initial notches are introduced by removing crossing bonds. Two residuals are therefore defined as

$$\begin{aligned}r_m &= W_{\text{ext}} - K - U - W_{\text{deg}}, \\ r_f &= W_{\text{ext}} - K - U - W_f = r_m + (W_{\text{deg}} - W_f).\end{aligned}\quad (18)$$

The plotted balance error is r_m/W_{ext} , which measures the mechanical accounting error, including the effects of time integration and work quadrature. The separate quantities $(W_{\text{deg}} - W_f)/W_f$ and $(W_{\text{deg}} - W_f)/W_{\text{ext}}$ measure the fracture-energy mismatch relative to the crack energy and the total input work, respectively. A small mechanical residual can coexist with a substantial fracture-energy mismatch. The two fracture-related quantities must not be added together in the same balance: doing so would count both the energy removed from the bonds and the crack energy assigned to the same damage evolution.

The hybrid driving force does not enforce $W_{\text{deg}} = W_f$. A common energy potential for the mechanical forces and the damage driving force would be required to establish a variational energy balance. Even then, an exact equality between the accumulated degradation work and crack energy requires examination of the evolution, constraints and numerical integration; it is not a consequence of the variational formulation alone.

3.5 Isotropy of the regularised crack

The crack measure of Eq. (9) was evaluated for the optimal profile $d = 1 - \sin(r/\ell)$ around straight cracks of every orientation $\phi \in [0^\circ, 60^\circ]$ (1° steps)

Table 1 Regularised crack length per unit length, Γ_d/L , of a straight crack in a first-neighbour triangular lattice, over all orientations ϕ . Spread: (max-min)/mean; the last column excludes the orientations parallel to a lattice row or to a bisector ($\phi = 0^\circ, 30^\circ, 60^\circ$). The last row is the orientation dependence of the energy dissipated by breaking first-neighbour bonds

	mean	min	max	spread	spread*
PF, $\ell/a = 1$	0.804	0.723	0.835	13.9%	2.6%
PF, $\ell/a = 2$	0.903	0.865	0.923	6.4%	1.2%
PF, $\ell/a = 4$	0.953	0.936	0.966	3.1%	0.6%
Bond breaking	—	—	—	14.0%	14.0%

* Excluding $\phi = 0^\circ, 30^\circ$ and 60° .

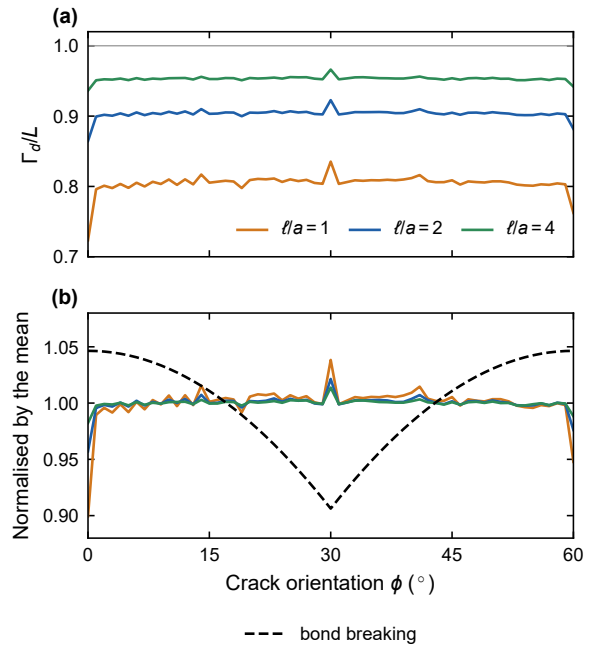


Fig. 3 Orientation dependence of the regularised crack length of a straight crack in a first-neighbour lattice: (a) Γ_d/L for three values of ℓ/a ; (b) the same values normalised by their mean, compared with the energy dissipated by breaking first-neighbour bonds

in a first-neighbour lattice. Table 1 and Fig. 3 compare the orientation dependence of the dissipated energy with that of bond breaking.

Three conclusions follow. First, the orientation dependence of the dissipated energy decreases as a/ℓ : its spread equals that of bond breaking for $\ell = a$, is halved for $\ell = 2a$ and is quartered for $\ell = 4a$.

Second, the spread is concentrated at the orientations parallel to a lattice row ($\phi = 0^\circ, 60^\circ$) or to a bisector ($\phi = 30^\circ$), where the nodes are aligned with the crack (Fig. 3b); for all the other orientations it is below 2.6%, 1.2% and 0.6%, whereas the energy of bond breaking varies smoothly with ϕ by 14%. The full-angle spread is, however, the relevant measure of isotropy, since a crack may follow any of these special orientations. Third, the lattice underestimates Γ_d by a bias that is also first order in a/ℓ and nearly orientation independent ($\approx 10\%$ at $\ell = 2a$), analogous to the well-known overestimation of G_c in finite-element phase-field models (Bourdin et al. 2000). It could be removed by using $G_f/\Gamma_d/L$ in Eq. (9); this correction is not applied in the computations reported here. Hence $\ell \geq 2a$ is recommended. The condition of Proposition 1 bounds ℓ from above, so the admissible window is $2a \leq \ell < 2\ell_{ch}/\pi$. For the Kalthoff steel ($\ell_{ch} = 5.92$ mm) this is $\ell < 3.77$ mm, which requires $a < 1.88$ mm.

3.6 Relation to the phase-field lattice model (PFLM)

The phase-field lattice model (PFLM) of Yue et al. (2023) also combines a lattice discretisation with the PF-CZM of Wu (2017). It uses a three-dimensional body-centred cubic lattice of axial bars, with the phase field interpolated along each bar and an equivalent bar toughness calibrated for a reference crack direction. In the current study, the mechanical model is a two-dimensional triangular Born-type lattice with first-neighbour normal and transverse interactions. The phase field is defined at the nodes, and its gradient energy is evaluated through the bond sum of Lemma 2, without calibration for a particular crack direction. The crack driving force is obtained from the maximum principal stress of the nodal strain tensor, rather than the axial stress of individual bars. This gives a hybrid formulation, whose energetic implications are assessed in the numerical examples. Whereas Yue et al. (2023) considered quasi-static problems with an implicit solution scheme, the current study uses explicit time integration for dynamic crack propagation and fragmentation. The dependence of the predictions on lattice orientation is assessed by rotating the lattice in

the Kalthoff–Winkler and SEN tests, complementing the orientation observations reported in the PFLM study.

4 Numerical examples

The performance of the proposed model is assessed with three examples, whose geometry and loading are shown in Fig. 4. Firstly, the Kalthoff–Winkler test is analysed, in which a crack propagates under dynamic mixed-mode loading. The second example is the quasi-static propagation of a mode I crack in a single-edge-notched (SEN) plate under tension, used by Chen et al. (2014) to assess the directional preference of the crack path in lattice spring models. Finally, the fragmentation of a thick cylinder under internal impulsive pressure is analysed, in which many cracks nucleate and interact. The parameters of the three examples are summarised in Table 2.

In the first two examples, the lattice is rotated by $\theta = 0^\circ, 15^\circ$ and 30° with respect to the specimen (Fig. 1b). Since the triangular lattice is 60° -periodic and mirror symmetric, $\theta \in [0^\circ, 30^\circ]$ spans every distinct orientation. For $\theta \neq 0^\circ$ no lattice row is aligned with the notch or the edges. The notch then removes the nodes within $\sqrt{3}a/4$ of the nominal segment, together with every bond crossing it, and the boundary conditions act on a band of width a along the corresponding edge. The results are presented in terms of the crack path, the crack speed, the energy balance and, for the SEN plate, the load–displacement response, and are compared with experimental results and with the numerical results reported by other authors, in particular with those obtained with the DM by Braun and Fernández-Sáez (2014, 2016).

The crack front is defined as the farthest point, from the notch tip, of the damaged region connected to it, i.e. of the nodes with d_i above a threshold that are joined to the notch tip by bonds whose two end nodes are also above it. Three thresholds, $d \geq 0.1, 0.5$ and 0.95 , are used; the threshold 0.5 is used for the kinematics and 0.95 for the crack path. All crack propagation speeds are normalised by the Rayleigh wave speed c_R of the reference material. Consistently with the post-processing of the numerical examples, it is

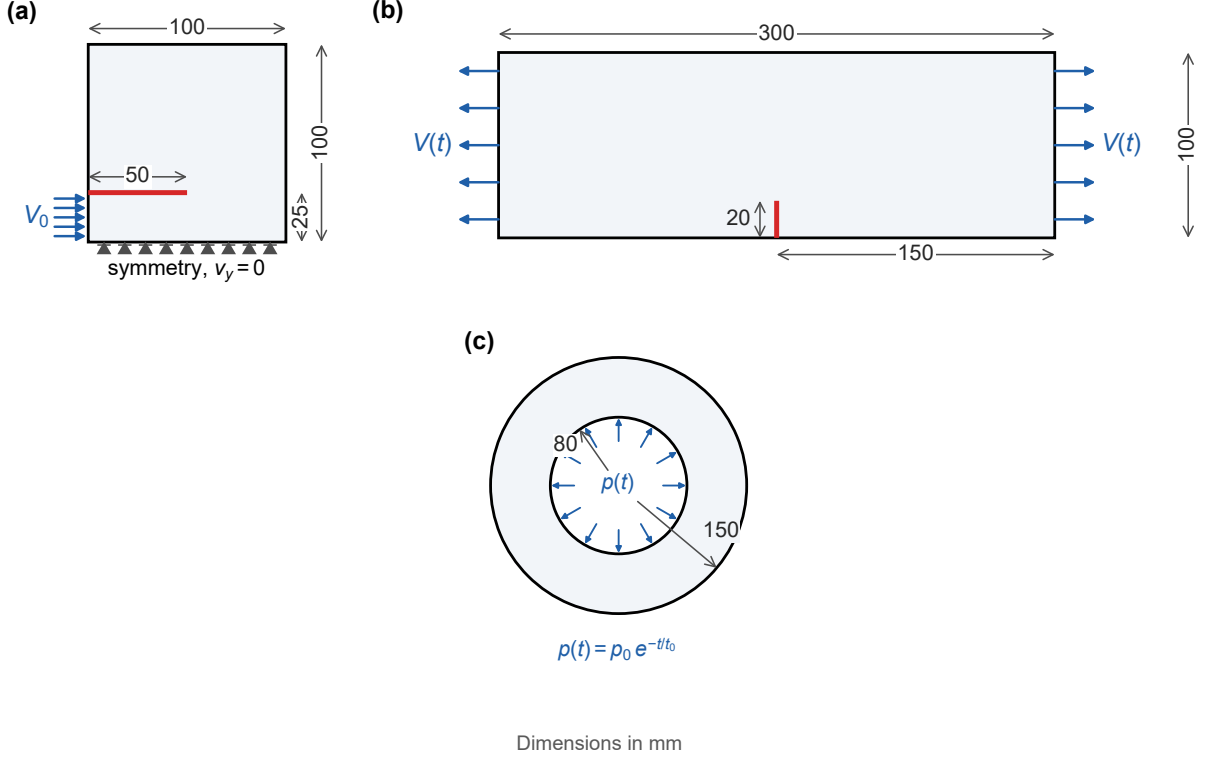


Fig. 4 Geometry and loading of the numerical examples (dimensions in mm; notches in red): (a) Kalthoff–Winkler test, upper half of the plate with a symmetry condition on the lower edge; (b) SEN plate under tension; (c) thick cylinder under internal impulsive pressure

Table 2 Material and discretisation parameters of the numerical examples. $\ell_{ch} = EG_f/f_t^2$; PS: plane stress; PE: plane strain

Example	E [GPa]	ν	ρ [kg/m ³]	f_t [MPa]	G_f [N/m]	ℓ_{ch} [mm]	State	a [mm]	ℓ [mm]	ℓ/ℓ_{ch}
Kalthoff–Winkler	190	0.30	8000	843.6	22 170	5.92	PE	1.25, 0.625	2.5, 1.25	0.42, 0.21
SEN tension	69	0.30	2700	200	4600 ^a	7.94	PS	2, 1, 0.5	4, 2	0.50, 0.25
Cylinder	210	0.30	7850	1050	20 000	3.81	PE	1, 0.5	2, 1	0.52, 0.26

^aNot given in the original benchmark; adopted in the current study (see text).

evaluated using

$$c_R \approx \frac{0.862 + 1.14\nu}{1 + \nu} c_t, \quad \hat{v} = \frac{v}{c_R}, \quad (19)$$

where c_t is given in Eq. (1). The same c_R is used for all lattice orientations and discretisations of each material. Mean and maximum speeds are reported as $\bar{v}/c_R$ and $v_{\max}/c_R$, respectively.

4.1 Edge-cracked plate under impulsive loading (Kalthoff–Winkler test)

The impact test of [Kalthoff and Winkler \(1987\)](#) is modelled as in [Braun and Fernández-Sáez \(2014, 2016\)](#). The upper half of the 100×200 mm plate is modelled with a symmetry condition ($v_y = 0$) on $y = 0$, a 50 mm notch at $y = 25$ mm, and a prescribed velocity $V_0 = 16.5$ m/s on the impacted edge below the notch (Fig. 4a). The material data are $E = 190$ GPa, $\nu = 0.30$, $\rho = 8000$ kg/m³,

$G_f = 22170$ N/m and $f_t = E\gamma_0 = 843.6$ MPa ($\gamma_0 = 4.44 \times 10^{-3}$), which give $\ell_{ch} = 5.92$ mm. The plane-strain thickness is $t = 9$ mm and the response is computed up to $75 \mu\text{s}$. For $\theta = 0^\circ$ the lattice coincides node by node with that used by [Braun and Fernández-Sáez \(2014, 2016\)](#).

The crack path is characterised by the nodes with $d_i \geq 0.95$. Two angles are reported: (i) the slope of a least-squares line fitted to these nodes within 25 mm of the notch tip; (ii) the median polar angle, about the notch tip, of the nodes lying in the annuli 2.5–10 mm and 10–20 mm. The experimental crack angle is approximately 70° ([Kalthoff and Winkler 1987](#)); the same value is used as reference in dynamic phase-field studies ([Borden et al. 2012](#)).

Table 3 and Fig. 5 summarise the results.

In all cases a single crack crosses the whole specimen, and no spurious secondary crack appears. The fitted angle varies between orientations by 1.6° for $a = 1.25$ mm and by 0.16° and 0.47° for $a = 0.625$ mm ($\ell = 2.5$ and 1.25 mm). The crack length and the regularised crack energy vary by less than 1% and 2%, respectively. The local angle in the first annulus is more sensitive: its spread between orientations is 5.5° for $a = 1.25$ mm and 5.1° and 3.5° for $a = 0.625$ mm ($\ell = 2.5$ and 1.25 mm), and the largest value corresponds to $\theta = 15^\circ$ in the three cases. This spread does not decrease with a/ℓ : on the finer lattice it is larger for $\ell/a = 4$ than for $\ell/a = 2$. It is smallest for the smallest ℓ/ℓ_{ch} , for which the damage band is narrowest.

With ℓ fixed, refining the lattice from $a = 1.25$ to 0.625 mm changes the fitted angle by less than 1° . Two spacings do not establish asymptotic convergence, particularly where phase-field iterations fail to reach tolerance. The length ℓ itself, however, matters. For $\ell = 2.5$ mm ($\ell/\ell_{ch} = 0.42$) the damage band is $\approx 3\ell$ wide: it is formed as the tip passes and does not widen afterwards. The crack leaves the notch at $\approx 69^\circ$ and bends to $\approx 66^\circ$. For $\ell = 1.25$ mm the band becomes a sharp crack, the fit reaches 69° with $R^2 \approx 0.99$, and the path agrees with the experimental 70° . The length ℓ also affects the energy, which is analysed below. The ℓ -insensitivity of the PF-CZM ([Wu and Nguyen 2018](#)) therefore holds here only for the path, and only at small ℓ/ℓ_{ch} ; $\ell \lesssim \ell_{ch}/4$ is a provisional choice suggested by this example, subject to solver-convergence verification.

These results can be compared with those reported for the same test. With the DM and a strain criterion, [Braun and Fernández-Sáez \(2014\)](#) obtained an initial crack angle of about 60° that increased to 70° at later times; the inclination depended on the lattice orientation, more strongly for rotations of 10° and 20° . With the bilinear softening law, [Braun and Fernández-Sáez \(2016\)](#) obtained an initial angle of about 65° increasing to 70° , on an unrotated lattice. XFEM and phase-field models predict angles between 65° and 70° ([Song et al. 2008](#); [Borden et al. 2012](#)), and the experimental angle is about 70° ([Kalthoff and Winkler 1987](#)). With $\ell = 2.5$ mm the phase-field DM gives 64 – 66° for every lattice orientation, at or slightly below the lower end of this range, and with $\ell = 1.25$ mm it is within 1° of the experimental value.

Figure 6 shows the velocity histories of the front tracked with $d \geq 0.5$, normalised by $c_R = 2799$ m/s. The curves include the initial stationary stage and the subsequent acceleration. The plotted window-averaged speeds never exceed the Rayleigh speed. Over $10 \leq L \leq 60$ mm, the mean speed is 0.68 – $0.74 c_R$ and the maximum is 0.78 – $0.86 c_R$ (Table 4). The speeds show limited sensitivity to lattice orientation and depend only slightly on a and ℓ . The reported initiation time, defined by the front exceeding $2a$, is approximately 20 – $23 \mu\text{s}$, compared with approximately $24 \mu\text{s}$ and $28 \mu\text{s}$ reported by [Braun and Fernández-Sáez \(2016\)](#) and [Braun and Fernández-Sáez \(2014\)](#), respectively. The propagation speeds are of the same order as those obtained with XFEM ([Belytschko et al. 2003](#)) and with the DM ([Braun and Fernández-Sáez 2014, 2016](#)).

The centred-window estimate is $v(t) = [L(t + \Delta T/2) - L(t - \Delta T/2)]/\Delta T$, with $\Delta T = 5 \mu\text{s}$, so each value is a window-averaged propagation speed. The initial position is extended constantly for $t < 0$ when evaluating the first windows. This averaging smooths the transition from zero velocity and does not resolve the exact instant of initiation. The histories end at $L = 60$ mm, before the front approaches the upper edge.

The reference curves permit a direct visual comparison of the velocity histories. The present sustained propagation speeds are higher than those of the earlier DM ([Braun and Fernández-Sáez 2014](#)), while remaining of the same order

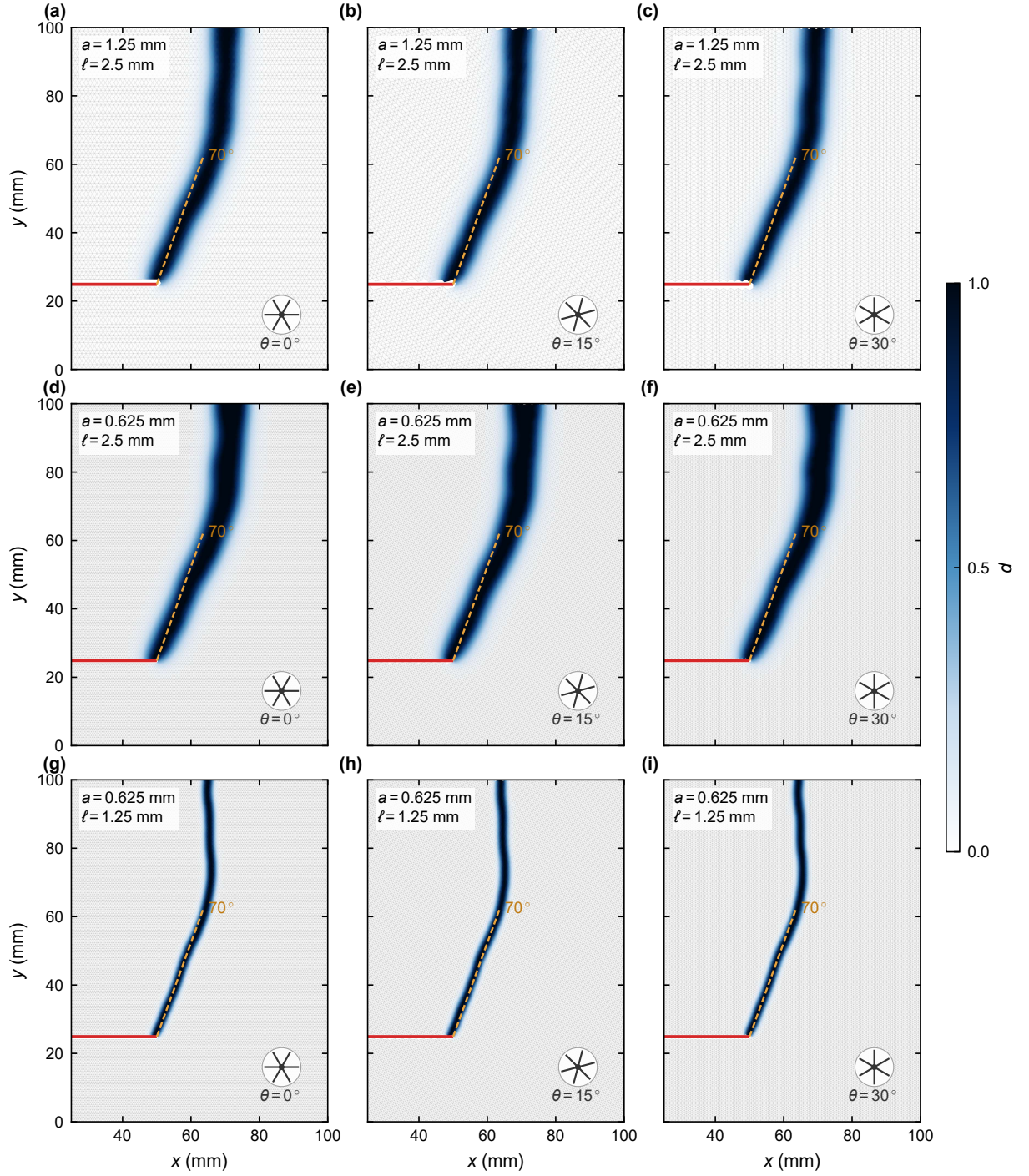


Fig. 5 Kalthoff-Winkler test at $t = 75 \mu\text{s}$: phase field d for (a)–(c) $a = 1.25 \text{ mm}$, $\ell = 2.5 \text{ mm}$; (d)–(f) $a = 0.625 \text{ mm}$, $\ell = 2.5 \text{ mm}$; and (g)–(i) $a = 0.625 \text{ mm}$, $\ell = 1.25 \text{ mm}$. The lattice orientations are $\theta = 0^\circ$, 15° and 30° from left to right in each row. Insets show the bond directions. Red line: notch; dashed line: experimental crack angle of 70° (Kalthoff and Winkler 1987). Light grey lines show the lattice bonds in the reference configuration

Table 3 Kalthoff–Winkler test at $t = 75 \mu\text{s}$ on lattices rotated by θ . Fit: line fitted within 25 mm of the notch tip (coefficient of determination R^2). Annuli: median polar angle of the crack nodes. L : distance from the notch tip to the farthest crack node. W_f : fracture energy, $G_f t \Gamma_d$

a [mm]	ℓ [mm]	θ [°]	Fit [°]	R^2	2.5–10 / 10–20 mm [°]	L [mm]	W_f [J]
1.25	2.5	0	65.6	0.89	68.8 / 67.9	78.3	25.0
1.25	2.5	15	65.4	0.90	74.3 / 69.1	78.1	24.8
1.25	2.5	30	64.0	0.92	69.5 / 67.7	78.0	24.7
0.625	2.5	0	64.7	0.91	69.1 / 66.6	79.0	27.1
0.625	2.5	15	64.5	0.91	73.1 / 67.5	78.5	26.9
0.625	2.5	30	64.5	0.92	68.0 / 66.1	79.2	26.9
0.625	1.25	0	68.9	0.98	72.5 / 69.7	76.6	19.4
0.625	1.25	15	69.4	0.98	74.1 / 71.4	76.5	19.2
0.625	1.25	30	69.3	0.99	70.7 / 69.8	76.5	19.2

Table 4 Kalthoff–Winkler test: crack-front kinematics (threshold 0.5) and energies at $75 \mu\text{s}$. t_{init} : time at which the front exceeds $2a$; $\bar{v}$: mean speed for $10 \leq L \leq 60$ mm; v_{max} : maximum over $5 \mu\text{s}$ windows; balance: $(W_{\text{ext}} - K - U - W_{\text{deg}})/W_{\text{ext}}$

a [mm]	ℓ [mm]	θ [°]	t_{init} [μs]	$\bar{v}/c_R$	v_{max}/c_R	W_{ext} [J]	$G_f t \Gamma_d$ [J]	W_{deg} [J]	balance [%]
1.25	2.5	0	21.4	0.70	0.80	237.1	25.0	24.2	-0.17
1.25	2.5	15	22.4	0.68	0.79	238.1	24.8	23.6	-0.16
1.25	2.5	30	22.9	0.68	0.78	237.3	24.7	23.5	-0.14
0.625	2.5	0	19.4	0.74	0.80	229.2	27.1	22.2	-0.09
0.625	2.5	15	19.4	0.73	0.81	229.4	26.9	21.9	-0.09
0.625	2.5	30	19.9	0.73	0.81	227.9	26.9	21.9	-0.08
0.625	1.25	0	20.4	0.72	0.85	250.9	19.4	26.3	-0.09
0.625	1.25	15	19.9	0.71	0.84	250.0	19.2	25.5	-0.09
0.625	1.25	30	19.9	0.71	0.86	249.2	19.2	25.7	-0.09

as the DEM and XFEM results (Kosteski et al. 2012; Belytschko et al. 2003). The reference histories exhibit larger short-time fluctuations. Their normalisation and temporal processing are retained from the source data; the comparison does not establish equivalence of front definitions or velocity-estimation windows across methods.

Figure 7 shows the kinetic energy K , the stored elastic energy U , the fracture energy W_f , the external work $W_{\text{ext}} = \int R V_0 dt$ (R : reaction on the impacted edge) and the balance error. The balance $W_{\text{ext}} = K + U + W_{\text{deg}}$ closes to within 0.2% at $75 \mu\text{s}$ (Table 4), which checks the mechanical energy accounting at that time. This does not verify phase-field convergence or equality of the two fracture energy measures; the transient balance error is larger.

The two measures of the fracture energy, however, differ. $W_{\text{deg}}/(G_f t \Gamma_d)$ is 0.95–0.97 for $a = 1.25$ mm, $\ell = 2.5$ mm; 0.82 for $a = 0.625$ mm, $\ell = 2.5$ mm; and 1.33–1.36 for $a = 0.625$ mm,

$\ell = 1.25$ mm. This is a consequence of the hybrid formulation: d is driven by the nodal Rankine energy, whereas the energy actually released is that of the degraded bonds, and their equality is not enforced (Section 3.4). The signed mismatch $(W_{\text{deg}} - W_f)/W_{\text{ext}}$ at $75 \mu\text{s}$ ranges from -2.2% to 2.8% across the retained cases, despite a mechanical balance error below 0.2%. For $\ell = 1.25$ mm the ratio increases after $60 \mu\text{s}$, when Γ_d no longer grows. This is probably the energy released by the bonds crossing the open crack, whose residual stiffness $\omega(\bar{d}_{ij})$, with $\bar{d}_{ij} < 1$, keeps decreasing while they undergo large openings. For reference, a straight crack of the final length releases $G_f t L \approx 15$ J, while $G_f t \Gamma_d = 19$ –27 J and $W_{\text{deg}} = 22$ –26 J. The excess is associated with the width of the damage band ($\Gamma_d/L = 1.25$ –1.72) and with the $O(a/\ell)$ bias of Table 1. Decreasing ℓ brings Γ_d closer to L but increases W_{deg} . These results motivate a common objective energy potential for

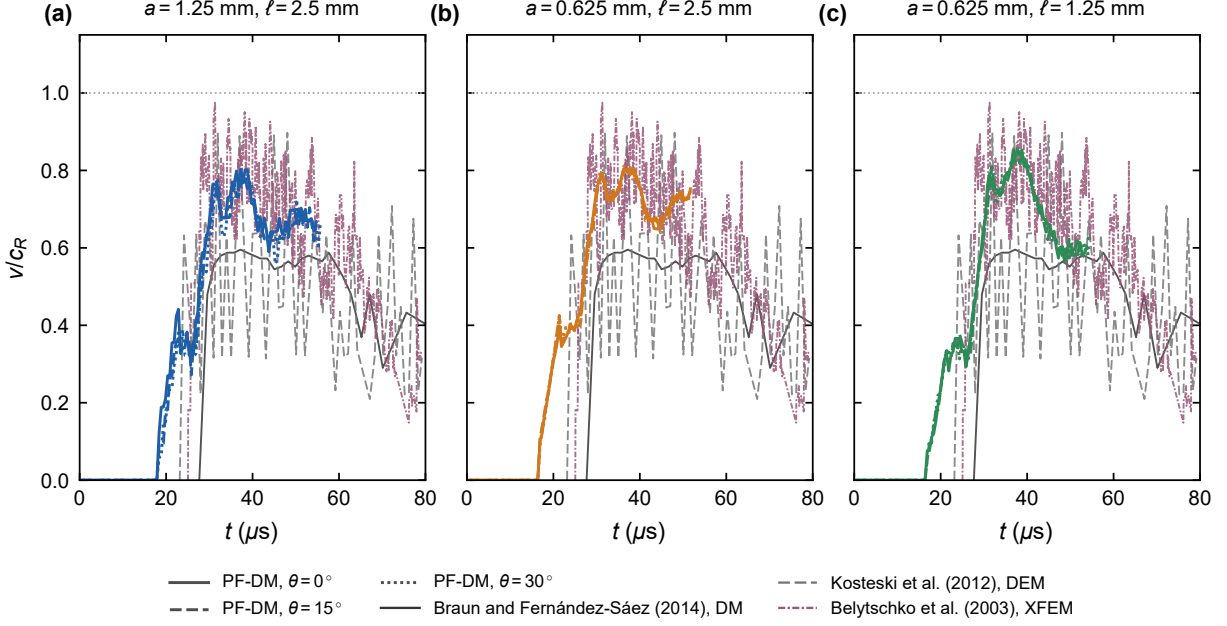


Fig. 6 Kalthoff–Winkler test: propagation speed normalised by the Rayleigh speed for (a) $a = 1.25$ mm, $\ell = 2.5$ mm; (b) $a = 0.625$ mm, $\ell = 2.5$ mm; and (c) $a = 0.625$ mm, $\ell = 1.25$ mm. The front is tracked with $d \geq 0.5$. Speeds are centred secant estimates over $5 \mu\text{s}$ windows, including the initial stationary stage and propagation until $L = 60$ mm. The dotted horizontal line denotes $v = c_R$; all plotted speeds remain below this line. Reference curves show the earlier DM (Braun and Fernández-Sáez 2014), DEM (Kosteski et al. 2012) and XFEM (Belytschko et al. 2003). Their published time coordinates and normalised velocities are retained without additional smoothing

the lattice and the damage evolution, followed by verification of its discrete energy balance.

For $a = 0.625$ mm and $\ell = 2.5$ mm the phase-field iteration reached its limit of 30 iterations in 29–37% of the steps; the results nevertheless coincide for the three orientations, but a tighter solver is advisable for production runs. In the other cases it converged at every step. The half-specimen symmetry condition is only approximate on rotated lattices, whose nodes do not lie on $y = 0$; no spurious damage was observed near that edge.

Overall, in this example the phase-field regularisation reduces the lattice-orientation dependence of the crack path. It does so using first neighbours only, with f_t and G_f as the only fracture parameters, and with no recalibration of Ω_{ij} or h_{ij} .

4.2 Single-edge-notched plate under tension

The quasi-static mode-I benchmark of Chen et al. (2014, Sect. 3.1) is used as a second test. The specimen is a 300×100 mm plate, analysed in plane stress with unit thickness, with a 20 mm notch

normal to the lower edge at mid-span (Fig. 4b). The left and right edges are pulled apart in the x direction with a velocity ramped to $V = 0.5$ m/s over $100 \mu\text{s}$, while v_y is left free. The material data of Chen et al. (2014) are $E = 69$ GPa, $\nu = 0.3$, $\rho = 2700$ kg/m³ and $f_t = 200$ MPa. They use a pure stress criterion and do not give a fracture energy, so $G_f = 4600$ N/m is adopted here, which gives $\ell_{ch} = 7.9$ mm. This value allows $\ell = 2$ mm $\approx \ell_{ch}/4$ with $\ell/a = 2$ on a lattice with $a = 1$ mm ($\approx 35\,000$ nodes). Because of this choice, the absolute loads are not comparable with those of Chen et al. (2014); the comparison concerns the crack path and the scatter between lattice orientations.

According to LEFM the crack runs straight from the notch tip to the upper edge. The angle $\pi/4$ used by Chen et al. (2014) is equivalent to $\theta = 15^\circ$ by the 60° periodicity of the lattice and the mirror symmetry of the specimen about the crack plane. The ratio of kinetic to stored elastic energy at peak load is $K/U = 1.5$ –5%. The loading is intended to be quasi-static, but a slower-loading study is required to quantify the remaining inertial

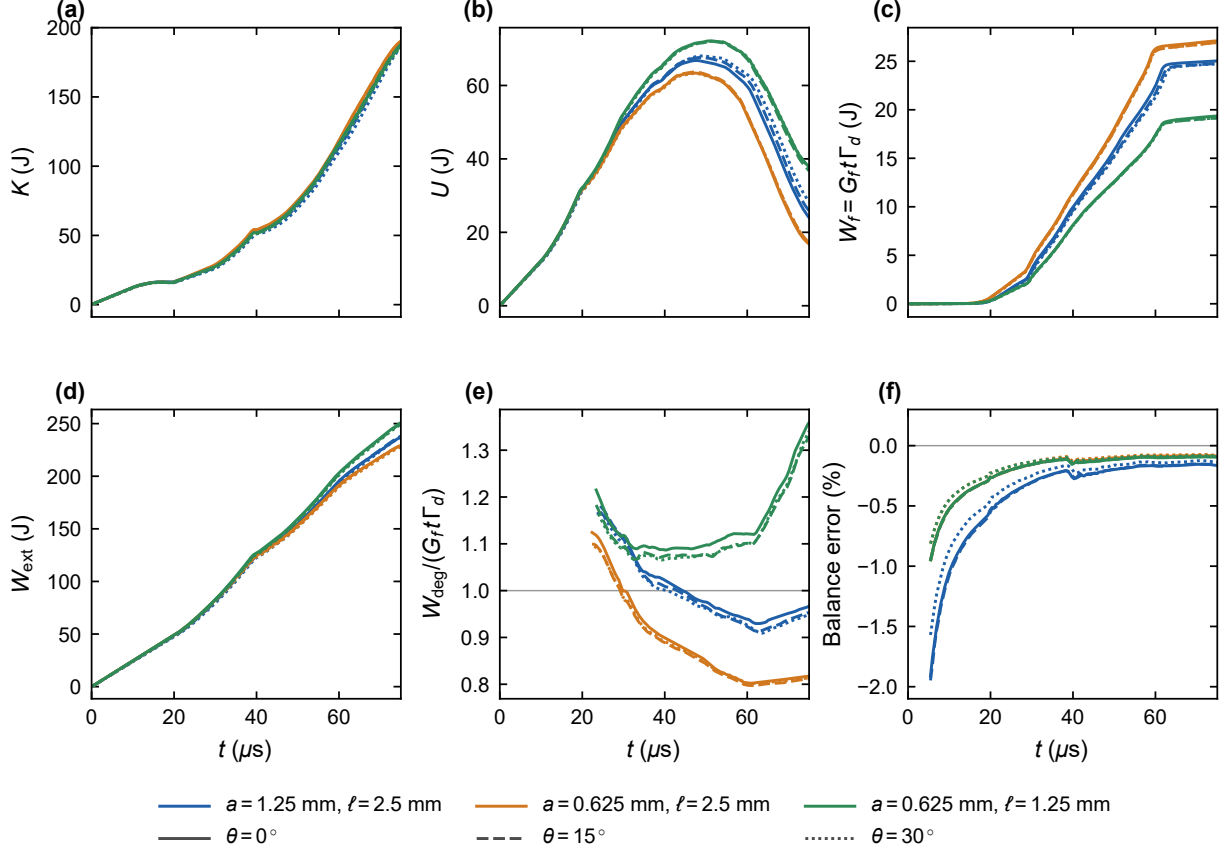


Fig. 7 Kalthoff–Winkler test, energy histories: (a) kinetic energy K ; (b) stored elastic energy U ; (c) fracture energy $W_f = G_f t \Gamma_d$; (d) external work W_{ext} ; (e) ratio $W_{\text{deg}}/(G_f t \Gamma_d)$; (f) balance error $(W_{\text{ext}} - K - U - W_{\text{deg}})/W_{\text{ext}}$

influence. The curves are cut at complete separation; after it the two halves rebound elastically and the reaction becomes compressive.

The crack path is described by the nodes with $d \geq 0.95$ connected to the notch tip. Three measures are reported: the angle with the vertical of the line $x(y)$ fitted to these nodes, and their maximum and root-mean-square (RMS) distance from the notch plane $x = 150$ mm.

Table 5 and Fig. 8 summarise the results. With $\ell = 2$ mm, the crack runs straight from the notch tip to the upper edge for every orientation (Fig. 8a–c). Its RMS distance from the notch plane is 0.53–0.58 mm, smaller than a , and its maximum distance is 1.0–1.6 mm. The largest maximum distance occurs for $\theta = 15^\circ$, the only orientation without a lattice line parallel or perpendicular to the crack. [Chen et al. \(2014\)](#) obtain comparable straightness only by including three or four neighbour shells: with first neighbours their

cracks deviate markedly for $\theta = \pi/12$, $\pi/6$ and $\pi/4$.

All the curves (Fig. 9) coincide up to $U \approx 0.18$ mm, so the elastic stiffness does not depend on the orientation. The oscillations are elastic waves of the explicit loading. With $\ell = 2$ mm, the three orientations give peak loads of 7.78–7.94 MN (a scatter of 2%) followed by a sudden brittle drop. For comparison, the scatter of the first-neighbour model of [Chen et al. \(2014\)](#) is of the order of 10–15%, and it is reduced to a few per cent only with three or four neighbour shells.

For $a = 2$ mm and $\ell = 4$ mm ($\ell/\ell_{ch} = 0.5$), the peak load is 7.48–7.55 MN. The crack remains approximately aligned with the notch plane, but its RMS deviation increases to 1.72–1.87 mm and its maximum deviation to 3.6–5.1 mm (Table 5). Both a and ℓ differ from the reference case, so this comparison does not isolate the influence of the

Table 5 SEN tension test on lattices rotated by θ . Path: maximum and RMS distance of the crack nodes from the notch plane (the fitted angle with the vertical is below 0.3° in every case). $F_{\max}$: peak reaction per unit thickness; $U_{\max}$: edge displacement at $F_{\max}$. Energies at complete separation; $G_f t L = 368$ J for the nominal 80 mm ligament; the tabulated L is measured on the discrete crack

a [mm]	ℓ [mm]	θ [$^\circ$]	max dev. [mm]	RMS dev. [mm]	$F_{\max}$ [MN]	$U_{\max}$ [mm]	$G_f t \Gamma_d$ [J]	$W_{\deg}$ [J]	balance [%]
1	2	0	1.5	0.53	7.94	0.209	485	647	-0.18
1	2	15	1.6	0.54	7.85	0.209	473	582	-0.16
1	2	30	1.0	0.58	7.78	0.204	477	587	-0.16
2	4	0	4.0	1.73	7.55	0.194	702	533	-0.18
2	4	15	5.1	1.87	7.48	0.194	695	528	-0.19
2	4	30	3.6	1.72	7.48	0.194	706	537	-0.20
0.5	2	0	2.0	0.62	7.32	0.190	484	458	-0.10
0.5	2	15	2.0	0.64	7.34	0.190	485	456	-0.10
0.5	2	30	2.3	0.69	7.37	0.190	494	456	-0.10

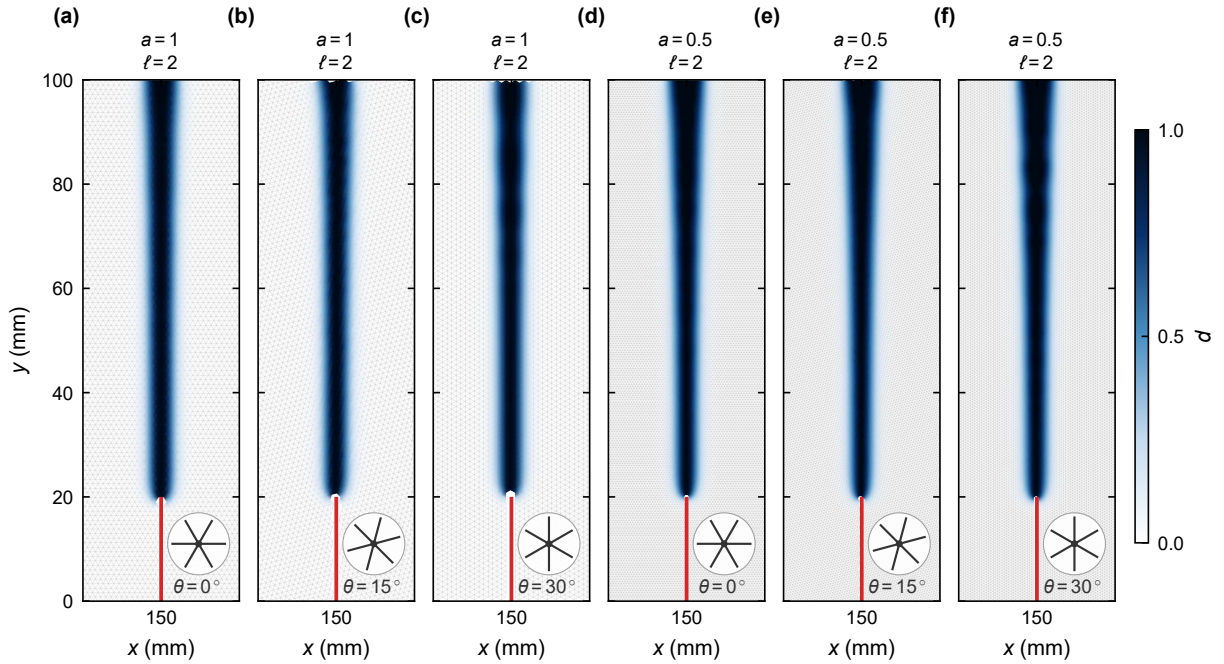


Fig. 8 SEN tension test, final state (central part of the specimen): phase field d for (a)–(c) $a = 1$ mm, $\ell = 2$ mm and $\theta = 0^\circ, 15^\circ$ and 30° ; (d)–(f) $a = 0.5$ mm, $\ell = 2$ mm, with the same orientations. Red line: notch. Light grey lines show the lattice bonds in the reference configuration

regularisation length. No refinement study at fixed $\ell = 4$ mm is reported.

At the recommended $\ell = 2$ mm, refining the lattice from $a = 1$ to 0.5 mm ($\ell/a = 2 \rightarrow 4$, Fig. 8d–f) leaves the path unchanged: straight, with RMS deviation 0.62 – 0.69 mm for the three orientations. The damage band becomes narrower and more sharply centred, and the solver converges at every step. The peak load decreases by

5 – 8% , to 7.32 – 7.37 MN, with a scatter between orientations of only 0.7% . The path remains approximately straight at both tested spacings, whereas the peak load is not yet fully converged at $\ell/a = 2$. In addition, the energy released by the degraded bonds becomes close to $G_f t \Gamma_d$ ($W_{\deg}/(G_f t \Gamma_d) = 0.92$ – 0.95 , against 1.2 – 1.3 for

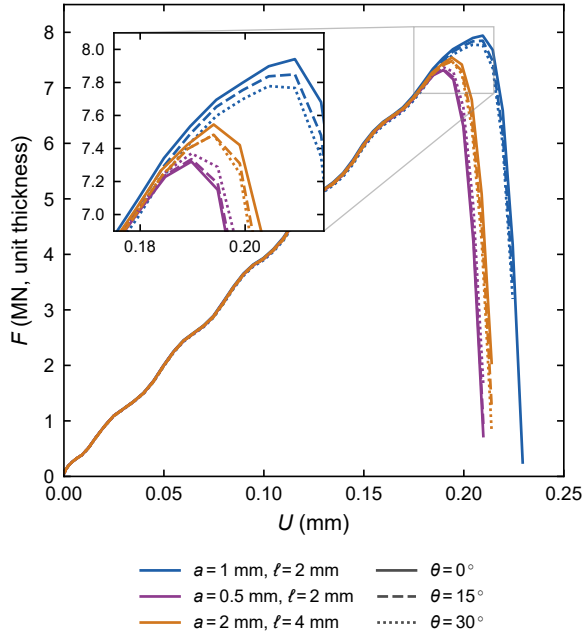


Fig. 9 SEN tension test: reaction F per unit thickness versus the displacement U of each loaded edge, up to complete separation. The inset shows the peak region

$a = 1$ mm). This suggests that $\ell/a \geq 4$ is preferable when the load level and the dissipated energy, and not only the path, are of interest.

The balance $W_{\text{ext}} = K + U + W_{\text{deg}}$ closes to within 0.3% in every case (Fig. 10). At the onset of propagation the stored energy is converted into kinetic energy (≈ 1000 J), as expected for unstable propagation under displacement control. As in the Kalthoff test, the energy released by the degraded bonds differs from $G_f t \Gamma_d$. The ratio $W_{\text{deg}}/(G_f t \Gamma_d)$ grows during propagation from 0.9 to 1.6 for $a = 1$ mm, $\ell = 2$ mm, and from 0.55 to 0.8 for $a = 2$ mm and $\ell = 4$ mm. Both measures exceed $G_f t L \approx 368$ J: by a factor 1.3 (Γ_d) and 1.6–1.8 (W_{deg}) for $a = 1$ mm, $\ell = 2$ mm.

The energy values in Table 5 are evaluated at the last saved state before separation, whereas Fig. 10 includes the subsequent rebound. At the tabulated state, $(W_{\text{deg}} - W_f)/W_{\text{ext}} = 5.6\text{--}8.2\%$ for $a = 1$ mm, $\ell = 2$ mm, and -9.9 to -9.7% for $a = 2$ mm, $\ell = 4$ mm. Refining to $a = 0.5$ mm at fixed $\ell = 2$ mm reduces its magnitude to 1.5–2.3%, with $W_{\text{deg}} < W_f$. Thus refinement improves the agreement of the two energy measures in this SEN comparison, although it does not establish their equality or an asymptotic convergence rate.

The velocity histories (Fig. 11) include the initial stationary stage. The crack does not advance until the peak load is reached, at $\approx 420\text{--}440$ μs , and then crosses the 80 mm ligament in ≈ 45 μs , as expected for an unstable propagation. For $a = 1$ mm and $\ell = 2$ mm, where the front is recorded every 0.5 μs , the mean speed of the front $d \geq 0.5$ is $0.54\text{--}0.60 c_R$ ($c_R = 2902$ m/s) and its maximum over 5 μs windows is $0.90\text{--}0.94 c_R$, with limited sensitivity to orientation. All speeds shown in Fig. 11 remain below the Rayleigh speed. The coarser temporal sampling in some cases requires wider windows, so their short-duration peaks are less well resolved; the plotted maxima should therefore not be compared as instantaneous speeds.

In summary, the SEN test confirms the Kalthoff results in a quasi-static mode-I setting. With $\ell \lesssim \ell_{ch}/4$, the phase-field DM with first neighbours only gives a straight crack and a peak load that are independent of the lattice orientation to within ≈ 1.5 mm and 2%. The path remains approximately straight at both spacings, and refining to $\ell/a = 4$ reduces both the peak load and the energetic inconsistency. Chen et al. (2014) need three or four neighbour shells to achieve the same independence. The energetic inconsistency of the hybrid driving force remains the main limitation.

4.3 Thick cylinder under internal impulsive pressure: fragmentation

In this example we consider the fragmentation of a thick cylinder due to an impulsive internal pressure. This problem was analysed with the DM by Braun and Fernández-Sáez (2016, Sect. 3.3), and similar problems have been simulated by other authors using different numerical techniques (Rabczuk and Belytschko 2004; Zhou and Molinari 2004; Song and Belytschko 2009). Unlike the previous examples, many cracks nucleate and propagate simultaneously, and none of them is initiated by a notch. The geometry and the loading are those of Song and Belytschko (2009) (Fig. 4c): the inner and outer radii are 80 mm and 150 mm, and the internal pressure decays exponentially with time,

$$p(t) = p_0 e^{-t/t_0}, \quad (20)$$

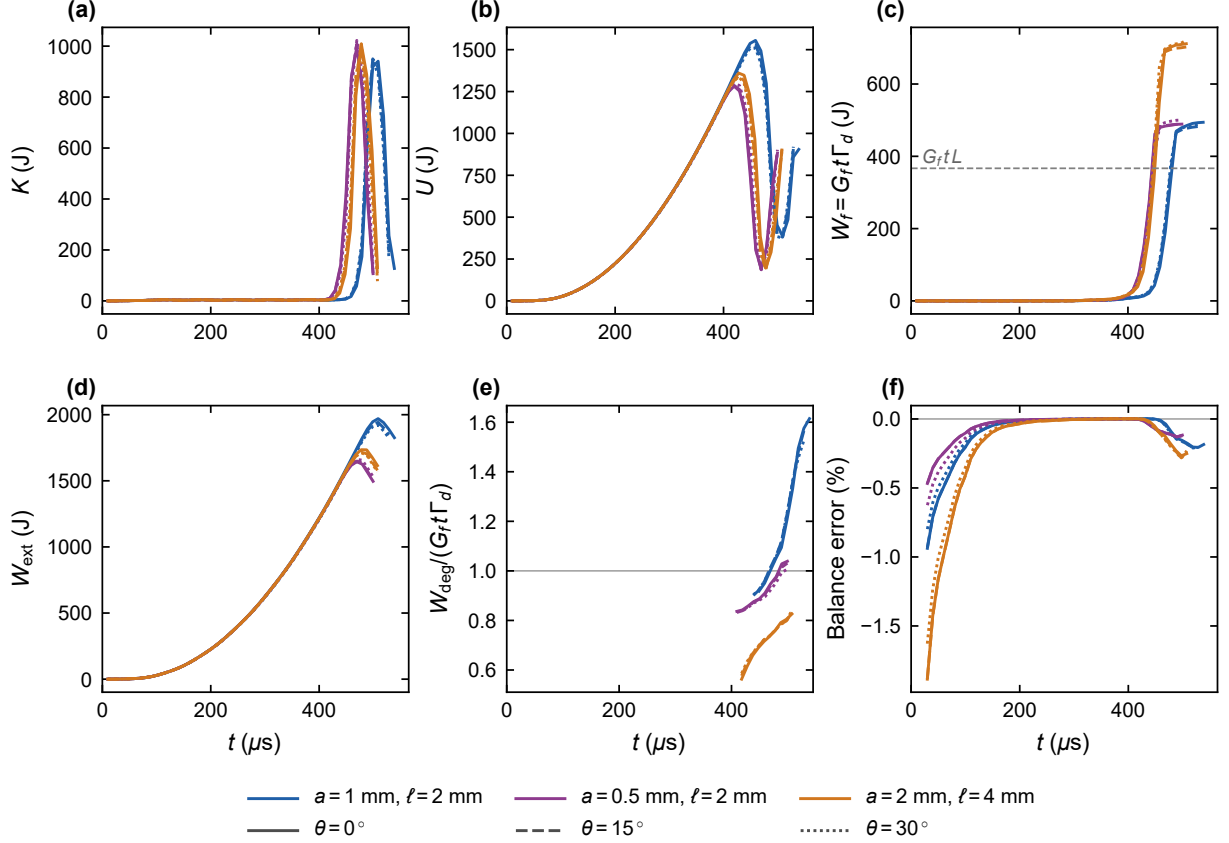


Fig. 10 SEN tension test, energy histories: (a) kinetic energy K ; (b) stored elastic energy U ; (c) fracture energy $W_f = G_f t \Gamma_d$ (dashed line: $G_f t L$ for a straight crack across the ligament); (d) external work W_{ext} ; (e) ratio $W_{\text{deg}}/(G_f t \Gamma_d)$; (f) balance error

with $p_0 = 400$ MPa and $t_0 = 0.1$ ms. The pressure is applied as radial forces, distributed equally among the nodes of the inner band. The material properties are $E = 210$ GPa, $\nu = 0.3$, $\rho = 7850$ kg/m³, critical strain $\gamma_0 = 0.005$ and $G_f = 20$ kN/m (Song and Belytschko 2009). The tensile strength is therefore $f_t = E\gamma_0 = 1.05$ GPa and $\ell_{ch} = 3.8$ mm. The problem is analysed in plane strain, with unit thickness, up to $t = 100$ μ s.

Two lattices are analysed: $a = 1$ mm with $\ell = 2$ mm (5.9×10^4 nodes, $a_1 = 2.43$) and $a = 0.5$ mm with $\ell = 1$ mm (2.3×10^5 nodes, $a_1 = 4.85$). In both cases $\ell/a = 2$, so that the lattice spacing and the regularisation length change together, and the two solutions correspond to $\ell/\ell_{ch} = 0.52$ and 0.26 , respectively. Note that $\ell = 2$ mm is close to the stability limit $a_1 > 2$ of Proposition 1. The phase-field iteration is limited to five iterations per step; this limit is reached in about half of the steps (49–51% for $a = 0.5$ mm

and 57% for $a = 1$ mm), and the iteration is then continued in the next step. The results of this example are therefore less tightly converged than those of the previous ones. Since the problem is axisymmetric, the lattice is the only source of perturbation. When the lattice is centred on a node of the axis, the discrete problem has the symmetries of the lattice, and the computed patterns preserve at least its mirror symmetries about the x and y axes. To assess the influence of this choice, the finer lattice is also computed with its nodes shifted by $(0.31a, 0.47a)$ with respect to the axis, which breaks these symmetries without changing the spacing.

The final crack patterns are shown in Fig. 12 and the main results in Table 6. In all cases a set of cracks nucleates on the inner surface and propagates towards the outer surface, which is reached by 12 of them in the centred lattices. The remaining cracks arrest at a distance of 20–50 mm from

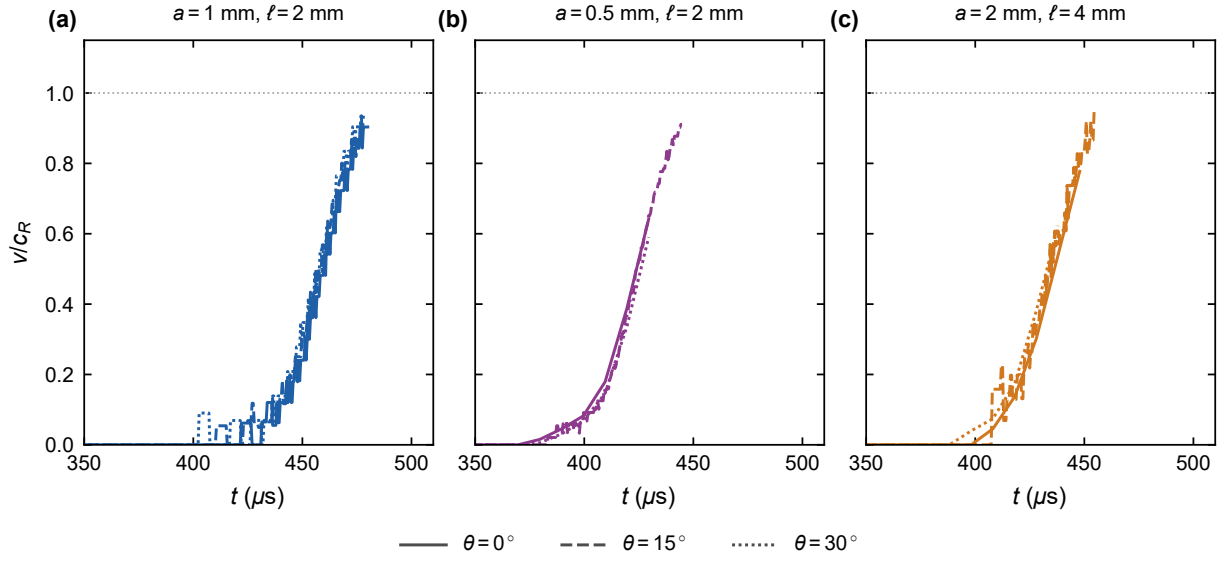


Fig. 11 SEN tension test: velocity of the front $d \geq 0.5$, normalised by c_R , for (a) $a = 1$ mm, $\ell = 2$ mm; (b) $a = 0.5$ mm, $\ell = 2$ mm; and (c) $a = 2$ mm, $\ell = 4$ mm. The initial stationary stage is included. Centred secants use approximately $5 \mu\text{s}$ windows where the front is recorded every $0.5 \mu\text{s}$, and approximately $20 \mu\text{s}$ windows where snapshots are spaced by $10 \mu\text{s}$. Both ends of each window precede $L = 70$ mm. The dotted line denotes $v = c_R$

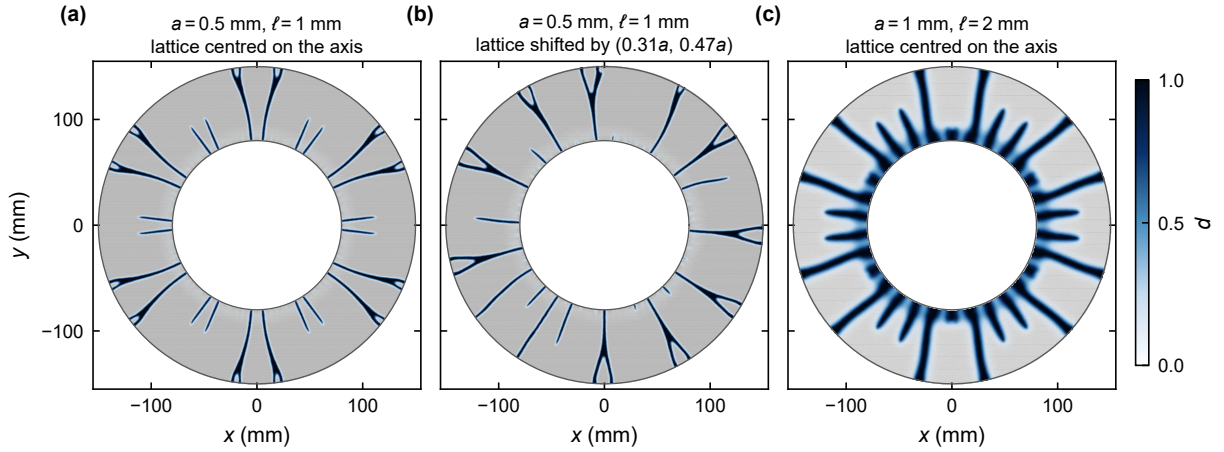


Fig. 12 Thick cylinder under internal impulsive pressure, $t = 100 \mu\text{s}$: phase field d for (a) $a = 0.5$ mm, $\ell = 1$ mm, lattice centred on the axis; (b) $a = 0.5$ mm, $\ell = 1$ mm, lattice shifted by $(0.31a, 0.47a)$; (c) $a = 1$ mm, $\ell = 2$ mm, lattice centred on the axis. Grey lines: inner and outer surfaces. Light grey lines show the lattice bonds in the reference configuration

the inner surface, as the pressure decays and the stress is released by the neighbouring cracks. The direction of propagation is predominantly radial, with a mean inclination with respect to the radial direction, measured between $r = 95$ mm and $r = 135$ mm, of 4 – 9° . With $a = 0.5$ mm most of the through cracks branch near the outer surface, giving wedge-shaped ends to the fragments. On the centred lattices the patterns are symmetric with respect to the x and y axes, a symmetry

inherited from the lattice. On the shifted lattice this symmetry disappears (Fig. 12b): the cracks are distributed irregularly around the cylinder, their spacing and mean obliquity change, and the number of fragments for the threshold $\bar{d}_{ij} \geq 0.9$ increases from 12 to 19, whereas the energies change by less than 3%. The fragment pattern of the centred lattice is therefore partly imposed by the symmetry of the lattice, while the global energetic response is not. With $\ell = 2$ mm the damage

Table 6 Thick cylinder at $t = 100 \mu\text{s}$. Fragments: connected components of more than 50 nodes after removing the bonds with $\bar{d}_{ij}$ above the threshold. Obliquity: mean angle between the cracks and the radial direction for $95 \leq r \leq 135 \text{ mm}$

	$a = 0.5 \text{ mm}$ centred	$a = 0.5 \text{ mm}$ shifted	$a = 1 \text{ mm}$ centred
Fragm., $\bar{d}_{ij} \geq 0.80$	24	22	12
Fragm., $\bar{d}_{ij} \geq 0.90$	12	19	12
Fragm., $\bar{d}_{ij} \geq 0.95$	1	12	12
Obliquity $[\circ]$	9.2	3.9	3.6
W_{ext} [kJ]	64.3	65.9	172.1
$G_f t \Gamma_d$ [kJ]	40.7	41.0	63.1
$W_{\text{deg}}/(G_f t \Gamma_d)$	0.93	0.94	1.30
Balance [%]	-0.41	-0.42	-0.15

bands are wider, and a ring of damage develops along the inner surface between the cracks.

The fragments are identified as the connected components of the lattice after removing the bonds with $\bar{d}_{ij} \geq 0.9$; components with fewer than 50 nodes are disregarded. The count is sensitive to the threshold (Table 6): on the centred fine lattice, with $\bar{d}_{ij} \geq 0.8$ the arrested cracks that join a through crack also separate fragments (24 fragments), while with $\bar{d}_{ij} \geq 0.95$ a single fragment is obtained, because the phase field along parts of the through cracks remains between 0.9 and 0.95. For $a = 1 \text{ mm}$ the count is 12 for all these thresholds. The values obtained, 12–19 fragments, are at the lower end of those reported by other authors. Rabczuk and Belytschko (2004) obtained around 20 fragments with a meshfree method, and Braun and Fernández-Sáez (2016) obtained between 15 and 21 fragments with the softening DM and between 21 and 42 with the DM of Braun and Fernández-Sáez (2014), increasing in both cases with the refinement of the lattice. The difference is related to the regularisation: the phase-field band of width $\sim 2\ell$ shields a region around each crack, and cracks closer than this distance merge or arrest. The two centred lattices, which change a and ℓ together, give the same count at the reference threshold, so the present results do not show the increase with refinement reported for other techniques (Zhou and Molinari 2004; Song and Belytschko 2009); a study at fixed ℓ would be required to assess it.

Regarding the crack direction, Song and Belytschko (2009) obtained radial cracks with XFEM, while Rabczuk and Belytschko (2004)

obtained oblique cracks that result in wedge-shaped fragments, and the softening DM gave an intermediate situation, with radial cracks and some oblique trajectories (Braun and Fernández-Sáez 2016). The phase-field DM also predicts an intermediate situation: the cracks are predominantly radial, but they become oblique and branch near the outer surface for the finer ℓ .

The energy histories are shown in Fig. 13. Before fracture, both lattices give the same response. The stored energy grows with the oscillations of the cylinder and reaches its maximum at $\approx 50 \mu\text{s}$, when the fracture starts. The fracture starts late because, with the stress criterion, the damage requires the maximum principal stress to reach f_t (Proposition 1). The static hoop stress on the inner surface for $p = p_0$ is $0.72 \text{ GPa} = 0.68 f_t$, so the fracture is produced by the dynamic amplification of the stress, which is maximum at the first peak of U . The fracture energy then grows rapidly until $\approx 75 \mu\text{s}$ and reaches 40.7 kJ for $a = 0.5 \text{ mm}$ and 63.1 kJ for $a = 1 \text{ mm}$. The difference is due to the ring of damage along the inner surface and to the wider bands for $\ell = 2 \text{ mm}$, which also reduce the stiffness of the inner part of the cylinder; the pressure then produces a larger work (172 kJ against 64 kJ at $t = 100 \mu\text{s}$), most of which becomes kinetic energy of the fragments. The energy balance $W_{\text{ext}} = K + U + W_{\text{deg}}$ closes to within 0.5%, and the ratio $W_{\text{deg}}/(G_f t \Gamma_d)$ is 0.93–0.94 for $a = 0.5 \text{ mm}$ and 1.30 for $a = 1 \text{ mm}$ at the end of the computation. The signed fracture-energy mismatch normalised by the external work is -4.4 to -3.7% for the two fine lattices and 11.0% for the coarse lattice. These differences are appreciably larger than the mechanical accounting error; the latter therefore cannot be used as evidence that the prescribed fracture energy is reproduced by bond degradation.

In summary, the phase-field DM reproduces the fragmentation of the cylinder with a crack pattern that is intermediate between the radial cracks of XFEM and the oblique cracks of meshfree methods, as the softening DM did. The number of fragments, 12–19, is at the lower end of the published solutions; it depends on the threshold used to define a broken bond and, on a lattice centred on the axis, on the symmetry of the lattice, which should therefore be broken in problems without a notch. These two coupled (a, ℓ) cases suggest

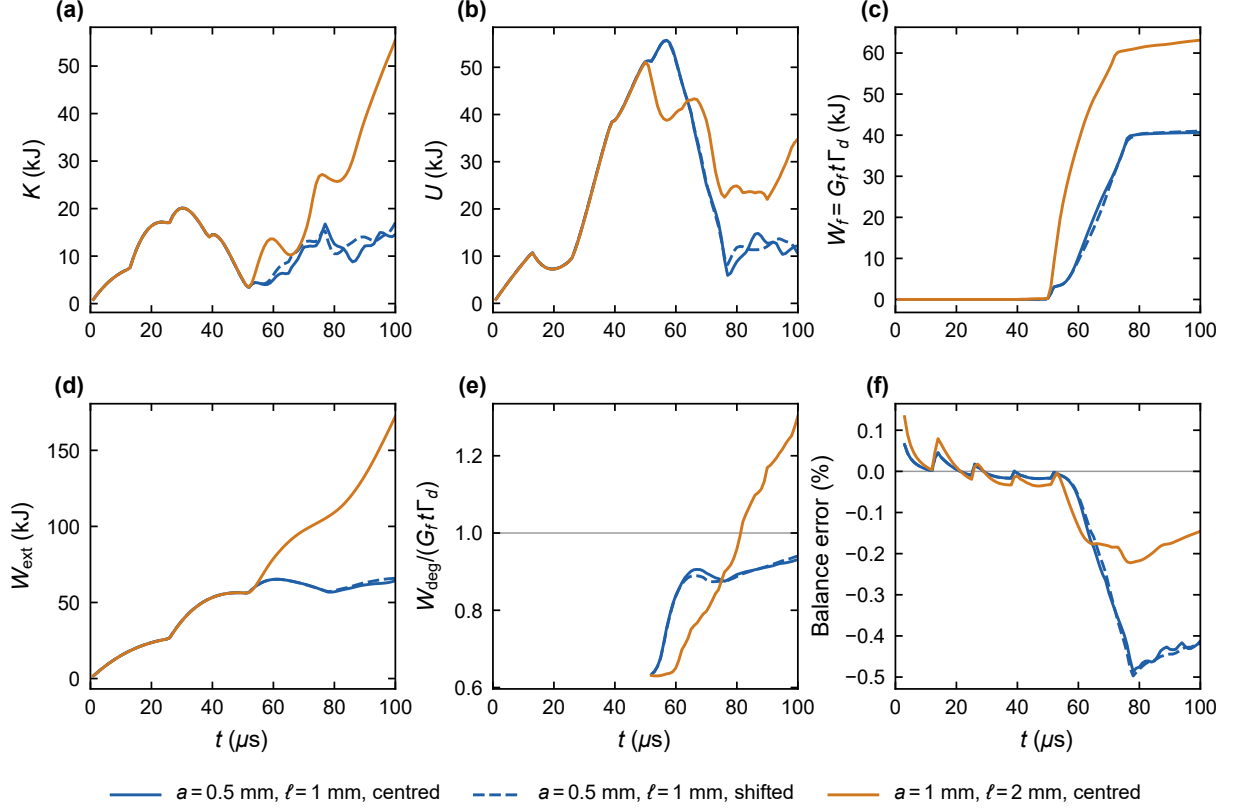


Fig. 13 Thick cylinder under internal impulsive pressure, energy histories: (a) kinetic energy K ; (b) stored elastic energy U ; (c) fracture energy $W_f = G_f t \Gamma_d$; (d) external work W_{ext} ; (e) ratio $W_{\text{deg}}/(G_f t \Gamma_d)$; (f) balance error

improved localisation at smaller ℓ/ℓ_{ch} , but do not isolate the influence of either parameter or establish a universal recommended ratio.

5 Conclusions

In the current study, we propose a phase-field regularisation of the 2D discrete model (DM) developed by [Braun and Fernández-Sáez \(2014\)](#), with the aim of reducing the dependence of the crack path on the orientation of the lattice. The elastic model is the first-neighbour DM, interpreted as a Born-type lattice with normal and transverse bond interactions. The crack is represented by a nodal phase field calibrated with the PF-CZM of [Wu \(2017\)](#), so that the model requires the same fracture parameters as the bilinear softening DM of [Braun and Fernández-Sáez \(2016\)](#): the tensile strength and the fracture energy.

The gradient energy of the phase field has been written as a sum over the lattice bonds that is

exact for linear fields, with natural boundary conditions on the free edges and on the crack faces. With this discretisation, the orientation dependence of the energy dissipated by a straight crack is reduced from 14% with bond breaking to 6% and 3% for $\ell = 2a$ and $\ell = 4a$, respectively, and to about 1% and 0.6% except for the orientations parallel to the lattice rows or bisectors. We have derived the tensile threshold and the condition $a_1 > p + a_2 + \frac{1}{2}$ for strict homogeneous local convexity at the history floor. This condition does not establish global stability. The crack driving force is computed from an objective nodal strain, since the transverse bond interaction of the DM is not invariant under rigid rotations.

In order to evaluate the proposed model, numerical examples have been analysed on lattices rotated by 0° , 15° and 30° , and the results have been compared with experimental results and with numerical results reported by other authors. The reported final mechanical energy balance errors are below 0.5%; transient errors are larger in

some examples. This accounting does not establish thermodynamic consistency.

In the Kalthoff–Winkler test, the phase-field DM predicts a single crack that crosses the specimen with a fitted inclination varying between the tested lattice orientations by 1.6° for $a = 1.25$ mm, and by 0.16° and 0.47° for $a = 0.625$ mm with $\ell = 2.5$ and 1.25 mm, respectively. With $\ell \lesssim \ell_{ch}/4$, the crack angle is 69° , in agreement with the experimental results of Kalthoff and Winkler (1987), and the crack speed is $0.71\text{--}0.72 c_R$, independent of the orientation. The dependence on the lattice orientation reported with the DM by Braun and Fernández-Sáez (2014) is substantially reduced for this problem.

In the single-edge-notched tension test, the phase-field DM predicts a straight crack for all the orientations, and the peak load varies by 2% and 0.7% between orientations for $a = 1$ mm and $a = 0.5$ mm, respectively. The mean crack speed is $0.54\text{--}0.60 c_R$. The crack path remains approximately straight at the tested lattice spacings, and the peak load and the energetic consistency improve with $\ell/a = 4$. These results are comparable to those obtained by Chen et al. (2014) with three or four shells of neighbours, while the present model only uses first neighbours.

In the fragmentation of a thick cylinder, the phase-field DM predicts predominantly radial cracks that become oblique and branch near the outer surface, an intermediate situation between the XFEM and meshfree solutions reported in the literature, as obtained with the softening DM. At the reference threshold of 0.90, the cylinder is identified as 12 to 19 connected components, at the lower end of the published solutions, since the regularised bands shield the neighbouring cracks. The number of fragments depends on the threshold used to define a broken bond and on the symmetry of a lattice centred on the axis. Between the centred and shifted fine lattices, the energies differ by less than 3%; changing a and ℓ together changes the external work much more substantially.

Compared with the PFLM of Yue et al. (2023), the proposed model permits positive bond coefficients over the admissible Poisson-ratio range, does not require an orientation-dependent calibration of the fracture energy, accounts for the multiaxial state of stress in the crack driving force, and is formulated for dynamic problems.

Its main limitation is the hybrid character of the crack driving force: the energy released by the degraded bonds differs from the energy of the regularised crack. Their agreement improves with refinement in the SEN test, but is not enforced by the formulation. The conclusions concern the prediction of crack paths and propagation speeds, their sensitivity to orientation within the Born-type lattice, and fragmentation. Solver-tolerance and time-step studies remain necessary to assess numerical robustness. Further work is required to develop a fully variational driving force based on an objective lattice energy.

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- **Funding:** No funding was received for this study.
- **Competing interests:** The author declares no competing interests.
- **Ethics approval:** Not applicable.
- **Author contributions:** The sole author was responsible for the conceptualisation, methodology, software, investigation, analysis, visualisation, and writing and revision of the manuscript.
- **Data availability:** The data supporting the findings of this study are presented in the article.
- **Code availability:** The simulation and post-processing codes will be provided as supplementary material.
- **Declaration of generative AI and AI-assisted technologies:** During the preparation of this work the author used Claude (Anthropic) in order to improve language and readability. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

References

- Agwai A, Guven I, Madenci E (2011) Predicting crack propagation with peridynamics: a comparative study. *International Journal of Fracture* 171(1):65–78. <https://doi.org/10.1007/s10704-011-9628-4>
- Ambati M, Gerasimov T, De Lorenzis L (2015) A review on phase-field models of brittle fracture

- and a new fast hybrid formulation. *Computational Mechanics* 55:383–405. <https://doi.org/10.1007/s00466-014-1109-y>
- Belytschko T, Black T (1999) Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering* 45:601–620. [https://doi.org/10.1002/\(SICI\)1097-0207\(19990620\)45:5<601::AID-NME598>3.0.CO;2-S](https://doi.org/10.1002/(SICI)1097-0207(19990620)45:5<601::AID-NME598>3.0.CO;2-S)
- Belytschko T, Chen H, Xu J, et al (2003) Dynamic crack propagation based on loss of hyperbolicity and a new discontinuous enrichment. *International Journal for Numerical Methods in Engineering* 58:1873–1905. <https://doi.org/10.1002/nme.941>
- Bleyer J, Roux-Langlois C, Molinari JF (2017) Dynamic crack propagation with a variational phase-field model: limiting speed, crack branching and velocity-toughening mechanisms. *International Journal of Fracture* 204:79–100. <https://doi.org/10.1007/s10704-016-0163-1>
- Bolander JE, Sukumar N (2005) Irregular lattice model for quasistatic crack propagation. *Physical Review B* 71:094106. <https://doi.org/10.1103/PhysRevB.71.094106>
- Borden MJ, Verhoosel CV, Scott MA, et al (2012) A phase-field description of dynamic brittle fracture. *Computer Methods in Applied Mechanics and Engineering* 217–220:77–95. <https://doi.org/10.1016/j.cma.2012.01.008>
- Born M, Huang K (1954) *Dynamical Theory of Crystal Lattices*. Oxford University Press, London
- Bourdin B, Francfort G, Marigo JJ (2000) Numerical experiments in revisited brittle fracture. *Journal of the Mechanics and Physics of Solids* 48:797–826. [https://doi.org/10.1016/S0022-5096\(99\)00028-9](https://doi.org/10.1016/S0022-5096(99)00028-9)
- Braun M, Aranda-Ruiz J (2021) Mixed mode crack propagation in polymers using a discrete lattice method. *Polymers* 13:1290. <https://doi.org/10.3390/polym13081290>
- Braun M, Ariza MP (2019) New lattice models for dynamic fracture problems of anisotropic materials. *Composites Part B: Engineering* 172:760–768. <https://doi.org/10.1016/j.compositesb.2019.05.082>
- Braun M, Ariza MP (2020) A progressive damage based lattice model for dynamic fracture of composite materials. *Composites Science and Technology* 200:108335. <https://doi.org/10.1016/j.compscitech.2020.108335>
- Braun M, Fernández-Sáez J (2014) A new 2D discrete model applied to dynamic crack propagation in brittle materials. *International Journal of Solids and Structures* 51:3787–3797. <https://doi.org/10.1016/j.ijsolstr.2014.07.014>
- Braun M, Fernández-Sáez J (2016) A 2D discrete model with a bilinear softening constitutive law applied to dynamic crack propagation problems. *International Journal of Fracture* 197:81–97. <https://doi.org/10.1007/s10704-015-0067-5>
- Braun M, González-Albuixech VF (2019) Analysis of the stress intensity factor dependence with the crack velocity using a lattice model. *Fatigue & Fracture of Engineering Materials & Structures* 42:1075–1084. <https://doi.org/10.1111/ffe.12971>
- Braun M, Iváñez I, Ariza MP (2021) A numerical study of progressive damage in unidirectional composite materials using a 2D lattice model. *Engineering Fracture Mechanics* 249:107767. <https://doi.org/10.1016/j.engfracmech.2021.107767>
- Braun M, Iváñez I, Ariza MP (2024) A discrete lattice model with axial and angular springs for modeling fracture in fiber-reinforced composite laminates. *European Journal of Mechanics - A/Solids* 104:105213. <https://doi.org/10.1016/j.euromechsol.2023.105213>
- Chen H, Liu Y (2016) A non-local 3D lattice particle framework for elastic solids. *International Journal of Solids and Structures* 81:411–420. <https://doi.org/10.1016/j.ijsolstr.2015.12.026>
- Chen H, Lin E, Jiao Y, et al (2014) A generalized 2D non-local lattice spring

- model for fracture simulation. *Computational Mechanics* 54:1541–1558. <https://doi.org/10.1007/s00466-014-1075-4>
- Cundall PA, Strack ODL (1979) A discrete numerical model for granular assemblies. *Géotechnique* 29:47–65. <https://doi.org/10.1680/geot.1979.29.1.47>
- Francfort G, Marigo JJ (1998) Revisiting brittle fracture as an energy minimization problem. *Journal of the Mechanics and Physics of Solids* 46:1319–1342. [https://doi.org/10.1016/S0022-5096\(98\)00034-9](https://doi.org/10.1016/S0022-5096(98)00034-9)
- Geelen RJM, Liu Y, Hu T, et al (2019) A phase-field formulation for dynamic cohesive fracture. *Computer Methods in Applied Mechanics and Engineering* 348:680–711. <https://doi.org/10.1016/j.cma.2019.01.026>
- Hofacker M, Miehe C (2012) Continuum phase field modeling of dynamic fracture: variational principles and staggered FE implementation. *International Journal of Fracture* 178:113–129. <https://doi.org/10.1007/s10704-012-9753-8>
- Hrennikoff A (1941) Solution of problems of elasticity by the framework method. *Journal of Applied Mechanics* 8:A169–A175. <https://doi.org/10.1115/1.4009129>
- Kalthoff JF, Winkler S (1987) Failure mode transition at high rates of shear loading. In: Chiem CY, Kunze HD, Meyer LW (eds) *Impact Loading and Dynamic Behavior of Materials*, vol 1. DGM Informationsgesellschaft, Oberursel, p 185–195
- Kosteski L, Barrios D’Ambra R, Iturrioz I (2012) Crack propagation in elastic solids using the truss-like discrete element method. *International Journal of Fracture* 174:139–161. <https://doi.org/10.1007/s10704-012-9684-4>
- Le QV, Bobaru F (2018) Surface corrections for peridynamic models in elasticity and fracture. *Computational Mechanics* 61:499–518. <https://doi.org/10.1007/s00466-017-1469-1>
- Mandal TK, Nguyen VP, Wu JY (2020) Evaluation of variational phase-field models for dynamic brittle fracture. *Engineering Fracture Mechanics* 235:107169. <https://doi.org/10.1016/j.engfracmech.2020.107169>
- Martín T, Español P, Rubio MA, et al (2000) Dynamic fracture in a discrete model of a brittle elastic solid. *Physical Review E* 61:6120–6131. <https://doi.org/10.1103/PhysRevE.61.6120>
- Miehe C, Hofacker M, Welschinger F (2010) A phase field model for rate-independent crack propagation: robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering* 199:2765–2778. <https://doi.org/10.1016/j.cma.2010.04.011>
- Moës N, Dolbow J, Belytschko T (1999) A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering* 46:131–150. [https://doi.org/10.1002/\(SICI\)1097-0207\(19990910\)46:1<131::AID-NME726>3.0.CO;2-J](https://doi.org/10.1002/(SICI)1097-0207(19990910)46:1<131::AID-NME726>3.0.CO;2-J)
- Ostoja-Starzewski M (2002) Lattice models in micromechanics. *Applied Mechanics Reviews* 55:35–60. <https://doi.org/10.1115/1.1432990>
- Rabczuk T, Belytschko T (2004) Cracking particles: a simplified meshfree method for arbitrary evolving cracks. *International Journal for Numerical Methods in Engineering* 61(13):2316–2343. <https://doi.org/10.1002/nme.1151>
- Schlangen E, Garboczi EJ (1997) Fracture simulations of concrete using lattice models: computational aspects. *Engineering Fracture Mechanics* 57:319–332. [https://doi.org/10.1016/S0013-7944\(97\)00010-6](https://doi.org/10.1016/S0013-7944(97)00010-6)
- Silling SA (2000) Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids* 48:175–209. [https://doi.org/10.1016/S0022-5096\(99\)00029-0](https://doi.org/10.1016/S0022-5096(99)00029-0)
- Silling SA, Askari E (2005) A meshfree method based on the peridynamic model of solid mechanics. *Computers & Structures* 83:1526–1535. <https://doi.org/10.1016/j.compstruc.2004.11.026>

- Song JH, Belytschko T (2009) Cracking node method for dynamic fracture with finite elements. *International Journal for Numerical Methods in Engineering* 77(3):360–385. <https://doi.org/10.1002/nme.2415>
- Song JH, Wang H, Belytschko T (2008) A comparative study on finite element methods for dynamic fracture. *Computational Mechanics* 42:239–250. <https://doi.org/10.1007/s00466-007-0210-x>
- Wu JY (2017) A unified phase-field theory for the mechanics of damage and quasi-brittle failure. *Journal of the Mechanics and Physics of Solids* 103:72–99. <https://doi.org/10.1016/j.jmps.2017.03.015>
- Wu JY, Nguyen VP (2018) A length scale insensitive phase-field damage model for brittle fracture. *Journal of the Mechanics and Physics of Solids* 119:20–42. <https://doi.org/10.1016/j.jmps.2018.06.006>
- Xu XP, Needleman A (1994) Numerical simulations of fast crack growth in brittle solids. *Journal of the Mechanics and Physics of Solids* 42:1397–1434. [https://doi.org/10.1016/0022-5096\(94\)90003-5](https://doi.org/10.1016/0022-5096(94)90003-5)
- Yue Q, Wang Q, Tian W, et al (2023) A phase-field lattice model (PFLM) for fracture problem: theory and application in composite materials. *Composite Structures* 323:117432. <https://doi.org/10.1016/j.compstruct.2023.117432>
- Zhou F, Molinari JF (2004) Dynamic crack propagation with cohesive elements: a methodology to address mesh dependency. *International Journal for Numerical Methods in Engineering* 59(1):1–24. <https://doi.org/10.1002/nme.857>

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