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Proximal Point Algorithms in Hadamard Spaces and Generalized Fermat-Torricelli Problem

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Abstract

In Hadamard spaces, we develop, in a stepwise manner, a double sum inexact cyclic proximal point algorithm and establish linear convergence of the sequences generated at each stage under metric subregularity assumptions on the underlying monotone operators. We further prove that the convergence becomes superlinear whenever either the regularity modulus degenerates or the proximal parameters vanish, provided the singularity set is a singleton. Using the geometry of weak sharp minima, we derive finite termination results for the proposed algorithm under suitable normal cone conditions. As an application, we investigate a new generalized Fermat–Torricelli problem in Hadamard spaces involving geodesic-preserving mappings and distances to convex subsets. Explicit resolvent formulas and numerical experiments are provided to illustrate and validate the theoretical convergence behaviour of the proposed algorithms.

Keywords: Hadamard space, proximal point algorithm, metric subregularity, normal cone, generalized Fermat-Torricelli problem.

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1 Introduction

The geometric optimization problem of locating a point that minimizes the sum of distances to certain given points has a long and rich history. One of the earliest versions was proposed and solved by Heron of Alexandria (10–70 AD), who considered finding a point on a straight line that minimizes the total distance to two fixed points on the same side of the line. This problem is known as the Heron problem.

In 1643, Pierre de Fermat introduced a variant: determining a point in the plane that minimizes the sum of its distances to three given points. Since Evangelista Torricelli was the first to solve it, the problem became known as the Fermat–Torricelli problem. The classical problem was later generalized by Jakob Steiner to configurations involving finitely many points in the plane.

Over the years, numerous extensions and variations of the Fermat–Torricelli problem have been developed. These models play an important role in optimization, network design, location science, and

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statistics, and they continue to attract significant interest from mathematicians worldwide. Refer to [3, 17, 34, 35, 43, 46] and its references, for instances.

In this paper, we develop a unified approach in the framework of Hadamard spaces by which we combine three convex minimization problems, namely the **weighted Fermat–Torricelli problem**, the **convex feasibility problem**, and the **best approximation problem** into a single problem, which we purport to solve. The choice of Hadamard space is for the reason of dealing with the problem in a general setting, which includes, as a special case, the Hilbert space and, consequently, $\mathbb{R}^n$ in particular. We introduce for our purpose the double sum structure and “**continuous geodesic-preserving maps**”, which are substitutions of “**bounded linear operators**” in the new framework of Hadamard spaces. In the following, we describe the development of this line of research technically to some extent. We then introduce the precise mathematical framework and present the key results that will be used in the subsequent discussion.

Let H be a real Hilbert space and $A : H \rightarrow 2^H$ a maximal monotone operator. Finding $v \in H$ with $0 \in Av$ is classical [6, 9, 20, 23, 32, 42], solved via the proximal point algorithm (PPA) of Martinet [32] and Rockafellar [42]. Bruck and Reich [8] extended the approach to accretive operators in Banach spaces, and it was first considered for monotone vector fields on Riemannian manifolds in [31]. Bačák [2] carried the method to Hadamard spaces via Yosida–Moreau regularization,

$$x_{n+1} = \arg \min_{y \in X} \left[f(y) + \frac{1}{2r_n} d^2(x_n, y) \right], \quad r_n > 0, \quad n \in \mathbb{N},$$

establishing Δ -convergence to a minimizer of f . Choleamjiak [14] then gave a Halpern-type modification with strong convergence in CAT(0) spaces. Building on the subdifferential theory of [21], Khatibzadeh and Ranjbar [24] extended monotone operators and resolvents to Hadamard spaces and established Δ - and strong-convergence results for the inexact PPA; when $A = \partial f$, their resolvent coincides with Bačák’s, $J_r x := \arg \min_{y \in X} [f(y) + \frac{1}{2r} d^2(x, y)]$. More recently, Tahernia, Moradi and Jafari [48] sharpened the existence theory in this setting, giving a necessary and sufficient condition for the zero set to be nonempty together with strong (metric) convergence.

A second, quantitative line of work asks not only whether such iterations converge, but how fast, and when they stop – a question also receiving renewed attention in Hilbert space, e.g. for Halpern and Tikhonov–Mann iterations [54, 55]. Kohlenbach, López-Acedo and Nicolae [49] introduced a *modulus of regularity* in Hilbert space unifying weak sharp minima, error bounds, metric subregularity, and Hölder regularity, from which rates of convergence and finite termination follow for the PPA and the cyclic algorithm. Leuştean and Sipoş [50] apply proof mining directly to the (unmodified) proximal point algorithm in CAT(0) spaces, obtaining rates of convergence when the objective is uniformly convex and rates of metastability when the space is totally bounded; Sipoş [51] extends this to an abstract framework covering *Halpern-type* and *Tikhonov-type* regularizations of the PPA, deriving uniform rates of metastability for both. Splitting-type proximal iterations – compositions of several resolvents – likewise already exist in Hadamard spaces: the backward-backward and barycentric schemes of Khatibzadeh and Moosavi [52], with weak ergodic and strong convergence, and a Halpern-type regularization of a resolvent composition (together with fixed points of strictly pseudocontractive mappings) by Banda, Abass, Aphane and Saidov [53], with strong convergence; neither addresses rate or finite termination questions, and neither treats the plain inexact cyclic composition with error tolerance ϵ_n used in Algorithms 1.2–1.3 below.

This paper’s contribution sits precisely where these two lines have not yet met – splitting algorithms on one side, quantitative rate/termination theory on the other – and is threefold.

1. **A double-sum inexact cyclic splitting PPA with rate and finite termination theory** (Algorithms 1.2–1.3, Section 6). The splitting iterations of [52, 53] give only existence-type convergence (the latter via Halpern regularization, not the plain inexact cyclic scheme used here), while the quantitative theory of [49–51] treats only single-operator Halpern/Tikhonov PPA. We couple a cyclic composition of resolvents – including a two-level, block-and-within-block double-sum version matched to the structure of the generalized Fermat–Torricelli problem above – with linear/superlinear convergence rates and finite termination.
2. **Metric subregularity formulated without an ambient linear or manifold structure** (Definition 2.1). The rate theory of [44, 45] on Hadamard manifolds relies on the exponential map to linearize locally; ours is defined directly via the tangent-cone distance $d_x(0_x, A(x))$ from Nikolaev’s tangent cones [37], so it applies to Hadamard spaces that are not manifolds – e.g. $\mathbb{R}$ -trees, Hilbert balls, and gluings thereof – where no exponential map exists.
3. **A generalized Fermat–Torricelli problem via geodesic-preserving maps** (Section 6, Problem (6.1)), extending [41] beyond the linear/Hilbert setting; this is the intended application driving Contribution 1.

This paper employs the framework of **tangent spaces**, or **tangent cones**, introduced by Nikolaev [37], rather than the **dual-space** approach considered in [21], thereby remaining consistent with the settings of Hilbert spaces and Hadamard manifolds. Owing to the favourable structure of these tangent spaces, the notion of monotone vector fields developed in [13] serves as a natural extension of the corresponding theory in Hilbert spaces as well as in Hadamard manifolds.

Motivated by [44, 45], in this paper, first, we discuss the different types of convergence of the following iteration in the context of Hadamard spaces which are well-defined by Proposition 2.5:

Let $A : X \rightrightarrows TX$ be a monotone operator satisfying the surjectivity condition with $A^{-1}(0) \neq \emptyset$. To approximate a singularity of A , we consider: for a given $x_0 \in X$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by the algorithm

$$\begin{aligned} \lambda_n d(y_n, x_n) \mu_{y_n, x_n} &\in A(y_n); \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned} \tag{1.1}$$

or, equivalently, with $r_n = \frac{1}{\lambda_n}$

$$\begin{aligned} y_n &= J_{r_n}^A(x_n); \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned}$$

by letting $n = 0, 1, 2, \dots$

Subsequently, we extend this framework to the inexact cyclic SPPA and investigate its convergence behaviour for the optimization problem $\operatorname{argmin}_{x \in C} \sum_{i=1}^m f_i(x)$ where $C(\subseteq X)$ is closed and convex, and every $f_i \in \Gamma_0$. The corresponding exact version of the algorithm was previously studied in [1].

For a given $x_0 \in X$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by the algorithm

$$\begin{aligned} y_n &= J_{r_n}^{f_m} \circ \dots \circ J_{r_n}^{f_1}(x_n); \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned} \tag{1.2}$$

by letting $n = 0, 1, 2, \dots$

Note that $x_n^{(0)} := x_n$, $x_n^{(i)} := J_{r_n}^{f_i}(x_n^{(i-1)})$, $i = 1, \dots, m$, $y_n := x_n^{(m)}$.

Furthermore, we address the requisite double sum inexact cyclic SPPA for the optimization problem $\operatorname{argmin}_{x \in C} \sum_{i=1}^m \sum_{j=1}^{m_i} f_{ij}(x)$ where $C(\subseteq X)$ is closed and convex, and every $f_{ij} \in \Gamma_0$.

For a given $x_0 \in X$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by the algorithm

$$\begin{aligned} x_n^{(i,0)} &:= x_n^{(i-1)}, \quad x_n^{(0)} := x_n; \\ x_n^{(i,j)} &:= J_{r_n}^{f_{i,j}}(x_n^{(i,j-1)}), \quad j = 1, 2, \dots, m_i; \\ y_n &:= x_n^{m, m_m}; \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned} \tag{1.3}$$

by letting $n = 0, 1, 2, \dots$

It should be noted that the surjectivity condition is essential to define a PPA. In contrast to the Hilbert space setting [33], where surjectivity is guaranteed for maximal monotone operators, we explicitly assume this condition here, since it remains an open question whether every maximal monotone operator $A : X \rightrightarrows TX$ satisfies the surjectivity condition.

The remainder of this paper is organized as follows.

Section 2 lays the geometric and analytical foundation required for our study. We begin with a concise review of Hadamard spaces and discuss several key notions, including tangent spaces, quasilinearization, and Δ -convergence of sequences. We then recall the essential tools from monotone operator theory that will be used throughout the paper, such as monotone operators, resolvents, and subdifferentials.

In Section 3, we investigate the behavior of the quasi-Fejér sequence generated by Algorithm (1.1) and establish its Δ -convergence in the CAT(0) setting.

Section 4 is devoted to the convergence rate analysis of Algorithm (1.1). In order to derive strong convergence properties, we extend the notion of metric subregularity to CAT(0) spaces. This framework enables us to prove that the proposed algorithm enjoys at least a superlinear rate of convergence.

In Section 5, we address the finite termination behavior of the algorithm. Motivated by the classical concept of weak sharp minima, we introduce a normal cone-based weak sharp minima-type condition in the CAT(0) framework. We show that this condition not only guarantees finite termination of the iterative process, but also implies metric subregularity.

Section 6 is dedicated to the generalized Fermat-Torricelli problem. After formulating the problem and discussing the existence of minimizers, we develop and analyze single-sum 1.2 and double-sum inexact cyclic SPPA 1.3 algorithms. Their convergence to the Fermat-Torricelli point, together with their finite termination properties, is studied in detail in the subsequent subsections.

Finally, in Section 7, we complement the theoretical developments with several numerical experiments, illustrating the performance and effectiveness of the proposed algorithms.

2 Preliminaries

Recall that a metric space (X, d) is considered geodesic if for each pair of points $x, y \in X$, there exists a mapping $\mu : [0, l] \rightarrow X$ (for some $l \geq 0$) with $\mu(0) = x, \mu(l) = y$ and a constant $\rho \geq 0$ such that $d(\mu(t), \mu(t')) = \rho|t - t'|$ for any $t, t' \in [0, l]$. We use the notation $(1 - t)x \oplus ty$ or $[[x, y]]$ for $\mu(t)$, where $x, y \in X, t \in [0, l]$. If $x = y$ we call μ , a zero geodesic at x .

If $l = 1$, we say that μ is normalized, and if $\rho = 1$, it is of unit speed. Note that in the normalized case $\rho = d(x, y)$. In the sequel, if the geodesic is either unique or the choice of geodesic is unimportant, we write $\mu_{x,y}$ to denote the normalized geodesic joining x and y .

Definition 2.1. A geodesic metric space (X, d) is said to be a CAT(0) space if it satisfies the following CN-inequality,

$$d^2(z, \mu(t)) \leq (1-t)d^2(z, x) + td^2(z, y) - t(1-t)d^2(x, y)$$

for each $x, y, z \in X, t \in [0, 1]$ and μ , a geodesic path from $x = \mu(0)$ to $y = \mu(1)$ in X .

Lemma 2.1. [19] Let (X, d) be a CAT(0) space. Then, for any $x, y, z \in X$ and $t \in [0, 1]$,

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

A complete CAT(0) space is called a **Hadamard space**.

2.1 Tangent Cones

Tangent cones, another name for tangent spaces, were first introduced in [37] (see also [7, 10]). For details, see [13]. Here, we give a brief discussion.

First, we need to define some notions.

Let (X, d) be a Hadamard space. For any $x, y, z \in X$, the geodesic triangle $\Delta \subset X$ is defined by $\Delta(x, y, z) := [[x, y]] \cup [[y, z]] \cup [[z, x]]$. Let $\mathbb{E}^2$ be the Euclidean plane with the usual norm. We define the Euclidean comparison triangle, or simply the comparison triangle $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z}) := \Delta(\bar{x}, \bar{y}, \bar{z})$ with $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ satisfying $\|\bar{x} - \bar{y}\| = d(x, y)$, $\|\bar{y} - \bar{z}\| = d(y, z)$ and $\|\bar{z} - \bar{x}\| = d(z, x)$.

Definition 2.2. For any $x, y, z \in X$, provided $y, z \in X \setminus \{x\}$, the comparison angle between y, z at x , denoted by $\bar{\angle}_x(y, z)$, is defined as:

$$\cos \bar{\angle}_x(y, z) := \frac{\langle \bar{y} - \bar{x}, \bar{z} - \bar{x} \rangle}{\|\bar{y} - \bar{x}\| \|\bar{z} - \bar{x}\|},$$

where $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{E}^2$ is the comparison triangle of the geodesic triangle $\Delta(x, y, z) \subset X$. Further, we set $\bar{\angle}_x(x, x) = 0$, and $\bar{\angle}_x(x, z) = \frac{\pi}{2}$ for $z \in X \setminus \{x\}$.

Definition 2.3. Given two geodesics μ_1, μ_2 on X with a common initial point $x_0 \in X$. The Alexandrov angle between the two geodesics is defined by

$$\alpha_{x_0}(\mu_1, \mu_2) := \lim_{t, t' \rightarrow 0^+} \bar{\angle}_{x_0}(\mu_1(t), \mu_2(t')).$$

Denote the set of all normalized geodesics with a common initial point $x_0 \in X$ by $\tilde{S}_{x_0}$. Then $\bar{\angle}_{x_0} := \alpha_{x_0}$ defines a pseudometric on $\tilde{S}_{x_0}$. The metric identification $(S_{x_0}, \angle_{x_0})$ of $(\tilde{S}_{x_0}, \bar{\angle}_{x_0})$ is known as the space of directions at x_0 . We write $\mu \equiv$ the equivalent class $[\mu]$, an element of S_{x_0} .

Now consider the space $[0, +\infty) \times S_{x_0}$ with the equivalence relation $\sim$ defined on it by: $(t, \mu_1) \sim (t', \mu_2)$ if and only if

- $t\chi(\mu_1) = t'\chi(\mu_2) = 0$ or
- $t\chi(\mu_1) = t'\chi(\mu_2) > 0$ and $\mu_1 = \mu_2$.

where χ is the zero geodesic indicator: $\chi(\mu) := 0$ if μ is a zero geodesic and $\chi(\mu) := 1$ otherwise.

Define the tangent cone at x_0 by $T_{x_0}X := [0, +\infty) \times S_{x_0} / \sim$. For simplicity we write $t\mu \equiv [(t, \mu)]_\sim$ for elements of $T_{x_0}X$.

Note that for any $t\mu_1, t'\mu_2 \in T_{x_0}X$, $d_{x_0}(t\mu_1, t'\mu_2) := \sqrt{t^2\chi(\mu_1) + t'^2\chi(\mu_2) - 2tt'\cos\angle_{x_0}(\mu_1, \mu_2)}$ is a metric on $T_{x_0}X$.

Proposition 2.1. [13] *If H is a Hilbert space and $x_0 \in H$, then $T_{x_0}H$ is isometric to H .*

Proposition 2.2. [13] *If M is a Hadamard manifold with the Riemannian tangent space $Tan_{x_0}M$ at $x_0(\in M)$, then $T_{x_0}M$ is isometric to $Tan_{x_0}M$.*

Analogous to manifolds, the tangent bundle TX of X is defined as $TX := \cup_{x_0} T_{x_0}X$. Furthermore, we employ the product

$$g_{x_0}(t\mu_1, t'\mu_2) := tt'\chi(\mu_1)\chi(\mu_2) \cos \angle_{x_0}(\mu_1, \mu_2)$$

On each tangent space $T_{x_0}X$, $0_{x_0} := 0\mu = t\mu_{x_0, x_0}$ (here, $\mu \in S_{x_0}$, $t > 0$) and $\|t\mu\|_{x_0} := d_{x_0}(0_{x_0}, t\mu) = t\chi(\mu)$.

Therefore, we have

$$g_{x_0}(t\mu_1, t'\mu_2) \leq \|t\mu_1\| \|t'\mu_2\|,$$

a Cauchy-Schwarz inequality analogue.

For convenience, we refer to the zero section $\{0_{x_0} : x_0 \in X\}$ of TX as $\mathbf{0}$.

2.2 Quasilinearization

In [5], Berg and Nikolaev have introduced the notion of quasilinearization for abstract metric spaces by introducing the quasi-inner product analogous to the inner product.

Let (X, d) be a Hadamard space. Each ordered pair (x, y) in X^2 is called a bound vector and is denoted by $\overrightarrow{xy}$, or $\overrightarrow{u}$. The zero bound vector at $x \in X$ is denoted by the symbol $\overrightarrow{0_x} := \overrightarrow{xx}$. An ordered pair of two bound vectors (a, b) and (c, d) in X^2 is said to be admissible if $b = c$ and thereafter we have the following definition of addition of a pair of admissible vectors

$$\overrightarrow{ab} + \overrightarrow{bd} = \overrightarrow{ad}.$$

The quasilinearization map $\langle \cdot, \cdot \rangle : X^2 \times X^2 \rightarrow \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d))$$

for all $a, b, c, d \in X$.

We say X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$$

for all $a, b, c, d \in X$.

Theorem 2.2. [5] *A geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.*

Lemma 2.3. [15] *Let (X, d) be a CAT(0) space. Then*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)^2 d^2(x, z) + t^2 d^2(y, z) + 2t(1-t) \langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

The next proposition establishes a relationship between the two previously defined notions: g_{x_0} and the quasi-inner product.

Proposition 2.3. [13] *For each $t, t' \geq 0$ and $x_0, x, y \in X$, the following inequality holds:*

$$g_{x_0}(td(x_0, x)\mu_{x_0, x}, t'd(x_0, y)\mu_{x_0, y}) \geq \langle t\overrightarrow{x_0x}, t'\overrightarrow{x_0y} \rangle.$$

2.3 Δ -Convergence

In [26], the notion of Δ -convergence was introduced by Kirk and Panyanak. Let (X, d) be an Hadamard space and $\{x_n\}$ be a bounded sequence in X . Then the asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf_{n \rightarrow +\infty} \{\limsup d(x, x_n) : x \in X\}$$

and the asymptotic center is the set;

$$A(\{x_n\}) = \{x \in X : \limsup_{n \rightarrow +\infty} d(x, x_n) = r(\{x_n\})\}.$$

A sequence $\{x_n\}$ in X is said to be Δ -converges to some $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$, for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Lemma 2.4. (*Opial Property*) [1] *Let $\{x_n\} \in X$ be a sequence Δ -converging to a point $x \in X$. Then we have*

$$\limsup_{n \rightarrow +\infty} d(x, x_n) < \limsup_{n \rightarrow +\infty} d(y, x_n)$$

for every $y \in X \setminus \{x\}$.

Theorem 2.5. [22] *Let (X, d) be an Hadamard space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ Δ -converges to some x in X if and only if $\limsup_{n \rightarrow +\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

Theorem 2.6. [1] *Every bounded sequence has a Δ -convergent subsequence.*

It should be noted that in general Hadamard spaces the following implications hold

$$\text{strong convergence} \Rightarrow \Delta\text{-convergence}.$$

The following lemma offers a sufficient condition for Δ -convergent sequences to become strongly convergent (convergent in metric).

Lemma 2.7. (*Kadec-Klee Property*) [25] *Let the sequence $\{x_n\}$ be Δ -convergent to $x \in X$ and $\lim_{n \rightarrow +\infty} d(x_n, x_0) = d(x, x_0)$ for some $x_0 \in X$, then $\{x_n\}$ strongly converges to x .*

Next, we have the following **demiclosedness principle**.

Theorem 2.8. [26] *Let $C \subseteq X$ be nonempty, closed and convex; $T : C \rightarrow C$ is nonexpansive. If the sequence $\{x_n\}$ be Δ -convergent to $x \in X$ and $\lim_{n \rightarrow +\infty} d(x_n, T(x_n)) = 0$, then $T(x) = x$.*

2.4 Monotone Operators

The monotone operators in Hadamard spaces will now be briefly covered. To define monotone operators, we mainly have two approaches: one using the quasi-inner product (see [24]) and another using the products on tangent cones (see [13]). Here we consider the second approach.

Let X be a Hadamard space with the tangent bundle TX . The set-valued operator $A : X \rightrightarrows TX$ satisfying $A_{x_0} \subseteq T_{x_0}X$ for all $x_0 \in X$ is called monotone if

$$g_x(\eta, \mu_{x,y}) + g_y(\nu, \mu_{y,x}) \leq 0$$

holds for every $(x, \eta), (y, \nu) \in \text{graph}(A)$, where $\text{graph}(A) := \{(u, \zeta) \in X \times TX : \zeta \in A(u)\}$. Furthermore, if $\text{graph}(A)$ is not properly contained in the graph of any other monotone operators, then the operator $A : X \rightrightarrows TX$ is considered maximal.

Proposition 2.4. [13] Let A be a monotone operator on X and $(x, \eta), (y, \nu) \in \text{graph}(A)$. Also, suppose that the representations $td(x, u)\mu_{x,u} = \eta$ and $t'd(y, v)\mu_{y,v} = \nu$ are satisfied for some $t, t' \geq 0$ and $u, v \in X$. Then the following holds:

$$\langle t\vec{xu}, \vec{xy} \rangle + \langle t'\vec{yv}, \vec{yx} \rangle \leq 0.$$

Definition 2.4. A monotone operator $A : X \rightrightarrows TX$ is said to satisfy the surjectivity condition if for any $t > 0$ and $x_0 \in X$, there exists a point $u \in X$ such that $td(x_0, u)\mu_{u,x_0} \in A(u)$.

Proposition 2.5. [13] Let A be a monotone operator on X that satisfies the surjectivity condition. Then for any $t > 0$ and $x \in X$, there exists a unique $y \in X$ such that $td(y, x)\mu_{y,x} \in A(y)$.

Proposition 2.6. [13] Let A be a monotone operator on X that satisfies the surjectivity condition. Then, A is maximally monotone.

2.5 Resolvents

In [13], the resolvents of monotone operators were defined, and some of their properties were discussed. One can see [24] also.

The resolvent of A of order $r > 0$ is the multivalued mapping $J_r : X \rightrightarrows X$ defined by $J_r(x) := \{z \in X : (1/r)d(z, x)\mu_{z,x} \in Az\}$ for every $x \in X$.

We first recall the definition of a firmly nonexpansive mapping and then state several useful properties of the resolvent.

Definition 2.5. Let $C \subseteq X$. A mapping $T : C \rightarrow C$ is said to be firmly nonexpansive if for any $x, y \in C$, the function $\Phi_{x,y}(t) := d(\mu_{x,T(x)}(t), \mu_{y,T(y)}(t))$ is nonincreasing on $t \in [0, 1]$.

Proposition 2.7. Let A be a monotone operator on X that satisfies the surjectivity condition. Then the following facts hold for any $r > 0$:

- (i) J_r is well-defined on X and is single-valued.
- (ii) J_r is nonexpansive.
- (iii) $\text{Fix}(J_r) = A^{-1}(\mathbf{0})$, where $\text{Fix}(J_r)$ is the set of all fixed points of J_r .
- (iv) If $0 < s \leq r$ and $x \in X$, then $J_r(x) = J_s(y)$ with $y := (1 - \frac{s}{r})J_r(x) \oplus \frac{s}{r}x$.
- (v) J_r is firmly nonexpansive.
- (vi) If $0 < s \leq r$ and $x \in X$, then $d(x, J_s(x)) \leq 2d(x, J_r(x))$.

Proof. We prove only part (vi), as the remaining assertions are proved in [13].

Define $u := (1 - \frac{s}{r})J_r(x) \oplus \frac{s}{r}x$.

Using (iv), $J_r(x) = J_s(u)$.

Since J_s is nonexpansive

$$d(J_r(x), J_s(x)) \leq d(u, x).$$

Then, by Lemma 2.1,

$$d(J_r(x), J_s(x)) \leq (1 - \frac{s}{r})d(J_r(x), x).$$

Therefore

$$\begin{aligned} d(x, J_s(x)) &\leq d(x, J_r(x)) + d(J_r(x), J_s(x)) \\ &\leq d(x, J_r(x)) + (1 - \frac{s}{r})d(J_r(x), x) \\ &\leq 2d(x, J_r(x)). \end{aligned}$$

□

2.6 Metric Projection

Let C be a nonempty, closed, and convex subset of a Hadamard space (X, d) and $P_C : X \rightarrow C$, be a mapping defined by

$$P_C(x) := \operatorname{argmin}_{y \in C} d(x, y)$$

for every $x \in X$, known as the metric projection of x onto C .

Theorem 2.9. [7] *The assertions listed below are true:*

(i) P_C is defined for all $x \in X$ and is single-valued.

(ii) given $x \notin C$, $y \in C$. If $y \neq P_C(x)$, then $\angle_{P_C(x)}(\mu_{P_C(x),x}, \mu_{P_C(x),y}) \geq \frac{\pi}{2}$, which further implies $g_{P_C(x)}(\mu_{P_C(x),x}, \mu_{P_C(x),y}) \leq 0$.

Theorem 2.10. [15] *Let C be a nonempty convex subset of a CAT(0) space X , $x \in X$ and $u \in C$. Then $u = P_C x$, the metric projection of x onto C if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{yu} \rangle \leq 0 \quad \forall y \in C.$$

2.7 Subdifferentials

The subdifferential of a proper function on Hadamard spaces was first introduced by Kakavandi and Amini [21] using a dual-space approach. Chaipuniya et al. [13] subsequently developed a tangent-cone-based formulation, which we adopt here. More details can be found in [29].

Let (X, d) be a Hadamard space and $f : X \rightarrow (-\infty, +\infty]$ be a proper function with an efficient domain $\mathcal{D}(f)$ where f gives only definite values, the subdifferential of f is the multivalued operator $\partial f : X \rightrightarrows TX$ defined by

$$\partial f(x) = \{x_0^* \in T_{x_0}X : f(x) \geq f(x_0) + d(x_0, x)g_{x_0}(x_0^*, \mu_{x_0,x})\} \quad (2.1)$$

when $x \in \mathcal{D}(f)$ and $\partial f(x) = \emptyset$, otherwise.

Definition 2.6. *A function $f : X \rightarrow (-\infty, +\infty]$ is called convex if $f((1-t)x \oplus ty) \leq (1-t)f(x) + tf(y)$ for any $x, y \in X$ and $t \in (0, 1)$.*

Let us denote the class of proper, convex, lower semicontinuous functions $f : X \rightarrow (-\infty, +\infty]$ by Γ_0 . The elements in ∂f are characterized by the following result.

Proposition 2.8. [13] *Let $f \in \Gamma_0$, $x_0, x \in X$ and $r > 0$ be given. Then*

$$\frac{1}{r}d(x_0, x)\mu_{x_0,x} \in \partial f(x_0) \iff x_0 = \operatorname{prox}_r(x) = \operatorname{argmin}_{y \in X} [f(y) + \frac{1}{2r}d^2(y, x)].$$

Theorem 2.11. [13] *If $f : X \rightarrow (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function on a Hadamard space X then $\partial f : X \rightrightarrows TX$ is a monotone operator.*

In fact ∂f is a maximal monotone operator.

We conclude this section with the following useful lemma.

Lemma 2.12. ([40], Lemma 2.2.2) *Let $\{a_n\} \subseteq (0, +\infty)$ and $\{b_n\} \subseteq (0, +\infty)$ be two sequences such that, for every $n = 0, 1, \dots$,*

$$a_{n+1} \leq (1 + b_n)a_n + b_n.$$

Assume that $\sum_n b_n < +\infty$. Consequently, $\{a_n\}$ is convergent.

3 Quasi-Fejér Monotonicity and Δ -Convergence

In this section, we study the convergence behaviour of quasi-Fejér monotone sequences. These sequences arise naturally in optimization, fixed-point theory, and nonlinear analysis, since many important iterative algorithms satisfy a quasi-Fejér type inequality rather than an exact Fejér monotonicity condition. The introduction of summable error terms makes this notion flexible enough to accommodate perturbations, approximation procedures, and inexact computations that frequently occur in practical algorithms.

Quasi-Fejér monotonicity provides a unified framework for the convergence analysis of a wide variety of iterative schemes, including proximal point methods, Mann and Ishikawa iterations, projection algorithms, splitting methods, and inertial or variable metric techniques. Owing to this broad applicability, the study of quasi-Fejér monotone sequences has become an important tool in modern nonlinear analysis and optimization theory. See [27, 39] and the references therein.

Definition 3.1. Let (X, d) be a metric space, let $C \subseteq X$ be a nonempty set, and let $G, H : [0, +\infty) \rightarrow [0, +\infty)$ be two given functions. Suppose that $\{\epsilon_n\} \subseteq [0, +\infty)$ satisfies $\sum_{n=0}^{\infty} \epsilon_n < +\infty$. A sequence $\{x_n\} \subseteq X$ is said to be (G, H) -quasi-Fejér monotone with respect to C if, for every $z \in C$ and every $n, m \in \mathbb{N}$, $H(d(z, x_{n+m})) \leq G(d(z, x_n)) + \sum_{i=n}^{n+m-1} \epsilon_i$.

And simply ‘quasi-Fejér’ if

$$d(z, x_{n+1}) \leq d(z, x_n) + \epsilon_n. \quad (3.1)$$

This specific form 3.1 is what we use in our Δ -convergence analysis.

Below are some useful lemmas that will be helpful throughout the paper.

Lemma 3.1. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by the Algorithm 1.1 and $z \in A^{-1}(0)$. Then the following assertions hold:

- (i) $d^2(y_n, z) + d^2(y_n, x_n) \leq d^2(x_n, z), \forall n = 0, 1, 2, \dots$,
- (ii) if $\{\epsilon_n\} \in [0, +\infty)$ satisfies $\sum_{n \geq 0} \epsilon_n < +\infty$ then the sequences $\{d(x_n, z)\}$ and $\{d(y_n, z)\}$ are convergent,
- (iii) if $\{\epsilon_n\} \in [0, +\infty)$ satisfies $\sum_{n \geq 0} \epsilon_n < +\infty$ then $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow +\infty$.

Proof. (i). Fix n , $\lambda_n d(y_n, x_n) \mu_{y_n, x_n} \in A(y_n)$.

Since $0_z \in A(z)$, from the monotonicity of A

$$g_{y_n}(\lambda_n d(y_n, x_n) \mu_{y_n, x_n}, \mu_{y_n, z}) + g_z(0_z, \mu_{z, y_n}) \leq 0$$

implies

$$\langle \overrightarrow{y_n x_n}, \overrightarrow{y_n z} \rangle \leq 0,$$

and therefore

$$d^2(y_n, z) + d^2(y_n, x_n) \leq d^2(x_n, z). \quad (3.2)$$

- (ii). From (i), $d(y_n, z) \leq d(x_n, z) \forall n$ and by $d(x_{n+1}, y_n) \leq \epsilon_n$,

$$d(x_{n+1}, z) - d(y_n, z) \leq \epsilon_n \quad (3.3)$$

$$d(x_{n+1}, z) - \epsilon_n \leq d(y_n, z) \leq d(x_n, z)$$

$$d(x_{n+1}, z) - d(x_n, z) \leq \epsilon_n.$$

Using Lemma 2.12, $\{d(x_n, z)\}$ is convergent and therefore $\{d(y_n, z)\}$ is also convergent.

(iii). From 3.3,

$$d(x_{n+1}, z) \leq d(y_n, z) + \epsilon_n.$$

Squaring gives

$$\begin{aligned} d^2(x_{n+1}, z) &\leq d^2(y_n, z) + 2\epsilon_n d(y_n, z) + \epsilon_n^2 \\ &\leq d^2(y_n, z) + 2M\epsilon_n + \epsilon_n^2 \quad (\text{for some } M > 0, \text{ since } \{d(y_n, z)\} \text{ is bounded}) \\ &\leq d^2(x_n, z) - d^2(x_n, y_n) + \eta_n \quad (\text{using 3.2}), \end{aligned}$$

where $\eta_n = 2M\epsilon_n + \epsilon_n^2$,
implies

$$d^2(x_n, y_n) \leq d^2(x_n, z) - d^2(x_{n+1}, z) + \eta_n,$$

Summing from 0 to N

$$\sum_{n=0}^N d^2(x_n, y_n) \leq d^2(x_0, z) - d^2(x_{N+1}, z) + \sum_{n=0}^N \eta_n.$$

Note that $\sum_{n=0}^{+\infty} \eta_n < +\infty$. Hence $\sum_{n=0}^{+\infty} d^2(x_n, y_n) < +\infty$ and therefore $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow +\infty$. $\square$

Proposition 3.1. *Let X be a complete CAT(0) space, and let $A : X \rightrightarrows TX$ be a monotone operator satisfying the surjectivity condition with $A^{-1}(0) \neq \emptyset$. Assume that $\{r_n\}$ is a sequence of positive real numbers defined by $r_n = \frac{1}{\lambda_n}$ where $0 < r \leq r_n$ for all n . Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by the Algorithm 1.1, and suppose that the error sequence $\{\epsilon_n\}$ satisfies $\sum_{n \geq 0} \epsilon_n < +\infty$. Then*

- (i) $\{x_n\}$ is quasi-Fejér monotone with respect to $A^{-1}(0)$,
- (ii) every Δ -cluster point of $\{x_n\}$ belongs to $A^{-1}(0)$, and
- (iii) the set of all Δ -cluster points of $\{x_n\}$ is a singleton.

Proof. (i). For any $z \in A^{-1}(0)$, we have

$$d(x_{n+1}, z) \leq d(x_n, z) + \epsilon_n,$$

implies quasi-Fejér monotonicity of the sequence $\{x_n\}$ with respect to $A^{-1}(0)$.

(ii). Let $\{x_{n_k}\}$ be a subsequence of the sequence $\{x_n\}$ Δ -converging to x^* . From Lemma 3.1,

$$d(x_{n_k}, y_{n_k}) \rightarrow 0 \text{ as } k \rightarrow +\infty$$

that is,

$$d(x_{n_k}, J_{r_{n_k}}(x_{n_k})) \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Also, $d(x_{n_k}, J_r(x_{n_k})) \leq 2d(x_{n_k}, J_{r_{n_k}}(x_{n_k}))$ implies $d(x_{n_k}, J_r(x_{n_k})) \rightarrow 0$ as $k \rightarrow +\infty$.

Hence, since J_r is nonexpansive, by demiclosedness of nonexpansive mappings, we have $x^* \in A^{-1}(0)$.

(iii). To show the uniqueness of asymptotic center, let us assume, for contradiction the subsequence $\{x_{n_k}\}$ Δ -converging to $c_1 \in A^{-1}(0)$ and some other subsequence $\{x_{m_k}\}$ Δ -converging to $c_2 \in A^{-1}(0)$ of the sequence $\{x_n\}$ with $c_1 \neq c_2$.

Since, $\{x_{n_k}\}$ Δ -converging to c_1 , by Lemma 2.4

$$\limsup_{k \rightarrow +\infty} d(x_{n_k}, c_1) < \limsup_{k \rightarrow +\infty} d(x_{n_k}, y) \quad \forall y (\neq c_1) \in X.$$

In particular for $y = c_2$

$$\limsup_{k \rightarrow +\infty} d(x_{n_k}, c_1) < \limsup_{k \rightarrow +\infty} d(x_{n_k}, c_2).$$

Because quasi-Fejér gives convergence of distances, we have

$$\lim_{n \rightarrow +\infty} d(x_n, c_1) = l_1; \quad \lim_{n \rightarrow +\infty} d(x_n, c_2) = l_2 \text{ for some } l_1, l_2.$$

Hence, along subsequences:

$$\lim_{k \rightarrow +\infty} d(x_{n_k}, c_1) = l_1; \quad \lim_{k \rightarrow +\infty} d(x_{n_k}, c_2) = l_2,$$

and

$$\lim_{k \rightarrow +\infty} d(x_{m_k}, c_1) = l_1; \quad \lim_{k \rightarrow +\infty} d(x_{m_k}, c_2) = l_2.$$

Therefore $l_1 < l_2$. Similarly, from the Δ -convergence of the subsequence $\{x_{m_k}\}$ to c_2 , we obtain $l_2 < l_1$, a contradiction. $\square$

Theorem 3.2. *Let X be a complete CAT(0) space, and let $A : X \rightrightarrows TX$ be a monotone operator satisfying the surjectivity condition with $A^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a sequence of positive real numbers such that $0 < r \leq r_n$ for all n where $r > 0$ is a constant and $r_n = \frac{1}{\lambda_n}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences generated by the Algorithm 1.1, and that $\sum_{n \geq 0} \epsilon_n < +\infty$. Then the sequence $\{x_n\}$ Δ -converges to some point $x^* \in A^{-1}(0)$.*

Proof. From Proposition 3.1(i), we have $\{x_n\}$ is quasi-Fejér monotone with respect to $A^{-1}(0)$.

Therefore, for any $z \in A^{-1}(0)$, the sequence $\{d(x_n, z)\}$ converges.

Hence, $\{x_n\}$ is bounded, which, by Theorem 2.6 implies the existence of a Δ -convergent subsequence of the sequence $\{x_n\}$.

Also, we have proved in Proposition 3.1(ii) that, if a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ is Δ -converging to $x^* \in X$, then $x^* \in A^{-1}(0)$.

Therefore, by the uniqueness of the asymptotic center, proved in 3.1(iii), we have $\{x_n\}$ Δ -converges to $x^* \in A^{-1}(0)$.

It is also worth noting that, since $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow +\infty$ by 3.1, the sequence $\{y_n\}$ Δ -converges to the same $x^* \in A^{-1}(0)$. $\square$

4 Metric Subregularity and Strong Convergence

In the following section, we extend the notion of metric subregularity for set-valued operators in order to establish the strong convergence of the proposed Algorithm 1.1, without requiring additional structural assumptions on the underlying space, such as local boundedness.

Metric subregularity, together with its equivalent calmness property for inverse mappings, has been extensively studied in linear spaces, where the terminology “error bound” is also frequently used. The work [4, 16, 18, 47] for linear spaces and [45] for Hadamard manifolds and the references therein provide a broad collection of results and applications. We now extend this notion of metric subregularity to the setting of Hadamard spaces.

Definition 4.1. *Let $A : X \rightrightarrows TX$ be a set-valued operator, let $k \geq 0$, and let $U \subseteq X$. A is said to be*

- **metrically subregular on U with modulus k if**

$$d(x, A^{-1}(0_x)) \leq k d_x(0_x, A(x))$$

for all $x \in U$.

- **metrically subregular** if, for every $\bar{x} \in A^{-1}(0_{\bar{x}})$, there exists a neighborhood $U_{\bar{x}} \subseteq X$ of $\bar{x}$ and a constant $k_{\bar{x}} > 0$ such that A is metrically subregular on $U_{\bar{x}}$ with modulus $k_{\bar{x}}$.
- **globally metrically subregular with modulus k** if

$$d(x, A^{-1}(0_x)) \leq k d_x(0_x, A(x))$$

for all $x \in X$.

Remark 1. Let $\bar{x} \in A^{-1}(0)$, and define

$$sreg_{\bar{x}}^A := \lim_{r \rightarrow 0} \sup_{x \in B(\bar{x}, r) - A^{-1}(0)} \frac{d(x, A^{-1}(0_{\bar{x}}))}{d_x(0_x, A(x))}.$$

Then A is metrically subregular if and only if $sreg_{\bar{x}}^A < +\infty$ for each $\bar{x} \in A^{-1}(0_{\bar{x}})$.

Throughout the rest of the paper, for a sequence $\{x_n\}$ in X , we take $\delta_n = \frac{\epsilon_n}{d(x_{n+1}, x_n)}$ for each $n \in \mathbb{N}$, and adopt the convention $\frac{0}{0} = 0$.

A lemma regarding the convergence rate of sequences produced by the Algorithm 1.1 is obtained below.

Lemma 4.1. Let $N \in \mathbb{N}, k > 0$ and $U \subseteq X$. Suppose that the operator A is metrically subregular on U with modulus k . Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by the Algorithm 1.1 such that $\{y_n : n \geq N\} \subseteq U$ and, assume that $\sup_{n \geq N} \delta_n < 1$.

Then, for every $n \geq N$, the following estimate holds

$$d(x_{n+1}, A^{-1}(0)) \leq \left(\frac{\delta_n}{1 - \delta_n} + \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} \right) d(x_n, A^{-1}(0)).$$

Proof. Let $z \in A^{-1}(0)$ be arbitrary; by 3.1(i), we have

$$d(x_n, y_n) \leq d(x_n, z).$$

Then, it follows that

$$d(x_n, y_n) \leq d(x_n, A^{-1}(0)). \quad (4.1)$$

Further,

$$\begin{aligned} d(x_{n+1}, y_n) &\leq \epsilon_n \\ &= \delta_n d(x_{n+1}, x_n) \\ &\leq \delta_n (d(x_{n+1}, y_n) + d(y_n, x_n)). \end{aligned}$$

This gives,

$$d(x_{n+1}, y_n) \leq \frac{\delta_n}{1 - \delta_n} d(x_n, y_n). \quad (4.2)$$

Combining 4.1 and 4.2

$$d(x_{n+1}, y_n) \leq \frac{\delta_n}{1 - \delta_n} d(x_n, A^{-1}(0)).$$

This implies that

$$\begin{aligned} d(x_{n+1}, A^{-1}(0)) &\leq d(x_{n+1}, y_n) + d(y_n, A^{-1}(0)) \\ &\leq \frac{\delta_n}{1 - \delta_n} d(x_n, A^{-1}(0)) + d(y_n, A^{-1}(0)). \end{aligned} \quad (4.3)$$

Since $\lambda_n d(y_n, x_n) \mu_{y_n, x_n} \in A(y_n)$ then, $d_{y_n}(0_{y_n}, A(y_n)) \leq \lambda_n d(x_n, y_n)$.
By assumption, for each $n \geq N$

$$\begin{aligned} d(y_n, A^{-1}(0)) &\leq k d_{y_n}(0_{y_n}, A(y_n)) \\ &\leq k \lambda_n d(x_n, y_n). \end{aligned}$$

Therefore

$$d(x_n, y_n) \geq \frac{1}{k \lambda_n} d(y_n, A^{-1}(0)) \quad \forall n \geq N_1.$$

Write $z = P_{A^{-1}(0)}(x_n)$, then

$$\begin{aligned} d^2(y_n, A^{-1}(0)) &\leq d^2(y_n, z) \\ &\leq d^2(x_n, z) - d^2(x_n, y_n) \\ &\leq d^2(x_n, A^{-1}(0)) - \frac{1}{k^2 \lambda_n^2} d^2(y_n, A^{-1}(0)). \end{aligned}$$

Hence

$$d(y_n, A^{-1}(0)) \leq \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} d(x_n, A^{-1}(0)). \quad (4.4)$$

Now, combining 4.3 and 4.4, we get the desired result. $\square$

Lemma 4.2. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by the Algorithm 1.1. Assume that*

$$\lim_{n \rightarrow +\infty} \frac{\epsilon_n}{d(x_n, x_{n+1})} = 0 \text{ and } \sup_n \lambda_n < +\infty. \quad (4.5)$$

Suppose further that there exists $k \in [0, +\infty)$, $N \in \mathbb{N}$, and a subset $U \subseteq X$ such that

- *A is metrically subregular on U with modulus k, and*
- *$\{y_n : n \geq N\} \subseteq U$.*

Then there exist a point $\bar{x} \in A^{-1}(0)$, a constant $q \in [\frac{1}{2}, 1)$ and a bounded sequence $\{a_n\} \subset (0, +\infty)$ satisfying

$$\limsup_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} \leq \frac{k \limsup_{n \rightarrow +\infty} \lambda_n}{q}$$

such that for every $n > N$,

$$d(x_{n+1}, x_n) \leq a_n (1 - q) q^{n-N} d(x_N, A^{-1}(0)).$$

Consequently, the sequence $\{x_n\}$ converges linearly to $\bar{x}$.

Proof. By assumption 4.5, there exists $q \in [\frac{1}{2}, 1)$ such that the following inequality holds for any $n \geq N$ (N can be taken to be large if necessary)

$$\delta_n \leq \frac{1}{2}, \quad \frac{\delta_n}{1 - \delta_n} + \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} < q.$$

Define the sequence $\{a_n\}$ by

$$a_n := \frac{4}{(1 - q) q^{n-N}} \prod_{l=N}^{n-1} \left(\frac{\delta_l}{1 - \delta_l} + \sqrt{\frac{\lambda_l^2 k^2}{1 + \lambda_l^2 k^2}} \right)$$

for any $n > N$.

Therefore $\{a_n\}$ is monotonically decreasing and bounded.

And

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} &= \frac{1}{q} \limsup_{n \rightarrow +\infty} \left(\frac{\delta_n}{1 - \delta_n} + \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} \right) \\ &= \frac{1}{q} \frac{k \limsup_{n \rightarrow +\infty} \lambda_n}{\sqrt{1 + k^2 \limsup_{n \rightarrow +\infty} \lambda_n^2}} \\ &\leq \frac{k \limsup_{n \rightarrow +\infty} \lambda_n}{q} \end{aligned}$$

for each $n > N$.

Next, we show that

$$d(x_{n+1}, x_n) \leq a_n(1 - q)q^{n-N}d(x_N, A^{-1}(0))$$

for each $n > N$.

Granting this, we have that for each $m \in \mathbb{N}$ and $n > N$

$$d(x_{n+m}, x_n) \leq a_n(1 - q)q^{n-N}(1 + q + \dots + q^{m-1})d(x_N, A^{-1}(0)), \quad (4.6)$$

which shows that sequence $\{x_n\}$ is Cauchy and therefore converges to some $\bar{x}$.

By the previous Lemma 4.1

$$d(x_{n+1}, A^{-1}(0)) \leq \left(\frac{\delta_n}{1 - \delta_n} + \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} \right) d(x_n, A^{-1}(0))$$

for each $n \geq N$.

Therefore, by induction, for each $n > N$

$$d(x_n, A^{-1}(0)) \leq \prod_{l=N}^{n-1} \left(\frac{\delta_l}{1 - \delta_l} + \sqrt{\frac{\lambda_l^2 k^2}{1 + \lambda_l^2 k^2}} \right) d(x_N, A^{-1}(0)).$$

By the definition of $\{a_n\}$, one has that

$$d(x_n, A^{-1}(0)) \leq \frac{(1 - q)a_n}{4} q^{n-N} d(x_N, A^{-1}(0)) \quad (4.7)$$

for each $n > N$. Therefore $d(x_n, A^{-1}(0)) \rightarrow 0$ as $n \rightarrow +\infty$ and by closedness of $A^{-1}(0)$, $\bar{x} \in A^{-1}(0)$.

Fix $n \geq N$ and let $z = P_{A^{-1}(0)}(x_n)$. Then we have

$$\begin{aligned} d(y_n, z) &\leq d(x_n, z) \\ &= d(x_n, A^{-1}(0)). \end{aligned}$$

Thus, using $d(x_{n+1}, y_n) \leq \epsilon_n$ and $\delta_n = \frac{\epsilon_n}{d(x_{n+1}, x_n)}$ we conclude that

$$\begin{aligned} d(x_{n+1}, z) &\leq d(x_{n+1}, y_n) + d(y_n, z) \\ &\leq \delta_n d(x_{n+1}, x_n) + d(x_n, z). \end{aligned}$$

Hence,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, z) + d(x_n, z) \\ &\leq \delta_n d(x_{n+1}, x_n) + 2d(x_n, z). \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{2d(x_n, z)}{1 - \delta_n} \\ &= \frac{2}{1 - \delta_n} d(x_n, A^{-1}(0)) \\ &\leq 4d(x_n, A^{-1}(0)). \end{aligned}$$

Combining this together with 4.7, we get

$$\frac{1}{4}d(x_{n+1}, x_n) \leq d(x_n, A^{-1}(0)) \leq \frac{(1-q)a_n}{4} q^{n-N} d(x_N, A^{-1}(0))$$

that is

$$d(x_{n+1}, x_n) \leq (1-q)a_n q^{n-N} d(x_N, A^{-1}(0))$$

for each $n > N$. □

Theorem 4.3. *Suppose that A is globally metrically subregular, and let $\{x_n\}$ be a sequence generated by the Algorithm 1.1 such that 4.5 holds. Then $\{x_n\}$ converges linearly to some point $\bar{x} \in A^{-1}(0)$. More precisely, there exists $q \in [\frac{1}{2}, 1)$ and a bounded sequence $\{a_n\} \subset (0, +\infty)$ satisfying*

$$\limsup_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} \leq \frac{k \limsup_{n \rightarrow +\infty} \lambda_n}{q}$$

such that $d(x_n, \bar{x}) \leq a_n q^{n-N} d(x_0, A^{-1}(0))$ for every $n \geq 1$. In particular, the sequence $\{x_n\}$ converges superlinearly to $\bar{x}$ whenever either $\lim_{n \rightarrow \infty} \lambda_n = 0$ or $sreg_{\bar{x}}^A = 0$, provided that $A^{-1}(0)$ is a singleton.

Proof. By Lemma 4.2, (applied to some $k > 0$ and $U := X$) the sequence $\{x_n\}$ converges to $\bar{x} \in A^{-1}(0)$. Letting $m \rightarrow +\infty$ to 4.6, we have

$$d(x_n, \bar{x}) \leq a_n q^{n-N} d(x_N, A^{-1}(0)) \text{ for any } n > N.$$

Also, by Lemma 4.1,

$$\frac{d(x_{n+1}, A^{-1}(0))}{d(x_n, A^{-1}(0))} \leq \left(\frac{\delta_n}{1 - \delta_n} + \sqrt{\frac{\lambda_n^2 k^2}{1 + \lambda_n^2 k^2}} \right).$$

Therefore, the sequence $\{x_n\}$ converges superlinearly to $\bar{x}$ whenever either $\lim_{n \rightarrow \infty} \lambda_n = 0$ or $sreg_{\bar{x}}^A = 0$, provided that $A^{-1}(0)$ is a singleton. □

5 Normal Cone and Weak Sharp Minima-type Condition

The aim of this section is to introduce the concept of weak sharp solutions for the problem of finding singularities of monotone operators in Hadamard spaces and to study the finite convergence behaviour of inexact proximal point algorithms under the weak sharpness property of the solution set. In doing so, we extend several classical results from the linear framework (see [11, 12]) and manifold setting (see [30, 36]) to the more general setting of Hadamard spaces.

Problem: Let $A : X \rightrightarrows TX$ be a (set-valued) monotone operator. Then consider the problem

$$0_x \in A(x) \tag{5.1}$$

where $0_x := 0\mu = t\mu_{x,x}$ (here, $\mu \in S_x$, the space of directions at x and $t > 0$).

Finite Termination: We say a sequence $\{x_n\}$ terminates in a finite number of steps if there exists a positive integer N such that $x_n = x_{n+1}$ and $x_n \in A^{-1}(0_{x_n})$ for all $n \geq N$.

The approximation of a smooth function and a smooth manifold by a linear function and a linear subspace, respectively, has historically been a key concept in smooth analysis. A family of linear functions approximates a function in the nonsmooth situation, whereas cones approximate a set.

Normal Cone: Let C be a nonempty, closed and convex subset of a Hadamard space X . Consider the indicator function

$$\delta_C : X \rightarrow [0, +\infty] \quad (5.2)$$

defined by

$$\begin{aligned} \delta_C(x) &= 0 \quad \text{if } x \in C \\ &= +\infty \quad \text{if } x \notin C. \end{aligned}$$

Then the normal cone to C at x is given by

$$\begin{aligned} N_C(x) &:= \partial \delta_C(x) \\ &:= \{r\mu_{x,z} \in T_x X : g_x(r\mu_{x,z}, d(x, y)\mu_{x,y}) \leq 0 \text{ for all } y \in C\} \end{aligned}$$

for some $z \in X$.

Proposition 5.1. [29] *If x is in the interior of C , then $N_C(x) = \{0_x\}$.*

More results on normal cones can be found in [29].

Definition 5.1. *Let $\bar{x} \in C$ and let $r, \alpha > 0$. We say that Problem 5.1 satisfies:*

- **weak sharp minima-type condition on $B(\bar{x}, r)$ with modulus α if**

$$\overline{B(0_z, \alpha)} \cap N_C(z) \subseteq A(z)$$

for each $z \in B(\bar{x}, r) \cap C$.

- **the local weak sharp minima-type condition** *if, for every $\bar{z} \in A^{-1}(0)$, there exists $r_{\bar{z}} > 0$ such that Problem 5.1 satisfies the weak sharp minima-type condition on $B(\bar{z}, r_{\bar{z}})$.*
- **the global weak sharp minima-type condition with modulus α** *if Problem 5.1 satisfies the weak sharp minima-type condition on the whole space X .*

Remark 2. *Let $C = A^{-1}(0)$. Suppose that Problem 5.1 satisfies a weak sharp minima-type condition on $B(\bar{x}, r)$. Then there exists $r_{\bar{x}} < \min\{1, r\}$ such that the operator A is metrically subregular on $B(\bar{x}, r_{\bar{x}})$; that is*

$$d(x, A^{-1}(0_x)) \leq \frac{1}{\alpha} d_x(0_x, A(x))$$

for every $x \in B(\bar{x}, r_{\bar{x}})$.

Proof. Let $x \in B(\bar{x}, r_{\bar{x}})$. Without loss of generality, we may assume that $d(x, C) > 0$.

Set $z = P_C(x)$. Then

- $d(x, z)\mu_{z,x} \in N_C(z)$, (by Theorem 2.9 (ii))
- $\|d(x, z)\mu_{z,x}\| = d(x, z) < 1$

implies

$$\alpha d(x, z) \mu_{z, x} \in \overline{B(0_z, \alpha)} \cap N_C(z).$$

Therefore

$$\alpha d(x, z) \mu_{z, x} \in A(z).$$

Since A is monotone, it follows that

$$g_z(\alpha d(x, z) \mu_{z, x}, \mu_{z, x}) + g_x(x^*, \mu_{x, z}) \leq 0$$

for any $x^* := td(x, p) \mu_{x, p} \in A(x)$.

By Proposition 2.3

$$\langle \alpha \vec{zx}, \vec{zx} \rangle + \langle t \vec{xp}, \vec{xp} \rangle \leq 0.$$

Implies

$$\alpha \langle \vec{zx}, \vec{zx} \rangle \leq t \langle \vec{xp}, \vec{xp} \rangle.$$

By Cauchy-Schwarz inequality

$$\begin{aligned} \alpha d^2(z, x) &\leq td(x, p)d(z, x), \\ \alpha d(z, x) &\leq d_x(0_x, x^*). \end{aligned}$$

Hence,

$$d(A^{-1}(0_x), x) \leq \frac{1}{\alpha} d_x(0_x, A(x)) \quad \forall x \in B(\bar{x}, r_{\bar{x}}).$$

□

Lemma 5.1. *Let $\bar{x} \in C$ and $\alpha > 0$, and let $r \in (0, +\infty]$. Assume that the Problem 5.1 satisfies a weak sharp minima-type condition on $B(\bar{x}, r)$ with modulus α . If $y \in B(\bar{x}, r)$, $t > 0$ and $z \in X$ satisfy $td(y, z) \mu_{y, z} \in A(y)$ and $td(y, z) < \alpha$ then $y \in C$.*

Proof. Suppose $y \notin C$.

Let $\bar{y} = P_C(y)$, then

$$g_{\bar{y}}(\mu_{\bar{y}, y}, \mu_{\bar{y}, x}) \leq 0 \quad \forall x \in C.$$

This implies

$$g_{\bar{y}}(\mu_{\bar{y}, y}, \mu_{\bar{y}, \bar{x}}) \leq 0 \text{ as } \bar{x} \in C.$$

In addition,

$$d(\bar{y}, y)d(\bar{y}, \bar{x})g_{\bar{y}}(\mu_{\bar{y}, y}, \mu_{\bar{y}, \bar{x}}) \leq 0.$$

By Proposition 2.3

$$\langle \vec{\bar{y}y}, \vec{\bar{y}x} \rangle \leq g_{\bar{y}}(d(\bar{y}, y) \mu_{\bar{y}, y}, d(\bar{y}, \bar{x}) \mu_{\bar{y}, \bar{x}}) \leq 0.$$

Consider

$$2\langle \vec{\bar{y}y}, \vec{\bar{y}x} \rangle = d^2(\bar{y}, \bar{x}) + d^2(y, \bar{y}) - d^2(y, \bar{x}),$$

implies

$$\begin{aligned} d^2(y, \bar{x}) &= d^2(\bar{y}, \bar{x}) + d^2(y, \bar{y}) - 2\langle \vec{\bar{y}y}, \vec{\bar{y}x} \rangle \\ &\geq d^2(\bar{y}, \bar{x}). \end{aligned}$$

Therefore,

$$d(\bar{y}, \bar{x}) \leq d(y, \bar{x}) < r.$$

Hence $\bar{y} \in B(\bar{x}, r) \cap C$.

Write $w = \frac{\alpha}{d(\bar{y}, y)} d(\bar{y}, y) \mu_{\bar{y}, y}$. Then w is well-defined as $y \notin C$ by assumption.

So $\|w\| = \alpha$, and therefore $w \in \overline{B(0_{\bar{y}}, \alpha)} \cap N_C(\bar{y})$.

Since Problem 5.1 satisfies a weak sharp minima-type condition on $B(\bar{x}, r)$ (with modulus α), therefore

$$w \in A(\bar{y}).$$

From monotonicity of A

$$\begin{aligned} \langle \frac{\alpha}{d(\bar{y}, y)} \vec{\bar{y}y}, \vec{\bar{y}y} \rangle + \langle t\vec{yz}, \vec{\bar{y}y} \rangle &\leq 0, \\ \frac{\alpha}{d(\bar{y}, y)} d^2(\bar{y}, y) &\leq td(y, z) d(\bar{y}, y), \\ td(y, z) &\geq \alpha, \text{ a contradiction.} \end{aligned}$$

□

Lemma 5.2. *Let $\bar{x} \in C$ and $\alpha > 0$, and $r \in (0, +\infty]$. Assume that the Problem 5.1 satisfies a weak sharp minima-type condition on the ball $B(\bar{x}, r)$ with modulus α . Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by the Algorithm 1.1, and suppose*

$$\limsup_{n \rightarrow +\infty} \delta_n < 1 \text{ and } \liminf_{n \rightarrow +\infty} \lambda_n d(x_n, x_{n+1}) = 0. \quad (5.3)$$

Assume further that there exists $N \in \mathbb{N}$ such that $\{y_n : n \geq N\} \subseteq B(\bar{x}, r)$. If $A(x)$ is a singleton for every $x \in C$, then the sequence $\{x_n\}$ terminates after finitely many steps.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by the Algorithm 1.1. Then

$$d(x_{n+1}, y_n) \leq \epsilon_n = \delta_n d(x_{n+1}, x_n),$$

implies

$$\begin{aligned} (1 - \delta_n) d(x_{n+1}, x_n) &\leq d(x_{n+1}, x_n) - d(x_{n+1}, y_n) \\ &\leq d(x_n, y_n) \\ &\leq d(x_{n+1}, x_n) + d(x_{n+1}, y_n) \\ &\leq (1 + \delta_n) d(x_{n+1}, x_n). \end{aligned}$$

This together with 5.3 implies that

$$\liminf_{n \rightarrow +\infty} \lambda_n d(x_n, y_n) = 0. \quad (5.4)$$

Then, there exists a subsequence $\{n_k\}$ of the sequence $\{n\}$ such that

$$\lim_{k \rightarrow +\infty} \lambda_{n_k} d(x_{n_k}, y_{n_k}) = 0.$$

Thus, there exists a positive integer $K \geq N$ such that for all $k \geq K$

$$y_{n_k} \in B(\bar{x}, r) \text{ and } \lambda_{n_k} d(x_{n_k}, y_{n_k}) < \alpha.$$

As $\lambda_{n_k} d(y_{n_k}, x_{n_k}) \mu_{y_{n_k}, x_{n_k}} \in A(y_{n_k})$, therefore, by previous Lemma 5.1

$$y_{n_k} \in C \quad \forall k \geq K.$$

Thus, there exists $\tilde{N} \geq n_k$ such that $y_{\tilde{N}} \in C$. By assumption $A(y_{\tilde{N}})$ is a singleton. Therefore, $0 = A(y_{\tilde{N}}) = \lambda_{\tilde{N}} d(y_{\tilde{N}}, x_{\tilde{N}}) \mu_{y_{\tilde{N}}, x_{\tilde{N}}}$, that is, $x_{\tilde{N}} = y_{\tilde{N}}$ as $\lambda_{\tilde{N}} > 0$. Thus, $x_{\tilde{N}} \in C$.

Below we show that $x_{n+1} = x_n$ for all $n \geq \tilde{N}$.

To proceed, we fix $n \geq \tilde{N}$.

Without loss of generality, we assume that $x_n \in C(= A^{-1}(0))$ and $\delta_n < 1$.

- By algorithm: $\lambda_n d(y_n, x_n) \mu_{y_n, x_n} \in A(y_n)$.
- By assumption: $0_{x_n} \in Ax_n$.

From monotonicity of A

$$g_{x_n}(0_{x_n}, \mu_{x_n, y_n}) + g_{y_n}(\lambda_n d(y_n, x_n) \mu_{y_n, x_n}, \mu_{y_n, x_n}) \leq 0,$$

implies

$$\begin{aligned} \lambda_n d(y_n, x_n) &\leq 0, \\ d(y_n, x_n) &= 0, \\ x_n &= y_n. \end{aligned}$$

Thus, we have $d(x_{n+1}, x_n) = d(x_{n+1}, y_n) \leq \epsilon_n$.

Then it follows that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{\epsilon_n}{d(x_{n+1}, x_n)} d(x_{n+1}, x_n) \\ &= \delta_n d(x_{n+1}, x_n). \end{aligned}$$

This, together with the assumption that $\delta_n < 1$, implies that $x_n = x_{n+1}$. □

Theorem 5.3. *Suppose that Problem 5.1 satisfies the global weak sharp minima-type condition with modulus $\alpha > 0$. Let $\{x_n\}$ be a sequence generated by the Algorithm 1.1. Assume that*

$$\limsup_{n \rightarrow +\infty} \delta_n < 1 \text{ and } \liminf_{n \rightarrow +\infty} \lambda_n d(x_n, x_{n+1}) = 0 \text{ hold.} \quad (5.5)$$

If $A(x)$ is a singleton for each $x \in C$, then the sequence $\{x_n\}$ terminates after finitely many iterations.

Proof. For any $\bar{x} \in C$, Problem 5.1 satisfies a weak sharp minima-type condition on $B(\bar{x}, r)$ (for any $r > 0$) with a uniform modulus α . Then, the conclusion follows from the previous lemma, Lemma 5.2. □

6 Application to Generalized Fermat-Torricelli Problem

First, we frame a new generalized Fermat-Torricelli problem in Hadamard spaces (the linear version of it can be found in [41]) as the minimization of a weighted sum of distances to convex sets through geodesic-preserving mappings. This model includes classical Fermat-Torricelli, convex feasibility, and best approximation problems as special cases. The section proves convexity and the existence of minimizers under suitable coercivity assumptions.

Problem: Let X, Y_i be Hadamard spaces for $i = 1, 2, \dots, m$ and $C \subseteq X, Q_{ij} \subseteq Y_i$ be nonempty, closed and convex subsets for $i = 1, 2, \dots, m; j = 1, 2, \dots, m_i$. Also, let $T_i : X \rightarrow Y_i$ be continuous affine maps for $i = 1, 2, \dots, m$, that is, for any $x, y \in X$ and $t \in [0, 1]$

$$T_i((1-t)x \oplus ty) = (1-t)T_i(x) \oplus tT_i(y)$$

for each i .

Let

$$f(x) = \sum_{i=1}^m \sum_{j=1}^{m_i} a_{ij} d(T_i(x), P_{Q_{ij}}^{Y_i}(T_i(x))), \quad (6.1)$$

where a_{ij} are positive real numbers and $P_{Q_{ij}}^{Y_i} : Y_i \rightarrow Q_{ij}$ are the metric projection of Y_i onto Q_{ij} for each i and j .

Our aim is to solve the constrained minimization problem

$$\min_{x \in C} f(x).$$

Further, the corresponding unconstrained minimization problem can be expressed as

$$\min_{x \in X} (f(x) + \delta_C(x)),$$

where δ_C is the indicator function defined in 5.2.

Some specific instances of this generalized Fermat-Torricelli problem are described below.

• **Weighted Fermat-Torricelli Problem:** If we choose

- $C = X = Y_i$ for each i ,
- $T_i(x) = x$ for each i and for all $x \in X$,
- $m_i = 1 \ \forall i$,
- $Q_{ij} = \{x_i\}$ for each i ,

in 6.1, we have the weighted Fermat-Torricelli problem

$$\min_{x \in X} \sum_{i=1}^m a_i d(x, x_i). \quad (6.2)$$

• **Convex Feasibility Problem:** For the following choices

- $C = X = Y_i$ for each i ,

made in 6.1, we have

$$\min_{x \in X} \sum_{i,j} a_{ij} d(T_i(x), P_{Q_{ij}}^X(T_i(x))). \quad (6.3)$$

Note that $minimum = 0$ if and only if $T_i(x) = P_{Q_{ij}}(T_i(x))$ for each i if and only if $T_i(x) \in \bigcap_j Q_{ij}$ for each i .

• **Best approximation:** Taking one pair from 6.3 with the particular choices

- $C = X = Y_i$,
- $a_i = 1$,

we get

$$\min_{x \in X} d(T(x), P_Q^X(T(x))). \quad (6.4)$$

Recall that the class of proper, convex, lower semicontinuous functions on X , was represented by Γ_0 .

Proposition 6.1. *The function $f : X \rightarrow (-\infty, +\infty]$ defined by 6.1 is in Γ_0 .*

Proof. Properness follows as f is the finite sum of nonnegative terms.

Lower semicontinuity of f can be concluded from the following facts (in fact, we prove f is continuous):

- projection onto a closed convex subset of a Hadamard space is nonexpansive, that is,

$$d(P_{Q_{ij}}^{Y_i}(y_1), P_{Q_{ij}}^{Y_i}(y_2)) \leq d(y_1, y_2)$$

for each $y_1, y_2 \in Y_i$;

- affine maps T_i 's between CAT(0) spaces are continuous by assumption;
- distance is continuous on CAT(0) spaces.

For convexity it is enough to check $g_{ij}(x) := d(T_i(x), P_{Q_{ij}}^{Y_i}(T_i(x)))$ is convex for each i and j .

Note that $d(y, P_{Q_{ij}}^{Y_i}(y)) = d(y, Q_{ij})$, implies $g_{ij}(x) = d(T_i(x), Q_{ij})$.

So we must check the convexity of

$$x \mapsto d(T_i(x), Q_{ij}).$$

For any $x_1, x_2 \in X$ and $t \in [0, 1]$, consider

$$\begin{aligned} g_{ij}((1-t)x_1 \oplus tx_2) &= d((1-t)T_i(x_1) \oplus tT_i(x_2), Q_{ij}) \\ &\leq (1-t)d(T_i(x_1), Q_{ij}) + td(T_i(x_2), Q_{ij}) \\ &= (1-t)g_{ij}(x_1) + tg_{ij}(x_2), \end{aligned}$$

for each i and j . □

A sufficient condition for the existence of a minimizer of the Fermat-Torricelli problem is established below.

Theorem 6.1. *Consider the Problem 6.1 and assume that there exists i_0 such that*

(F1) *for some $x_0 \in X$,*

$$d(x_n, x_0) \rightarrow +\infty \implies d(T_{i_0}(x_n), T_{i_0}(x_0)) \rightarrow +\infty,$$

(F2) *for every sequence $\{y_n\} \subseteq Y_{i_0}$,*

$$d(y_n, y_0) \rightarrow +\infty \implies \max_{1 \leq j \leq m_{i_0}} d(y_n, Q_{ij}) \rightarrow +\infty,$$

for some $y_0 \in Y_{i_0}$.

Then, f is coercive on X , that is,

$$d(x_n, x_0) \rightarrow +\infty \implies f(x_n) \rightarrow +\infty,$$

which further implies the existence of a minimizer of the Fermat-Torricelli problem.

Proof. Let sequence $\{x_n\} \subseteq X$ such that $d(x_n, x_0) \rightarrow +\infty$.

From 6.1, we have

$$f(x_n) \geq \sum_{j=1}^{m_{i_0}} a_{i_0j} d(T_{i_0}(x_n), Q_{i_0j}) \geq \min_j a_{i_0j} \cdot \max_j d(T_{i_0}(x_n), Q_{i_0j})$$

Thus, it suffices to prove that

$$\max_j d(T_{i_0}(x_n), Q_{i_0j}) \rightarrow +\infty.$$

Apply (F2) to $y_n := T_{i_0}(x_n)$, and get f is coercive.

Therefore, by applying Lemma 2.2.19 [1] we ensure the existence of a minimizer. $\square$

Now we study the inexact cyclic splitting proximal point algorithms (SPPA) and establish linear and superlinear convergence results using metric subregularity assumptions. Under stronger conditions, finite termination of the algorithm is obtained. These results are also extended to a double-sum version of the SPPA, which are required to solve the proposed Problem 6.1.

6.1 Inexact Cyclic SPPA

For a given $x_0 \in X$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by the algorithm

$$\begin{aligned} y_n &= J_{r_n}^{f_m} \circ \dots \circ J_{r_n}^{f_1}(x_n); \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned}$$

by letting $n = 0, 1, 2, \dots$

Note that $x_n^{(0)} := x_n$, $x_n^{(i)} := J_{r_n}^{f_i}(x_n^{(i-1)})$, $i = 1, \dots, m$, $y_n := x_n^{(m)}$.

Lemma 6.2. *Let (X, d) be a Hadamard space and $f = \sum_{i=1}^m f_i$, where each f_i is in Γ_0 with $C := \operatorname{argmin} f \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$ be generated by the Algorithm 1.2 and define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$. Suppose, there exists $k \in [0, +\infty)$, $N \in \mathbb{N}$, and $U \subseteq X$ such that*

- ∂f_m is metrically subregular on U with modulus $k > 0$,
- $\{y_n : n \geq N\} \subseteq U$,
- $d(x, C) \leq r d(x, C_m)$, $\forall x \in U$, for some $r > 0$, where $C_m := (\partial f_m)^{-1}(0)$,
- $\sup_{n \geq N} \delta_n < 1$.

Then, for each $n \geq N$,

$$d(x_{n+1}, C) \leq \left(\frac{\delta_n}{1 - \delta_n} \sqrt{m} + \sqrt{\frac{\lambda_n^2 r^2 k^2}{1 + \lambda_n^2 r^2 k^2}} \right) d(x_n, C).$$

Proof. Let $z \in C$. For each proximal step $x_n^{(i)} = J_{r_n}^{f_i}(x_n^{(i-1)})$

$$d^2(x_n^{(i)}, z) + d^2(x_n^{(i)}, x_n^{(i-1)}) \leq d^2(x_n^{(i-1)}, z) \text{ (using Lemma 3.1(i)).}$$

Taking sum over $i = 1$ to m

$$d^2(y_n, z) + \sum_{i=1}^m d^2(x_n^{(i)}, x_n^{(i-1)}) \leq d^2(x_n, z). \quad (6.5)$$

Therefore,

$$\begin{aligned} d(y_n, C) &\leq d(x_n, C); \\ \sum_{i=1}^m d^2(x_n^{(i)}, x_n^{(i-1)}) &\leq d^2(x_n, C). \end{aligned} \quad (6.6)$$

By triangle inequality

$$d(x_n, y_n) \leq \sum_{i=1}^m d(x_n^{(i)}, x_n^{(i-1)}).$$

Further using Cauchy-Schwarz inequality

$$d(x_n, y_n) \leq \sqrt{m} d(x_n, C).$$

Therefore we have

$$d(x_{n+1}, y_n) \leq \frac{\delta_n}{1 - \delta_n} \sqrt{m} d(x_n, C).$$

Since ∂f_m is metrically subregular on U with modulus k

$$\begin{aligned} d(y_n, C_m) &\leq k d_{y_n}(0, \partial f_m(y_n)) \\ &= k \lambda_n d(y_n, x_n^{(m-1)}). \end{aligned} \quad (6.7)$$

Also from 6.5

$$d^2(y_n, C) + d^2(y_n, x_n^{(m-1)}) \leq d^2(x_n, C). \quad (6.8)$$

Combining 6.7 and 6.8 with the assumption $d(x, C) \leq r d(x, C_m)$, $\forall x \in U$ we have

$$d(y_n, C) \leq \sqrt{\frac{\lambda_n^2 r^2 k^2}{1 + \lambda_n^2 r^2 k^2}} d(x_n, C).$$

Lastly using

$$\begin{aligned} d(x_{n+1}, C) &\leq d(x_{n+1}, y_n) + d(y_n, C) \\ &\leq \frac{\delta_n}{1 - \delta_n} \sqrt{m} d(x_n, C) + \sqrt{\frac{\lambda_n^2 r^2 k^2}{1 + \lambda_n^2 r^2 k^2}} d(x_n, C). \end{aligned}$$

Hence

$$d(x_{n+1}, C) \leq \left(\frac{\delta_n}{1 - \delta_n} \sqrt{m} + \sqrt{\frac{\lambda_n^2 r^2 k^2}{1 + \lambda_n^2 r^2 k^2}} \right) d(x_n, C).$$

□

Theorem 6.3. Let (X, d) be a Hadamard space and $f = \sum_{i=1}^m f_i$, where each f_i is in Γ_0 with $C := \operatorname{argmin} f \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$ be generated by the Algorithm 1.2 and define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$. Suppose, there exists $k \in [0, +\infty), N \in \mathbb{N}$, and $U \subseteq X$ such that

- ∂f_m is metrically subregular on U with modulus $k > 0$,
- $\{y_n : n \geq N\} \subseteq U$,
- $d(x, C) \leq rd(x, C_m), \forall x \in U$, for some $r > 0$, where $C_m := (\partial f_m)^{-1}(0)$,
- $\sup_{n \geq N} \delta_n < 1$,

and 4.5 holds. Then $\{x_n\}$ converges linearly to a point $\bar{x} \in C$. In particular, the sequence $\{x_n\}$ converges superlinearly to $\bar{x}$ whenever either $\lim_{n \rightarrow +\infty} \lambda_n = 0$ or $k = 0$ or $r = 0$, provided that C is a singleton.

Proof. The proof mimics Lemma 4.2. □

Theorem 6.4. Let (X, d) be a Hadamard space and $f = \sum_{i=1}^m f_i$, where each f_i is in Γ_0 with $C := \operatorname{argmin} f \neq \emptyset$ and $\bar{x} \in C$. Let the sequences $\{x_n\}, \{y_n\}$ be generated by the Algorithm 1.2.

Define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$ and assume

- there exists $\alpha > 0$ and $r > 0$ such that $\overline{B(0_z, \alpha)} \cap N_{C_m}(z) \subseteq \partial f_m(z)$ for all $z \in B(\bar{x}, r) \cap C_m$,
- $\{y_n : \text{for sufficiently large } n\} \subseteq B(\bar{x}, r)$
- there exists $t > 0$ such that $d(x, C) \leq td(x, C_m), \forall x \in B(\bar{x}, r)$, where $C_m := (\partial f_m)^{-1}(0)$,
- ∂f_m is singleton for each $x \in B(\bar{x}, r) \cap C_m$,
- $\limsup_{n \rightarrow +\infty} \delta_n < \frac{1}{2}$ and $\liminf_{n \rightarrow +\infty} \lambda_n d(x_n, x_{n+1}) = 0$

such that 4.5 holds. Then $\{x_n\}$ terminates in a finite number of iterations.

Proof. By assumption, $y_n \in B(\bar{x}, r)$ for sufficiently large n .

Again, from 5.4

$$\liminf_{n \rightarrow +\infty} \lambda_n d(x_n, y_n) = 0,$$

then there exists a subsequence $\{n_k\}$ such that

$$\lambda_{n_k} d(x_{n_k}, y_{n_k}) \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Choose $z = P_C(x_n)$ for a fix n .

By Theorem 6.3

$$d(x_n, z) = d(x_n, C) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence, by 6.6

$$\sum_{i=1}^m d^2(x_n^{(i)}, x_n^{(i-1)}) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Therefore $\forall i$

$$d(x_n^{(i)}, x_n^{(i-1)}) \rightarrow 0. \tag{6.9}$$

In particular

$$d(y_n, x_n^{(m-1)}) \rightarrow 0.$$

Since λ_n is bounded

$$\lambda_n d(y_n, x_n^{(m-1)}) \rightarrow 0.$$

Thus, along the subsequence $\{n_k\}$

$$\lambda_{n_k} d(y_{n_k}, x_{n_k}^{(m-1)}) < \alpha \text{ for sufficiently large } k.$$

Also, from the last proximal step

$$\lambda_{n_k} d(y_{n_k}, x_{n_k}^{(m-1)}) \mu_{y_{n_k}, x_{n_k}^{(m-1)}} \in \partial f_{y_{n_k}}.$$

Therefore, by Lemma 5.1

$$y_{n_k} \in C_m \quad \forall k \geq K,$$

for some $K \in \mathbb{N}$.

Thus, there exists $\tilde{N} \geq n_k$ such that $y_{\tilde{N}} \in B(\bar{x}, r) \cap C_m$. By assumption $\partial f_m(y_{\tilde{N}})$ is a singleton. Therefore, $0 = \partial f_m(y_{\tilde{N}}) = \lambda_{\tilde{N}} d(y_{\tilde{N}}, x_{\tilde{N}}^{(m-1)}) \mu_{y_{\tilde{N}}, x_{\tilde{N}}^{(m-1)}}$, that is, $x_{\tilde{N}}^{(m-1)} = y_{\tilde{N}}$ as $\lambda_{\tilde{N}} > 0$.

Thus, $x_{\tilde{N}}^{(m-1)} \in C_m$.

Below we show that $x_{n+1}^{(m-1)} = x_n^{(m-1)}$ for all $n \geq \tilde{N}$.

To proceed, we fix $n \geq \tilde{N}$.

Without loss of generality, we assume that $x_n^{(m-1)} \in C_m (= \partial f_m^{-1}(0))$ and $\delta_n < 1$.

- By algorithm: $\lambda_n d(y_n, x_n^{(m-1)}) \mu_{y_n, x_n^{(m-1)}} \in \partial f_m(y_n)$.
- By assumption: $0_{x_n^{(m-1)}} \in \partial f_m x_n^{(m-1)}$.

From monotonicity of ∂f_m

$$g_{x_n^{(m-1)}}(0_{x_n^{(m-1)}}, \mu_{x_n^{(m-1)}}, y_n) + g_{y_n}(\lambda_n d(y_n, x_n^{(m-1)}) \mu_{y_n, x_n^{(m-1)}}, \mu_{y_n, x_n^{(m-1)}}) \leq 0,$$

implies

$$\begin{aligned} \lambda_n d(y_n, x_n^{(m-1)}) &\leq 0, \\ d(y_n, x_n^{(m-1)}) &= 0, \\ x_n^{(m-1)} &= y_n. \end{aligned}$$

Thus, we have $d(x_{n+1}, x_n) \leq d(x_{n+1}, y_n) + d(y_n, x_n) = \varepsilon_n + d(x_n^{(m-1)}, x_n) \leq 2\varepsilon_n$ (using 6.9).

Then it follows that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{2\varepsilon_n}{d(x_{n+1}, x_n)} d(x_{n+1}, x_n) \\ &= 2\delta_n d(x_{n+1}, x_n). \end{aligned}$$

This, together with the assumption that $\delta_n < \frac{1}{2}$, implies that $x_n = x_{n+1}$. □

6.2 Double Sum Inexact Cyclic SPPA

For a given $x_0 \in X$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by the algorithm

$$\begin{aligned} x_n^{(i,0)} &:= x_n^{(i-1)}, \quad x_n^{(0)} := x_n; \\ x_n^{(i,j)} &:= J_{r_n}^{f_{i,j}}(x_n^{(i,j-1)}), \quad j = 1, 2, \dots, m_i; \\ y_n &:= x_n^{m, m_m}; \\ d(x_{n+1}, y_n) &\leq \epsilon_n, \end{aligned}$$

by letting $n = 0, 1, 2, \dots$

Define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$.

Theorem 6.5. *Let (X, d) be a Hadamard space and $f = \sum_{i=1}^m \sum_{j=1}^{m_i} f_{i,j}$, where each $f_{i,j}$ is in Γ_0 with $C := \operatorname{argmin} f \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\}$ be generated by the Algorithm 1.3 and define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$. Assume*

- $\partial f_{m, m_m}$ is metrically subregular on U with modulus $k > 0$,
- $\{y_n : n \geq N\} \subseteq U$,
- there exists $r > 0$ such that $d(x, C) \leq rd(x, C_{m, m_m})$, $\forall x \in U$, where $C_{m, m_m} := (\partial f_{m, m_m})^{-1}(0)$,
- $\sup_{n \geq N} \delta_n < 1$,

for some $N \in \mathbb{N}$ such that 4.5 holds. Then $\{x_n\}$ converges linearly to a point $\bar{x} \in C$. In particular, the sequence $\{x_n\}$ converges superlinearly to $\bar{x}$ whenever either $\lim_{n \rightarrow +\infty} \lambda_n = 0$ or $k = 0$ or $r = 0$, provided that C is a singleton.

Proof. The proof is identical to the proof of Theorem 6.3. □

Theorem 6.6. *Let (X, d) be a Hadamard space and $f = \sum_{i=1}^m \sum_{j=1}^{m_i} f_{i,j}$, where each $f_{i,j}$ is in Γ_0 with $C := \operatorname{argmin} f \neq \emptyset$ and $\bar{x} \in C$. Let the sequences $\{x_n\}, \{y_n\}$ be generated by the Algorithm 1.3. Define $\delta_n := \frac{\epsilon_n}{d(x_{n+1}, x_n)}$ and assume*

- there exists $\alpha > 0$ and $r > 0$ such that $\overline{B(0_z, \alpha)} \cap N_{C_{m, m_m}}(z) \subseteq \partial f_{m, m_m}(z)$ for all $z \in B(\bar{x}, r) \cap C_{m, m_m}$,
- $\{y_n : \text{for sufficiently large } n\} \subseteq B(\bar{x}, r)$
- there exists $t > 0$ such that $d(x, C) \leq td(x, C_{m, m_m})$, $\forall x \in B(\bar{x}, r)$, where $C_{m, m_m} := (\partial f_{m, m_m})^{-1}(0)$,
- $\partial f_{m, m_m}$ is singleton for each $x \in B(\bar{x}, r) \cap C_{m, m_m}$,
- $\limsup_{n \rightarrow +\infty} \delta_n < \frac{1}{2}$ and $\liminf_{n \rightarrow +\infty} \lambda_n d(x_n, x_{n+1}) = 0$

such that 4.5 holds. Then $\{x_n\}$ terminates in a finite number of iterations.

Proof. The proof is the same as that of Theorem 6.4. □

For our numerical investigations, the next subsection will be useful.

6.3 Metric Subregularity of the Subdifferential

A Hadamard space X is required to satisfy the following geodesic extension property in order to define a negative geodesic on it: For each unit speed geodesic $\mu : [0, l] \rightarrow X$, if there is an isometry $\gamma : \mathbb{R} \rightarrow X$ such that $\mu(t) = \gamma(t)$ whenever $t \in [0, l]$, we say that the Hadamard space X has the geodesic extension property.

Definition 6.1. Assume the Hadamard space X has the geodesic extension property. Let $x_0, x \in X$. Then there exists a point $y \in X$ (not necessarily unique) such that $\mu_{x,y}(\frac{1}{2}) = x_0$. The negative geodesic of $\mu_{x_0,x}$ is then defined by $-\mu_{x_0,x} := \mu_{x_0,y}$.

Proposition 6.2. [13] Let X be a Hadamard space with the geodesic extension property. Then

$$g_{x_0}(-\mu_{x_0,x}, \mu_{x_0,y}) \leq -g_{x_0}(\mu_{x_0,x}, \mu_{x_0,y})$$

for any $x_0, x, y \in X$.

In a Hadamard space X with geodesic extension property we try to determine whether metric subregularity is satisfied by the subdifferential ∂f_i of each f_i in the weighted Fermat-Torricelli problem 6.2. First observe

$$\partial f_i(x) = \begin{cases} \{-a_i \mu_{x,x_i}\} & \text{for } x \neq x_i, \\ \frac{B(0_{x_i}, a_i)}{B(0_{x_i}, a_i)} & \text{for } x = x_i. \end{cases}$$

Metric subregularity check: Note that $0_x \in \partial f_i(x) \iff x = x_i$ implies $\partial f_i^{-1}(0) = \{x_i\}$.

Case(i): If $x \neq x_i$ then

$$d_x(0_x, \partial f_i(x)) = \|-a_i \mu_{x,x_i}\| = a_i.$$

So the required inequality becomes

$$d(x, x_i) \leq ka_i. \tag{6.10}$$

Case(ii): If $x = x_i$ then the required inequality holds trivially.

Take any $r > 0$, $U_i := B(x_i, r)$. Therefore, for any $x \in U_i$ if we choose $k = \frac{r}{a_i}$ 6.10 holds good. This implies the metric subregularity of the subdifferential ∂f_i on U_i .

7 Numerical Experiments

The final section addresses the generalized Fermat-Torricelli Problem 6.1 numerically. We take $X = \mathbb{R}_{++} := (0, +\infty)$ with $\langle u, v \rangle_x = \frac{1}{x^2}uv$; by [36] this is a Hadamard manifold of zero sectional curvature, and by [38] $(1-t)x \oplus ty = x^{1-t}y^t$, so (X, d) is a complete CAT(0) space with $d(x, y) = |\ln x - \ln y|$.

The weighted Fermat-Torricelli Problem 6.2 reads $\min_{x>0} \sum_{i=1}^m a_i |\ln x - \ln x_i|$.

Example 7.1. Take $m = 3$, $x_1 = e^{-2}$, $x_2 = e^3$, $x_3 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, so

$$f(x) = |\ln x + 2| + 2|\ln x - 3| + 4|\ln x|, \quad C = \{1\}.$$

We verify Theorem 6.3: (i) by Subsection 6.3, ∂f_3 is metrically subregular on $U = B(1, r)$ ($r > 0$) with $k = r/4$; (ii) since a_3 dominates, $C = \{1\} = C_3$, so $d(x, C) \leq \rho d(x, C_3)$ on U with $\rho = 1$; (iii) with $r_n = n + 2$, $\lambda_n = 1/(n + 2)$, $\delta_n = 1/(n + 2)^2$, $\sup_n \delta_n = 1/4 < 1$; (iv) choosing x_{n+1} on $[x_n, y_n]$ with $d(x_{n+1}, y_n) = \frac{\delta_n}{2} d(x_n, y_n)$ and $\epsilon_n = \delta_n d(x_{n+1}, x_n)$ gives $\epsilon_n/d(x_n, x_{n+1}) = \delta_n \rightarrow 0$ and $\sup_n \lambda_n \leq \frac{1}{2} < \infty$, so (4.5) holds; (v) $y_n \rightarrow 1$, so $\{y_n\} \subseteq U$ eventually.

The resolvents are

$$J_r^{f_i}(x) = \begin{cases} xe^{-a_i r}, & \ln x - \ln x_i > a_i r, \\ x_i, & |\ln x - \ln x_i| \leq a_i r, \\ xe^{a_i r}, & \ln x - \ln x_i < -a_i r, \end{cases}$$

and $y_n = J_{r_n}^{f_3} \circ J_{r_n}^{f_2} \circ J_{r_n}^{f_1}(x_n)$. Starting from $x_0 = e^5$, Table 1 lists the iterates, while Figure 1 illustrates their metric convergence and verifies the relevant parameter and inexactness conditions of Theorem 6.3. Since $\lambda_n \rightarrow 0$, Theorem 6.3 gives superlinear convergence to $x^* = 1$, confirmed by

$$x_{10} = 1.0000000000000000, \quad d(x_{10}, x^*) = 3.0644928445 \times 10^{-18}.$$

To validate Theorem 6.4, take the best approximation Problem 6.4.

Example 7.2. With $T(x) = x$, $Q = [2, +\infty)$: $f(x) = \max\{0, \ln 2 - \ln x\}$, $C = [2, +\infty)$. Take $\bar{x} = 3$, $r = 0.4$, so $B(\bar{x}, r) = (3e^{-0.4}, 3e^{0.4}) \subset (2, +\infty)$ and every $z \in B(\bar{x}, r) \cap C$ has $z > 2$.

We verify Theorem 6.4: (i) for $z > 2$, $z \in \text{int}(C)$ so $N_C(z) = \{0_z\}$, and f is locally 0 there so $\partial f(z) = \{0_z\}$; hence $B(0_z, \alpha) \cap N_C(z) = \{0_z\} = \partial f(z)$ for all $\alpha > 0$ and $z \in B(\bar{x}, r) \cap C$; (ii) with $r_n = 0.5$, $\lambda_n = 2$, $\delta_n = 1/(n+2)^2$: $\limsup_n \delta_n = 0 < \frac{1}{2}$, $\sup_n \lambda_n = 2 < \infty$; (iii) $C = (\partial f)^{-1}(0)$ gives $d(x, C) \leq \rho d(x, (\partial f)^{-1}(0))$ with $\rho = 1$; (iv) $\partial f(z) = \{0_z\}$ is a singleton on $B(\bar{x}, r) \cap C$.

The resolvent is

$$J_r^f(x) = \begin{cases} xe^r, & x < 2e^{-r}, \\ 2, & 2e^{-r} \leq x \leq 2, \\ x, & x > 2. \end{cases}$$

From $x_0 = 0.5$, put $y_n = J_{r_n}^f(x_n)$, $D_n = d(x_n, y_n)$; for $D_n > 0$ choose x_{n+1} beyond y_n on the ray from x_n with $d(y_n, x_{n+1}) = \frac{\delta_n}{2(1-\delta_n)} D_n$ and $\epsilon_n = \delta_n d(x_n, x_{n+1})$, so $d(x_{n+1}, y_n) \leq \epsilon_n$ and $\epsilon_n/d(x_n, x_{n+1}) = \delta_n \rightarrow 0$.

Table 2 lists the iterates, while Figure 2 illustrates the finite termination of the sequence and verifies the relevant inexactness, parameter, and neighborhood conditions of Theorem 6.4. Since $d(y_3, \bar{x}) = 0.3964 \dots < 0.4$, $y_n \in B(\bar{x}, r)$ for $n \geq 3$; since $x_3 = 2.018196346599555 > 2$, $x_3 \in C$ and $J_{0.5}^f(x_3) = x_3$, so $x_n = x_3$ for all $n \geq 3$ and $\liminf_n \lambda_n d(x_n, x_{n+1}) = 0$. Hence all hypotheses of Theorem 6.4 hold and the sequence terminates finitely:

$$N = 3, \quad x_3 = 2.018196346599555 \in C.$$

Table 1: Iterations to the FT point (Ex. 7.1)

n	x_n
0	148.4131591025766
1	1.868245957432222
2	1.035332076609055
3	1.001085658345274
4	1.000021701624366
5	1.000000301408224
6	1.000000003075594
7	1.000000000024028
8	1.000000000000148
9	1.000000000000001
10	1.000000000000000

Table 2: Finite termination (Ex. 7.2)

n	x_n	y_n
0	0.5000000000000000	0.824360635350064
1	0.896000912827878	1.477255763546054
2	1.524148895228694	2.000000000000000
3	2.018196346599555	2.018196346599555

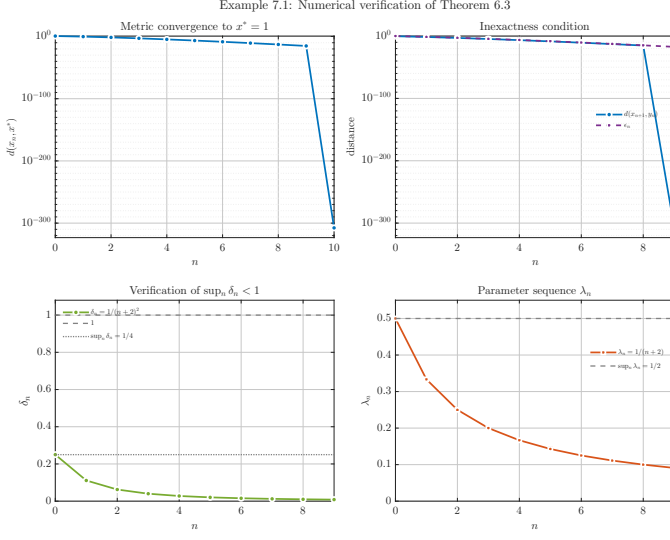


Figure 1: Numerical verification of Theorem 6.3 for Example 7.1.

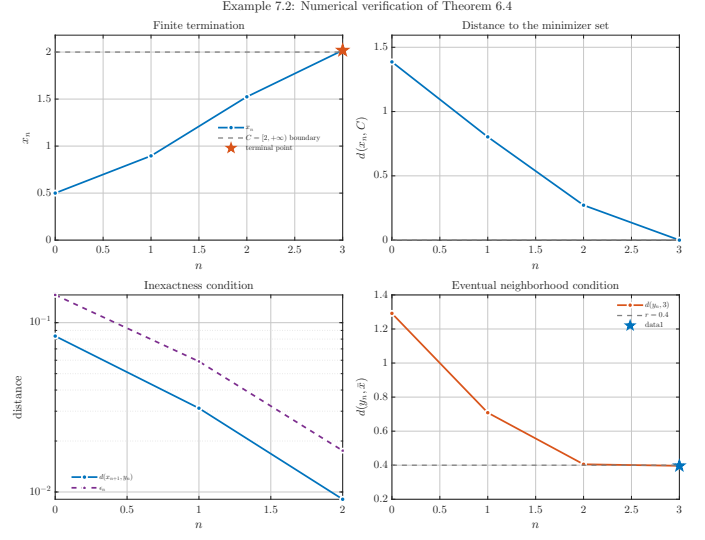


Figure 2: Numerical verification of Theorem 6.4 and finite termination for Example 7.2.

8 Conclusions and Remarks

Hadamard space extends Hilbert-space geometry to nonlinear settings while preserving key features such as geodesics, convexity, and unique metric projections, making it a natural framework for optimization beyond linear spaces. Motivated by this, we extend the generalized Fermat–Torricelli problem of [41] from Hilbert spaces to Hadamard spaces by replacing bounded linear operators with continuous geodesic-preserving mappings.

The formulation matters concretely: Problem (6.1) recovers all three sub-problems as special cases (equations (6.2)–(6.4)), so a single algorithm—the double-sum SPPA of Section 6.2 – covers tasks usually solved separately; and geodesic-preserving maps are the structural substitute required in the absence of a global linear structure, letting the convexity, coercivity, and existence-of-minimizer results of Section 6 go through, while the explicit resolvents of Section 6.3 keep the problem computationally tractable. Algorithmically, we developed an inexact double-sum cyclic SPPA with superlinear convergence under metric subregularity of $\partial f_{m,m_m}$ alone, and finite termination under a weak sharp minima-type condition; both were applied to approximate the Fermat–Torricelli point and confirmed numerically. Notably, the underlying Hadamard space need not be locally compact.

A natural direction for future work is to extend this framework to split feasibility problems with multiple output sets in Hadamard spaces.

Conflict of Interest

The authors declare that they have no conflict of interest.

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